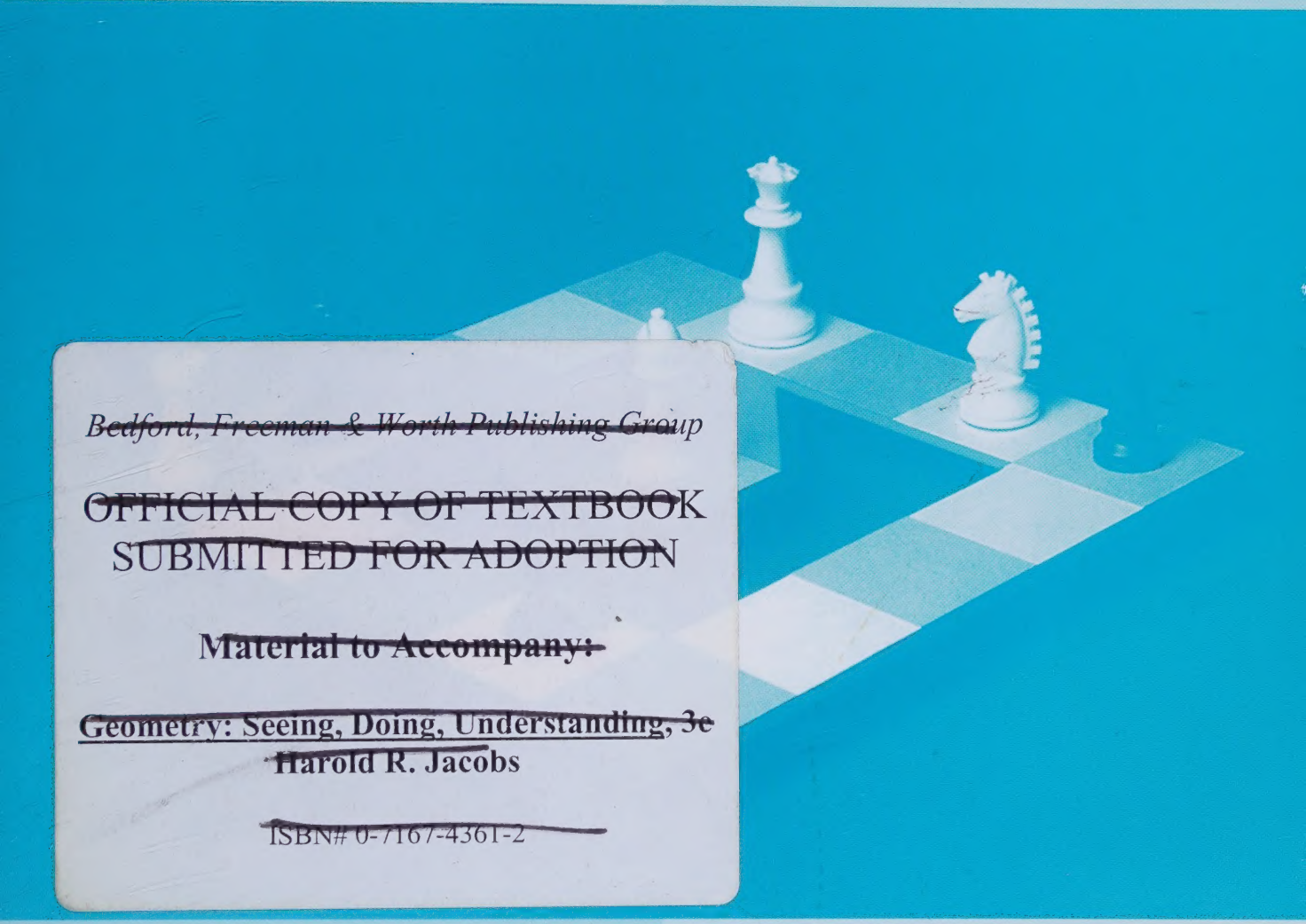
Teacher's Guide for

GEOMETRY

Seeing, Doing, Understanding

Third Edition



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Harold R. Jacobs

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Harold R. Jacobs

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INTRODUCTORY COMMENTS

Many years ago, Dr. Morris Kline wrote about teaching mathematics, and his thoughts still hold true today:

> I believe that the chief problem in teaching mathematics is motivation. It is difficult for beginners in mathematics . . . to see just why one should study the subject . . . . Vague answers, such as that scientists and engineers use it . . . or that it is needed to get into college, don't help at all. The motivation was sadly deficient in the traditional curriculum, and is entirely absent in the modern mathematics curriculum. By utilizing real problems chosen from the world in which the student lives and involving phenomena which he himself experiences we may be able to motivate the study of mathematics . . . .
>
> The method recommended to supply motivation . . . gives meaning to mathematics. We would all agree that we want our subject to make sense to the students. The road to making sense is through the senses, and this means physical problems and situations . . . .*

The purpose of this teacher's guide is to provide you with ideas on how to motivate your students to want to learn geometry and on how to help them enjoy the subject. I have found that most students do not judge the value of what they are learning by the immediate practical use they may see in it, but by whether it is interesting to them. Students who are enjoying what they are learning do not continually ask why they have to learn it or what good it is.

First, a comment about the textbook itself. I am aware of the problems of students who are poor readers and have tried, in writing my book, to present the material in a way that will help such students improve their reading skills. Every high school course for college-bound students should provide this opportunity. The difficulty with many mathematics textbooks is that they are either so dull or so condescending that many students do not want to read them, and therefore they become too dependent on their teachers. It is my hope that your students will find the anecdotes and illustrations sufficiently entertaining that they will be encouraged to read the lessons. I have made every effort to present geometry in such a way that students will be able to teach themselves any lessons they have missed.

This guide contains daily lesson plans; in addition to presenting many practical applications and historical anecdotes, I have included puzzles with surprising answers and problems that are merely whimsical. Reduced transparency masters appear with the lesson plans for reference, and the transparency masters themselves are available from the publisher on a CD-ROM.

I have tried to include sufficient detail in the lesson plans for this guide to give a clear picture of some of the things that I do in my own classroom. However, what works well for me may not for you, just as I might have difficulty in succeeding

*Morris Kline, "A Proposal for the High School Mathematics Curriculum," *The Mathematics Teacher,* April 1966, pages 329–330.

with some of the things that you do in your classroom. If you are a beginning teacher, you may be tempted to try to follow these lesson plans rigidly. If so, I urge you not to take them too seriously. I am a firm believer in continually trying out new ideas and in modifying or discarding those that don't turn out well.

Most classes will cover one lesson each day. You will note from some of the lesson plans, however, that a few of the lessons may take two days. You may decide that more than one day should be devoted to some of the other lessons as well. Some teachers provide their students with a complete list of assignments and due dates at the beginning of a course. I think that this is unwise. It is better to be flexible and to modify one's plans whenever the students seem to be finding the work especially difficult. You may want to devote some days to laboratory activities or to algebra review or to doing an assignment together in small groups in class.

For most classes, the normal assignment might consist of a selection of exercises from Sets I and II. The exercises in Set I are ordinarily the easiest and emphasize the basic ideas of the lesson. The Set III exercises are meant to be optional and I give extra credit to those students who do them correctly. Set III exercises should be sufficiently interesting so that brighter students will want to do them; although some of the exercises are fairly easy, others are quite challenging.

At the end of this guide is a section of comments concerning the content of the lessons. Written by Dr. Peter Renz, these comments provide a broad overview of the course.

If in teaching this course you have any questions, criticisms, or suggestions, I would be most happy to hear from you. Correspondence should be addressed to me in care of W. H. Freeman and Company, 41 Madison Avenue, New York, New York 10010.

Harold R. Jacobs

RECOMMENDED BOOKS

Artmann, Benno. *Euclid—The Creation of Mathematics.* Springer, 1999.

Banchoff, Thomas F. *Beyond the Third Dimension.* Scientific American Library, 1990.

Beyer, Jinny. *Designing Tessellations.* Contemporary Books, 1999.

Burton, David M. *The History of Mathematics—An Introduction.* Allyn and Bacon, 1985.

Coxeter, H. S. M. *Introduction to Geometry,* Second Edition. John Wiley and Sons, 1969.

Coxeter, H. S. M., and S. L. Greitzer. *Geometry Revisited.* Mathematical Association of America, 1975.

Devlin, Keith. *Mathematics—The Science of Patterns.* Scientific American Library, 1994.

Eves, Howard W. *An Introduction to the History of Mathematics,* Sixth Edition. W. B. Saunders Company, 1990.

Eves, Howard W. *Fundamentals of Modern Elementary Geometry.* Jones and Bartlett, 1992.

Eves, Howard W. *College Geometry.* Jones and Bartlett, 1995.

Field, J. V. *The Invention of Infinity—Mathematics and Art in the Renaissance.* Oxford University Press, 1997.

Frederickson, Greg N. *Dissections—Plane and Fancy.* Cambridge University Press, 1997.

Gardner, Martin. *The Scientific American Book of Mathematical Puzzles and Diversions.* Simon and Schuster, 1959.

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Gardner, Martin. *The Unexpecting Hanging and Other Mathematical Diversions.* Simon and Schuster, 1969.

Gardner, Martin. *Martin Gardner's Sixth Book of Mathematical Games from Scientific American.* W. H. Freeman and Company, 1971; University of Chicago Press, 1984.

Gardner, Martin. *Mathematical Carnival.* Alfred A. Knopf, 1975.

Gardner, Martin. *Mathematical Magic Show.* Alfred A. Knopf, 1977.

Gardner, Martin. *Mathematical Circus.* Alfred A. Knopf, 1979.

Gardner, Martin. *Wheels, Life, and Other Mathematical Amusements.* W. H. Freeman and Company, 1983.

Gardner, Martin. *Knotted Doughnuts and Other Mathematical Entertainments.* W. H. Freeman and Company, 1986.

Gardner, Martin. *Time Travel and Other Mathematical Bewilderments.* W. H. Freeman and Company, 1988.

Gardner, Martin. *Penrose Tiles to Trapdoor Ciphers.* W. H. Freeman and Company, 1989.

Gardner, Martin. *Fractal Music, Hypercards and More . . .* W. H. Freeman and Company, 1992.

Gardner, Martin. *The Last Recreations.* Copernicus, 1997.

Gerdes, Paulus. *Geometry from Africa.* Mathematical Association of America, 1999.

Greenberg, Marvin Jay. *Euclidean and Non-Euclidean Geometries,* Third Edition. W. H. Freeman and Company, 1993.

Hansen, Vagn Lundsgaard. *Geometry in Nature.* A K Peters, 1993.

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Heilbron, J. L. *Geometry Civilized.* Clarendon Press, 1998.
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Hilbert, David, and Stefan Cohn-Vossen. *Geometry and the Imagination.* AMS Chelsea, 1991.
Hildebrandt, Stefan, and Anthony Tromba. *The Parsimonious Universe.* Copernicus, 1996.
Holden, Alan. *Shapes, Space, and Symmetry.* Columbia University Press, 1971.
Kappraff, Jay. *Connections—The Geometric Bridge between Art and Science.* McGraw-Hill, 1991.
Kline, Morris. *Mathematics in Western Culture.* Oxford University Press, 1953.
Kline, Morris. *Mathematics and the Physical World.* Thomas Y. Crowell, 1959; Dover, 1981.
National Council of Teachers of Mathematics. *Learning and Teaching Geometry, K–12,* 1987 Yearbook. National Council of Teachers of Mathematics, 1987.
Ogilvy, C. Stanley. *Excursions in Geometry.* Oxford University Press, 1969.
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Senechal, Marjorie, and George Fleck, editors. *Shaping Space.* Birkhäuser, 1988.
Steinhaus, H. *Mathematical Snapshots,* Third Edition. Oxford University Press, 1969.
Trudeau, Richard J. *The Non-Euclidean Revolution.* Birkhäuser, 1987.
Valens, Evans G. *The Number of Things: Pythagoras, Geometry, and Humming Strings.* E. P. Dutton and Company, 1964.
Van der Waerden, B. L. *Geometry and Algebra in Ancient Civilizations.* Springer-Verlag, 1983.
Wells, David. *The Penguin Dictionary of Curious and Interesting Geometry.* Penguin Books, 1991.
Wells, David. *You Are a Mathematician.* John Wiley and Sons, 1995.
Wenninger, Magnus J. *Polyhedron Models.* Cambridge University Press, 1971.
Wilson, Robin J. *Stamping Through Mathematics.* Springer, 2001.

ANSWERS TO EXERCISES

Chapter 1, Lesson 1

Set I (pages 10–11)

Ch'ang-an, now called Sian, was one of two great capital cities of ancient China. It was built on the bank of the Wei River on a main trade route to Central Asia.

The city measured 5 miles from north to south and 6 miles from east to west and was surrounded by a wall 17.5 feet high. Its population in the eighth century is estimated to have been 2 million.

More about Ch'ang-an can be found in chapter 5 of *Ancient China,* by Edward H. Schafer (*Great Ages of Man* series, Time-Life Books, 1967).

1. Collinear.

•2. 12 cm.

3. 6 cm.

4. 4 cm.

•5. About 1.3 cm.

6. 10 cm.

•7. 3 mi. ($\frac{6 \text{ mi}}{2} = 3 \text{ mi.}$)

8. 0.5 mi. ($\frac{6 \text{ mi}}{12 \text{ cm}} = \frac{0.5 \text{ mi}}{\text{cm}}$.)

9. 5 mi. ($10 \text{ cm} \times \frac{0.5 \text{ mi}}{\text{cm}} = 5 \text{ mi.}$)

10. No, because its sides are not all equal.

11. The colored regions (the Palace and Imperial Cities, the two markets, and the park).

12. 22 mi. ($44 \text{ cm} \times \frac{0.5 \text{ mi}}{\text{cm}} = 22 \text{ mi.}$)

13. Coplanar.

•14. Line segments.

15. A line segment is part of a line and has a measurable length. A line is infinite in extent.

16. 25.

• Answers to exercises marked with a bullet are also given in the textbook.

Set II (page 11)

The figure in this exercise set is a "nine-three" configuration, one of three such configurations in which nine lines and nine points are arranged so that there are three lines through every point and there are three points on every line. It and other configurations of projective geometry are discussed in chapter 3 of *Geometry and the Imagination,* by David Hilbert and Stefan Cohn-Vossen (A.M.S. Chelsea Publishing, 1990).

17. Nine.

18. Points that lie on a line.

19. Nine. They are E-F-I, A-B-C, A-D-I, G-H-I, B-C-D, B-E-H, D-F-H, A-C-F, C-E-G.

20. Lines that contain the same point.

•21. Three. (AC, AF, AI.)

22. Three. (AG, BD, BH.)

23. Three. (AF, BD, CG.)

24. Three. (Three lines are concurrent at every lettered point of the figure.)

•25. Six; AB, AG, AC, AF, AD, AI.

26. Six; BA, BC, BD, BE, BH, BG.

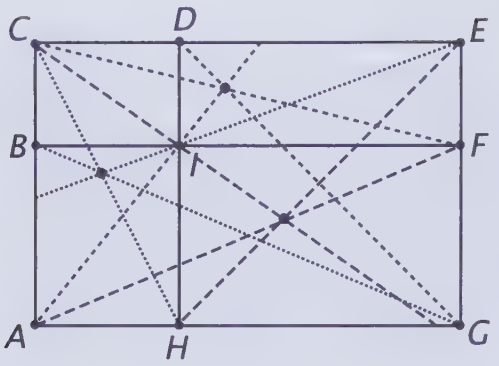
27. Six; CB, CD, CE, CG, CF, CA.

Set III (pages 11–12)

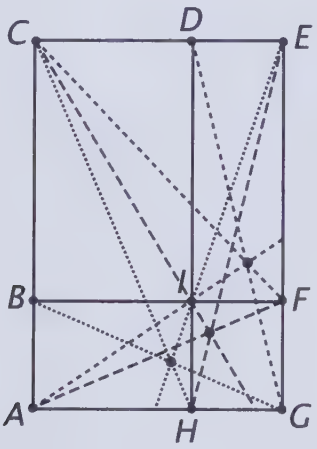
The first three figures in this exercise illustrate a special case of the dual of Pappus's theorem. As long as two sets of three lines in the figure are parallel, as in the first three figures, the added sets of lines are concurrent. A figure such as the last one in the exercise suggests that, when the lines are not parallel, the concurrence is lost. However, the two sets of three segments in the original figure must not necessarily belong to parallel lines. If they belong to concurrent sets of lines, the added sets of lines also are concurrent.

In the exercises, three sets of concurrent lines are named: AI, CF, DG; AF, CI, EH; and BG, CH, EI. There are three more sets what are not mentioned, of which one is AD, BE, GI. The other two sets, AE, BD, FH and BH, CG, DF, are more elusive (1) because, for certain spacings of the lines in the original figure, they are parallel rather than concurrent and (2) because, for the cases in which they *are* concurrent, the points of concurrency lie outside the figure.

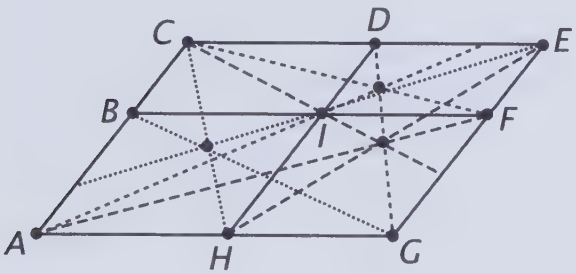
1.

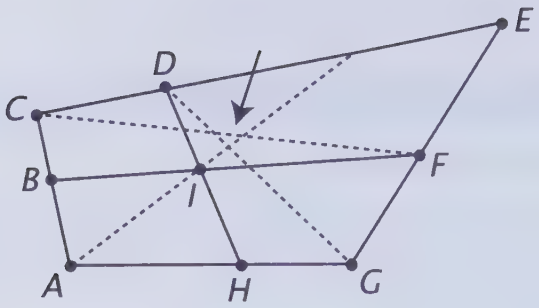


2. They seem to be concurrent.
3. These lines also seem to be concurrent.
4. These lines also seem to be concurrent.
5. *Example figure:*



6. The three sets of lines again seem to be concurrent.
7. *Example figures:*





(Just one set of the three lines is shown for the figure above, and the lines are clearly not concurrent.)

8. As long as the line segments in the original figure consist of two parallel (or concurrent) sets, the three sets of lines seem to be concurrent. Otherwise, they do not.

Chapter 1, Lesson 2

Set I (pages 15–16)

In the problems about the constellations, it is interesting to realize that, although the stars that form them give the illusion of being both coplanar and close together, they are certainly not coplanar and not necessarily near one another. The angles that are measured in the exercises are merely the angles between the lines in our map of the sky and not the actual angles between the lines in space.

Triangle Measurements.

1. AB.
•2. $\angle C$.
3. AC.
4. $\angle B$.
5. AB = 5.1 cm, AC = 2.6 cm, BC = 3.7 cm.
•6. Rays AC and AB.
7. $\angle A = 45°$.
8. *(Student drawing.)*
9. Rays BA and BC.
•10. $\angle B = 30°$.
11. Rays CA and CB.
12. $\angle C = 105°$.

Constellation Angles.

•13. Collinear.

14. *(Student guess.)* ($\angle B$ is actually smallest.)

15. *(Student guess.)* ($\angle 2$ is largest.)

•16. $\angle 1 = 67°$.

17. $\angle B = 60°$.

18. $\angle 2 = 84°$.

19. $\angle R = 71°$.

20. *(Student drawing.)*

21. $\angle D = 143°$.

•22. $\angle I = 117°$.

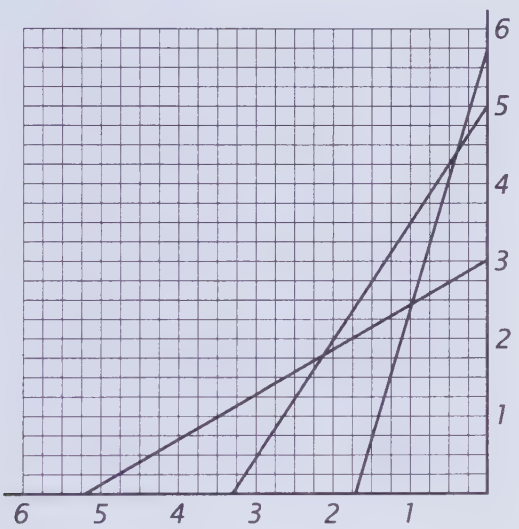
23. $\angle P = 100°$.

Set II (pages 16–17)

According to *Rules of Thumb,* by Tom Parker (Houghton Mifflin, 1983), "the base of a ladder should be 30 percent of its height from the base of the wall it is leaning against."

The Sliding Ladder.

24.



•25. Approximately $1\frac{3}{4}$ ft.

26. 74°.

27. 16°.

•28. Approximately $3\frac{1}{4}$ ft.

29. 56°.

•30. 34°.

31. Approximately $5\frac{1}{4}$ ft.

32. 30°.

33. 60°.

34. 45°. (The sum of the two angles seems to always be 90°.)

Set III (page 17)

More information on speculations about Mars can be found in the entertaining chapter titled "The Martian Dream" in *Pictorial Astronomy,* by Dinsmore Alter, Clarence H. Cleminshaw, and John G. Philips (Crowell, 1974).

Measuring Mars.

Dividing 24° into 360°, we get 15; so the distance around Mars is 15 times the distance between the two cities:

$$15 \times 880 \text{ miles} = 13{,}200 \text{ miles}.$$

Chapter 1, Lesson 3

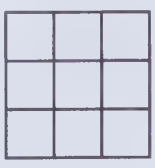
Set I (page 21)

Pentominoes were invented by Solomon Golomb, a long-time professor of mathematics and electrical engineering at the University of Southern California, who introduced them in a talk in 1953 at the Harvard Mathematics Club. His book titled *Polyominoes* (Princeton University Press, 1994) is the definitive book on the part of combinatorial geometry included in recreational mathematics. It is interesting to note that Arthur C. Clarke, the author of *2001—A Space Odyssey,* originally intended to use Pentominoes as the game played with the computer HAL before chess was substituted for it instead.

The problems about the Texas ranches were inspired by the fact that people sometimes naively assume that the size of a property can be measured by the time that it takes to walk around it. Proclus wrote about this assumption in 450 A.D. in his commentary on the first book of Euclid.

The Perimeter and Area of a Square.

1.



•2. 9 in^2.

•3. 12 in.

•4. 36 in^2.

•5. 24 in.

6. 1 ft^2.

7. 4 ft.

8. 144 in^2.

9. 48 in.

10. x^2 square units.

11. $4x$ units.

Pentominoes.

12. A polygon is a two-dimensional figure that is bounded by line segments.

•13. Piece I because it has four sides.

14. 5 square units.

15. Piece P. (It has a perimeter of 10 units; all the other pieces have perimeters of 12 units.)

Texas Ranches.

•16. Ranch A. (Ranch A needs 36 miles of fencing, whereas ranch B needs 32 miles).

17. Ranch B. (Ranch B has an area of 60 square miles, whereas ranch A has an area of 56 square miles).

18. Area.

Set II (page 22)

A book filled with fascinating information about the Great Pyramid is *Secrets of the Great Pyramid,* by Peter Tompkins (Harper & Row, 1971).

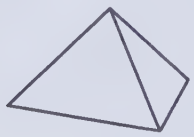
A Model of the Great Pyramid.

19. 64°.

20. Yes. (The sides opposite the 58° angles appear to be about 9.4 cm each.)

21. *(Model of Great Pyramid.)*

22. *Example figure:*



•23. Polygons.

24. A polyhedron.

•25. Eight.

26. Five.

27. Four.

•28. Three.

29. Five.

•30. A line.

31. *(Student estimate.)* (Calculations using trigonometry give a height of about 6.25 cm.)

•32. 75.6 ft.

33. *(Student estimate.)* (6.25×75.6 gives a height of approximately 470 ft.)

34. A view from directly overhead.

Set III (page 23)

The square-to-triangle puzzle was created, as the footnote on page 23 states, by Henry Dudeney, England's greatest innovator of mathematical puzzles. In *The Canterbury Puzzles,* he told of exhibiting it to the Royal Society of London in 1905 in the form of a model made of four mahogany pieces hinged together in a chain. When closed in one direction, the pieces formed the triangle and, when closed in the other direction, they formed the square. The version illustrated on page 23 of the text is slightly altered to make it easier for beginning geometry students to draw. Martin Gardner gives directions for constructing the figure exactly on pages 33 and 34 of *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961).

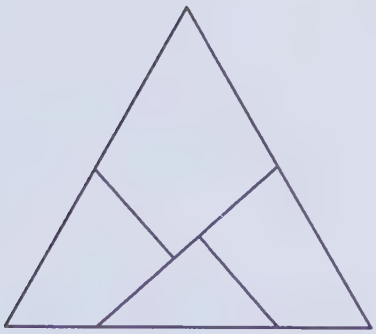
A Square Puzzle.

1. 60°.

2. $\angle BFP = 79°$ and $\angle AEH = 49°$.

3. $\angle BEH = 131°$ and $\angle EHA = 41°$.

4. PF and PG.

5. 

6. About 21.3 cm.

7. 60°.

8. The area of the triangle is equal to that of the square because they are made of the same pieces.

9. 196 cm^2.

10. It is longer. (The perimeter of the triangle is about 64 cm, whereas the perimeter of the square is 56 cm.)

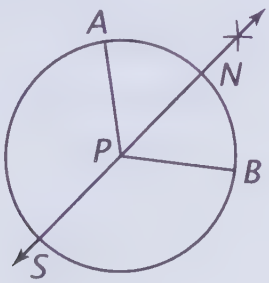
Chapter 1, Lesson 4

An excellent history and mathematical treatment of sundials is *Sundials—Their Theory and Construction,* by Albert E. Waugh (Dover, 1973).

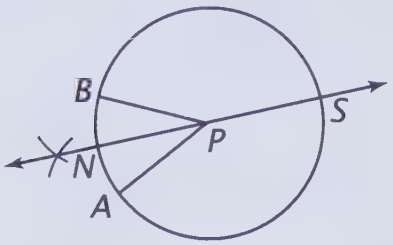
The most common type of sundial is the horizontal one and is the kind depicted on the Kellogg's Raisin Bran box. For this type of sundial to tell time accurately, the angle of its gnomon must be equal to the latitude of the location in which the dial is used. The angle of the gnomon printed on the Kellogg box has a measure of about 39°, evidently chosen because that is a good approximation of the average latitude of the United States. Seattle has a latitude of almost 48°, whereas Miami's latitude is less than 26°; in locations such as these, the claim that "now you can tell time outdoors" with the cereal box sundial is less than accurate!

Set I (pages 26–27)

Finding North by Shadows.

1. 

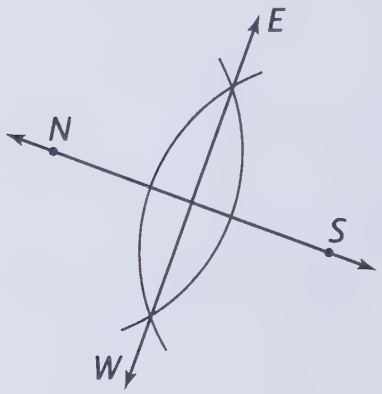
2. *(Student answer.)* (∠APB should be twice ∠APN.)

3. 

4. *(Student answer.)* (∠APB should be twice ∠APN.)

•5. Noncollinear.

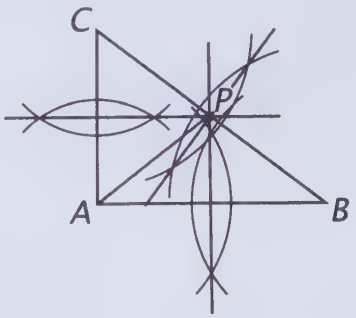
6. Coplanar.

7. 

8. *(Student answer.)* (The angles formed by NS and EW have measures of 90°.)

9. *(Student answer.)* (The sum of all four angles is 360°.)

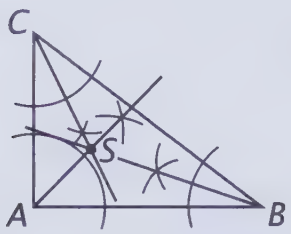
Bisectors in a Triangle.

10. 

•11. The bisecting lines seem to be concurrent.

12. They seem to meet on side CB.

13. The distances seem to be equal.

14. 

15. They seem to be concurrent.

•16. No.

Set II (pages 27–29)

About Sundials.

•17. West.

18. North.

19. South.

20. Noon.

•21. About 6 A.M. and 6 P.M.

22. ∠IOJ.

•23. OI bisects ∠HOJ.

24. No. ∠POQ and ∠QOR do not appear to be equal.

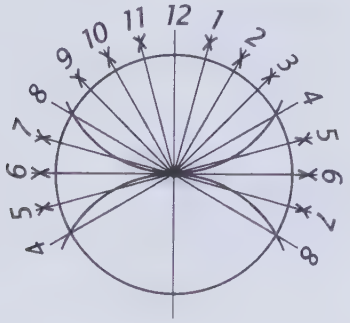
•25. ∠DOB and ∠EOA.

26. ∠POL.

27. ∠COB, ∠PON and ∠POQ.

28. ∠EOM.

29. *Example figure:*



30. Collinear means that there is a line that contains the points.

•31. 60°.

32. 30°.

•33. 180°.

34. 90°.

35. 15°.

36. 4 minutes. ($\frac{60 \text{ minutes}}{15^\circ} = \frac{4 \text{ minutes}}{1^\circ}$.)

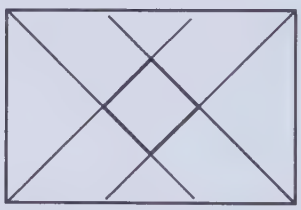
Set III (page 29)

This discovery exercise is adapted from a proof exercise (3.5.18) in *Geometry Civilized,* by J. L. Heilbron (Oxford University Press, 1998). Every geometry teacher should have a copy of this beautifully written and illustrated book. Dr. Heilbron was formerly Professor of History and the History of Science and Vice Chancellor of the University of California at Berkeley and is currently Senior Research Fellow at Worcester College, Oxford University, and at the Oxford Museum for the History of Science.

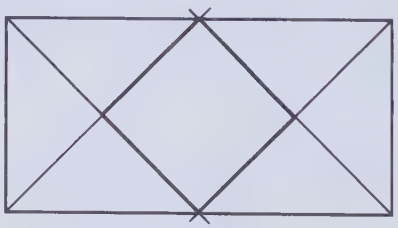
Angle Bisectors in a Rectangle.

1. 

2. A square.

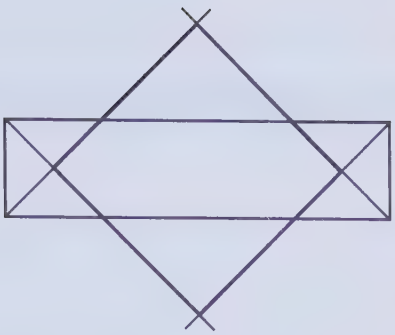
3. 

Again, the lines seem to form a square.

4. 

The lines seem to form a square.

5.



The lines still seem to form a square.

6. Yes. The more elongated the rectangle, the larger the square.

7. If the original rectangle were square in shape, the polygon would shrink to a point.

Chapter 1, Lesson 5

Set I (pages 31–32)

The incorrect Egyptian formula for finding the area of a quadrilateral was evidently based on the notion of multiplying the averages of the lengths of the opposite sides. It is easy to show (as the students will be asked to do later in the course) that this method gives the correct answer if the quadrilateral is a rectangle. For all other quadrilaterals, the method gives an area that is too large.

The Egyptian Tax Assessor.

•1. 100 square units. ($\frac{1}{4}20 \cdot 20 = 100$.)

2. 88 square units. ($\frac{1}{4}16 \cdot 22 = 88$.)

3. 98 square units. ($\frac{1}{4}14 \cdot 28 = 98$.)

4. Ramses.

5. Yes. ($11 \cdot 8 = 88$.)

•6. It contains 80 square units. ($10 \cdot 8 = 80$.)

7. It contains 108 square units. ($9 \cdot 12 = 108$.)

•8. 54 square units. ($\frac{1}{2}108 = 54$.)

9. It contains 60 square units. ($5 \cdot 12 = 60$.)

10. 30 square units. ($\frac{1}{2}60 = 30$.)

•11. It contains 84 square units. ($54 + 30 = 84$.)

12. Ramses is cheated the most. He pays the highest tax, yet actually has the smallest property. (See the chart below.)

	Assessment Area	*Actual Area*
Ramses	100	80
Cheops	88	88
Ptahotep	98	84

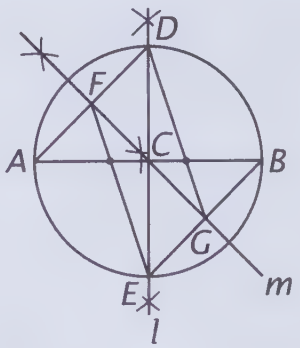
Set II (pages 32–34)

We will return to the trisection of line segments and angles later in the course. The method for trisecting an angle presented here is surely the first one to be thought of by most people encountering the problem for the first time. For small angles, it is a fairly good approximation but, for an angle of 60°, the error just begins to become detectable through measurement and, for obtuse angles, its failure becomes very obvious.

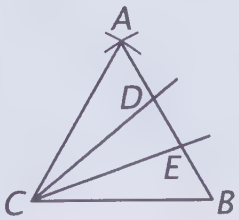
The classic book on angle trisection for many years was *The Trisection Problem,* by Robert C. Yates, originally published in 1942 and reprinted in 1971 by N.C.T.M. as a volume in the *Classics in Mathematics Education* series. The definitive book is currently *A Budget of Trisections,* by Underwood Dudley (Springer Verlag, 1987). Dr. Dudley includes some additional information on the subject in "Trisection of the Angle" in his book titled *Mathematical Cranks* (M.A.A., 1992). All make fascinating reading.

Trisection Problems.

13.



14.



15. An accurate drawing and careful measurements suggest that ∠ACD, ∠DCE, and ∠ECB are about 19°, 22°, and 19°, respectively.

Set III (page 34)

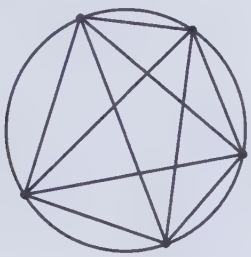
This problem was discovered by Leo Moser, who, in a letter to Martin Gardner, said that he thinks it was first published in about 1950. The "obvious" formula for the number of regions, $y = 2^{n-1}$, where n is the number of points, is wrong. The correct formula,

$$y = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$$

is surprisingly complicated. For more on this problem, see the chapter titled "Simplicity" in *Mathematical Circus* by Martin Gardner (Knopf, 1979).

Cutting Up a Circle.

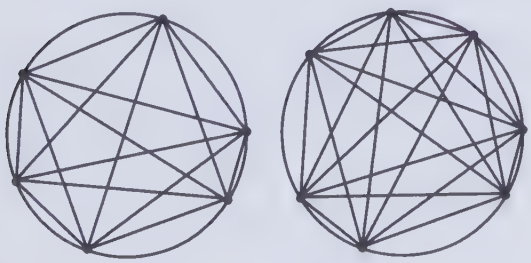
1. *Example figure:*



2. 16.
3. 2.
4. 4.
5.

No. of points connected	2	3	4	5
No. of regions formed	2	4	8	16

6. It doubles.
7. 32.
8. 64.
9. *Example figures:*



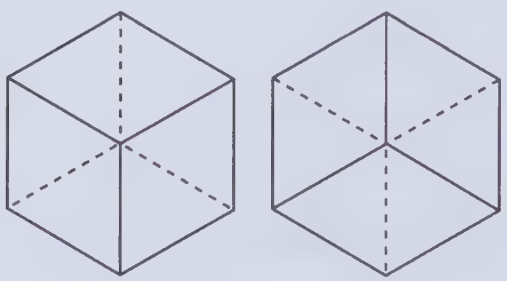
10. No. (Six points result in at most 31 regions, and seven points in at most 57 regions.)

Chapter 1, Review

Set I (pages 36–37)

The transparent view of the cube used in exercises 15 through 17 is named after psychologist Hertha Kopfermann, who studied it in 1930. An interesting discussion of our perception of this figure appears on pages 27 and 28 of *Visual Intelligence,* by Donald D. Hoffman (Norton, 1998).

1. Line segments.
•2. Polygons (or squares).
3. A polyhedron.
4. 3.
•5. 12.
6. 3.
7. 6.
•8. Coplanar.
9. 9.
•10. PQ.
11. NO.
12. OP.
•13. T.
14. U and Q.
15. Triangles, quadrilaterals, and a hexagon. (Students who are aware of concave polygons may notice more than one hexagon and even some heptagons!)
16.



17. 3.

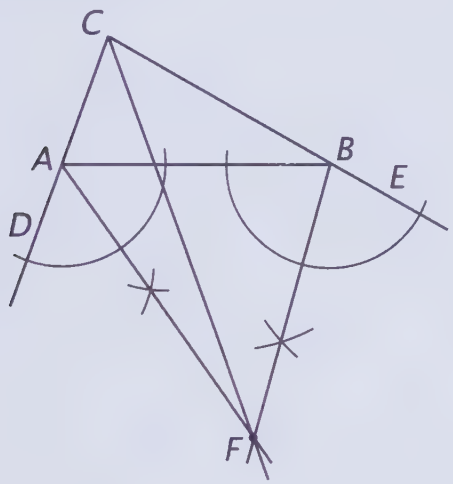
Set II (pages 37–38)

Exercises 19 through 26 are based on the fact that every triangle has three excenters as well as an incenter. Each excenter is the point of concurrence of the lines bisecting exterior angles at two vertices of the triangle (∠DAB and ∠ABE in the figure) and the line that bisects the (interior) angle at the third vertex (∠C in the figure). Point F is one of the excenters of the triangle.

Exercises 27 through 30 are based on the fact that, when the midpoints of the sides of any quadrilateral are connected in order with line segments, the quadrilateral that results is a parallelogram. Furthermore, its perimeter is equal to the sum of the length of the diagonals of the original quadrilateral. These facts are easily proved and will be established later in the course.

•18. Constructions.

19.



•20. ∠DAB = 110°.

21. ∠ABE = 150°.

22. ∠DCE = 80°.

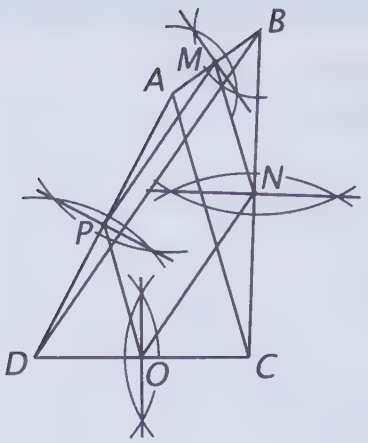
23. ∠DAF = ∠FAB = 55°.

24. ∠ABF = ∠FBE = 75°.

•25. ∠DCF = ∠FCE = 40°.

26. Line CF seems to bisect ∠DCE.

27.



28. The opposite sides of the quadrilateral appear to be equal (and parallel).

29. (Answer depends on student's drawing.)

30. (Answer depends on student's drawing but should be close to the answer to exercise 29.)

Set III (page 38)

The story of Queen Dido and Carthage is told on pages 44–47 of *Mathematics and Optimal Form*, by Stefan Hildebrandt and Anthony Tromba (Scientific American Library, 1985) and on pages 64–71 of the revised and enlarged version of this book titled *The Parsimonious Universe* (Copernicus, 1996).

•1. 1,200 yd. ($4 \cdot 300 = 1{,}200$.)

2. 90,000 yd^2. ($300^2 = 90{,}000$.)

3. 500 yd. ($900 - 2 \cdot 200 = 500$.)

4. 1,400 yd. ($2 \cdot 200 + 2 \cdot 500 = 1{,}400$.)

5. 100,000 yd^2. ($200 \cdot 500 = 100{,}000$.)

6. *(Student drawings.)* (Students who have made several drawings may suspect that the length along the river should be twice the side width to get as much land as possible. In this case, the numbers are 450 yards and 225 yards and the area is 101,250 square yards.)

Algebra Review (page 40)

•1. Associative for addition.

•2. Commutative for multiplication.

•3. Definition of subtraction.

•4. Identity for addition.

•5. Definition of division.

•6. Inverse for multiplication.

•7. 1,728. ($12 \cdot 12 \cdot 12 = 1{,}728$.)

•8. 12,000. ($12 \cdot 10 \cdot 10 \cdot 10 = 12{,}000$.)

•9. 20. ($5 + 15 = 20$.)

•10. –75.

•11. 225. ($289 - 64 = 225$.)

•12. 81. ($9^2 = 81$.)

•13. 49π.

•14. 14π.

•15. $5x$.

•16. x^5.

•17. $5x$.

•18. x^5.

•19. $2x + 3y$.

•20. x^2y^3.

•21. $x^2 + y^3$.

•22. $x^3 + y^2$.

•23. $x + 2y$.

•24. xy^2.

•25. $7x^2$.

•26. $7x^2$.

•27. $5x^2$.

•28. $5x^2$.

•29. $4x^2 + 3x$.

•30. $3x^3 + 2x^2 + x$.

•31. x^{14}.

•32. x^{14}.

•33. $6x^5$.

•34. $6x^5$. ($2x^3 \cdot 3x^2 = 6x^5$.)

•35. $36x^2$. ($6x \cdot 6x = 36x^2$.)

•36. $72x^5$. ($8x^3 \cdot 9x^2 = 72x^5$.)

•37. $4x + 4y$. ($3x + 2y + x + 2y = 4x + 4y$.)

•38. $4x + 4y$.

•39. $2x$. ($3x + 2y - x - 2y = 2x$.)

•40. $2x$.

•41. 0.

•42. 1.

•43. $5x + 10y$.

•44. $3x + 4y$.

•45. $5x - 10y$.

•46. $3x - 4y$.

•47. $31xy$. ($28xy + 3xy = 31xy$.)

•48. $25xy$. ($28xy - 3xy = 25xy$.)

•49. $7 + x^3 + x^5$.

•50. $10x^8$.

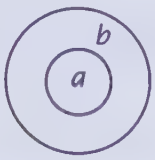
Chapter 2, Lesson 1

Set I (pages 43–44)

•1. A conditional statement.

2. "*a*" represents the hypothesis and "*b*" represents the conclusion.

3.



A circle labeled *a* inside another circle labeled *b*.

•4. You live in the Ozarks.

5. You live in the United States.

•6. Yes.

7. No.

8. No. If it is cold outside, it isn't necessarily snowing.

9. No.

10. It is cold outside if it is snowing.

11. Yes.

12. Statements 1 and 3.

13. Yes.

•14. If an animal is a koala bear, then it eats only eucalyptus leaves.

15. If the cat is in the birdcage, then it isn't there to sing.

16. If money grew on trees, then Smokey Bear wouldn't have to do commercials for a living.

17. If you are an architect, then you use geometry.

•18. If I don't understand, then I ask questions.

19. If there is a fire, then use the stairs instead of the elevator.

20. If you are a vampire, then you do not cast a shadow. (Or, If you cast a shadow, then you are not a vampire.)

Set II (pages 44–45)

•21. If at first you don't succeed, then try again.

22. Try again if at first you don't succeed.

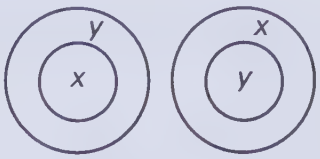
23. Region 1.

•24. Regions 1 and 2.

25. Regions 2 and 3.

26. Region 3.

27.



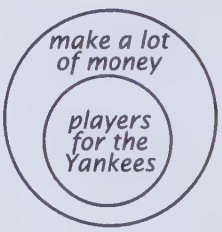
If *x*, then *y* If *y*, then *x*

•28. The second.

29. The first.

30. Two.

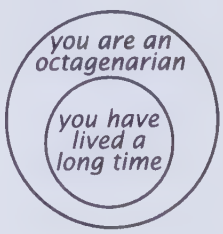
31.



32. Statements a and d.

33. Statements a and d.

34.



•35. Statements c and d.

36. Statements a and b.

Set III (page 45)

The Animal Mind, the source of the information for the Set III exercises, is a fascinating book. This paragraph from the dust jacket describes it well:

"In this engaging volume, James and Carol Gould go in search of the animal mind. Taking a fresh look at the evidence on animal capacities for perception, thought, and language, the Goulds show how scientists attempt to distinguish actions that go beyond the innate or automatically learned. They provide captivating, beautifully-illustrated descriptions of a number of clever and curious animal behaviors—some revealed to be more or less pre-programmed, some seemingly proof of a well-developed mental life."

1. The red monkey shape.
2. In the left-hand column.
3. The red square.
4. If Sarah takes the banana, then Mary will *not* give chocolate to Sarah.

Chapter 2, Lesson 2

Set I (pages 47–48)

Exercises 1 through 4 have an interesting source. In his book titled *Euclid and His Modern Rivals* (1885), Charles L. Dodgson (Lewis Carroll) criticized some of the geometry books of the time. One of them, *Elementary Geometry, following the Syllabus prepared by the Geometrical Association,* by J. M. Wilson (1878) contained on page 17 the following "question":

"State the fact that all 'geese have two legs' in the form of a Theorem." Carroll wrote: "This I would not mind attempting; but, when I read the additional request, to 'write down its converse *theorem*,' it is so powerfully borne in upon me that the writer of the Question is probably himself a biped, that I feel I must, however reluctantly, decline the task."

1. If a creature is a goose, then it has two legs.
2. 

•3. If a creature has two legs, then it is a goose.

4. No.

•5. You are not more than six feet tall.

6. If you are not more than six feet tall, then you are an astronaut.

7. No.

8. No.

•9. If you do not know how to reason deductively.

10. If you cannot comprehend geometry, then you do not know how to reason deductively.

11. You do not know how to reason deductively.

12. If and only if.

•13. They have the same meaning.

14. If and only if.

15. The fear of peanut butter sticking to the roof of your mouth.

16. If you are afraid of peanut butter sticking to the roof of your mouth, then you have arachibutyrophobia.

17. Yes. It must be true because the sentence is a definition.

18. Common, proper, improper.

19. If a fraction is a vulgar fraction, then it is a common fraction that is either proper or improper.

20. If a fraction is a common fraction that is either proper or improper, then it is a vulgar fraction.

•21. Yes. The converse must be true because this is a definition.

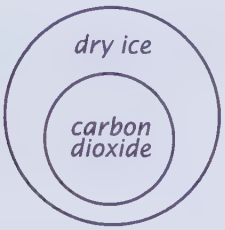
22. "New Year's Day" does not mean the same thing as "holiday." Other days are holidays as well, so its converse is not true.

23. Yes. "New Year's Day" means the same thing as "the first day of the year." The converse is true.

Set II (pages 48–49)

The wolf-pack definition in exercises 36 and 37 is an amusing example of recursion. An entire chapter of Clifford Pickover's book titled *Keys to Infinity* (Wiley, 1995) deals with the subject of "recursive worlds."

24. Yes.
25. No.
•26. No.
27. The second sentence. (Only if it is your birthday do you get some presents.)
•28. It is the converse.
29.



30. It is dry ice if and only if it is frozen carbon dioxide.
31. The teacher thought you misspelled "if."
•32. Sentence 2. (If it is a detective story, then it is a whodunit.)
33. Yes.
34. If a car is a convertible, then it has a removable top. If a car has a removable top, then it is a convertible.
•35. It is the converse.
36. Two or more.
37. Because, according to the definition, there is no limit to the number of wolves in a wolf pack.

Set III (page 49)

Since its creation in the early 1970s, Peter Wason's puzzle has appeared in puzzle books in a variety of forms. In *The Math Gene* (Basic Books, 2000), Keith Devlin remarks that most people find the puzzle extremely hard. Devlin restates the problem as a party attended by young people, some drinking alcohol and some soft drinks, some legally and some not. Recast in this format, the problem seems quite simple. Devlin remarks:

"The Wason test is famous because the outcome is so dramatic. A change in presentation turns a difficult problem that most people get wrong into an extremely simple one that most easily get right. The fact is that people reason much better about familiar, everyday objects and circumstances than they do about abstract objects in unfamiliar settings, even if the logical structure of the task is the same."

Turn over the blue card to see if there is a circle on the other side. Turn over the card with the triangle to see if the other side is blue. (It doesn't matter what is on the other side of the card with the circle. For example, the possibility that the other side is red doesn't contradict the statement

> If a card is blue on one side, it has a circle on the other side.

Chapter 2, Lesson 3

Set I (pages 52–53)

1. If you keep quiet, others will never hear you make a mistake.
   If others never hear you make a mistake, they will think you are wise.
•2. If you keep quiet, others will think you are wise.
3. $a \rightarrow b$, $b \rightarrow c$. Therefore, $a \rightarrow c$.
•4. All Greeks are humans.
   All humans are mortals.
   Therefore, all Greeks are mortals.
5. All Greeks are statues.
   All statues are mortals.
   Therefore, all Greeks are mortals.
6. Yes.
•7. No.
•8. The hypothesis.
9. The conclusion.
10. $a \rightarrow b$, $b \rightarrow c$, $c \rightarrow d$. Therefore, $a \rightarrow d$.
11. The third: If you live where it is cold, you see a lot of penguins.
•12. It also may be false.
13. If Captain Spaulding is in the jungle, he can't play cards.
14. Yes.
15. If NASA launched some cows into space, they would be the herd shot around the world.
•16. A theorem.

17. If you go to Dallas, you will take a plane.
If you take a plane, you will go to the airport.
If you go to the airport, you will see all the cabs lined up.
If you see all the cabs lined up, you will see the yellow rows of taxis.

•18. If you go to Dallas, you will see the yellow rows of taxis.

19. If a duck had sore lips, he would go to a drugstore.
If he went to a drugstore, a duck would ask for some Chapstick.
If he asked for some Chapstick, a duck would ask to have it put on his bill.

20. *Second statement:* If they wanted to take along some food, they would try to carry on six dead raccoons.
Last statement: If the flight attendant objected, they would be told that there is a limit of two carrion per passenger.

21. *First statement:* If a group of chess players checked into a hotel, they would stand in the lobby bragging about their tournament victories.
Third statement: If the manager asked them to leave, they would ask why.

Set II (pages 53–54)

Dice Proof.

•22. 21. $(1 + 2 + 3 + 4 + 5 + 6 = 21.)$

23. 7. $(\frac{21}{3} = 7.)$

24. 2.

25. 6.

Matchbox Proof.

•26. Red. Because the label is wrong.

27. A red marble and a white marble.

28. Two white marbles.

Tick-tack-toe Proofs.

29.



30. If your opponent marks an X in the upper left, then you will mark an O in the lower right.
If you mark an O in the lower right, then your opponent will mark an X in the lower center.
If your opponent marks an X in the lower center, you will mark an O in the upper center.

31. *(Student drawings.)*

32. You will win the game because, no matter where you mark your next O, your opponent will then be forced to allow you to set a winning trap!

Set III (page 54)

Word ladders were invented by Lewis Carroll in 1877 and have been a popular form of word play ever since. An entire chapter in Martin Gardner's *The Universe in a Handkerchief—Lewis Carroll's Mathematical Recreations, Games, Puzzles, and Word Plays* (Copernicus, 1996) deals with word ladders (called "doublets" by Carroll).

Another interesting reference is *Making the Alphabet Dance*, by Ross Eckler (St. Martin's Press, 1996). At the beginning of chapter 4, titled "Transforming One Word into Another", Eckler remarks: "By drawing on ideas from mathematics . . . [word play] gains additional respectability and legitimacy."

1. *Example solution:*
LESS, LOSS, LOSE, LOVE, MOVE, MORE.

2. *Example solution:*
HEAD, HEAR, HEIR, HAIR, HAIL, TAIL.

Chapter 2, Lesson 4

Set I (pages 57–58)

It is interesting to note that, like that of Rube Goldberg, Samuel Goldwyn's name has entered the dictionary. According to *The Random House Dictionary of the English Language*, a "Goldwynism" means "a phrase or statement involving a humorous and supposedly unintentional misuse of idiom."

•1. Contradiction.

2. "In–direct."

•3. Suppose that it would not speak foul language.

4. Suppose that he has control over his pupils.
5. Suppose that it does not lead to a contradiction.
•6. No.
7. Yes.
8. No.
9. Indirect.
•10. It is the opposite of the theorem's conclusion.
•11. The earth is not flat.
12. Suppose that the earth is flat.
13. The stars would rise at the same time for everyone, which they do not.
•14. That it is false.

Set II (pages 58–59)

The puzzles about the weights are adapted from ideas in A. K. Dewdney's book titled *200% of Nothing* (Wiley, 1993). The subtitle of the book is *An Eye-Opening Tour through the Twists and Turns of Math Abuse and Innumeracy.* Dewdney uses the idea of equal partition of numbers in a section of the book titled "What Do Mathematicians Do?"

Trading Desks Proof.

•15. *Beginning assumption:*
Suppose that the pupils can obey the teacher.
Contradiction:
There are 13 pupils at the black desks but only 12 brown desks.
Conclusion:
The pupils can't obey the teacher.

Ammonia Molecule Proof.

16. *Beginning assumption:*
Suppose that the atoms of an ammonia molecule are coplanar.
Contradiction:
Each bond angle is 107°.
Conclusion:
The atoms of an ammonia molecule are not coplanar.

Poker Proof.

17. *Beginning assumption:*
Suppose that someone holds a royal flush.
Contradiction:
You have all the "10" cards.
Conclusion:
In this deal of the cards, no one holds a royal flush.

Balanced Weights Proofs.

18. Puzzle 1. One set is 2, 4, 6, 13; the other set is 7, 18.
19. If a puzzle of this type has a solution, then each set will weigh half the total weight.
20. *Beginning assumption:*
Suppose that there is a solution.
Contradiction:
The sum of all of the weights is odd.
Conclusion:
There is no solution.

Athletes Puzzle.

21. There is only 1 football player and consequently there are 99 basketball players. If there were two or more football players, then, given any two of the athletes, both could be football players. This contradicts the fact that, given any two of the athletes, at least one is a basketball player. Therefore, the assumption that there are two or more football players is false, and so there is only one football player.

Set III (page 59)

A good discussion of David L. Silverman's puzzle about the list of true and false statements appears on page 79 of *Knotted Doughnuts and Other Mathematical Entertainments,* by Martin Gardner (W. H. Freeman and Company, 1986). In its original form, the puzzle contained 1,969 statements (to celebrate the new year of 1969) comparable to those in the list in Set III. The only true statement in the original version was the 1,968th one.

This sort of puzzle suggests other possibilities that students might enjoy exploring. What if the word "false" were changed into "true" in all of the statements? What if the word "exactly" were omitted?

1. No, because any two statements on the list contradict each other.
2. No, because then the list would contain exactly zero false statements.
3. No.
4. No.
5. One, because the preceding conclusions eliminate all of the other possibilities. The true statement must be C.

Chapter 2, Lesson 5

Set I (pages 61–63)

The dog Rin-Tin-Tin was Warner Brothers' first big money-making star. His movie career began in 1922, and his salary at one time was $1,000 a week. According to Clive Hirschhorn's *The Warner Brothers Story,* Rin-Tin-Tin's 19 films kept the studio prosperous during the silent area and saved many theaters from closure. He was voted most popular film performer in 1926!

1. No (because a period has dimensions.)
2. No (because even an atom has dimensions.)
- •3. You go around in circles.
4. Star, tail.
5. Because Rin-Tin-Tin was a movie star who had a tail.
- •6. Two points determine a line.
7. Because it is taut and hence straight like a line.
- •8. Theorems.
9. A statement that is assumed to be true without proof.
10. No.

Duck Pond Table.

11. Three noncollinear points determine a plane.
- •12. The points.
13. The plane.
14. The table top might tip over.
15. Noncollinear.
16. If and only if.
17. If points are coplanar, then there is a plane that contains all of them. If there is a plane that contains all of them, then points are coplanar.
- •18. Converse.

Intersecting Cubes.

19. True.
- •20. True.
21. True.
- •22. False.
23. False.
24. True.
25. True.

Set II (pages 63–64)

Japanese Weights.

- •26. A picul is 1,600 taels.
27. A picul is 16,000 momme.
28. A momme is one-tenth of a tael.
29. No, because we don't know what any of these units are.
30. Yes. An elephant weighs more than one picul.
31. Because "the load carried by a man" is too vague; some men can carry much heavier loads than others.

Credit Card Rules.

- •32. Statements 1 and 3.
33. Transaction finance charge and supercheck.
34. Statements 2, 4, and 5.
35. Statements 2 and 4. (If you go over your credit limit, you will be charged a fee. If you are charged a fee, the fee will be added to your balance.)
36. If you go over your credit limit, a fee will be added to your balance.

Crooked Lines.

- •37. Two points determine a line.
38. Yes.

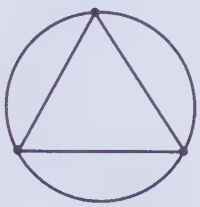
39. We did not define the word "line" (nor did we define the word "straight").

40. A theorem is a statement that is proved.

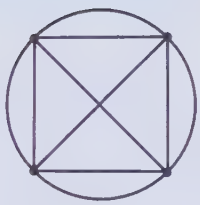
41. *Beginning assumption:*
Suppose two lines can intersect in more than one point.
Contradiction:
Two points determine a line.
Conclusion:
Two lines cannot intersect in more than one point.

Set III (page 64)

1. 3 lines.



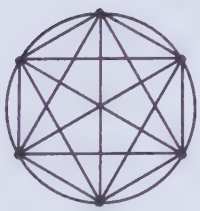
2. 6 lines.



3. 10 lines.



4. 15 lines.



5. 20 points: 190 lines.

Some students who answer this correctly will probably do it by observing the pattern of increasing differences.

Number of points	2	3	4	5	6
Number of lines	1	3	6	10	15
		2	3	4	5

Each additional point results in one more line than the point before it.

Other students may deduce a formula for the number of lines, l, in terms of the number of points, p:

$$l = \frac{p(p-1)}{2}$$

(This is the formula for the binomial coefficient: $\binom{p}{2}$.)

Chapter 2, Lesson 6

Set I (pages 67–68)

Sherman Stein makes some scathing remarks in his book titled *Strength in Numbers* (Wiley, 1996) about the "tear off the corners" of a triangle procedure illustrated in exercises 8 through 10.

He writes:

"Here is how one of the seventh-grade texts has the pupils 'discover' that the sum of the three angles of a triangle is always 180°. Under the heading 'Work Together,' it has these directions:

Begin with a paper triangle that is a different shape from those of the other members of your group. Number the angles of the triangle and tear them off the triangle. Place the three angles side-by-side so that pairs of angles are adjacent and no angles overlap.

What seems to be true? Compare your results with those of your group. Make a conjecture.

On the very next page, facing these directions, we find, 'In the Work Together activity you discovered that the following statement is true.' In boldface follows, 'The sum of the measures of the angles of a triangle is 180°.'

What kind of discovery or constructing knowledge out of experience is that? From what I know about pupils, they would glance at the bold type on the next page and stop experimenting."

•1. The length of the hypotenuse of a right triangle.

2. The speed of light.

3. The circumference of a circle.

4. The length of one of the other sides of a right triangle.

•5. The area of a circle.

6. The Pythagorean Theorem.

7. The squares on the sides contain 16, 9, and 25 small squares; 16 + 9 = 25.

8. The sum of the angles of a triangle is 180°.

•9. No.

10. No.

Target Circles.

11. 2, 4, 6, 8, and 10 units.

•12. 2π, 4π, 6π, 8π, and 10π units.

•13. 9π square units. ($\pi 3^2 = 9\pi$.)

14. 9π square units. ($\pi 5^2 - \pi 4^2 = 9\pi$.)

15. It is because the purple region appears to many people to be much larger than the yellow region.

16. 100. 36 + 64 = 100 by the Pythagorean Theorem.

17. 144. 169 – 25 = 144 by the Pythagorean Theorem.

•18. 30°. (The Triangle Angle Sum Theorem.)

19. 45°.

20. $(90 - n)°$.

•21. 70°. ($180 - 40 = 140$, $\frac{140}{2} = 70$.)

22. 65°. ($180 - 50 = 130$, $\frac{130}{2} = 65$.)

23. $(\frac{180-n}{2})°$.

•24. No.

25. This is the converse of the Pythagorean Theorem, and the converse of a true statement isn't necessarily true. (We will see later that this particular converse is true, but to show why requires a separate argument.)

Set II (pages 69–70)

David Gale, Professor Emeritus of Mathematics at the University of California at Berkeley recalls the rotating pencil demonstration in exercises 38 through 40 as his first encounter with a mathematical proof. Another student showed it to him when he was in the fifth grade. (*Tracking The Automatic Ant*, by David Gale, Springer, 1998).

Eye Pupil.

•26. Approximately 50 mm^2. ($\pi 4^2 = 16\pi \approx 50$.)

27. Approximately 3 mm^2. ($\pi 1^2 = \pi \approx 3.14$.)

28. Approximately 16 times as much.

Triangle Angle Sum Theorem.

29. 90°.

30. 60°.

31. No, because then the third angle would have a measure of 0°.

•32. Yes, because the sum of the other two angles could be 1°.

33. Yes, because the sums of the angles in the two triangles are the same.

The Stretched Cord.

•34. a^2.

35. $2a^2$.

36. $2a^2$.

37. By the Pythagorean Theorem.

Pencil Experiment.

•38. To the left.

39. That the pencil has turned through 180°.

40. The sum of the angles of a triangle is 180°.

Aryabhata on the Circle.

41. Yes. $a = \frac{1}{2}c\frac{1}{2}d = \frac{1}{2}(2\pi r)\frac{1}{2}(2r) = \pi r^2$.

42. 3.1416. If the circumference is $8(4 + 100) + 62{,}000 = 62{,}832$ and the diameter is 20,000, then, because $c = \pi d$, $62{,}832 = \pi(20{,}000)$; so $\pi = \frac{62{,}832}{20{,}000} = 3.1416$.

43. Dilcue's answer is a good one because not only is PIE an old way to spell PI but his answer reflected in a vertical mirror is 3.14.

Set III (page 70)

Greg N. Frederickson, the author of *Dissections, Plane and Fancy* (Cambridge University Press, 1997), is a professor of computer science at Purdue University. His book is the definitive work on the amazing topic of geometric dissections and should be on the bookshelf of every geometry classroom.

When the three pieces are arranged as in the second figure, they seem to form a square whose side is the hypotenuse of one of the right triangles. When the pieces are arranged as in the fourth figure, they seem to form two squares whose sides are equal to the other two sides of the right triangle. The total area of the three pieces is the same no matter how they are arranged; so the two arrangements illustrate the Pythagorean Theorem.

Chapter 2, Review

Set I (pages 71–73)

A. Bartlett Giamatti was president of Yale University and, later, of the National Baseball League. He also served briefly as Commissioner of Major League Baseball until his death in 1989. *Take Time for Paradise* is a nice little book on sports and play.

The Morton Salt Company originally considered the slogan "Even in rainy weather, it flows freely" to promote the message that its salt would pour easily in damp weather. They then settled on a variation of the old proverb: "It never rains but it pours" for their slogan and have been using it on their packages since 1914.

The SAT problem was a multiple-choice question in its original format, and the problem asked only for x.

The judge in the trial excerpt was Judge Ito, and the conversation took place in Los Angeles on July 26, 1995, in the case of The People of the State of California vs. Orenthal James Simpson.

•1. If something is a limerick, then it has five lines.

2. If I perfect my perpetual motion machine, then I will make a fortune.

3. If something is a toadstool, then it is not edible.

4. $a \rightarrow b$, $b \rightarrow c$; therefore, $a \rightarrow c$.

5. If we have known freedom, then we fear its loss.

6. One or both of the premises being false.

7. If you are a daredevil, then you are recklessly bold.
If you are recklessly bold, then you are a daredevil.

8. Each is the converse of the other.

•9. Collinear.

10. Two.

•11. A postulate.

12. An Euler diagram.

•13. If it rains, then it pours.

14. It pours if it rains.

15. If it pours, then it rains.

16. No.

17. 100°. (180 – 48 – 32 = 100.)

•18. 50°. ($2x = 100$, $x = 50$.)

19. 98°. (180 – 50 – 32 = 98.)

•20. Three noncollinear points determine a plane.

21. If the feet of the four legs were not coplanar, the tripod might tip back and forth.

Geometry in Spanish.

22. In every right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

•23. Square.

24. Equal.

25. They are defined by other words, which results in going around in circles.

26. Point, line, plane.

27. Because we must have a starting point to avoid going around in circles. These words are especially simple and so serve as good starting points. (Note: The answer is *not* that we cannot define them.)

Courtroom Questions.

28. The witness thought π might be 2.12 or 2.17. The witness also said that pi times 2.5 squared is 19 but, rounded to the nearest whole number, the answer would be 20.

29. No. The judge guessed that pi was 3.1214, perhaps influenced by the witness's guess of 2.12, and 3.1214 is too small.

30. No.

Set II (pages 73–74)

In his book titled *Science Awakening* (Oxford University Press, 1961), B. L. van der Waerden said of the Egyptians' method for finding the area of a circle: "It is a great accomplishment of the Egyptians to have obtained such a good approximation. The Babylonians, who had reached a much higher stage of mathematical development, always used $\pi = 3$."

Although it is unlikely that your students will be surprised that the circumference/perimeter ratio of the circle to its circumscribed square is equal to the area/area ratio, this relation is remarkable. It holds true even if the circle and square are stretched into an ellipse and rectangle. John Pottage wrote a nice dialogue between Galileo's characters Simplicio and Segredo about this topic in the first chapter of his *Geometrical Investigations—Illustrating the Art of Discovery in the Mathematical Field* (Addison-Wesley, 1983).

The planet so small that someone can walk around it in 20 minutes is included in Lewis Carroll's *Sylvie and Bruno Concluded* (1893). More about Carroll's mathematical recreations can be found in Martin Gardner's *The Universe in a Handerchief* (Copernicus, 1996). Carroll wrote of the planet: "There had been a great battle . . . which had ended rather oddly: the vanquished army ran away at full speed, and in a very few minutes found themselves face-to-face with the victorious army, who were marching home again, and who were so frightened at finding themselves between *two* armies, that they surrendered at once!"

31. Postulates.

•32. Theorems.

33. Direct and indirect.

•34. 144 square units. ($12^2 = 144$.)

35. 256 square units. ($16^2 = 256$.)

36. 400 square units. ($144 + 256 = 400$.)

•37. 20 ft. ($x^2 = 16^2 + 12^2 = 400$, $x = \sqrt{400} = 20$.)

38. $(\frac{16}{9}r)^2 = (\frac{256}{81})r^2$; so π would be equal to $\frac{256}{81}$, or approximately 3.16.

39. They would be too large.

Panda Proof.

40. *Second statement:* If he had a sandwich, he would take out a gun.
Fourth statement: If he shot the waiter, he would leave without paying.

•41. Direct.

Clue Proof.

42. *Beginning assumption:* Suppose that Colonel Mustard didn't do it.
Missing statement: If Miss Scarlet did it, then it was done in the dining room.
Contradiction: It happened after 4 P.M.
Conclusion: Colonel Mustard did it.

43. Indirect.

Circle in the Square.

•44. $2\pi r$.

45. $8r$.

46. $\frac{2\pi r}{8r} = \frac{\pi}{4}$. ($\frac{\pi}{4}$ is approximately 0.785.)

47. πr^2.

•48. $4r^2$.

49. $\frac{\pi r^2}{4r^2} = \frac{\pi}{4}$.

50. Approximately 1,680 feet. If someone can walk three miles in an hour, they can walk one mile in 20 minutes. Because $c = \pi d$,

$$d = \frac{c}{\pi} = \frac{5{,}280}{\pi} \approx 1{,}680 \text{ feet.}$$

Algebra Review (page 76)

•1. $10x$.

•2. $16x^2$.

•3. $6x + 5$.

•4. $4x - 5$.

•5. $6x^2$.

•6. $-6x^2$.

•7. $9x + 4y$.

•8. $3x + 2y$.

•9. $5x + 15$.

•10. $44 - 11x$.

•11. $18x + 6$.

•12. $40 - 56x$.

•13. $9x^2 - 18x$.

•14. $20x + 6x^2$.

•15. $x^2y + xy^2$.

•16. $2x^2 - 4x + 8$.

•17. $5x - 3 = 47$,
$5x = 50$,
$x = 10$.

•18. $9 + 2x = 25$,
$2x = 16$,
$x = 8$.

•19. $4x + 7x = 33$,
$11x = 33$,
$x = 3$.

•20. $10x = x + 54$,
$9x = 54$,
$x = 6$.

•21. $6x - 1 = 5x + 12$,
$x - 1 = 12$,
$x = 13$.

•22. $2x + 9 = 7x - 36$,
$-5x + 9 = -36$,
$-5x = -45$,
$x = 9$.

•23. $8 - x = x + 22$,
$8 - 2x = 22$,
$-2x = 14$,
$x = -7$.

•24. $3x - 5 = 10x + 30$,
$-7x - 5 = 30$,
$-7x = 35$,
$x = -5$.

•25. $4(x - 11) = 3x + 16$,
$4x - 44 = 3x + 16$,
$x - 44 = 16$,
$x = 60$.

•26. $x(x + 7) = x^2$,
$x^2 + 7x = x^2$,
$7x = 0$,
$x = 0$.

•27. $5(x + 2) = 2(x - 13)$,
$5x + 10 = 2x - 26$,
$3x + 10 = -26$,
$3x = -36$,
$x = -12$.

•28. $6(4x - 1) = 7(15 + 3x)$,
$24x - 6 = 105 + 21x$,
$3x - 6 = 105$,
$3x = 111$,
$x = 37$.

•29. $8 + 2(x + 3) = 10$,
$8 + 2x + 6 = 10$,
$2x + 14 = 10$,
$2x = -4$,
$x = -2$.

•30. $x + 7(x - 5) = 2(5 - x)$,
$x + 7x - 35 = 10 - 2x$,
$8x - 35 = 10 - 2x$,
$10x - 35 = 10$,
$10x = 45$,
$x = 4.5$.

•31. $4(x + 9) + x(x - 1) = x(6 + x)$,
$4x + 36 + x^2 - x = 6x + x^2$,
$3x + 36 = 6x$,
$36 = 3x$,
$x = 12$.

•32. $2x(x + 3) + x(x + 4) = 5x(x + 2) - 2x^2$,
$2x^2 + 6x + x^2 + 4x = 5x^2 + 10x - 2x^2$,
$3x^2 + 10x = 3x^2 + 10x$,
$0 = 0$;
so x can be any number.

Chapter 3, Lesson 1

Set I (pages 80–81)

Zero among all numbers is surely the cause of the most confusion. Ask anyone other than someone knowledgeable in mathematics what number one divided by zero is or what zero divided by zero is, and you will probably not get a meaningful answer. The problem, of course, is that there is no number equal to $\frac{n}{0}$ if $n \neq 0$, whereas quantities that approach $\frac{0}{0}$ may yield any number in the limit, a fact of which beginning calculus students become aware when they first encounter l'Hôpital's Rule. That is why $\frac{0}{0}$ itself must remain undefined as well.

Charles Seife remarks in his book titled *Zero—The Biography of a Dangerous Idea* (Viking, 2000): "If you wantonly divide by zero, you can destroy the entire foundation of logic and mathematics. Dividing by zero once—just one time—allows you to prove, mathematically, anything at all in the universe."

•1. Subtraction.

•2. Substitution.

3. Multiplication.

4. Substitution.

5. Division.

6. $a + b + c + d = b + e + f + g$.

7. Subtraction.

8. Subtraction.

9. Substitution.

•10. Addition.

11. Substitution.

12. Division (or multiplication by $\frac{1}{2}$).

13. 2.

•14. No number (because there is no number when multiplied by 0 that gives 6.)

15. 0.

16. Any number!

17. Any number can replace ?, although this fact makes the first equation seem strange for numbers other than 0 or 1.

18. Substitution.

•19. If $a = b$ and $c \neq 0$, then $\frac{a}{c} = \frac{b}{c}$.

If $c = d$, then $\frac{a}{c} = \frac{b}{d}$ by substitution.

Correcting Mistakes.

20. In the first step, 7 was subtracted from the left side of the equation but added to the right side. The equation should be $2x = 12$.

•21. Substitution.

22. Indirect.

Set II (pages 81–82)

We will return to the "sum of the area of the crescents equal to the area of the triangle" problem later in the course.

The figure from the Chinese text illustrating the Pythagorean Theorem is, of course, the one from Euclid's *Elements*. Although the Chinese text dates back only about 400 years, the Pythagorean Theorem may have been known in China long before Pythagoras was born. The theorem appears in the oldest Chinese mathematics book known, the *Chou Pei*, possibly written as long ago as 1100 B.C.

Triangle and Crescents.

23. They are equal.

24. By subtracting II + IV from both sides of the equation I + II + IV + V = II + III + IV, we get I + V = III.

Binomial Square.

•25. a^2 and b^2.

•26. $2ab$.

27. $a + b$.

•28. $(a + b)^2$.

29. $a^2 + 2ab + b^2$.

30. $(a + b)^2 = a^2 + 2ab + b^2$.

Chinese Proof.

31. The Pythagorean Theorem.

•32. Substitution.

33. Substitution.

34. Addition.

35. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

Circle Formula.

36. $a = \pi r^2$ and $r = \frac{d}{2}$; so $a = \pi(\frac{d}{2})^2 = \frac{\pi}{4}d^2$.

Quadratic Formula.

37. $ax^2 + bx + c = 0$.

•38. Division.

39. Subtraction.

40. Addition.

•41. A direct proof.

42. Substitution.

43. $x = \frac{-5 \pm \sqrt{25 - 24}}{4}$, $x = \frac{-5 \pm 1}{4}$;

$x = \frac{-5 + 1}{4} = \frac{-4}{4} = -1$;

$x = \frac{-5 - 1}{4} = \frac{-6}{4} = -1.5$.

Dilcue's Pie.

44. The area and circumference of the pie were evidently the same number.

45. If $\pi r^2 = 2\pi r$, then $\frac{\pi r^2}{\pi r} = \frac{2\pi r}{\pi r}$, so $r = 2$. The radius of the pie must have been 2 feet. The circumference must have been $2\pi(2) = 4\pi$ (or approximately 12.6) feet.

46. 4π (or approximately 12.6) *square* feet.

Set III (page 83)

There are quite a number of fallacious algebraic "proofs" based on dividing by zero in a disguised form. Carroll's version, accompanied by his charming whimsy, is undoubtedly the best—especially because it results in such an unexpected answer to one of the first multiplication problems that every child encounters.

Because $x = y$, $x - y = 0$. In dividing each side of the equation by $(x - y)$, we are dividing by 0, so the rest does not follow. This is especially obvious if we substitute 0 for $(x - y)$ in the equation $2(x + y)(x - y) = 5(x - y)$. It is certainly true that $2 \times 2 \times 0 = 5 \times 0$, but not that $2 \times 2 = 5$.

Chapter 3, Lesson 2

Set I (pages 86–88)

Although the triple jump was included in the first modern Olympics in 1896, it was probably not part of the ancient Greek Olympics. The triple jump is now thought to have originated as a result of a misunderstanding. The Greeks added the lengths of the three best jumps in the long-jump competition, which may have led to the idea that they were performing a "triple" jump. The world record for the triple jump is currently a little more than 18 meters.

The closest footwear that we currently have to "seven league boots" was reported in the July 15, 2000, issue of *New Scientist* magazine. A team of Russian engineers has developed fuel-powered boots that enable the wearer to jump over 6-foot walls, take 12-foot steps, and walk at speeds of 35 miles an hour!

Hotels and apartment buildings commonly have floor numbers that skip from 12 to 14, owing mainly to the possible superstition of potential residents. Even room numbers ending in 13 are often omitted.

1. The points on a line can be numbered so that positive number differences measure distances.

2. A point is between two other points on the same line iff its coordinate is between their coordinates.

3. If A-B-C, then AB + BC = AC.

•4. The definition of betweenness of points.

5. The Betweenness of Points Theorem.

Triple Jump.

6. 18 meters. (58 – 40 = 18.)

•7. 52. (47 + 5 = 52.)

8. 6 meters. (58 – 52 = 6.)

•9. BC + CD = BD, or 5 + 6 = 11.

10. AB + BD = AD, or 7 + 11 = 18.

11. AC + CD = AD, or 12 + 6 = 18.

Eclipse Problems.

12. Collinear.

13. Figure B.

14. 93,240,000 mi. (SE + EM = SM, 93,000,000 + 240,000 = 93,240,000.)

•15. 92,760,000 mi. (SM + ME = SE, SM + 240,000 = 93,000,000, SM = 92,760,000.)

Seven League Boots.

16. B, 11; C, 18; D, 25; E, 32; F, 39.

•17. 18.

18. 39.

19. How far (in leagues) the person is from home.

Skyscraper.

•20. 9 – 2 = 7.

21. 15 – 5 = 10.

22. No. The second answer is wrong because the ruler omits 13. (The skyscraper does not have a "13th" floor.)

Football Field.

23. A, 25; B, 35.

24. AB = 35 – 25 = 10.

25. The lines are not numbered like a ruler because to most numbers there correspond two points, not one.

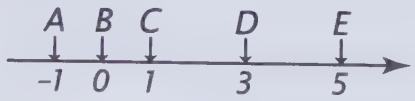
•26. A, –25; B, 15.

27. AB = 15 – (–25) = 40.

28. Yes.

SAT Problem.

29.



•30. B.

31. 6 units. [5 – (–1) = 6.]

Set II (pages 88–90)

When we observe stars at night, any three stars that seem to be collinear almost certainly are not. Such stars in space generally determine the vertices of a triangle that we happen to be seeing "on edge."

At first glance, the arrangement of the numbers on the yardstick in exercise 48 does not seem especially remarkable. The story behind them, however, is rather intriguing. Martin Gardner in his book titled *The Magic Numbers of Dr. Matrix* (Prometheus Books, 1985) noted: "Henry E. Dudeney, in problem 180, *Modern Puzzles* (1926), asked for the placing of eight marks on a 33-inch ruler and gave 16 solutions. Dudeney undoubtedly believed that at least nine marks were required for any ruler longer than 33 inches." Gardner reported that the yardstick problem of exercise 46 was solved by John Leech in 1956 and first published in a paper titled "On the Representation of 1, 2, . . . , *n* Differences" in the *Journal of the London Mathematical Society*. It has since been proved that Leech's solution for marking the yardstick with only 8 marks is the only one possible.

Pole Vaulter.

32. A point is between two other points on the same line iff its coordinate is between their coordinates.

33. The pole is bent; so the points are not collinear.

34. If A-B-C, then AB + BC = AC.

•35. AB + BC is greater than AC.

36. No.

Ladder Problem 1.

37. *(Student answer.)* ("Yes" seems reasonable.)

•38. AC = AB + BC.

39. BD = BC + CD.

40. Substitution.

41. Subtraction.

Ladder Problem 2.

42. B, 23; C, 35; D, 47; E, 59; F, 71; G, 83; H, 95; I, 107; Y, 118.

43. 107 – 11 = 96 and 8 × 12 = 96.

44. $2 \times 11 + 8 \times 12 = 22 + 96 = 118$.

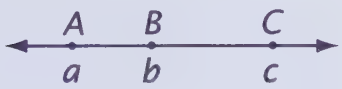
Stars Proof.

45. Proof.
Suppose that star Y is between star X and star Z.
If X-Y-Z, then XY + YZ = XZ.
If XY + YZ = XZ, then 7.2 + 9.8 = 16.6.
This contradicts the fact that 17.0 ≠ 16.6.
Therefore, our assumption is false and
star Y is not between star X and star Z.

Signal Problem.

46. If you don't change your speed, you will travel $3 \times 44 = 132$ feet in 3 seconds. Because A-B-C, AB + BC = AC; so $x + 60 = 132$ and $x = 72$. You can be 72 feet from the intersection.

47.



Proof.
(1) A-B-C. (The hypothesis.)
(2) $a > b > c$. (Definition of betweenness of points and this case.)
(3) $AB = a - b$ and $BC = b - c$. (Ruler Postulate.)
(4) $AB + BC = (a - b) + (b - c) = a - c$. (Addition.)
(5) $AC = a - c$. (Ruler Postulate.)
(6) AB + BC = AC. (Substitution.)

Yardstick Distances.

48.

1	AB, IJ	19	BF
2	BC	20	AF
3	AC, CD	21	DG
4	GH, HI	22	EI
5	BD, HJ	23	EJ
6	AD	24	CG
7	DE, EF, FG	25	DH
8	GI	26	BG
9	GJ	27	AG
10	CE	28	CH
11	FH	29	DI
12	BE	30	BH, DJ
13	AE	31	AH
14	DF, EG	32	CI
15	FI	33	CJ
16	FJ	34	BI
17	CF	35	AI, BJ
18	EH	36	AJ

Set III (page 90)

It is always fun to see good mathematics problems posed and solved in unexpected places. The oar problem is adapted from an example in the entry on "Oars" in John Lord's fascinating book titled *Sizes—The Illustrated Encyclopedia* (HarperCollins, 1995).

1. Half of CB is 21 inches and half of the overlap is 2 inches; so AB and CD should each be 21 + 2 = 23 inches.

2. Because $AB = 7x = 23$, $x = \frac{23}{7}$. Because $AF = 25x$, $AF = 25(\frac{23}{7}) \approx 82$ inches. (Notice that the length of the paddle is not needed.)

3. Because 82 inches ≈ 6.8 feet, you should probably order 7-foot oars.

Chapter 3, Lesson 3

Set I (pages 93–95)

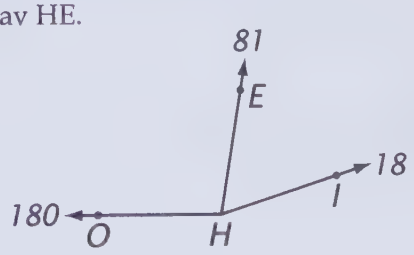
More examples of rotations in sports include somersaults in gymnastics and spins in figure skating. Single, double, and triple axels in skating are named for their inventor, Norwegian skater Axel Paulsen.

1. The rays in a half-rotation can be numbered from 0 to 180 so that positive number differences measure angles.

2. A ray is between two others in the same half-rotation iff its coordinate is between their coordinates.

3. If OA-OB-OC, then $\angle AOB + \angle BOC = \angle AOC$.

4. Rav HE.



5. $\angle OHE = 99°$, $\angle EHI = 63°$, and $\angle OHI = 162°$.

•6. The definition of betweenness of rays.

7. The Protractor Postulate.

•8. Rays.

9. Its sides.

10. ∠A, ∠CAB, ∠PAC.

Danger Area.

•11. The coordinates.

12. 75°.

•13. Acute.

14. One revolution. (The player jumps, spins around once, and releases the ball before landing.)

15. The diver turns one and a half rotations, or 540°, during the dive.

•16. 15. (One each at A, E, and G, and three each at B, C, D, and F.)

17. All 15 of them.

18. ∠CBG, ∠CDE, ∠CFE, ∠CFG.

19. ∠ABG, ∠BCF, ∠DCF, ∠ADE, ∠E, ∠G.

20. No. We do not know from what perspective the photograph was taken.

Bubble Angles.

21. Right.

•22. Obtuse.

23. 90°. ($\frac{360}{4} = 90$.)

24. 120°. ($\frac{360}{3} = 120$.)

Cactus Spokes.

•25. 14.4°. (There are 25 spokes and hence 25 such angles; $\frac{360}{25} = 14.4$.)

•26. Yes.

27. Its coordinate is between their coordinates. (CS-CP-CK because 43.2 < 57.6 < 86.4.)

28. 14.4°. (57.6 – 43.2 = 14.4.)

•29. 28.8°. (86.4 – 57.6 = 28.8.)

30. 43.2°. (86.4 – 43.2 = 43.2.)

•31. ∠SCP + ∠PCK = ∠SCK.

32. The Betweenness of Rays Theorem.

33. If OA-OB-OC, then ∠AOB + ∠BOC = ∠AOC.

Set II (pages 95–97)

Runway Numbers.

34. 18.

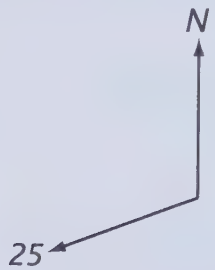
•35. 27.

36. They are $\frac{1}{10}$th of the numbers on the protractor.

37. 23. (50° + 180° = 230°.)

•38. They are the same.

39. 110°. (360° – 250°.)



40. The left runway and the right runway.

Pool Ball Angles.

•41. Betweenness of Rays Theorem. (If OA-OB-OC, then ∠AOB + ∠BOC = ∠AOC.)

42. Substitution.

43. Subtraction.

NBC Peacock.

•44. 20°. ($\frac{180}{9} = 20$.)

45. 18°. ($\frac{180}{10} = 18$.)

46. 15°. ($\frac{180}{12} = 15$.)

47. 16.363636 . . .° (or $16\frac{4}{11}$°). ($\frac{180}{11} = 16\frac{4}{11}$.)

Minutes and Seconds.

48. An hour.

49. 3,600. (1° = 60' = 60 × 60" = 3,600".)

•50. 360.

•51. 15. ($\frac{360°}{24 \text{ hours}}$.)

52. 15.

53. 15.

54. *Proof.*
(1) OA-OB-OC. (The hypothesis.)
(2) $a < b < c$. (Definition of betweenness of rays and this case.)
(3) $\angle AOB = b - a$ and $\angle BOC = c - b$. (Protractor Postulate.)
(4) $\angle AOB + \angle BOC = (b - a) + (c - b) = c - a$. (Addition.)
(5) $\angle AOC = c - a$. (Protractor Postulate.)
(6) $\angle AOB + \angle BOC = \angle AOC$. (Substitution.)

Set III (page 97)

This exercise is another example of the need for deductive reasoning and proof. No matter how accurately the figure may be drawn, measurements of $\angle EFB$ can neither give its exact value nor reveal why it should be 20°. J. L. Heilbron calls it "The Tantalus problem" after Tantalus, the character in Greek mythology from which the word "tantalize" is derived. The name is fitting because the prospective solver of the problem is "tormented with something desired but out of reach."

A nice discussion of the problem and a similar but easier one are included on pages 292–295 of Heilbron's *Geometry Civilized* (Clarendon Press, 1998). More interesting information about the Tantalus problem is included in *Tracking the Automatic Ant*, by David Gale (Springer, 1998). See pages 123–125 on "Configurations with Rational Angles" and the research done on the problem by Armando Machado on pages 227–232.

1. *(Student drawing of figure.)*
2. $\angle AGB = 50°$, $\angle AGF = 130°$, $\angle BGF = 130°$, $\angle C = 20°$.
3. On an accurate drawing, it should appear to be about 20°.

Chapter 3, Lesson 4

Set I (pages 100–101)

The arrow figure in exercises 12 through 14 is a variation of the "Muller-Lyer illusion." It was discovered in 1899 by C. H. Judd, who mentioned it in an article on geometrical illusions published in *Psychological Review*. Other variations of this illusion are included in *Can You Believe Your Eyes?* by J. R. Block and Harold E. Yuker (Brunner/Mazel, 1992).

Origami is the subject of one of Martin Gardner's early "Mathematical Games" columns in *Scientific American* (July 1959), reprinted in *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961). As a child, Peter Engel was inspired by this column and has written a wonderful book on the subject, *Folding the Universe* (Vintage Books, 1989). Beyond including detailed procedures for folding a large variety of figures from simple to extremely complex, Engel relates the history of origami and its connections with many other fields. For example, in a section of the book titled "The Psychology of Invention," Engel includes a summary of French mathematician Jacques Hadamard's remarks on the subject of invention in mathematics. *Folding the Universe* is without doubt the definitive book on origami for the mathematics classroom.

1. Points that are on the same line.
2. Lines that contain the same point.
•3. Figures that can be made to coincide (or fit exactly together).
4. A drawing made with a straightedge and compass.
5. A statement formed by interchanging the hypothesis and conclusion of another statement.
6. A theorem that is easily proved as a consequence of another statement.
•7. Then it divides it into two equal segments.
8. If a point divides a line segment into two equal segments, then it is the midpoint of it.
•9. Each is the converse of the other.
10. A line segment has exactly one midpoint.
11. An angle has exactly one ray that bisects it.

Arrow Illusion.

12. (Most people would say M.)
13. No, because a line segment has exactly one midpoint.
14. AN = NB.

Origami Duck.

•15. No.

16. Yes.

17. ∠BAD.

18. ∠BCD, ∠ACD, and ∠BCA.

•19. 45°.

20. 22.5°.

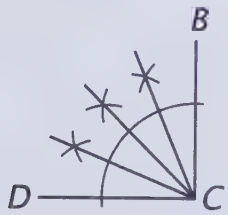
21. 45°.

•22. 67.5°.

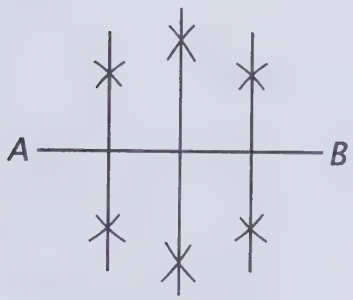
23. 67.5° (because in triangle DFC, ∠D = 90° and ∠FCD = 22.5°).

Bisection Constructions.

24.



25.



Clock Puzzle.

•26. The midpoint of a line segment divides it into two equal segments.

27. 1.5.

•28. 1.5.

29. D, 4.5; E, 6; F, 7.5.

30. 7.5 seconds.

Set II (pages 102–104)

Students taking chemistry might enjoy identifying the atoms in the acetylene molecule: two carbon atoms (points B and C in the figure) and two hydrogen atoms (points A and D). According to Linus Pauling, the H–C bond length is 1.1 angstroms and the C–C bond length is 1.2 angstroms.

Euclid defined an angle as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line"; so, for Euclid, straight angles are not considered.

David W. Henderson, in his valuable book titled *Experiencing Geometry* (Prentice Hall, 2001), offers some important insights with regard to the question, "What is an angle?" He writes:

"I looked in all the plane geometry books in the university library and found their definitions for 'angle.' I found nine different definitions! Each expressed a different meaning or aspect of 'angle' . . . .

There are at least three different perspectives from which one can define 'angle,' as follows:

a *dynamic* notion of angle—angle as *movement*;

angle as *measure*; and,

angle as *geometric shape* . . . .

Each of these perspectives carries with it methods for checking angle congruency. You can check the congruency of two dynamic angles by verifying that the actions involved in creating or replicating them are the same. If you feel that an angle is a measure, then you must verify that both angles have the same measure. If you describe angles as geometric shapes, then one angle should be made to coincide with the other using isometries in order to prove angle congruence."

Yen Measurement.

•31. 2 cm.

•32. 2π (or about 6.28). ($c = \pi d = 2\pi$.)

33. π (or about 3.14).

Acetylene Proofs.

34. Addition.
Betweenness of Points Theorem.
Substitution.

35. Suppose that B is the midpoint of AC.
The midpoint of a line segment divides it into two equal segments.
Substitution.
The fact that AC > 2AB.
B is not the midpoint of AC.

Miter Joint Proof.

36. (1) PA-PC-PB. (By hypothesis.)
    (2) $\angle 1 + \angle 2 = \angle APB$. (Betweenness of Rays Theorem.)
    (3) PB bisects $\angle APB$. (By hypothesis.)
    (4) $\angle 1 = \angle 2$. (If a line bisects an angle, it divides the angle into two equal angles.)
    (5) $\angle 1 + \angle 1 = \angle APB$; so $2\angle 1 = \angle APB$. (Substitution.)
    (6) $\angle 1 = \frac{\angle APB}{2}$. (Division.)

Angle Definition.

37. No. The second figure looks like a line.

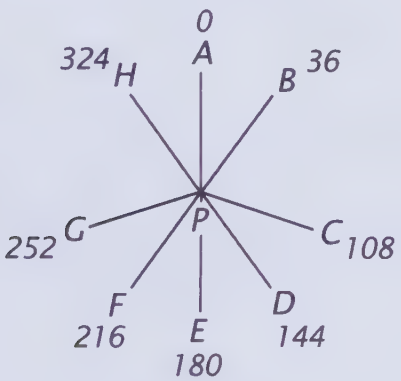
38. Unless it is marked on the angle, we don't know where it is.

39. It is hard to tell because we don't know on which side of the line to draw them.

40. It seems as if a straight angle has two bisectors, one on either side of the line.

Passion Flower.

41.



•42. 180.

43. 72°. ($\frac{360}{5} = 72$.)

•44. They are each 36°. ($\frac{72}{2} = 36$.)

45. PB, 36; PC, 108; PD, 144; PF, 216; PG, 252; PH, 324.

46. No, because the rays do not lie in a "half-rotation."

47. Yes, because it divides $\angle GPC$ into two equal angles.

Set III (page 104)

In the chapter on Mascheroni constructions in *Mathematical Circus* (Knopf, 1979), Martin Gardner wrote:

"It is often said that the ancient Greek geometers, following a tradition allegedly started by Plato, constructed all plane figures with a compass and a straightedge (an unmarked ruler). This is not true. The Greeks used many other geometric instruments, including devices that trisected angles. They did believe, however, that compass-and-straightedge constructions were more elegant than those done with other instruments. . . .

In later centuries geometers amused themselves by imposing even more severe restrictions used in construction problems. The first systematic effort of this kind is a work ascribed to the 10th-century Persian mathematician Abul Wefa, in which he described constructions possible with the straightedge and a 'fixed compass,' later dubbed the 'rusty compass.'"

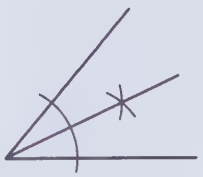
That it is the 120° angle that can be most easily bisected with a rusty compass will be confirmed in a later lesson.

Rusty Compass Constructions.

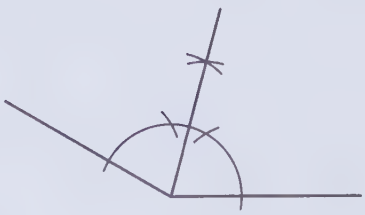
1.



2.



3.



4. An angle of 120°.

Chapter 3, Lesson 5

Set I (pages 107–108)

•1. 90°.

2. 180°.

3. A *compliment*.

Safe Angle.

•4. 87°.

5. They are complementary.

6. 177°.

7. They are supplementary.

•8. 30°. ($\frac{360}{12} = 30$.)

•9. 150°.

10. 60°.

11. From signs of the zodiac (astrological signs).

12.

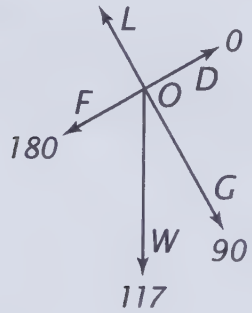


13. 15°. ($\frac{90}{6} = 15$.)

14. They are multiples of 15°.

Skier Forces.

15.



•16. ∠DOW + ∠WOF = ∠DOF.

17. ∠DOW = 117 – 0 = 117°;
∠WOF = 180 – 117 = 63°;
∠DOF = 180 – 0 = 180°.

18. They are supplementary.

•19. 27°. (∠WOG = 117 – 90 = 27°.)

20. They are complementary.

21. ∠WOG, ∠WOF.

22. ∠LOW, ∠DOW.

Triangle in the Woods.

23. The word supplementary is defined in reference to two angles, not three.

24. No. If two of its angles were supplementary, their sum would be 180°; there would be no degrees left for the third angle.

25. Yes. Two angles could have a sum of 90°; the third angle would consequently be a right angle.

Hammer Throw.

•26. Betweenness of Rays Theorem.

27. A right angle is equal to 90°.

•28. Substitution.

29. If a line bisects an angle, it divides the angle into two equal angles.

30. Substitution.

31. Division.

Set II (pages 108–109)

The Arctic Circle problems are based on information in Ernest Zebrowski's *History of the Circle—Mathematical Reasoning and the Physical Universe* (Rutgers University Press, 1999). Zebrowski writes:

"Shown on every globe and drawn (in projection) on every world map, the Arctic Circle connects points around Earth at 66.54° north latitude, and the Antarctic Circle is at 66.54° south latitude. But why would anyone attach special significance to such an unusual number? Clearly, it must not be the number itself that is important, but rather some geographical concept that gives rise to this numerical value. In fact, the latitude of the Arctic Circle is but an artifact of a geometrical relationship, and it is this underlying geometry that is significant.

We've already seen that the Tropic of Capricorn is a circle of latitude where the sun is directly overhead at noon around the date of December 22, and that this happens at a latitude of 23.46° south of the equator. On that day of the

year, the sun's path is quite low in the sky in the Northern Hemisphere, and the farther north we go, the lower the path of the sun. At points 90° north of the Tropic of Capricorn, we reach a circle of latitude where the sun barely peeks over the horizon at noon at the winter solstice. This is the Arctic Circle. North of this circle, the sun does not rise at all during the first day of winter. The Arctic Circle has a latitude of 23.46° south latitude, and Earth's spherical geometry requires that there be a 90° angle between the two.

Meanwhile . . . we find a circle of latitude around the South Pole that basks in sunlight all twenty-four hours of the day when locations within the Arctic Circle are in twenty-four hour darkness. Six months later, of course, the roles of the Arctic and Antarctic Circles are reversed."

Clock Hands.

32. 8:30.



33. 60°.

34. If the big hand were on the 6, the small hand would be halfway from 8 to 9.

•35. 75°.

Complement and Supplement.

36.

∠A	∠C	∠S
10°	80°	170°
20°	70°	160°
30°	60°	150°

•37. ∠S seems to be 90° larger than ∠C.

38. $\angle C = (90 - x)°$; $\angle S = (180 - x)°$.

39. $\angle S - \angle C = (180 - x)° - (90 - x)° = 90°$, or $\angle S = \angle C + 90°$.

Earth Angles.

40. They are complementary.

41. On the horizon.

•42. They are supplementary.

43. The sun never rises.

44. OB bisects ∠AOC. ∠1 = ∠2 because complements of the same angle are equal.

45. Yes. ∠1 + ∠AOC = 90° and ∠AOC = ∠1 + ∠2; so ∠1 + ∠1 + ∠2 = 90°. Also ∠1 = ∠2; so 3∠1 = 90° and so ∠1 = 30°. Therefore ∠2 = 30° and ∠AOC = 60°.

46. *Proof.*
    (1) ∠1 and ∠2 are supplements of ∠3. (Given.)
    (2) ∠1 + ∠3 = 180°; ∠2 + ∠3 = 180°. (If two angles are supplementary, their sum is 180°.)
    (3) ∠1 + ∠3 = ∠2 + ∠3. (Substitution.)
    (4) ∠1 = ∠2. (Subtraction.)

Set III (page 109)

The angles of the squares are each 90°. Three squares and three star points surround a point just below the center of the figure.

Because the sum of the three angles of the squares is 270°, the sum of the three star points is 360° – 270° = 90°. The angles at the points of the stars are evidently each 30°.

Chapter 3, Lesson 6

Set I (pages 112–114)

The perception example of exercises 3 through 6 is derived from the work of computer scientist A. Guzmán as reported in *Human Information Processing—An Introduction to Psychology,* by Peter H. Lindsay and Donald A. Norman (Harcourt Brace Jovanovich, 1977). Guzmán analyzed pictures of overlapping figures and discovered that, wherever three lines meet to form a linear pair in a figure, it usually indicates that one object is in front of another. This rule is just one of a number of geometric rules that we automatically use in interpreting line drawings.

A rule of thumb followed by civil engineers in applying linear pairs to the construction of road intersections is that the smaller of the two angles in which two streets meet at a "T intersection" should not be less than 80°.

•1. Because they are vertical angles.

2. Because they form linear pairs.

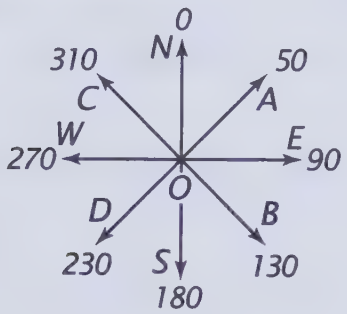
3. As a four-sided figure in front of a triangle.

4. A, B, D, E; C, B, I, H; D, F, G; I, F, E.

5. $\angle CBA$ and $\angle ABI$ (ABH); $\angle HIA$ and $\angle AIB$ (AIC); $\angle DFE$ and $\angle EFG$.
6. No.
•7. They are a linear pair and they are supplementary.
8. 125°.
9. $(180 - x)°$.
10. $\angle 1$.

Sun Directions.

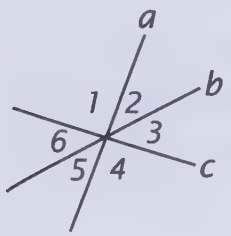
11.



•12. 80°. (130 – 50 = 80.)
13. 80°.
•14. They are vertical angles.
15. 100°.
•16. No.
17. If two angles are equal, then they are vertical angles.
18. No.
19. No.
20. Yes.

Ship Position.

21. Concurrent.
22. *(Figure similar to the one below. Measurements of angles will vary.)*



•23. Two.
24. 180°.
25. Yes. Because $\angle 1 + \angle 2 + \angle 3 = 180°$ and $\angle 5 = \angle 2$, $\angle 1 + \angle 3 + \angle 5 = 180°$ by substitution.
•26. 180°.
27. The sum of the angles of a triangle is 180°.
28. 180°.
29. These angles are vertical angles with the angles of the triangle and so equal to them.
•30. These angles are linear pairs; so they are supplementary.
31. Addition.
32. 360°.
33. Because $\angle 2 + \angle 3 + \angle 5 + \angle 6 + \angle 8 + \angle 9 = 540°$ and $\angle 3 + \angle 6 + \angle 9 = 180°$, $\angle 2 + \angle 5 + \angle 8 = 360°$ by subtraction.

Set II (pages 114–115)

The bending of light in a gravitational field certainly complicates our ability to perceive the physical universe. In the part titled "The Rise of Geometrical Analysis" in their book *Mathematics in Civilization* (Dover, 1984), H. L. Resnikoff and R. O. Wells, Jr., write:

"All terrestrial experiments appear to show that light traveling in a vacuum follows a straight-line path, and it was natural to assume that this would be true of light passing through any region in which there is a gravitational force . . . .

The fundamental idea underlying Einstein's model, that the force of gravity acts by creating curvature in space . . . suggests that the paths of light-rays should be deformed from straight lines wherever the geodesics due to gravity are not straight. In particular, this model suggests that the path of a light particle should bend toward the more curved regions (in physical terms, toward the *sources* of gravitational force). For a light-ray just grazing the limb of the sun the Einstein model predicts that the direction of the ray will be bent toward the sun by an angle of 1.75". This prediction has been verified . . . ."

Steep Slope.

34.

∠1	∠2	∠3	∠4
25°	65°	115°	65°
33°	57°	123°	57°
48°	42°	138°	42°

35. $\angle 4 = \angle 2$.

•36. Vertical angles are equal.

•37. $\angle 3 = \angle 1 + 90°$, or $\angle 3 - \angle 1 = 90°$.

38. $\angle 2 = (90 - x)°$;
$\angle 3 = [180 - (90 - x)]° = (90 + x)°$;
$\angle 4 = (90 - x)°$.

39. $(90 + x)° - x° = 90°$.

Hedge Shears.

40. $\angle ESR$ and $\angle HSA$.

41. SA is always between SE and SR;
SR is always between SA and SH.

42. $\angle ESA$ and $\angle HSR$.

43. *Proof.*
(1) $\angle ESR = \angle HSA$. (Given.)
(2) $\angle ESA + \angle ASR = \angle ESR$;
$\angle ASR + \angle HSR = \angle HSA$. (Betweenness of Rays Theorem.)
(3) $\angle ESA + \angle ASR = \angle ASR + \angle HSR$. (Substitution.)
(4) $\angle ESA = \angle HSR$. (Subtraction.)

Bent Light Ray.

44. There are 60 minutes in 1 degree and 60 seconds in 1 minute.

•45. 179° 59' 58.25".

Pinhole Camera.

46. 45°. ($\angle 1 = \angle 2$ and $\angle 1 + \angle 2 = 90°$.)

47. *Proof.*
(1) $\angle 1$ and $\angle 2$ are vertical angles. (Given.)
(2) $\angle 1 = \angle 2$. (Vertical angles are equal.)
(3) $\angle 1$ and $\angle 2$ are complementary. (Given.)
(4) $\angle 1 + \angle 2 = 90°$. (If two angles are complementary, their sum is 90°.)
(5) $\angle 1 + \angle 1 = 90°$ so $2\angle 1 = 90°$. (Substitution.)
(6) $\angle 1 = 45°$. (Division.)

SAT Problem.

48. The angles labeled $(5z)°$ and $(6z + 4)°$ form a linear pair; so $(5z) + (6z + 4) = 180$, $11z = 176$, and $z = 16$. The angles labeled $(5z)°$ and $(8w)°$ also form a linear pair; so $5z + 8w = 180$. Because $z = 16$, $5(16) + 8w = 180$, $8w = 100$, and so $w = 12.5$.

Set III (page 116)

The Chinese proof of the vertical angle theorem is identical with that given by Euclid (Book I, Proposition 15).

Euclid's equivalent of our linear pair theorem is his Proposition 13, which states: "If a straight line set up on a straight line makes angles, it will make either two right angles or angles equal to two right angles." Hence the appearance of "$2\angle R$" in the Chinese proof rather than reference to either "supplements" or "180°".

1. AB and CD intersect at O.
2. $\angle AOD$ and $\angle BOC$ are vertical angles, $\angle AOC$ and $\angle BOD$ are vertical angles.
3. To prove.
4. Proof.
5. The measures of two right angles, or 180°.
6. The angles in a linear pair are supplementary.
7. Substitution.
8. Subtraction.

Chapter 3, Lesson 7

Set I (pages 119–120)

It is interesting to note that the scuba diver figure comes from a section titled "Geometrical Optics" in a physics textbook (Maschuri L. Warren's *Introductory Physics*, W. H. Freeman and Company, 1979). The author points out that Isaac Newton used five postulates about the geometry of optics as well as geometric constructions in his analysis of the behavior of light. Newton's book, *Opticks*, was published in 1704 and was immediately accepted as the definitive description of light. Newton modeled *Opticks*, as well as his most famous work, the *Principia*, after Euclid's *Elements* by starting with definitions and postulates and then proceeding to develop the subject matter through a logical sequence of theorems.

In working the exercises on the Necker cube, your students may discover it "flipping back and forth" and consequently be ready for the ambiguity of exercise 16. Donald D. Hoffman includes a long discussion of this illusion in *Visual Intelligence—How We Create What We See* (Norton, 1998). Hoffman remarks that, although the picture "feels flat," it "looks like a cube." The viewer "constructs the cube" by following certain "rules" critical to vision. He says:

"No one teaches you these rules. Instead, you acquire them early in life in a genetically predetermined sequence that requires, for its unfolding, visual experience."

The difficulty with the Necker cube is that these rules produce two conflicting but equally coherent interpretations of it as a figure in three dimensions.

•1. Two angles are complementary if their sum is 90°.

2. Two lines are perpendicular if they form a right angle.

3. All right angles are equal.

•4. Supplements of the same angle are equal.

5. A line bisects an angle if it divides it into two equal angles.

•6. If the angles in a linear pair are equal, then their sides are perpendicular.

7. Two lines are parallel iff they lie in the same plane and do not intersect.

8. They may not lie in the same plane. They may lie in lines that intersect.

9. They are concurrent.

•10. Vertical angles are equal.

11. The angles in a linear pair are supplementary.

12. Five.

Cube Illusion.

13. DC, HG, and EF.

•14. DH, AE, and, if perspective is taken into account, HG, EF.

15. (There are four: DA, HE, AB, and EF.)

16. Either one, depending on how you look at the cube.

Mud Cracks.

•17. They are a linear pair.

18. They are supplementary.

19. 90°.

20. A right angle.

•21. They are perpendicular.

22. If the angles in a linear pair are equal, then their sides are perpendicular.

Set II (pages 120–121)

The waterfall exercise comes, of course, from the fact that, if a line is perpendicular to a plane, it is perpendicular to every line in the plane that passes through the point of intersection. This definition will be considered formally in the chapter on geometric solids.

A Deceptive Figure.

23. Yes. AO ⊥OD because ∠1 is a right angle.

•24. ∠1 and ∠3 look like vertical angles, yet they are not equal.

25. ∠1 and ∠4 look like a linear pair as do ∠1 and ∠2, yet they are not supplementary.

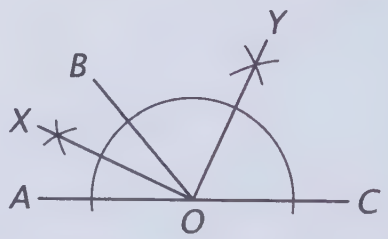
26. Except for AO and OD, the lines that look perpendicular do not form right angles.

27. No. If they were vertical angles, the figure would contain two pairs of opposite rays and all four angles would be right angles.

Bisecting a Linear Pair.

28. 130°.

29.



•30. ∠BOX, 25°; ∠BOY, 65°.

31. 90°.

32. The angles in a linear pair are supplementary.

•33. 180°.

34. 90°.

35. A right angle.

36. Because it has a measure of 90°.

•37. They are perpendicular.

38. Because they form a right angle.

39. $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 360°$ so $90 + n + 2n + 3n = 360$, $6n = 270$, and $n = 45$.

40. $\angle 1 = 90°$, $\angle 2 = 45°$, $\angle 3 = 90°$, $\angle 4 = 135°$.

41. They are equal.

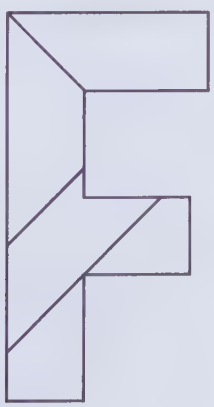
42. They are supplementary.

43. *Proof.*
(1) WA ⊥ AB and WA ⊥ AC. (Given.)
(2) ∠WAB and ∠WAC are right angles. (Perpendicular lines form right angles.)
(3) ∠WAB = ∠WAC. (All right angles are equal.)

44. *Proof.*
(1) ∠1 and ∠2 are complementary. (Given.)
(2) ∠1 + ∠2 = 90°. (The sum of two complementary angles is 90°.)
(3) PB-PC-PD. (Given.)
(4) ∠BPD = ∠1 + ∠2. (Betweenness of Rays Theorem.)
(5) ∠BPD = 90°. (Substitution.)
(6) ∠BPD is a right angle. (A 90° angle is a right angle.)
(7) AB ⊥ PD. (Two lines that form a right angle are perpendicular.)

Set III (page 122)

The FD Puzzle was sold by the Perry and Elliott Company of Massachusetts and is a variation of the well-known "T-Puzzle," which will be explored in the lesson on trapezoids.



Chapter 3, Review

Set I (pages 124–125)

•1. Yes. E-F-H because $6.1 < 7.7 < 10.8$.

2. Point D is closer because CB = 1.7 and CD = 1.6.

3. No. AE = 6.1 and EI = 5.9.

Sliding Bevel.

4. They are always supplementary.

•5. $(180 - x)°$.

6. ∠BEV is obtuse and ∠VEL is acute.

7. 90°.

•8. The handle is perpendicular to the blade.

9. If the angles in a linear pair are equal, their sides are perpendicular.

Old Quadrant.

10. 47°. (65 – 18 = 47.)

•11. CE-CS-CD.

12. 23.5°.

•13. 41.5.

14. 65 – 41.5 = 23.5.

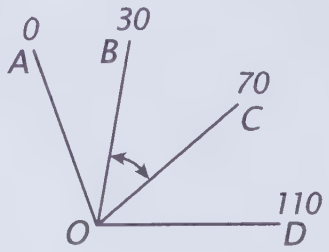
Ollie's Logic.

15. Yes. It is true that either two lines intersect or they do not.

16. If two lines intersect, they must form a right angle to be perpendicular. If two lines do not intersect, they must lie in the same plane to be parallel.

SAT Problem.

17.



•18. 110.

19. 30.

20. 40°.

Shoe Laces.

21. They are vertical angles and they are equal.

22. 45°.

•23. They would be perpendicular.

24. They would be equal.

25. Supplements of the same angle are equal.

Axioms and Corollaries.

26. An axiom is like a postulate. It is a statement that is assumed to be true without proof.

27. A corollary is a theorem that can be easily proved as a consequence of another statement that is already accepted.

•28. Any number is equal to itself.

29. A line segment has *exactly one midpoint*.

30. An angle has exactly one ray *that bisects it*.

31. All right angles *are equal*.

Set II (pages 126–128)

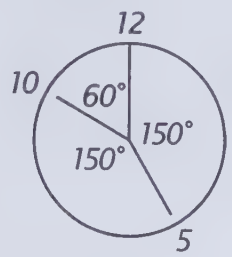
The chair-design exercises are from the field of biotechnology. According to *Human Factors in Engineering and Design*, by Mark S. Sanders and Ernest J. McCormick (McGraw-Hill, 1992),

"The field of human factors—referred to as *ergonomics* in Europe and elsewhere—deals with the consideration of human characteristics, expectations, and behaviors in the design of the things people use in their work and everyday lives and of the environments in which they work and live."

Your students may be surprised by the extensive use of mathematics, including geometry and statistics, in this field.

Clock Problems.

32. 150° at 5 o'clock; 60° at 10 o'clock.

33. 

$60° = 360° - 2(150°)$.

Stilt Distances.

•34. $(6 + x)$ ft.

35. x ft.

36. *Proof.*
(1) S-F-T and F-T-H. (Given.)
(2) SF + FT = ST and FT + TH = FH. (Betweenness of Points Theorem.)
(3) ST = FH. (Given.)
(4) SF + FT = FT + TH. (Substitution.)
(5) SF = TH. (Subtraction.)

Chair Design.

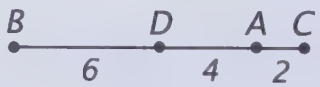
37. $\angle 2 = (90 - x)°$.

•38. $\angle AOB = (90 - x + y)°$.

39. *Proof.*
Suppose that $\angle AOB$ is a right angle.
$\angle AOB = 90°$. *(A right angle equals 90°.)*
$90 - x + y = 90$. *(Substitution.)*
$-x + y = 0$. *(Subtraction.)*
$y = x$. *(Addition.)*
This contradicts the fact that if x is between 5 and 8 degrees and y is between 20 and 25 degrees, $y \neq x$.
Therefore, our assumption is false and *$\angle AOB$ cannot be a right angle.*

40. No. [$\angle AOB = (90 - x + y)°$. To find the smallest possible value of $\angle AOB$, let $x = 8$ and $y = 20$. $\angle AOB = 90 - 8 + 20 = 102°$. $\angle AOB$ is at least 102°.]

Another SAT Problem.

41. 

42. 4 units.

Graphite and Diamond.

•43. Hexagons.

44. 720°.
$(\angle 1 = \frac{360°}{3} = 120°; 6\angle 1 = 6(120°) = 720°.)$

45. *Proof.*
(1) $\angle 1 + \angle 2 + \angle 3 = 360°$ and all of the angles in the figure are equal. (Given.)
(2) $\angle 1 + \angle 1 + \angle 1 = 360°$; so $3\angle 1 = 360°$. (Substitution.)
(3) $6\angle 1 = 720°$. (Multiplication.)

•46. Six.

47. ∠AOB, ∠BOC, ∠COD, ∠AOD, ∠AOC, ∠BOD.

48. *Proof.*
Suppose that ∠AOB and ∠BOC are a linear pair.
∠AOB and ∠BOC are supplementary. *(The angles in a linear pair are supplementary.)*
∠AOB + ∠BOC = 180°. *(The sum of two supplementary angles is 180°.)*
This contradicts the fact that ∠AOB + ∠BOC = 109.5° + 109.5° = 219°.
Therefore, our assumption is false and *∠AOB and ∠BOC cannot be a linear pair.*

49. No. Any three points are always coplanar. (If they are collinear, they lie in many different planes. If they are noncollinear, they determine a plane.)

March Times.

•50. 150.

51. 3,000. (100·30 = 3,000.)

52. $30n$.

•53. 120 steps per minute.

54. 3,600 inches per minute. (120·30 = 3,600.)

55. Approximately 3.4 miles per hour. (3,600 inches per minute = 300 feet per minute = 18,000 feet per hour ≈ 3.4 miles per hour.)

•56. 6,480 inches per minute. (180·36 = 6,480.)

57. No. Twice 3,600 is 7,200, not 6,480.

Algebra Review (page 130)

•1. $6x + 1$.

•2. $2x - 4y$.

•3. $4x^2 + 8$.

•4. $3x^2 + 6x - 5$.

•5. $12x - 3$.

•6. $x^3 + 4x^2 - x - 8$.

•7. $14x - 5y$.

•8. $7x + 4$.

•9. $6x$.

•10. $-x^2 - 11x$.

•11. $2x^2 + 10x - 1$.

•12. $2x^3 + y^3$.

•13. $x^2 + 14x + 24$.

•14. $6x^2 - 7x - 20$.

•15. $16x^2 - 1$.

•16. $x^3 - 10x^2 - 23x + 6$.

•17. $4y^3 - 8y^2 + 13y - 5$.

•18. $x^3 + y^3$.

•19. $5x^3 + 3x^2 - 20x - 12$.

•20. $(x^2 - x - 6)(x + 6) = x^3 + 5x^2 - 12x - 36$.

•21. $10x - 35 - 10x - 14 = -49$.

•22. $(10x - 1)(10x - 1) = 100x^2 - 20x + 1$.

•23. $(x + 4)(x + 4)(x + 4) = (x^2 + 8x + 16)(x + 4) = x^3 + 12x^2 + 48x + 64$.

•24. $x^2 - (x^2 + 4x - 5) = -4x + 5$.

Chapter 4, Lesson 1

Set I (pages 134–136)

David Wells tells about Nicholas Saunderson in *The Penguin Book of Curious and Interesting Mathematics* (Penguin Books, 1997). Although blind from the age of 12, Saunderson became a noted mathematician and was appointed to the professorship first held by Sir Isaac Newton at Cambridge University. His pin board, now known as the geoboard, is popular even today.

Pin-Board Geometry.

1. The axes.
2. The origin.

•3. A and E.

4. D.

•5. B.

6. G.

•7. A (4, 3), C (–3, –4).

8. D (6, 4), E (4, –2), F (–2, 0), G (0, 6).

•9. On the y-axis.

10. On the x-axis.

Grids.

11. The x-axis.

•12. Two.

13. To remind them to read (list) the x-coordinate before the y-coordinate.

14. (154, 263).

15. Both are divided into sections by parallel and perpendicular lines.

Birds and Bats.

•16. The fourth (IV).

17. The second and third (II and III).

18. The parrots.

•19. The hummingbirds.

20. The storks.

21. The albatrosses.

22. Approximately (–2.5, 0) and (–35, 0).

An Early Graph.

•23. Point A.

24. Their x-coordinates are negative numbers and their y-coordinates are zero. (Also, the x-coordinate of D is less than the x-coordinate of P.)

25. Both are negative. (Also, the x-coordinate of T is equal to the x-coordinate of P.)

Set II (pages 136–138)

Although we can show at this point that the current arrows have equal lengths, we have not yet learned how to prove them parallel. Some of your students may know that such arrows represent a vector, a quantity with magnitude and direction.

The Japanese home problems provide a preview of the translation transformation, a topic to be explored later.

It is amusing to realize that computer users are unknowingly selecting coordinates every time they click the mouse. Without a coordinate system, the mouse couldn't operate as it does.

It will not be long until we can prove the conjecture suggested in exercise 49—that it is sufficient to know that the sides of two triangles are equal in order to know that they are congruent.

Students intrigued by the "distances in different dimensions" problems will be interested in Thomas Banchoff's book titled *Beyond the Third Dimension,* cited in the footnote in the text. Banchoff, Professor of Mathematics at Brown University, is a specialist in the geometry of higher dimensions and has created award-winning films on the subject.

Tidal Current.

26. They seem to be parallel and to be equal in length.

•27. A (4, 1), B (6, 6).

$AB = \sqrt{(6-4)^2+(6-1)^2} = \sqrt{4+25} = \sqrt{29} \approx 5.39.$

28. C (–4, –2), D (–2, 3).

$CD = \sqrt{[-2-(-4)]^2+[3-(-2)]^2} = \sqrt{4+25} = \sqrt{29}.$

29. E (0, –6), F (2, –1).

$EF = \sqrt{(2-0)^2+[-1-(-6)]^2} = \sqrt{4+25} = \sqrt{29}.$

30. Yes (assuming that all of the arithmetic has been done correctly!)

Japanese Floor Plan.

•31. A_1 (5, 2), A_2 (15, 2).

32. B_1 (7, 6), B_2 (17, 6).

33. The x-coordinates are 10 units larger; the y-coordinates are the same.

34. The floor plan would appear (10 units) above the post pattern.

Computer Screen.

•35. (900, 200).

36. (100, 800).

•37. 250 mm. ($AC = \sqrt{(900-100)^2+(800-200)^2} = \sqrt{640{,}000+360{,}000} = \sqrt{1{,}000{,}000} =$ 1,000 pixels. $1{,}000 \times 0.25$ mm = 250 mm.)

38. 250 mm. ($BD = \sqrt{(100-900)^2+(800-200)^2}$, etc. The same distance as AC.)

39. The eBay icon.

Distance Formulas.

40. $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

•41. $\sqrt{3^2+3^2} = \sqrt{18}$, or $3\sqrt{2}$.

42. Yes. ($\sqrt{(x_2-x_1)^2} = |x_2-x_1|$), the positive difference of the coordinates.)

43. $\sqrt{(4-1)^2} = \sqrt{3^2} = 3$.

44. $\sqrt{3^2+3^2+3^2} = \sqrt{27}$, or $3\sqrt{3}$.

45. $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2}$.

46. $\sqrt{3^2+3^2+3^2+3^2} = \sqrt{36} = 6$.

Triangle Coordinates.

47. $AB = \sqrt{20^2+15^2} = \sqrt{400+225} = \sqrt{625} = 25$; $BC = 20$; $AC = 15$.

48. $CD = \sqrt{16^2+12^2} = \sqrt{256+144} = \sqrt{400} = 20$; $OC = \sqrt{9^2+12^2} = \sqrt{81+144} = \sqrt{225} = 15$; $OD = 25$.

49. It seems as if the tracing would fit because the sides of the triangles are equal.

Set III (page 138)

A Circle Challenge.

To find out, we need to find the distances of the six points from the center of the circle.

$OA = \sqrt{63^2+16^2} = \sqrt{4{,}225} = 65$.

$OB = \sqrt{25^2+60^2} = \sqrt{4{,}225}$.

$OC = \sqrt{(-36)^2+54^2} = \sqrt{4{,}212} \approx 64.9$.

$OD = \sqrt{(-52)^2+39^2} = \sqrt{4{,}225}$.

$OE = \sqrt{(-33)^2+(-56)^2} = \sqrt{4{,}225}$.

$OF = \sqrt{20^2+(-62)^2} = \sqrt{4{,}244} \approx 65.1$.

The radius of the circle is 65 so point C is inside the circle and point F is outside.

Chapter 4, Lesson 2

Set I (pages 141–143)

Chapter 11 of *Mathematical Gems II*, by Ross Honsberger (M.A.A., 1976) deals with the art-gallery problem. Honsberger reports:

"At a conference in Stanford in August, 1973, Victor Klee asked the gifted young Czech mathematician Václav Chvátal (University of Montreal) whether he had considered a certain problem of guarding the paintings in an art gallery. The way the rooms in museums and galleries snake around with all kinds of alcoves and corners, it is not an easy job to determine the minimum number of guards that are necessary to survey the entire building. The guards are to remain at fixed posts, but they are able to turn around on the spot. The walls of the gallery are assumed to be straight. In a short time, Chvátal had the whole matter figured out. He showed that if the gallery has n walls (i.e., if the floor plan is an n-gon), then, whatever its zig-zag shape, the minimum number of guards needed is never more than $\left[\frac{n}{3}\right]$, the integer part of $\frac{n}{3}$."

The symmetry of the art gallery in exercises 20 through 24 makes four guards sufficient. (Note that one guard standing at one of the corners of the central room can see all of it as

well as one entire wing.)

To realize that the "magic magazine" polygon of exercise 25 can be divided into *any* number of congruent polygons is surprising at first but obvious when you see how!

Connect the Dots.

1. The sides.
•2. The vertices.
3. 32.
•4. A 32-gon.
5. No. The first and last line segments would intersect only one other line segment.
•6. That they are congruent.
7. That they are equal.

Paper-Clip Patent.

•8. They are right angles (and equal).
9. 45°.
10. They are vertical angles.
•11. Line segments AB and EF intersect two others but not at their endpoints.

Origami Duck.

•12. ΔAEG.
13. ΔACB.
14. ΔCGF, ΔCGE, ΔCBE.
15. ΔACF.
16. ΔCEF.
•17. CBEG.
18. DCEF.
19. ABCF.

Art Gallery.

20. Two.
21. 16.
22. A 16-gon.
23. 16.
24. Four (a guard could stand in each corner of the central square.)

Puzzle Figure.

•25. An octagon (an 8-gon).
26. Hexagons (6-gons).
•27. D.
28. B.
29. H.
30. ABHJGD ↔ HBCFIE.
•31. H.
32. C.
33. F.
•34. HBCFIE ↔ MNIEHK.
35. HBCFIE ↔ GLMKHJ.
36. MNIEHK ↔ GLMKHJ.

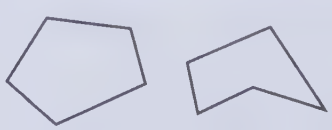
Set II (pages 143–144)

In *The Penguin Book of Curious and Interesting Geometry* (Penguin Books, 1991), David Wells says of the tessellation pattern of hexagons and stars:

"The regular polygons which tessellate individually can be transformed into tessellations of polygons and stars by moving them slightly apart, and then dividing the space between the tiles. This can be interpreted as a *hinged tessellation*. Each side of a star which is not a side of a hexagon is a strap, hinged at both ends, which holds two hexagons together. As the hexagons separate further, the star becomes fatter, then momentarily appears as a large equilateral triangle, and finally as a hexagon, identical to the original hexagon."

Tessellation Pattern.

37. Both have six sides (or both are hexagons). (Also, both are equilateral, symmetric, etc.)
38. No (as the tessellation shows).
39. Concave.
•40. Convex.
41. *(Drawings of convex and concave pentagons.)*



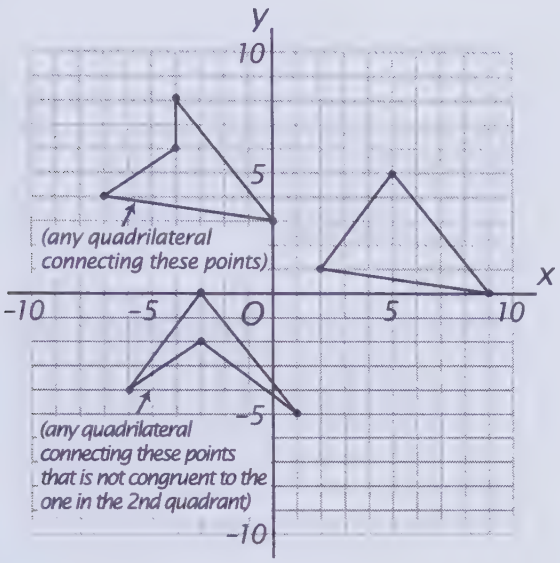
42. *(Drawings of convex and concave quadrilaterals.)*



43. No.

Polygons on a Grid.

44.



•45. No.

46. Yes, if they connected them in a different order.

47. Three.

Dividing a Rectangle.

•48. 21 square units.

49. 3.5 square units. $(\frac{21}{6})$.

50. That congruent polygons have equal areas.

51. Yes. Each piece contains 3.5 square units.

52. 45°, 90° (two angles), 135°.

•53. No.

54. No.

Proof Without a Figure.

55. (1) Given.
(2) If two triangles are congruent, their corresponding sides and angles are equal.
(3) Substitution.
(4) If their corresponding sides and angles are equal, two triangles are congruent.

Set III (pages 144–145)

In chapter 3, titled "Mandelbrot's Fractals," of *Penrose Tiles to Trapdoor Ciphers* (W. H. Freeman and Company, 1989), Martin Gardner explains that the polygons considered in the Set III exercises are among the "monster curves" discovered in the second half of the nineteenth century. David Hilbert created open-ended versions of these figures, and Waclaw Sierpinski conceived of the polygon equivalents.

1. 16.
2. 64.
3. 256 and 1,024. (Each one has four times the number of sides of the preceding one.)
4. 4,194,304.
5. The nth one would have 4^{n+1} sides. (The first one has 4^2 sides, the second has 4^3 sides, the third has 4^4 sides, the fourth has 4^5 sides, and the tenth has 4^{11} sides.)

Chapter 4, Lesson 3

Set I (pages 148–149)

According to the U.S. Coast Guard and U.S. Navy, the Bermuda, or Devil's, Triangle is "an imaginary area located off the southeastern Atlantic coast of the United States which is noted for a high incidence of unexplained losses of ships, small boats, and aircraft." The vertices of the triangle are generally taken as Bermuda, Miami, Florida, and San Juan, Puerto Rico. A sensationalized book on the Triangle was a bestseller in 1974 (*The Bermuda Triangle,* by Charles Berlitz.)

The exercises concerning intersections of a line with a triangle were inspired by the following postulate of Pasch: "If a line in the plane of a triangle intersects one side, it also intersects another side." Benno Artmann tells of Pasch in *Euclid—The Creation of Mathematics* (Springer, 1999):

"Euclid had employed the notions of ordering and betweenness of points on a line intuitively . . . . The subject of ordering was taken up by Moritz Pasch in a book *Lectures on Recent Geometry* in 1882. His axioms about betweenness essentially completed the axiomatization of plane geometry. (Pasch had such an acute sense for fine points in logic that in his later days he was the foremost expert on the bylaws of his university in Giessen. His son-in-law C. Thaer produced the standard

German translation of Euclid."

Bermuda Triangle.

•1. BF and BP.

2. FB.

•3. ∠B.

4. ∠F and ∠B.

French Postulate.

•5. SAS.

6. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent.

7. CAC.

8. ACA.

9. If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent.

Congruence Correspondences.

•10. TR = AN.

•11. ∠I = ∠G.

12. IT = GA.

13. LS = AG.

14. ∠E = ∠N.

•15. Two triangles congruent to a third triangle are congruent to each other.

Triangle Names.

16. All the sides are equal. (Some students may guess: All the angles are equal.)

•17. All the angles are acute.

18. Two sides are equal. (The labels imply that equilateral triangles are not isosceles, although this is not generally considered to be the case.)

19. One of the angles is obtuse.

•20. None of the sides are equal.

21. One of the angles is a right angle.

Intersecting a Triangle.

•22. All three.

23. Two.

24. Two.

25. No.

26. Yes.

Set II (pages 149–150)

In his book titled *Triangles—Getting Ready for Trigonometry* (Crowell, 1962), Henry M. Neely wrote about the eye-focusing problem:

"Every waking moment, while your eyes are open and you are looking at something, you are solving triangles . . . . Did you ever see a little baby reach out and try to grasp its mother's finger? It doesn't do very well. The baby will reach out in the general direction of the finger and then will swing its little hand around until its sense of touch finishes the job. The trouble is that the baby is working with a calculating machine that has not yet been completely hooked up.

The baby's eyes can measure the angles fairly well and so it is able to detect the general direction of the finger. But its eyes cannot focus to the correct distance because the brain's memory circuit has not yet stored the most important bit of information—the length of the base line.

Two angles are not enough to solve a triangle. You must also have the length of at least one side."

The astronomy exercise shows how the base line AB is measured in preparation for solving a second triangle with AB as a side and a point on the moon as its third vertex. Points A and B mark the locations of two widely separated observatories on the same meridian: Stockholm (60° N) at A and the Cape of Good Hope (34° S) at B.

Eye Focus Triangle.

27.



28. 5 inches (measured from the drawing.)

•29. 35°.

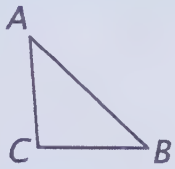
30. Yes, because 80° + 65° + 35° = 180°.

•31. They get larger.

32. It gets smaller.

Astronomy Problem.

33.



•34. About 5.9 cm.

35. We used 1 cm to represent 1,000 miles; so 5.9 cm represents 5,900 miles.

36. $\angle A \approx 43°$, $\angle B \approx 43°$.

37. They seem to be equal.

Molecule Path.

•38. Perpendicular lines form right angles.

39. All right angles are equal.

40. The reflexive property.

•41. ASA.

42. They are equal.

Pond Problem.

43. They mean that point O is between points X and A and is also between points Y and B.

•44. Both are pairs of opposite rays.

45. Vertical angles are equal.

46. SAS.

47. The distance across the pond, XY, is equal to the distance from B to A. The distance BA can be measured or paced off.

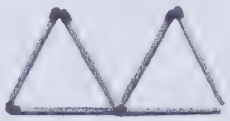
Set III (page 150)

Match Puzzles.

1. One way is shown below.



2. One way is shown below.



3. One way is shown below.

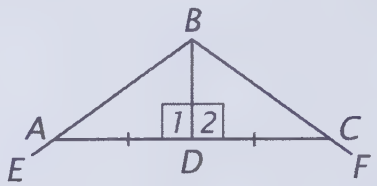


Chapter 4, Lesson 4

Set I (page 153)

Roof Plan.

1.



•2. The midpoint of a line segment divides it into two equal segments.

3. Perpendicular lines form right angles.

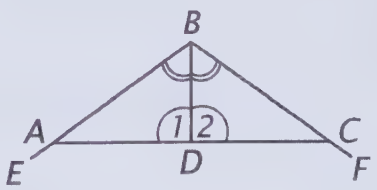
4. All right angles are equal.

5. Reflexive.

6. SAS.

•7. Corresponding parts of congruent triangles are equal.

8.

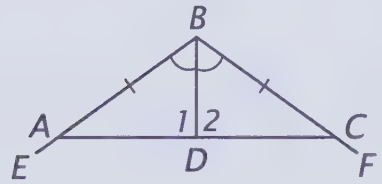


9. Reflexive.

•10. ASA.

11. Corresponding parts of congruent triangles are equal.

12.



13. If a line bisects an angle, it divides it into two equal angles.
14. Reflexive.
15. SAS.
16. Corresponding parts of congruent triangles are equal.
•17. If the angles in a linear pair are equal, then their sides are perpendicular.

Repeating Patterns.

18. Squares.
•19. Hexagons.
•20. $\Delta ABX \cong \Delta CBX$.
21. ASA.
22. $\Delta CBX \cong \Delta ECY$.
23. SAS.
24. Two triangles congruent to a third triangle are congruent to each other.
25. Corresponding parts of congruent triangles are equal.

Set II (pages 154–155)

The Muddy Boggy River is in Oklahoma.

Strange Results.

•26. No. ΔBDC is larger than ΔADB.
27. ASA does not apply, because, in ΔADB, DB is not included by $\angle A$ and $\angle ADB$.
28. AB is less than AC.
29. SAS does not apply, because the equal sides in the two triangles are not included by the equal angles.

Congruence Proofs.

30. *Proof.*
(1) HN = NR. (Given.)
(2) NO bisects $\angle HNR$. (Given.)
(3) $\angle HNO = \angle RNO$. (If an angle is bisected, it is divided into two equal angles.)
(4) NO = NO. (Reflexive.)
(5) $\Delta HNO \cong \Delta RNO$. (SAS.)

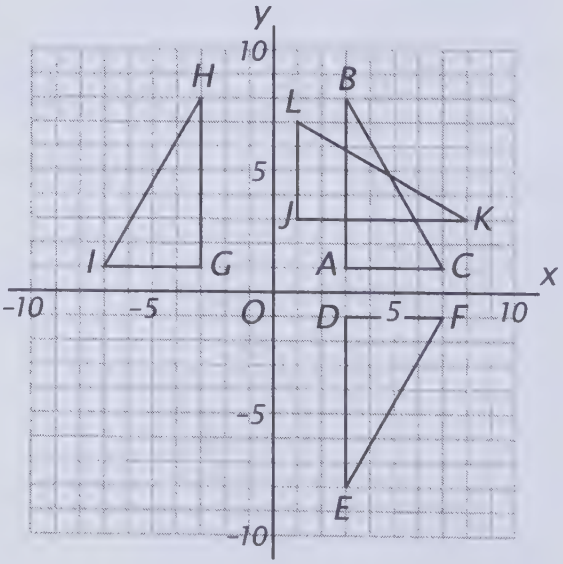
31. *Proof.*
(1) $\angle BGU$ and $\angle EGL$ are vertical angles. (Given.)
(2) $\angle BGU = \angle EGL$. (Vertical angles are equal.)
(3) BG = GE and UG = GL. (Given.)
(4) $\Delta BUG \cong \Delta ELG$. (SAS.)
(5) BU = LE. (Corresponding parts of congruent triangles are equal.)

32. *Proof.*
(1) $\angle C = \angle O$. (Given.)
(2) $\angle R$ and $\angle N$ are supplements of $\angle 1$. (Given.)
(3) $\angle R = \angle N$. (Supplements of the same angle are equal.)
(4) CR = ON. (Given.)
(5) $\Delta CRE \cong \Delta ONT$. (ASA.)

33. *Proof.*
(1) $\angle T$ and $\angle 2$ are complements of $\angle 1$. (Given.)
(2) $\angle T = \angle 2$. (Complements of the same angle are equal.)
(3) TA = AU and TU = UB. (Given.)
(4) $\Delta TAU \cong \Delta UAB$. (SAS.)
(5) AU = AB. (Corresponding parts of congruent triangles are equal.)

Grid Exercise.

34.



•35. AB = 7, AC = 4, BC = $\sqrt{65} \approx 8.1$.

36. The fourth (IV).

•37. ΔABC and ΔDEF would coincide.

•38. The x-coordinates are equal; the y-coordinates are opposites.

39. DE = 7, DF = 4, EF = $\sqrt{65}$.

•40. SAS.

41. The second (II).

42. G(–3, 1); H(–3, 8); I(–7, 1).

43. The x-coordinates are opposites; the y-coordinates are equal.

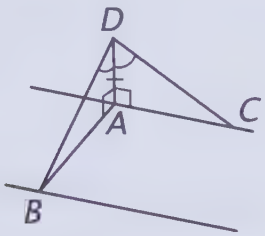
•44. Yes.

45. The x- and y-coordinates are reversed.

46. Yes. Fold the paper so that the x-axis coincides with the y-axis.

River Problem.

47.



48. Alice adjusted the rim of her hat to a point directly across the river on the opposite bank. She then turned and looked to see where the corresponding point would be along the bank on her side of the river. The triangles are congruent by ASA, so AC = AB.

Set III (page 156)

The Handcuffed Prisoners problem is adapted from Kirkman's schoolgirls problem. Both versions are discussed at length by Martin Gardner in "Dinner Guests, Schoolgirls, and Handcuffed Prisoners" in his book titled *The Last Recreations* (Copernicus, 1997). The following information is taken from Gardner's chapter.

The Reverend Thomas Kirkman, a nineteenth-century amateur British mathematician, first published his schoolgirls problem in 1847. He stated the problem in the following way: "Every day of the week a teacher takes 15 schoolgirls on a walk. During the walk the girls are grouped in triplets. Can the teacher construct the triplets so that after the seven walks each pair of girls has walked in the same triplet once and only once?"

Any solution to the problem is a "Steiner triple system," named after nineteenth-century Swiss geometer Jacob Steiner, who investigated it and other problems like it belonging to the area of combinatorics called "block-design theory." The solution for the "nine prisoners in four days" version of the problem is unique.

Dudeney's actual puzzle of the nine prisoners is a more complicated version of the schoolgirls problem in that only pairings of prisoners handcuffed *to each other* were not to be repeated.

Gardner cites quite a number of articles in various mathematics journals about these problems, with titles such as "Kirkman's Schoolgirls in Modern Dress" *(The Mathematical Gazette)*, "The Nine Prisoners Problem" *(Bulletin of the Institute of Mathematics and Its Applications)*, and "Generalized Handcuffed Designs" *(Journal of Combinatorial Theory)*.

The Problem of the Handcuffed Prisoners.

The four arrangements are:

123, 456, 789;
167, 258, 349;
148, 269, 357;
159, 247, 368.

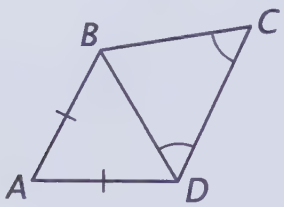
Chapter 4, Lesson 5

Set I (pages 159–160)

The figure from Euclid's *Elements* illustrates his proof of the isosceles triangle theorem (Book I, Proposition 5). Amazingly, this theorem was as far as the study of geometry went in some of the universities of the Middle Ages. The name commonly associated with it, *pons asinorum*, Latin for "bridge of asses" or "bridge of fools" may derive from the appearance of Euclid's diagram. In his *History of Mathematics—An Introduction* (Allyn & Bacon, 1985), David M. Burton describes it as resembling "a trestle-bridge so steep that a horse could not climb the ramp, though a sure-footed animal such as an ass could. Perhaps only the sure-footed student could proceed beyond this stage in geometry."

The proof by Pappus (c. 300 A.D.) is much simpler than that by Euclid and is the one

presented in the text. One of the first computer programs developed to prove theorems of geometry in 1959 "rediscovered" Pappus's proof.

1. 

2. ∠ADB.

•3. ∠ABD and ∠ADB.

•4. BD.

5. BD and BC.

•6. No.

7. Isosceles.

•8. If two sides of a triangle are equal, the angles opposite them are equal.

9. ∠A.

10. SAS.

11. Corresponding parts of congruent triangles are equal.

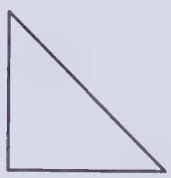
"Isosceles."

•12. That it has at least two equal sides.

13. Yes. A triangle all of whose sides are equal has at least two equal sides.

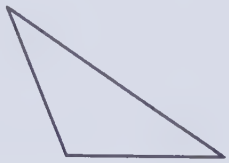
14. They are opposite the legs.

15. *Example answer:*

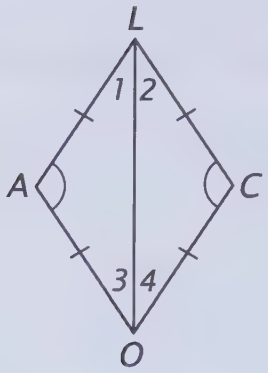


16. Not possible.

17. *Example answer:*



Alcoa Building.

18. 

19. Isosceles.

20. If two sides of a triangle are equal, the angles opposite them are equal.

•21. Addition.

•22. Betweenness of Rays Theorem.

•23. Substitution.

24. SAS.

25. Corresponding parts of congruent triangles are equal.

26. It bisects them.

Shuffleboard Court.

27. It appears to be perpendicular to them and to bisect them.

•28. They appear to be parallel.

29. Isosceles.

•30. $\angle S = \angle B = 72°$. ($\frac{180-36}{2} = \frac{144}{2} = 72$.)

31. Approximately 72°.

32. Approximately 108°. (180 – 72 = 108.)

Set II (pages 160–162)

In *The Architecture of Molecules* (W. H. Freeman and Company, 1964), Linus Pauling discusses the bond angle in the water molecule and similar molecules. The sixth column of elements in the periodic table begins with oxygen, followed by sulfur, selenium, and tellurium. The bond angle for water (H_2O) is 104.5°. For hydrogen sulfide (H_2S), it is 92.2°; for hydrogen selenide (H_2Se), it is 91°; and, for hydrogen telluride (H_2Te), it is 90°. Because the bond angle of hydrogen sulfide is fairly close to 90°, assuming that the triangle is

a right triangle in calculating the hydrogen–hydrogen distance gives a fairly good result: 1.90 angstroms. The Pythagorean Theorem is, of course, a special case of the Law of Cosines. Applying the Law of Cosines to the obtuse triangle gives a distance of 1.93 angstroms.

33. *Theorem 10.* If two angles of a triangle are equal, the sides opposite them are equal.
Given: In ΔEFG, $\angle E = \angle F$.
Prove: $GE = GF$.
Proof.
(1) In ΔEFG, $\angle E = \angle F$. (Given.)
(2) $EF = FE$. (Reflexive.)
(3) $\angle F = \angle E$. (Substitution.)
(4) $\Delta EFG \cong \Delta FEG$. (ASA)
(5) $GE = GF$. (Corresponding parts of congruent triangles are equal.)

34. *Corollary.* An equilateral triangle is equiangular.
Given: ΔABC is equilateral.
Prove: ΔABC is equiangular.
Proof.
(1) ΔABC is equilateral. (Given.)
(2) $BC = AC$ and $AC = AB$. (All of the sides of an equilateral triangle are equal.)
(3) $\angle A = \angle B$ and $\angle B = \angle C$. (If two sides of a triangle are equal, the angles opposite them are equal.)
(4) $\angle A = \angle C$. (Substitution.)
(5) ΔABC is equiangular. (A triangle is equiangular if all of its angles are equal.)

35. *Corollary.* An equiangular triangle is equilateral.
Given: ΔABC is equiangular.
Prove: ΔABC is equilateral.
Proof.
(1) ΔABC is equiangular. (Given.)
(2) $\angle A = \angle B$ and $\angle B = \angle C$. (All of the angles of an equiangular triangle are equal.)
(3) $BC = AC$ and $AC = AB$. (If two angles of a triangle are equal, the sides opposite them are equal.)
(4) $BC = AB$. (Substitution.)
(5) ΔABC is equilateral. (A triangle is equilateral if all of its sides are equal.)

Hydrogen Sulfide.

36. Isosceles.

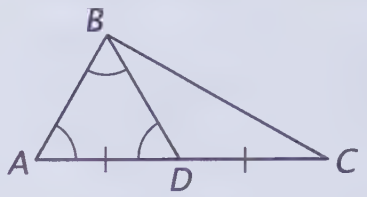
37. 43.9°.

•38. 92.2°. ($180 - 2 \cdot 43.9 = 92.2$.)

•39. Obtuse.

40. The Pythagorean Theorem applies only to right triangles, and this triangle is obtuse. (However, it is close to being a right triangle; so the Pythagorean Theorem gives a good approximation: $H_1H_2 \approx 1.9$ angstroms.)

41.



42. ΔBDC is isosceles. Because ΔABD is equiangular, it is also equilateral; so $BD = AD$. D is the midpoint of AC; so $AD = DC$. It follows that $BD = DC$ by substitution and that $\angle DBC = \angle DCB$ because the sides opposite them are equal. (It is also possible to find the measures of the angles by using the Triangle Angle Sum Theorem.)

Star Puzzles.

•43. A pentagon.

•44. Because its sides intersect in points other than their endpoints.

45. Yes. The triangles ABC, BCD, CDE, DEA, and EAB are congruent by SAS. The line segments that form the red star are corresponding parts of these triangles.

46. No. They don't look equal and the figure is said to be accurately drawn.

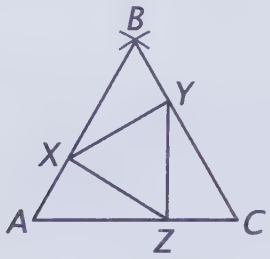
47. Yes. The triangles ACD, BDE, CEA, DAB, and EBC would be congruent by SAS. The sides of the blue polygon are corresponding parts of these triangles.

SAT Problem.

48. $y = 20°$. ΔPQT and ΔTRS are isosceles; so $\angle PTQ = \angle P = 40°$ and $\angle RTS = \angle S = 40°$. So $\angle PTS = 40° + y° + 40°$. Also $\angle PTS + \angle P + \angle S = 180°$; so $\angle PTS = 100°$. Therefore, $100 = 80 + y$; so $y = 20$.

Drawing Exercise.

49.



•50. They seem to be congruent.

51. Because ΔABC is equilateral, it is also equiangular; so ∠A = ∠B = ∠C; AX = BY = CZ = 1 inch and XB = YC = ZA = 2 inches; so the triangles are congruent by SAS.

•52. It seems to be equilateral (and equiangular).

53. Because ΔAXZ ≅ ΔBYX ≅ ΔCZY, XY = YZ = XZ (corresponding parts of congruent triangles are equal).

Set III (page 162)

Escher's reptile design can be extended to fill the plane. Among the many geometric patterns underlying it is the fact that the points at which the heads of three lizards touch are the vertices of equilateral triangles.

1. Three.
2. 120°.
3. It is the mirror image of the lizards in Escher's drawing.

Chapter 4, Lesson 6

Set I (pages 165–166)

Richard Pawley wrote an article in *The Mathematics Teacher* (May 1967) concerning the conditions under which two triangles can have five parts of one of them equal to five parts of the other and yet not be congruent. The simplest pair of such triangles whose sides are integers have sides of lengths 8-12-18 and 12-18-27. (The pair used in exercises 9 and 10 have sides of lengths 343-392-448 and 392-448-512.) An analysis of such triangles requires a knowledge of the triangle inequality theorem as well as of similarity. The golden ratio even enters into the picture!

Force Triangles.

•1. SSS.

•2. Corresponding parts of congruent triangles are equal.

Plumb Level.

3. SSS.

4. Corresponding parts of congruent triangles are equal.

•5. They are a linear pair.

6. If the angles in a linear pair are equal, their sides are perpendicular.

7. Line BC is horizontal as long as the weight hangs over the midpoint of segment BC.

8. Yes. They would be congruent by SAS.

Equal Parts.

9. No. One triangle is larger than the other.

•10. Five.

•11. No.

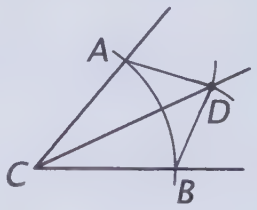
12. No.

13. No.

14. Yes. If they have six pairs of equal parts, they must be congruent by SSS.

Angle Bisection.

15.



•16. CA = CB and AD = BD.

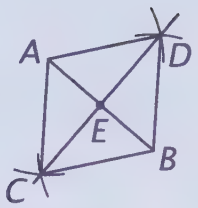
17. Reflexive.

•18. SSS.

19. Corresponding parts of congruent triangles are equal.

Segment Bisection.

20.



21. AD = BD = AC = BC.

22. SSS.

23. Corresponding parts of congruent triangles are equal.

•24. SAS. (AC = BC, ∠ACD = ∠BCD, CE = CE.)

25. Corresponding parts of congruent triangles are equal.

Set II (pages 166–167)

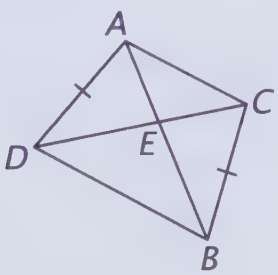
In *Geometry and the Imagination*, David Hilbert refers to the study of linkages as "a part of kinematics that is intimately connected with elementary metric geometry." He continues:

"The term 'plane linkage' applies to every plane system of rigid rods which are interconnected, or connected with fixed points of the plane about which they are free to turn, in such a way that the system is capable of certain motions in the plane. The simplest mechanism of this kind consists of a single rod with one end attached to a point of the plane; in effect, this is a compass. Just as the free end of a compass traces out a circle, so the points of the rods in all other plane linkages move on algebraic curves, i.e. curves satisfying an algebraic equation in a system of Cartesian coordinates. Conversely, it is possible to find a system of jointed rods that is suitable for the construction (at least piecemeal) of any given algebraic curve no matter how complicated."

The linkage considered in exercises 36 through 43 is Peaucellor's inversor, a device that links circular and linear motion.

Linkage Problem 1.

26.



27. They seem to be congruent.

28. Yes. ΔABC ≅ ΔCDA by SSS.

•29. ∠ADC = ∠CBA because they are corresponding parts of congruent triangles.

30. No. E does not appear to be the midpoint of either line segment.

31. Yes. ∠AEC = ∠DEB because they are vertical angles.

32. ΔDAB ≅ ΔBCD (by SSS).

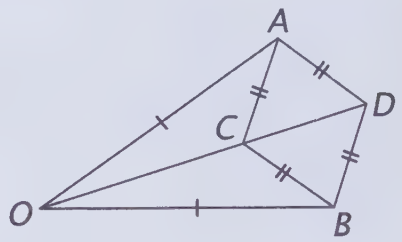
33. ∠DAB = ∠BCD.

34. ΔAED ≅ ΔCEB by ASA. ∠ADC = ∠CBA was shown in exercise 29, AD = CB was given, and ∠DAB = ∠BCD was shown in exercise 33.

35. They are isosceles. Each has a pair of equal angles: in ΔAEC, ∠CAB = ∠ACD because ΔABC ≅ ΔCDA, shown in exercise 28; in ΔDEB, ∠ABD = ∠CDB, because ΔDAB ≅ ΔBCD, shown in exercise 32.

Linkage Problem 2.

36.



37. They seem to be collinear.

•38. SSS.

39. They are corresponding parts of ΔAOC and ΔBOC.

•40. OC bisects ∠AOB.

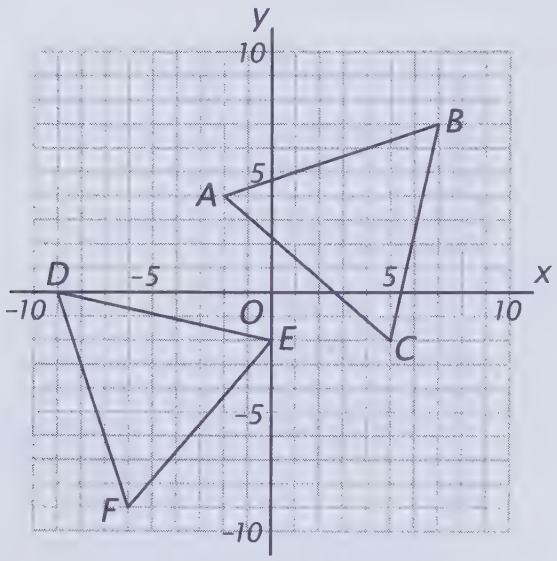
41. OD bisects ∠AOB because ΔAOD ≅ ΔBOD (SSS).

•42. Because an angle has exactly one ray that bisects it (the corollary to the Protractor Postulate).

43. They are collinear.

Grid Exercise.

44.



45. It looks as if it might be equilateral (and equiangular).

46. $AB = \sqrt{81+9} = \sqrt{90}$.
$BC = \sqrt{4+81} = \sqrt{85}$.
$AC = \sqrt{49+36} = \sqrt{85}$.

47. *(Student answer.)* (ΔABC is isosceles.)

48. $DE = \sqrt{81+4} = \sqrt{85}$.
$EF = \sqrt{36+49} = \sqrt{85}$.
$DF = \sqrt{9+81} = \sqrt{90}$.

49. ΔABC and ΔDEF are congruent by SSS.

50. $\angle A = \angle B = \angle D = \angle F$.

51. $\angle E = \angle C$.

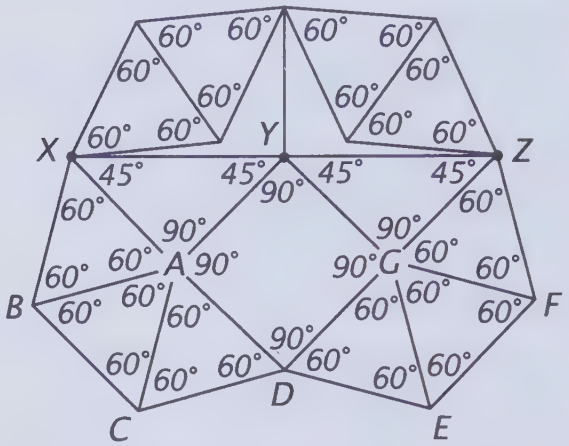
Set III (page 168)

The puzzle of bracing the square was devised by Raphael M. Robinson of the University of California at Berkeley. Robinson's solution for bracing the square in the plane required 31 rods. When Martin Gardner included it in his "Mathematical Games" column, 57 readers sent in a solution using 25 rods, and about 20 other readers discovered the solution illustrated in the Set III exercise. Gardner remarked that solutions sent in with fewer than 23 rods turned out to be either nonrigid or geometrically inexact. His *Sixth Book of Mathematical Games from Scientific American*, cited in the text, includes several illustrations of alternative solutions proposed for the puzzle, some correct and some incorrect.

Bracing a square in space turns out to be surprisingly simple: eight rods can be attached to it, four "above" and four "below" to form a regular octahedron.

Bracing the Square.

1.



2. Because $\angle XYA = 45°$, $\angle AYG = 90°$, and $\angle GYZ = 45°$, $\angle XYZ = 180°$. X, Y, and Z must be collinear because $\angle XYZ$ is a straight angle.

Chapter 4, Lesson 7

Set I (pages 172–173)

In the law of reflection, the angles of incidence and reflection are considered to be the angles formed by the ray of light with the normal to the mirror. Because your students are unlikely to know this, they are as likely to copy $\angle 1$ as $\angle 2$; the end result, of course, is the same.

•1. A compass.

•2. Distances between pairs of points.

3. Yes.

4. No.

•5. A line segment is bisected.

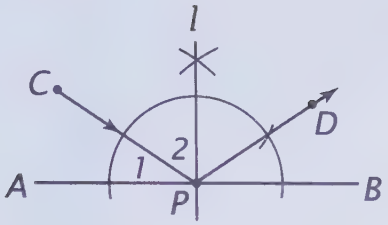
6. An angle is copied.

7. An angle is bisected.

8. A triangle is copied.

Light Reflection.

9.



10. Line l is perpendicular to line AB.

•11. $\angle 2$ is complementary to $\angle 1$.

•12. Copying an angle.

Bridge Girder.

13.



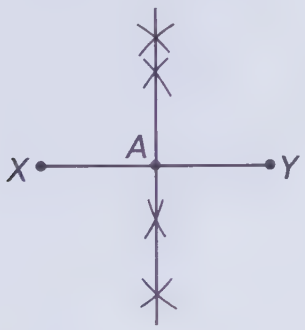
14. SSS.

15. An equilateral triangle is equiangular, and corresponding parts of congruent triangles are equal.

16. The two lines appear to be parallel.

Equidistant Points.

17.



•18. It means equally distant.

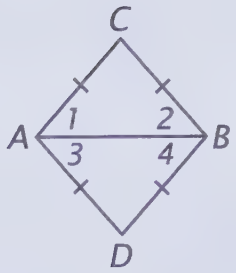
19. One.

20. Infinitely many.

21. On the line constructed in exercise 17 to find the midpoint of segment XY (the perpendicular bisector of XY).

Car Jack.

22.



23. Isosceles.

24. If two sides of a triangle are equal, the angles opposite them are equal.

•25. Addition.

•26. Substitution.

27. SSS.

28. Corresponding parts of congruent triangles are equal.

29. It bisects them.

Set II (pages 174–175)

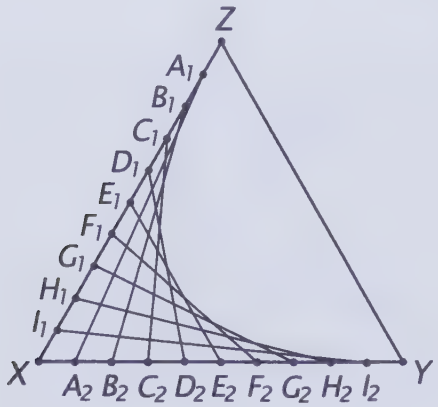
The surprise in the equilateral triangle construction is the illusion of a curve; specifically, a parabola. The line segments form its envelope.

The triangle suggested by Vitruvius for use in designing stair steps is the 3-4-5 right triangle. We can't prove that such a triangle is a right triangle at this point, however, because we have not yet established the converse of the Pythagorean Theorem.

An amazing story goes with the guitar fret construction exercise. In *Another Fine Math You've Got Me Into . . .* (W. H. Freeman and Company, 1992), Ian Stewart reports that the inventor of the construction, Daniel Strähle, explained it in an article in the *Proceedings of the Swedish Academy* in 1743. One of the members of the academy, a noted mathematician, checked the construction and claimed that it was far more inaccurate than a musician would find acceptable. More than two centuries later (1957), J. M. Barbour of Michigan State University discovered that this noted mathematician had made a mistake. In fact, his mistake was so large that it could have easily been discovered by measuring the drawing! (He computed the measure of $\angle AIU$ as 49°14', whereas it is actually 33°32'.) Strähle, the man with no mathematical training, was vindicated and the expert discovered to be wrong!

A Construction with an Unexpected Result.

30.



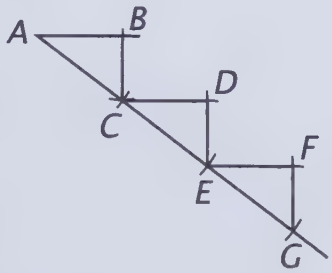
31. A curve.

32. ΔXI_1I_2.

33. SAS.

Stair Design.

34.



35. Right triangles.

•36. SSS.

•37. Height = 7.5 in; width = 10 in.

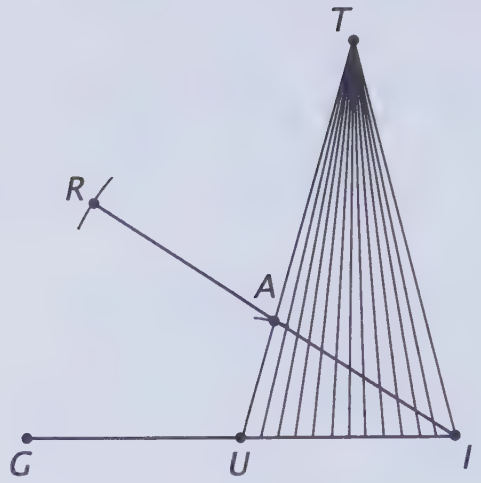
(10 ft = 120 in; $\frac{120 \text{ in}}{16 \text{ steps}} = 7.5$ in per step;

3 units = 7.5 in; so 1 unit = 2.5 in and
4 units = 10 in.)

38. 13 ft, 4 in. (16 × 10 in = 160 in = 13 ft, 4 in.)

Guitar Frets.

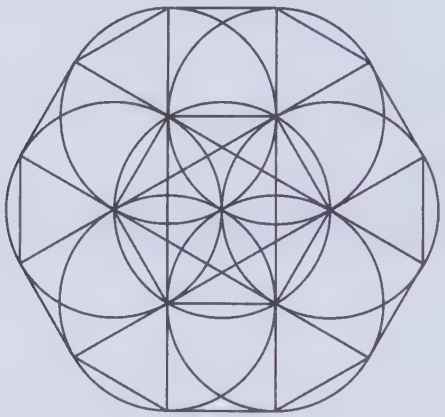
39.



40. They are not evenly spaced (they are spaced more closely together toward the body of the guitar).

Set III (page 175)

1.



2. 6.

3. 14.

4. 12.

Chapter 4, Review

Set I (pages 176–178)

Approximate angles of repose of various materials can be found in handbooks for architects, civil engineers, and contractors. Such angles are especially important in estimating the stability of excavations. Round gravel, for example, has an angle of repose of about 30°, whereas "angular" gravel has an angle of repose of about 38°.

Polar-Bear Pattern.

1. It is not a polygon, because two of its sides are not line segments; they are curved.

•2. Because they have the same size and shape (they can be made to coincide).

3. The bears fill the plane.

Turkish Geometry.

•4. They are all equal.

5. Two of them are equal.

6. None of them are equal.

From Euclid.

7. Statement 3.

•8. Statement 4.

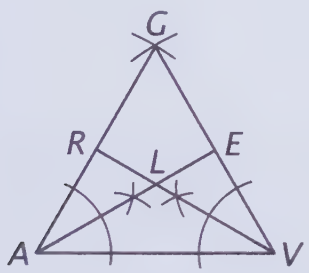
9. A postulate is a statement that is assumed to be true without proof; a theorem is a statement that is proved.

10. Substitution.

11. Statement 2.

Angles of Repose.

12.



•13. An equilateral triangle is equiangular.

14. Division.

•15. Substitution.

16. If two angles of a triangle are equal, the sides opposite them are equal.

•17. 60°. (By the Triangle Angle Sum Theorem.)

18. 30°.

•19. 120°.

20. 90°.

•21. 30°.

22. 90°.

23. 45°.

24. Yes, because we can construct a 90° angle and we can bisect it to construct a 45° angle.

Congruence in 3-D.

•25. SSS.

26. Corresponding parts of congruent triangles are equal.

27. $\angle ABD$ and $\angle ACD$. (They are the other angles opposite sides marked with three tick marks.)

28. $\angle BAD$, $\angle BCD$, and $\angle ADC$. (They are the other angles opposite sides marked with two tick marks.)

Set II (pages 178–180)

One of the elementary figures of projective geometry is the "complete quadrilateral," the subject of exercises 35 and 36. Four lines in a plane, no two of which are parallel and no three of which are concurrent, intersect in six points. The six points determine three line segments that are called the "diagonals" of the quadrilateral. That the midpoints of these diagonals are always collinear is a theorem of projective geometry.

The emblem showing the compass and carpenter's square is a registered trademark of the Masons. It appears on the door of Masonic temples and has been used by the organization since 1801. The G is said to stand for *God* and *geometry*.

SAT Problem.

•29. 90°. [$x + y + (x - y) = 180$, $2x = 180$, $x = 90$.]

30. 90°.

31. 90°.

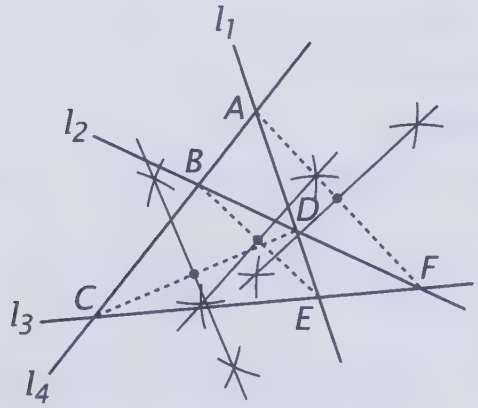
32. The value of x does not depend on the value of y.

•33. 45°.

34. The triangle cannot be equilateral, because $y = 90°$.

Lines and Midpoints.

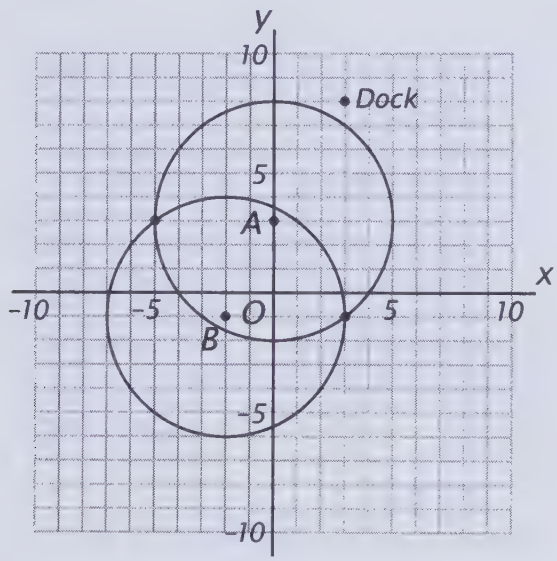
35.



36. They appear to be collinear.

Ollie's Mistake.

37.



•38. $\sqrt{4+16} = \sqrt{20}$ (or $2\sqrt{5} \approx 4.5$).

39. Two.

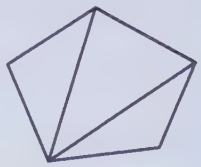
•40. At (–5, 3).

41. At (3, –1).

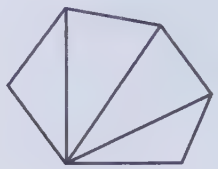
42. The distance between the dock at (3, 8) and (–5, 3) is $\sqrt{64+25} = \sqrt{89}$. The distance between the dock at (3, 8) and (3, –1) is 9; so Ollie's money is at (3, –1).

Rigid Structures.

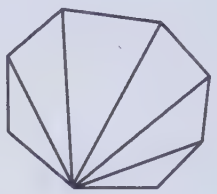
43. *Example answer:*



44. *Example answer:*



45. *Example answer:*



•46. Seven.

47. $n - 3$.

Pool Proof.

48. (1) Given.
(2) Betweenness of Points Theorem.
(3) Given.
(4) Perpendicular lines form right angles.
(5) All right angles are equal.
(6) Reflexive.
(7) SAS.
(8) Corresponding parts of congruent triangles are equal.
(9) Substitution (steps 2 and 8).

Masonic Tools.

49. First use the scale on the square to mark two points X and Y on the sides of the angle at the same distance from the vertex (so that XB = YB). Then position the square so that the distances from its corner to these two points are equal. $\Delta BXP \cong \Delta BYP$ by SSS; so $\angle ABP = \angle CBP$.

50. Because BP bisects the angle, $\angle ABP = \angle CBP$. Also BX = BY and BZ = BZ; so $\Delta BXZ \cong \Delta BYZ$ by SAS. $\angle 1$ and $\angle 2$ are a linear pair and $\angle 1 = \angle 2$ (they are corresponding parts of the congruent triangles); so $XY \perp BP$ because, if the angles in a linear pair are equal, their sides are perpendicular.

51. *Proof.*
    (1) $\angle A = \angle 1$ and $\angle 2 = \angle C$. (Given.)
    (2) BD = AB and CD = BD. (If two angles of a triangle are equal, the sides opposite them are equal.)
    (3) AB = CD. (Substitution.)

52. *Proof.*
    (1) O is the midpoint of XY. (Given.)
    (2) OX = OY. (The midpoint of a line segment divides it into two equal segments.)
    (3) $\angle 1 = \angle 2$ and OA = OB. (Given.)
    (4) $\Delta AOX = \Delta BOY$. (SAS)
    (5) $\angle A = \angle B$. (Corresponding parts of congruent triangles are equal.)

Algebra Review (page 182)

•1. $2(3x - 1)$.

•2. $3(2x - 1)$.

•3. $2(3x - 2)$.

•4. $4(x + 3y)$.

•5. $x(x + 10)$.

•6. $x^2(3x - 2)$.

•7. $5(3\pi - 8)$.

•8. $2x(4 + y)$.

•9. $x(a + b - c)$.

•10. $2\pi r(h + r)$.

•11. $(x + 4)(x + 4)$ or $(x + 4)^2$.

•12. $(x + 4)(x - 4)$.

•13. $(x + 3)(x + 10)$.

•14. $(x - 2)(x + 15)$.

•15. $(x + 7)(x - 6)$.

•16. $(2x - 1)(x - 7)$.

•17. $(3x - 5)(x + 2)$.

18. $x(x - 4y)$.

19. $(x - 2y)(x - 2y)$ or $(x - 2y)^2$.

20. $(x + 2y)(x - 2y)$.

21. $\pi(a^2 - r^2) = \pi(a + r)(a - r)$.

22. $5x(x^2 - 4) = 5x(x + 2)(x - 2)$.

23. $x(x^2 + x - 12) = x(x + 4)(x - 3)$.

24. $z(2x - 3)$.

25. $(x + 2)(2x - 3)$.

Chapter 5, Lesson 1

Set I (pages 186–187)

A chapter of Lawrence Wright's *Perspective in Perspective* (Routledge & Kegan Paul, 1983) is titled "Grand Illusions" and concerns various tricks played by architects with perspective. Plato mentions the use of such tricks in the design of Greek temples such as the Parthenon. Giotto's campanile in Florence is unusual in its use of "counter-perspective." The tower widens measurably upward, and the upper levels become successively taller toward the top. From the ground, they can look almost equal.

Hat Illusion.

•1. Either $h > w$, $h = w$, or $h < w$.

2. The "three possibilities" property.

3. *(Student answer.)* (For most people, the hat appears to be taller than it is wide, and so $h > w$.)

4. The hat is 1.7 cm high and its brim is 1.8 cm wide.

Pecking Order.

5. Hen A will always peck hen C.

•6. The transitive property.

Equalities and Inequalities.

7. The "three possibilities" property.

•8. Multiplication.

9. Subtraction.

10. Substitution.

•11. Addition.

12. Division.

Deceiving Appearances.

13. Isosceles. (Also acute.)

•14. AB > BC.

•15. Substitution.

16. Scalene. (Also acute.)

17. DF < DE.

18. The transitive property.

19. Because one has two equal sides and the other does not. (One is isosceles, the other scalene.)

Perspective Inequalities.

•20. $\angle APB > \angle HPI$.

21. $\angle BPC > \angle CPD$.

•22. $\angle PIH < \angle PBA$.

23. $\angle PHG < \angle PGF$.

24. $\angle PHI > \angle PGH$ and $\angle PGH > \angle PFG$; so $\angle PHI > \angle PFG$.

25. So that the upper levels would appear to be the same height when viewed from the ground.

Set II (pages 187–189)

SAT Problem.

•26. Transitive.

27. Addition.

28. Substitution. (Substituting AC for AB + BC and CE for CD + DE in exercise 27.)

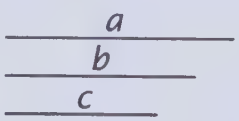
29. Could be false. (To see why, imagine that DE is much longer than it is.)

•30. The "whole greater than part" theorem.

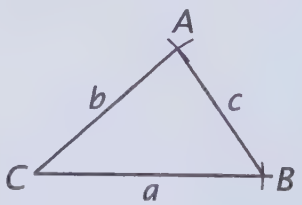
31. Transitive. (AB < BC is given and BC < BD in exercise 30.)

Scalene Triangle.

32.



33.



•34. $\angle A \approx 83°$, $\angle B \approx 56°$, $\angle C \approx 41°$.

35. $\angle A > \angle B > \angle C$.

36. Yes. The order corresponds to the order of the lengths of the opposite sides.

Rectangle Inequalities.

•37. 9 units. ($2 \cdot 4 + 2 \cdot 0.5 = 9$.)

38. Yes. The "whole greater than part" theorem.

39. 2 square units. ($4 \cdot 0.5 = 2$.)

40. No. This area problem shows why: $4 > 0$ and $0.5 > 0$, but $4 \times 0.5 = 2 < 4$.

Vertical Lines and Angles.

•41. That the line is "upright" (contains the center of the earth).

42. The sides of one angle must be opposite rays to the sides of the other angle. (It must also be true that two angles are equal if they are vertical angles.)

43. Yes: $\angle 1$ and $\angle 2$ are not vertical angles. If they were vertical angles, they would be equal. Because AB = AC, $\angle 2 = \angle 3$ (if two sides of a triangle are equal, the angles opposite them are equal). But $\angle 1 < \angle 3$; so $\angle 1 < \angle 2$ (substitution).

44. No. Because $\angle 2 = \angle 3$ (AB = AC) and $\angle 2 = \angle 4$, $\angle 3 = \angle 4$. The fact that two angles are equal does not mean, however, that they are necessarily vertical angles.

45. (1) Given.
(2) Addition.
(3) Given.
(4) Addition.
(5) Transitive (steps 2 and 4).

46. (1) Given.
(2) Subtraction.
(3) Given.
(4) Substitution.
(5) Addition.

Optics Figure.

47. (1) Given.
(2) Betweenness of Rays Theorem.
(3) Protractor Postulate.
(4) The "whole greater than part" theorem.

48. *Proof.*
(1) A-B-C. (Given.)
(2) AB + BC = AC. (Betweenness of Points Theorem.)
(3) AC > AB. ("Whole greater than part.")
(4) $\angle ADB = \angle DAB$. (Given.)
(5) AB = DB. (If two angles of a triangle are equal, the sides opposite them are equal.)
(6) AC > DB. (Substitution.)

Set III (page 189)

In *Human Information Processing,* Peter H. Lindsay and Donald A. Norman deal with the logic of choice. Concerning the three statements of the Set III exercises, they write:

"Almost without exception, people agree that their own decision processes ought to be logical. Moreover, formal theories of decision making assume logical consistency: Preferences among objects ought to be consistent with one another. If A is preferred to B and B to C, then logically A should be preferred to C. Transitivity of this sort is a basic property that just ought to hold anytime different objects are compared to one another. . . .

These three rules constitute a sensible set of postulates about decision behavior. Indeed, if the rules are described properly, so that the mathematical framework is removed, they simply sound like common sense. Those who study decision making, however, have discovered that there is a difference between the rules that people believe they follow and the rules that they actually do follow. Thus, although these three assumptions form the core of most decision theories, it is also realized that they do not always apply. There are some clear examples of instances in which each rule is violated."

(The authors then give anecdotal examples of violations for each of the three rules.)

Martin Gardner treated the subject of nontransitive paradoxes in one of his "Mathematical Games" columns. It is included in his *Time Travel and Other Mathematical Bewilderments* (W. H. Freeman and Company, 1988).

1. If you prefer an apple to a banana and a banana to a cookie, then you prefer an apple to a cookie.

2. The transitive property.

3. If you have no preference between an apple and a banana or between a banana and a cookie, then you have no preference between an apple and a cookie.

4. Substitution.

5. If you have no preference between an apple and a banana and you prefer a cookie to nothing at all, then you would prefer having an apple and a cookie to having just a banana.

6. Yes.
If $C > 0$, then $A + C > A$. (Addition.)
But $A = B$, so $A + C > B$. (Substitution.)

Chapter 5, Lesson 2

Set I (pages 192–193)

Aristotle included a section on the causes of the rainbow in *De Meteorologia*. According to Carl Boyer, Aristotle's work is "the first truly systematic theory of the rainbow that has come down to us." Boyer comments on the soundness of some of Aristotle's geometrical arguments and says: "Had his successors continued his work at the same high level, the story of the rainbow might not have been such a tale of frustration as it was destined to be."

Garage Door.

•1. $\angle BCY$, $\angle BAX$, $\angle CAD(\angle YAD)$.

2. It gets larger.

•3. It gets smaller.

4. It gets smaller.

5. If two sides of a triangle are equal, the angles opposite them are equal.

•6. An exterior angle of a triangle is greater than either remote interior angle.

7. Vertical angles are equal.

8. They are always equal. Because $\angle B = \angle BAY$ and $\angle BAY = \angle DAX$, $\angle B = \angle DAX$ by substitution.

Exterior Angles.

•9. $\angle 2$, $\angle 5$, $\angle 8$.

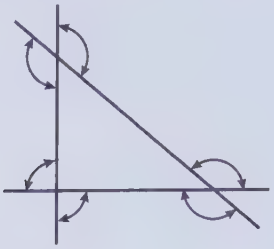
•10. Two.

11. Six.

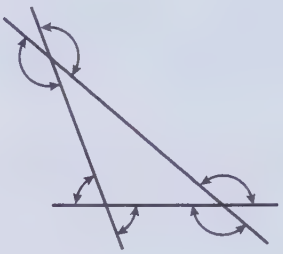
12. Yes. If the triangle is equiangular, its exterior angles must all be equal because they are supplements of equal angles.

13. No. The exterior angles at each vertex are vertical angles and vertical angles are equal.

14. *Example figure:*



15. *Example figure:*



•16. Two.

17. Two.

18. Six.

Rainbow.

•19. ΔROS.

•20. $\angle ROA > \angle SRO$ and $\angle ROA > \angle S$.

21. ΔROC.

22. $\angle ROS > \angle ORC$ and $\angle ROS > \angle RCO(\angle RCS)$.

23. ΔRCS and ΔRCO.

24. $\angle RCA > \angle SRC$, $\angle RCA > \angle S$, $\angle RCA > \angle ORC$, $\angle RCA > \angle ROC$.

•25. No. It does not form a linear pair with an angle of the triangle.

Lines and Angles.

•26. 360°.

27. 1,080°. ($3 \cdot 360 = 1{,}080$.)

•28. 180°.

29. 180°.

30. 720°. ($1{,}080 - 180 - 180 = 720$.)

31. It indicates that the sum is 720°.

Set II (pages 193–194)

We are indebted to Proclus for his commentary on the first book of Euclid's *Elements*. It and the work of Pappus are our two main sources of information on the history of Greek geometry. Proclus defended Euclid from the charge that he proved things that had no need of proof. (More on this is included in Lesson 4 of this chapter.)

Exterior Angle Theorem.

•32. A line segment has exactly one midpoint.

33. The Ruler Postulate (or by Construction 1, to copy a line segment).

34. SAS.

35. Corresponding parts of congruent triangles are equal.

36. Vertical angles are equal.

•37. The "whole greater than part" theorem.

38. Substitution.

Angle Sum.

•39. Two points determine a line.

40. An angle is an exterior angle of a triangle if it forms a linear pair with an angle of the triangle.

•41. The angles in a linear pair are supplementary.

42. If two angles are supplementary, their sum is 180°.

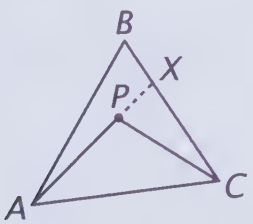
•43. An exterior angle of a triangle is greater than either remote interior angle.

44. Addition.

45. Substitution.

Angle in a Triangle.

46.



47. $\angle APC > \angle AXC$ and $\angle AXC > \angle B$ because an exterior angle of a triangle is greater than either remote interior angle. So $\angle APC > \angle B$ by the transitive property.

Proclus's Proof.

48. If two sides of a triangle are equal, the angles opposite them are equal.

49. $\angle 1 = \angle C$ and $\angle 2 = \angle A$.

50. The Exterior Angle Theorem ($\angle 1$ is an exterior angle of ΔPBC and $\angle 2$ is an exterior angle of ΔPBA.)

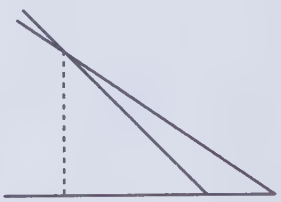
51. Indirect.

Set III (page 194)

In his *History of Mathematics*, David Eugene Smith includes a paragraph on "drumhead trigonometry." He wrote: "The continual warfare of the Renaissance period shows itself in many ways in the history of mathematics . . . . One of them is related to the subject now under consideration. Several writers of the 16th century give illustrations of the use of the drumhead as a simple means of measuring angles of elevation in computing distances to a castle or in finding the height of a tower." The illustration is from S. Belli's *Libro del Misvrar con la Vista (Book of Measuring with Eyesight)*, published in Venice in 1569.

Drumhead Geometry.

1.



2. (About 4 in; more precisely, about $4\frac{1}{8}$ in.)

3. About 80 ft.

Chapter 5, Lesson 3

Set I (pages 197–198)

There are now many more "anamorphic artists" painting streets than pictures. The idea in using anamorphic figures as traffic markers is clearly not to make them difficult to recognize but

rather just the opposite. The driver of a moving car who is looking far ahead sees the markers from a sharp angle and hence sees them regain their "normal" shape. An entire "Mathematical Games" column focuses on various types of anamorphic art (*Scientific American*, January 1975). It is included in Martin Gardner's *Time Travel and Other Mathematical Entertainments* (W. H. Freeman and Company, 1988).

"Triangles in perspective" is an interesting topic. Suppose we have some wire-frame models of triangles of various shapes. What kinds of shadows can they cast? Can the shadow of an equilateral triangle look scalene? Can the shadow of an isosceles triangle look equilateral? Can the shadow of an obtuse triangle look acute? Does the shadow of a triangle always look like a triangle? What determines the smallest shadow that a triangle can cast?

1. Bicycle lane.
2. So that it can be seen more easily to a driver viewing it "on edge."

•3. Each is the converse of the other.

•4. $BC < AC$.

5. The "three possibilities" property.
6. $\angle A = \angle B$.
7. If two sides of a triangle are equal, the angles opposite them are equal.

•8. $\angle A > \angle B$.

9. $\angle A < \angle B$.

•10. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

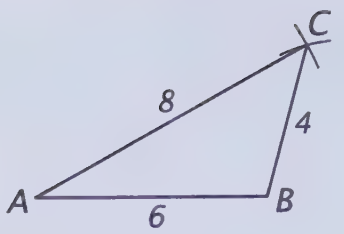
11. $\angle A > \angle B$.
12. $BC > AC$.
13. Indirect.

Triangle Drawing 1.

•14. $\angle B$.

15. $\angle A$.

16.



17. Obtuse. (Also, scalene.)

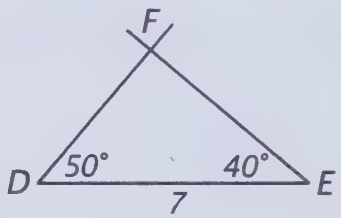
•18. $\angle A \approx 29°$, $\angle B \approx 104°$, $\angle C \approx 47°$.

Triangle Drawing 2.

•19. DE.

20. DF.

21.



•22. DF ≈ 4.5 cm, EF ≈ 5.4 cm.

Triangle in Perspective.

•23. $\angle X > \angle Y$.

24. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

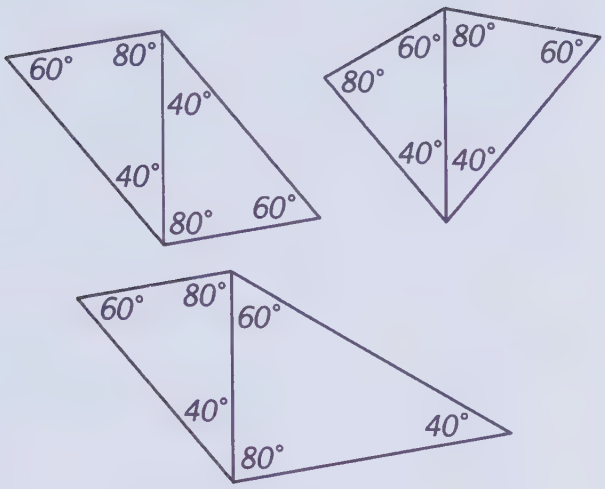
25. $\angle Z > \angle Y$.

26. $\angle Z > \angle X$ (given) and $\angle X > \angle Y$ (exercise 23); so $\angle Z > \angle Y$ by the transitive property.

27. $XY > XZ$.

28. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

Set II (pages 198–199)



The exercises on the pairs of "not quite equilateral" triangles are more challenging than they first appear. Some comparable drawings (above) in which the angles are 40°, 60°, and 80° instead are revealing. (Although the triangles in the second and third pairs are not congruent, they are obviously similar. We will return to these figures in the chapter on similarity.)

Folding Experiment.

29. *(Triangle cut out and folded).*

•30. ∠BDE.

31. ∠BDE is an exterior angle of ΔDEC.

32. ∠BDE > ∠C.

33. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

•34. BE bisects ∠ABC.

35. ΔABE ≅ ΔDBE (SAS); so ∠BDE = ∠A (corresponding parts of congruent triangles are equal).

Not Quite Equilateral.

•36. Yes. They are congruent by ASA.

37. AC.

38. BD.

39. They are equal because corresponding parts of congruent triangles are equal.

40. FG.

41. GH.

•42. FG < GH because both are sides of ΔFGH in which GH is the longest side.

43. No. They can't be congruent because FG and GH are corresponding parts of the triangles but they are not equal.

44. Yes. They are not congruent, because the shortest side of ΔIJK is IJ and the shortest side of ΔJKL is JK. IJ < JK because they are both sides of ΔIJK in which IJ is the shortest side. As with the preceding pair of triangles, these triangles cannot be congruent, because IJ and JK are corresponding parts of the triangles but they are not equal.

•45. An equilateral triangle is equiangular.

46. Betweenness of Rays Theorem.

•47. The "whole greater than part" theorem.

48. Substitution.

49. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

50. Because ∠PXA is an exterior angle of ΔPXY, ∠PXA > ∠PYX. Because PX ⊥ AB, ∠PXA and ∠PXY are right angles; so ∠PXA = ∠PXY = 90°. Therefore, in ΔPXY, 90° > ∠PYX (substitution) and ∠PXY = 90°; so ∠PXY > ∠PYX (substitution again). It follows that PY > PX because, if two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

Set III (page 199)

The message says "HELLO." (As Martin Gardner explains, "hold the page horizontally [with the bottom of the page] near the tip of your nose, close one eye, and read the message on a sharp slant.")

Chapter 5, Lesson 4

Set I (pages 202–203)

In *Euclid—The Creation of Mathematics* (Springer, 1999), Benno Artmann quotes Proclus on the Triangle Inequality Theorem:

"The Epicureans are want to ridicule this theorem, say it is evident even to an ass and needs no proof

...they make [this] out from the observation that, if hay is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side and not by way of the other two sides."

Artmann adds some pertinent remarks of his own:

"(The Epicureans of today might as well add that one could see the proof on every campus where people completely ignorant of mathematics traverse the lawn in the manner of the ass.) Proclus replies rightly that a mere perception of the truth of a theorem is different from a scientific proof of it, which moreover gives reason why it is true. In the case of Euclid's geometry, the triangle inequality can indeed be derived from the other (equally plausible) axioms. On the other hand, the Epicureans win in the modern theory of metric spaces, where the triangle inequality is the fundamental axiom of the whole edifice."

The transits of Venus (the occasions when Venus is between the sun and Earth) are few and far between. In fact, there have been only six of them since the invention of the telescope (1631, 1639, 1761, 1769, 1874, 1882). The next transits will be on June 8, 2004, and June 6, 2012. In the past, these alignments were used to determine the distance between Earth and the sun by timing the beginning and ending of a transit from widely separated geographical locations, but such methods are now obsolete. More information on the transits of Venus can be found in *June 8, 2004—Venus in Transit*, by Eli Maor (Princeton University Press, 2000).

Donkey Sense.

1. The sum of any two sides of a triangle is greater than the third side.

•2. $DP + PH > DH$.

3. $DH + HP > DP$; $PD + DH > PH$.

4. A postulate.

5. No.

Earth, Sun, and Venus.

•6. They are collinear.

•7. Yes; 160 million miles. $(93 + 67 = 160.)$

8. Yes; 26 million miles. $(93 - 67 = 26.)$

9. Not possible.

Spotter Problem.

•10. $PA + PB > 12$, $PA + PC > 12$, and $PB + PC > 12$.

11. $(PA + PB) + (PA + PC) + (PB + PC) > 36$.

12. $2PA + 2PB + 2PC > 36$.

13. $PA + PB + PC > 18$.

14. It is more than 18 km.

SAT Problem.

15. No. The sides of the triangle cannot be 2, 7, and 3, because $2 + 3 < 7$.

16. One triangle. The only integer that will work for x is 6.

Distance and Collinearity.

•17. A, B, and C are not collinear.

18. In ΔABC, $AB + BC > AC$. The sum of any two sides of a triangle is greater than the third side.

•19. $AB + BC = AC$.

20. A, B, and C must be collinear.

The Third Side.

21. Yes. Because the triangle is isosceles, the length of the third side must be either 4 or 9. It can't be 4, because $4 + 4 < 9$; so it must be 9.

22. Yes. From the Triangle Inequality Theorem we know that, if x is the length of the third side, then $5 + 7 > x$, and $5 + x > 7$. So $2 < x < 12$.

23. If the length of the third side is x, then either $x^2 = 6^2 + 8^2$ or $6^2 + x^2 = 8^2$. So either $x^2 = 36 + 64 = 100$ and $x = 10$ or $36 + x^2 = 64$ so that $x^2 = 28$ and $x = \sqrt{28} \approx 5.3$.

Set II (pages 203–204)

Heron's Proof.

•24. An angle has exactly one line that bisects it.

25. If an angle is bisected, it is divided into two equal angles.

26. An exterior angle of a triangle is greater than either remote interior angle.

•27. Substitution.

•28. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

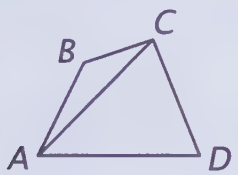
29. Addition.

30. Betweenness of Points Theorem.

31. Substitution.

Quadrilateral Inequality.

32.



Proof.
(1) ABCD is a quadrilateral. (Given.)
(2) Draw AC. (Two points determine a line.)
(3) AB + BC > AC and AC + CD > AD. (The sum of any two sides of a triangle is greater than the third side.)
(4) AB + BC + CD > AC + CD. (Addition.)
(5) AB + BC + CD > AD. (Transitive.)

Light Path.

•33. The sum of any two sides of a triangle is greater than the third side.

34. Betweenness of Points Theorem.

•35. Substitution.

36. SAS.

37. Corresponding parts of congruent triangles are equal.

38. Substitution.

39. Vertical angles are equal.

40. Corresponding parts of congruent triangles are equal.

41. Substitution.

42. AY + YB > AX + XB (the statement in exercise 38).

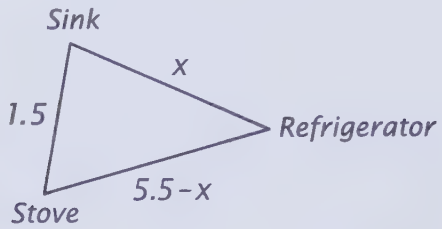
43. $\angle 1 = \angle 3$ (the statement in exercise 41).

Work Triangle.

•44. 5.5 m. (7 – 1.5 = 5.5.)

45. (5.5 – *x*) m.

46.



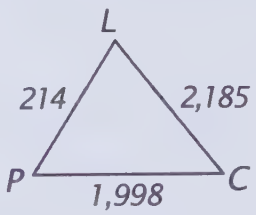
47. (1) $x + 1.5 > 5.5 - x$; so $2x > 4$, and so $x > 2$.
(2) $x + (5.5 - x) > 1.5$; so $5.5 > 1.5$ (which doesn't tell us anything about x).
(3) $1.5 + (5.5 - x) > x$; so $7 - x > x$; so $7 > 2x$, and so $x < 3.5$.

48. It should be more than 2 m but less than 3.5 m.

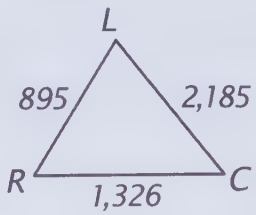
Set III (page 205)

We are, of course, assuming that our plane geometry is a reasonably good approximation for what is actually a problem in spherical geometry. The distances in this problem are "air distances" as reported in the *World Almanac*. The incorrect number is the distance between Paris and Rome; it is actually 690 miles.

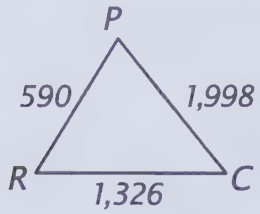
1. *(London, Paris, and Cairo)*



2. *(London, Rome, and Cairo)*



3. *(Paris, Rome, and Cairo)*



4. The London-Paris-Rome triangle, because $214 + 590 = 804 < 895$.
   The Paris-Rome-Cairo triangle, because $590 + 1{,}326 = 1{,}916 < 1{,}998$.

5. The length that the two impossible triangles have in common is 590, which suggests that the distance between Paris and Rome is wrong.

Chapter 5, Review

Set I (pages 206–208)

The Gateway Arch in St. Louis, at 630 ft, is approximately twice as tall as the Statue of Liberty (305 ft) and half as tall as the Empire State Building (1,250 ft). Designed by Eero Saarinen in the shape of an inverted catenary, the arch was completed in 1965. The cross sections of its legs are equilateral triangles with sides 54 ft long at ground level, tapering to 17 ft at the top. Like the top-hat illusion in Lesson 1, the arch gives the impression of being taller than it is wide; the two dimensions are actually the same.

1. In the tree trunk.
2. From the right edge at a sharp angle.
3. Diamond will scratch glass.
4. The transitive property.

Gateway Arch.

•5. The "three possibilities" property.

6. *(Student answer.)* (Most people see the arch as looking taller than it is wide.)

•7. Both dimensions are 2.5 in.

8. Each is 630 ft. ($2.5 \times 252 = 630$.)

Portuguese Theorem.

9. The Exterior Angle Theorem.

10. An exterior angle of a triangle is greater than either remote interior angle.

11. An exterior angle of a triangle is an angle that forms a linear pair with an angle of the triangle.

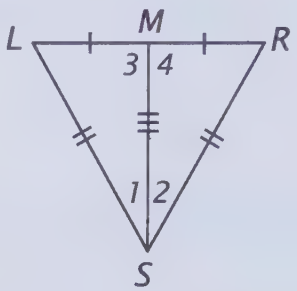
•12. One angle has a measure of 179° and the other two angles are each less than 1°.

13. One angle is a right angle and the other two angles are acute.

14. *Every* triangle has several exterior angles that are obtuse; so this doesn't tell us anything.

Soccer Angle.

15.



•16. The midpoint of a line segment divides it into two equal segments.

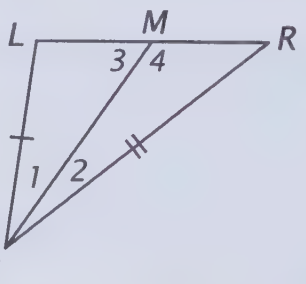
•17. Reflexive.

18. SSS.

19. Corresponding parts of congruent triangles are equal.

•20. If the angles in a linear pair are equal, their sides are perpendicular.

21.



•22. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

23. An exterior angle of a triangle is greater than either remote interior angle.

•24. Transitive.

25. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

Roman Column.

•26. The small angles each appear to be equal to 10°.

27. They increase as you look upward.

28. So that the sections would look equal to someone standing at the base of the column.

•29. An exterior angle of a triangle is greater than either remote interior angle.

30. Transitive.

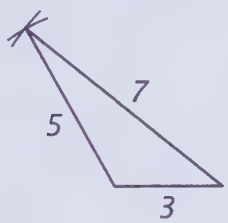
Set II (pages 208–209)

Integers and Triangles.

31. $1 + 3 < 5$ and $2 + 4 = 6$. For a triangle to be possible, both of these sums would have to be *greater* than the third length.

32. They are collinear.

33. *Example figure:*



•34. $n > 2$, ($2n + 2 > n + 4$, $n > 2$.)

35. Yes. We've just shown that a triangle is possible as long as $n > 2$.

36. An equilateral triangle.

Different Definition.

•37. One.

38. Our definition says that a triangle is isosceles if it has *at least* two equal sides; so the third side could also be equal.

•39. Yes.

40. (1) If the triangle is equilateral, all three sides can be 10.
    (2) If the triangle has exactly two equal sides and the side of length 10 is one of them, then the second side is 10 and the third side is less than 20.
    (3) If the triangle has exactly two equal sides and the side of length 10 is not one of them, then each of the other sides must be more than 5.

Screen Display.

•41. SSS.

42. Corresponding parts of congruent triangles are equal.

•43. Betweenness of Rays Theorem.

44. The "whole greater than part" theorem.

45. Substitution.

46. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

SAT Problem.

47. $x + (x - 2) > 7 - x$.
    $x + (7 - x) > x - 2$.
    $(x - 2) + (7 - x) > x$.

48. $2x - 2 > 7 - x$, so $3x > 9$; so $x > 3$.
    $7 > x - 2$; so $9 > x$, and so $x < 9$.
    $5 > x$; so $x < 5$.
    Therefore, $3 < x < 5$.

49. *Proof.*
    (1) $AB > AC$. (Given.)
    (2) $\angle C > \angle B$. (If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.)
    (3) $\angle A$ and $\angle B$ are complementary. (Given.)
    (4) $\angle A + \angle B = 90°$. (The sum of two complementary angles is 90°.)
    (5) $\angle A + \angle C > \angle A + \angle B$. (Addition.)
    (6) $\angle A + \angle C > 90°$. (Substitution.)

50. *Proof.*
(1) XB = XC. (Given.)
(2) $\angle C = \angle XBC$. (If two sides of a triangle are equal, the angles opposite them are equal.)
(3) $\angle ABC = \angle ABX + \angle XBC$. (Betweenness of Rays Theorem.)
(4) $\angle ABC > \angle XBC$. (The "whole greater than part" theorem.)
(5) $\angle ABC > \angle C$. (Substitution.)
(6) AC > AB. (If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.)

Algebra Review (page 210)

•1. $\frac{32}{40} = \frac{4(8)}{5(8)} = \frac{4}{5}$.

•2. $\frac{6x}{16} = \frac{2(3x)}{2(8)} = \frac{3x}{8}$.

•3. $\frac{x}{x^2} = \frac{1(x)}{x(x)} = \frac{1}{x}$.

•4. $\frac{5(x+1)}{10} = \frac{5(x+1)}{5(2)} = \frac{x+1}{2}$.

•5. $\frac{4x+4}{7x+7} = \frac{4(x+1)}{7(x+1)} = \frac{4}{7}$.

•6. $\frac{x-7}{3x-21} = \frac{x-7}{3(x-7)} = \frac{1}{3}$.

•7. $\frac{x+5}{x^2-25} = \frac{x+5}{(x-5)(x+5)} = \frac{1}{x-5}$.

•8. $\frac{x^2-xy}{xy-y^2} = \frac{x(x-y)}{y(x-y)} = \frac{x}{y}$.

•9. $\frac{x^2+5x+6}{10x+20} = \frac{(x+2)(x+3)}{10(x+2)} = \frac{x+3}{10}$.

•10. $\frac{x^2+x-20}{x^2-x-30} = \frac{(x+5)(x-4)}{(x+5)(x-6)} = \frac{x-4}{x-6}$.

•11. $\frac{1}{x} = \frac{4(1)}{4(x)} = \frac{4}{4x}$.

•12. $\frac{1}{x} = \frac{(x^3)(1)}{(x^3)(x)} = \frac{x^3}{x^4}$.

•13. $\frac{3}{2-x} = \frac{(-1)(3)}{(-1)(2-x)} = \frac{-3}{x-2}$.

•14. $\frac{4(2)}{4(3)}$ and $\frac{3(3)}{3(4)}$, $\frac{8}{12}$ and $\frac{9}{12}$.

•15. $\frac{7(4)}{7(x)}$ and $\frac{2}{7x}$, $\frac{28}{7x}$ and $\frac{2}{7x}$.

•16. $\frac{y(1)}{y(x)}$ and $\frac{x(1)}{x(y)}$, $\frac{y}{xy}$ and $\frac{x}{xy}$.

•17. $\frac{3(x)}{3(x+2)}$ and $\frac{2}{3x+6}$, $\frac{3x}{3x+6}$ and $\frac{2}{3x+6}$.

•18. $\frac{5(x+1)}{(x-1)(x+1)}$ or $\frac{5x+5}{x^2-1}$ and $\frac{x}{(x-1)(x+1)}$ or $\frac{x}{x^2-1}$.

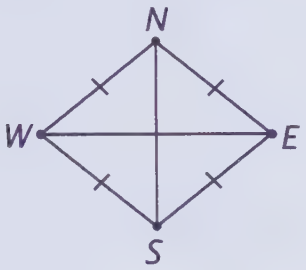
•19. $\frac{x(x)}{x(yz)}$ and $\frac{y(y)}{y(xz)}$, $\frac{x^2}{xyz}$ and $\frac{y^2}{xyz}$.

Chapter 6, Lesson 1

Set I (pages 214–215)

The exercises on folding a paper chain suggest some possible exploration on the student's part. Cutting and manipulating a paper chain of the type illustrated reveals examples of how the strip could have been folded any number of times from three (a "binary" folding approach) to seven (an "accordion" folding approach). In general, 2^n penguins require at least n folds but could be produced by as many as $2^n - 1$ folds. Even exploring the physical limit on the possible value of n for the binary-folding approach yields an answer that surprises most students.

Rope Trick.

1. 

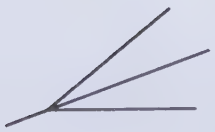
2. Two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

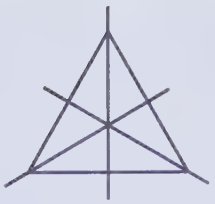
3. Yes.

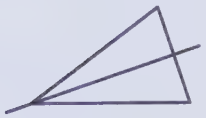
•4. Yes, because W and E are equidistant from the endpoints of NS.

Lines of Symmetry.

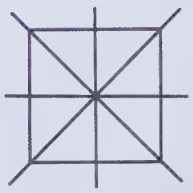
5. 

6. 

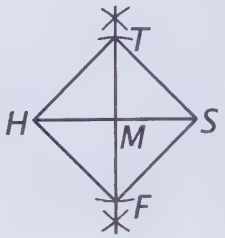
7. 

8. 

9. 

10. 

Baseball Diamond.

11. 

•12. 6 cm.

•13. 90 ft. ($6 \cdot 15 = 90$.)

14. 127.5 ft. ($8.5 \cdot 15 = 127.5$.)

•15. In the west.

16. So that the sun doesn't get in the batter's eyes.

17. Because, as he faces the batter, his left arm is on the south.

Paper Chain.

18. *(Student answer.)* (Any number from three to seven is correct.)

•19. Seven.

20. Four times. ($2^4 = 16$. Unlike exercise 18, this exercise asks for the *minimum* rather than for simply the possible number of folds.)

Stealth Bomber.

•21. Line l is the perpendicular bisector of ST.

•22. If a line segment is bisected, it is divided into two equal segments.

23. Perpendicular lines form right angles.

24. All right angles are equal.

25. Reflexive.

26. SAS.

27. Corresponding parts of congruent triangles are equal.

Set II (pages 215–217)

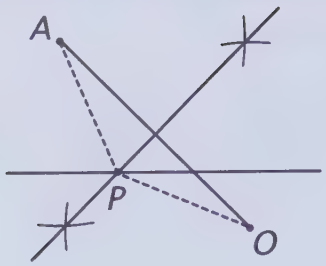
The "clairvoyance" test problem is another version of the surfer's puzzle but with a regular pentagon instead of an equilateral triangle. An explanation of why the sum of the five distances is independent of the choice of the point is based on area. For this figure, the "exact" sum is $\frac{55}{8 \tan 36°}$, or approximately 9.46 inches. We will return to the surfer's puzzle later.

In his book titled *Inversions* (Byte Books, 1981), Scott Kim wrote of his amazing PROBLEM/SOLUTION figure: "Reflecting on this PROBLEM yields a surprising SOLUTION. Notice that reflection about a diagonal axis preserves a pen angle of 45 degrees, so that calligraphic conventions are maintained."

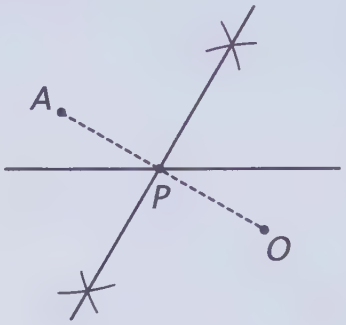
The figure drawn by Kepler is the horoscope that he cast for Albrecht von Wallenstein, one of the most famous military commanders of his time. He was Emperor Ferdinand II's major general during Europe's Thirty Year War. Charles Blitzer in his book titled *Age of Kings* (Time-Life Books, 1967) wrote that, although hundreds of books have been written about Wallenstein, "the most illuminating analysis of this complex man comes from a horoscope cast for him [on his 25th birthday] by the great 17th century astronomer, Johannes Kepler."

Race to the Fence.

28.

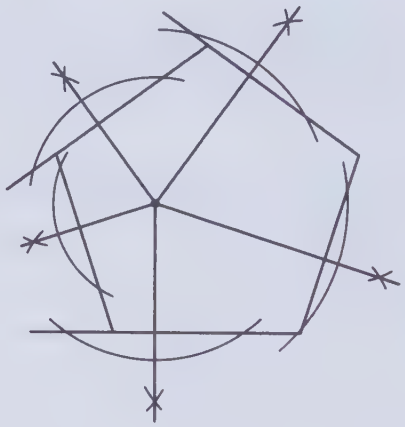


29. It would have been the point midway between them.



Clairvoyance Test.

30. *Example answer:*



•31. 9.5 in.

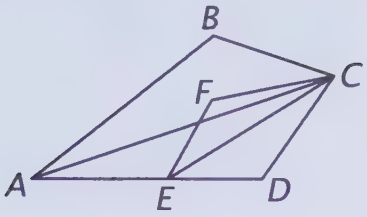
32. Because the answer is predictable; it doesn't depend on the choice of the point. The "clairvoyant" can predict the subject's answer.

Linkage Problem.

33. Because points A and C are equidistant from the endpoints of BD (two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.)

•34. C and E; D and F.

35.



•36. Because they are corresponding parts of congruent triangles (ΔABC ≅ ΔADC by SSS).

37. Yes. ∠B = ∠D (exercise 36) and ∠F = ∠D because ΔEFC ≅ ΔEDC; so ∠B = ∠F by substitution.

One Problem, Two Solutions.

38. At a 45° angle from the lower left to the upper right.

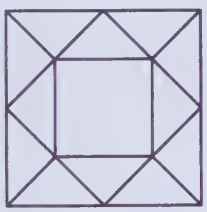
•39. N.

40. T.

41. The word SOLUTION is readable on both sides of the symmetry line.

Kepler's Diagram.

42. *(There are various ways to construct this figure.)*



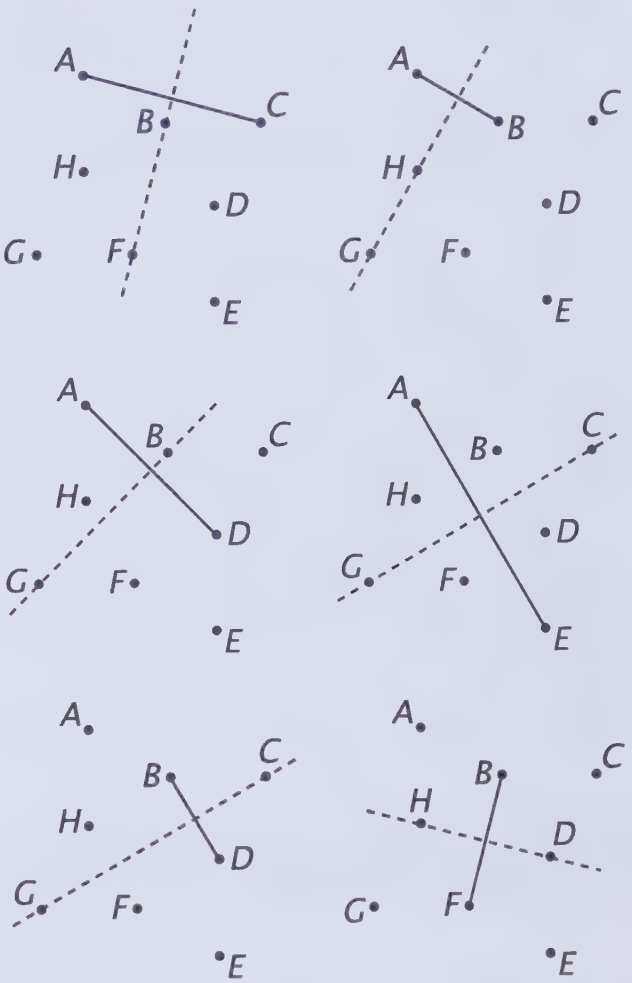
•43. Three.

44. Isosceles right triangles.

45. 16.

Set III (page 218)

In the reference cited in the footnote, Martin Gardner reports that Hallard Croft, a mathematician at Cambridge University, "asked if there existed a finite set of points on the plane such that the perpendicular bisector of the line segment joining any two points would always pass through at least two other points of the set." The figure in the Set III exercise is the solution using only eight points discovered by Leroy Kelly of Michigan State University. (The four central points are the vertices of a square, and the four outer points are the vertices of four equilateral triangles erected on the sides of the square.) Although there are 28 possible pairings of the eight points, the symmetries of the figure are such that only six of them are distinct. These six possibilities are illustrated in the figures below.



An Unusual Set of Points.

Every perpendicular bisector passes through two other points of the figure.

Chapter 6, Lesson 2

Set I (pages 221–222)

The figure for exercises 17 through 23 illustrates a theorem of projective geometry. The theorem states that, if the vertices of a [self-intersecting] hexagon lie alternately on two lines and if two pairs of opposite sides are respectively parallel, then the third pair of opposite sides also are parallel. If the two lines are parallel, this theorem

is easily proved. In contrast, the fact that it is true even if the lines are not parallel seems surprising from the perspective of Euclidean geometry, and how it might be proved is not obvious.

•1. (1) $\angle 1 = \angle 2$.
(2) $\angle 2 = \angle 3$.
(3) $\angle 1 = \angle 3$.
(4) $a \parallel b$.

2. (1) $\angle 1$ and $\angle 2$ are supplementary.
(2) $\angle 2$ and $\angle 3$ are supplementary.
(3) $\angle 1 = \angle 3$.
(4) $a \parallel b$.

3. (1) $a \perp c$ and $b \perp c$.
(2) $\angle 1$ and $\angle 2$ are right angles.
(3) $\angle 1 = \angle 2$.
(4) $a \parallel b$.

Folded Paper.

•4. Alternate interior angles.

5. Equal alternate interior angles mean that lines are parallel.

•6. In a plane, two lines perpendicular to a third line are parallel.

Snake Track.

•7. Corresponding angles.

8. They are parallel.

•9. Equal corresponding angles mean that lines are parallel.

10. Interior angles on the same sides of the transversal.

11. They are supplementary.

12. They are parallel.

13. Supplementary interior angles on the same side of a transversal mean that lines are parallel.

I-Beam.

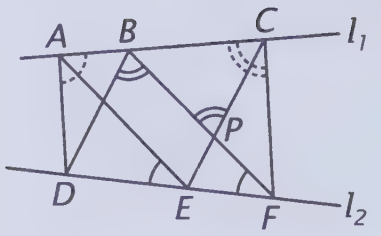
•14. No.

15. Yes.

16. Yes.

Parallel Lines.

17.



•18. AE || BF.

19. Equal corresponding angles mean that lines are parallel.

20. DB || EC(PC).

21. Equal alternate interior angles mean that lines are parallel.

22. AD || CF.

23. Supplementary interior angles on the same side of a transversal mean that lines are parallel.

Set II (pages 223–224)

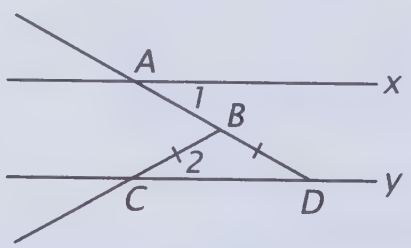
Drafting Triangles.

24. $\Delta ABC \cong \Delta FED$ because the drafting triangles are identical; so $\angle A = \angle F$. AB || EF because equal alternate interior angles mean that lines are parallel.

25. $\angle BCA = \angle EDF$; so BC || DE because equal alternate interior angles mean that lines are parallel.

Optical Illusion.

26.



•27. No. These angles are not formed by a transversal. They could be equal and yet lines x and y could intersect either to the left or to the right.

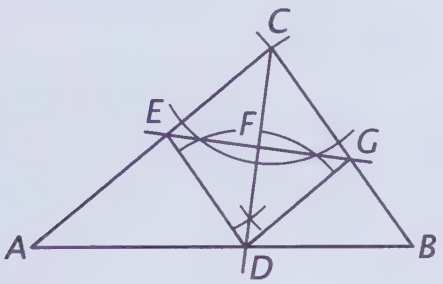
28. Yes. If BD = BC, $\angle 2 = \angle BDC$ (if two sides of a triangle are equal, the angles opposite them are equal). Because $\angle 1 = \angle 2$, $\angle 1 = \angle BDC$ (substitution). So $x \parallel y$ (equal alternate interior angles mean that lines are parallel).

Different Proofs.

29. (2) Two lines are perpendicular if they form a right angle.
    (3) In a plane, two lines perpendicular to a third line are parallel.

30. (2) All right angles are equal.
    (3) Equal alternate interior angles mean that lines are parallel.

Construction Exercise.

31. 

•32. Four.

33. ASA ($\angle ECF = \angle GCF$, $CF = CF$, $\angle CFE = \angle CFG$.)

34. Corresponding parts of congruent triangles are equal.

•35. SAS. [$CF = FD$ and $CF \perp FD$ (construction), $EF = FG$ (exercise 34).]

36. Corresponding parts of congruent triangles are equal.

37. Equal alternate interior angles mean that lines are parallel.

•38. No.

39. *Proof.*
    (1) $AE = AD$ and $\angle E = \angle BCE$. (Given.)
    (2) $\angle E = \angle ADE$. (If two sides of a triangle are equal, the angles opposite them are equal.)
    (3) $\angle ADE = \angle BCE$. (Substitution.)
    (4) $AD \parallel BC$. (Equal corresponding angles mean that lines are parallel.)

40. *Proof.*
    (1) $AB = CD$ and $AD = BC$. (Given.)
    (2) $BD = BD$. (Reflexive.)
    (3) $\Delta ABD \cong \Delta CDB$. (SSS).
    (4) $\angle ABD = \angle CDB$. (Corresponding parts of congruent triangles are equal.)
    (5) $AB \parallel DC$. (Equal alternate interior angles mean that lines are parallel.)

Set III (page 224)

The cross-bar figures are included in a section titled "The Importance of Rules" in the chapter on human perception in Lindsay and Norman's *Human Information Processing.* They write:

"When a rectilinear object is viewed straight on, the intersecting lines formed by its contours form right angles. As the object is tilted and rotated in space, the angles in the retinal image diverge from right angles. The degree of divergence depends on the orientation of the object in space. By interpreting that divergence from right angles as a result of depth, distance information can be extracted."

What Do You See?

1. *(Student answer.)* (The top bar.)

2. *(Student answer.)* (The bottom bar.)

3. The first figure is viewed "in two dimensions," seen as lying in the plane of the paper. The second figure is viewed "in three dimensions," seen as in perspective on the side of a box.

4. Yes. A third possibility is that the two cross bars do not lie in a common plane, in which case both could be perpendicular to the vertical bar.

5. It doesn't follow that they are parallel. Our theorem says that *in a plane,* two lines perpendicular to a third line are parallel.

Chapter 6, Lesson 3

Set I (pages 226–228)

Euclid's assumption that "if two lines form interior angles on the same side of a transversal whose sum is less than 180°, then the two lines meet on that side of the transversal" is his fifth postulate (his version of the Parallel Postulate).

In his commentary on Euclid's *Elements,* Sir Thomas Heath wrote: "When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validity of his whole system of geometry, was really indemonstrable.

From the very beginning, as we know from Proclus, the Postulate was attacked as such, and attempts were made to prove it as a theorem or

to get rid of it by adopting some other definition of parallels; while in modern times the literature of the subject is enormous."

An entire chapter of David Henderson's valuable book titled *Experiencing Geometry* (Prentice Hall, 2001) is on the subject of parallel postulates. In his historical notes, Henderson remarks:

"Playfair's Parallel Postulate got its current name from the Scottish mathematician, John Playfair (1748–1819), who brought out successful editions of Euclid's *Elements* in the years following 1795. After 1800 many commentators referred to Playfair's [parallel] postulate (PPP) as the best statement of Euclid's [fifth] postulate (EFP), so it became a tradition in many geometry books to use PPP instead of EFP."

Henderson goes on to explain that in *absolute geometry*, the part of geometry that can be developed without using a parallel postulate, Playfair's postulate is equivalent to Euclid's. He then points out:

"Because absolute geometry was the focus of many investigations the statement 'PPP is equivalent to EFP' was made and is still being repeated in many textbooks and expository writings about geometry even when the context is not absolute geometry . . . . EFP is true on spheres and so cannot be equivalent to PPP, which is clearly false."

Optical Illusion.

•1. 12.

•2. If they form a right angle (or if they form four equal angles).

3. If they lie in the same plane and do not intersect.

4. *(Student answer.)* (Possibly white circles or squares at the "breaks" in the grid. Also broad white diagonal strips.)

Windshield Wipers.

•5. $\angle 3$.

•6. Equal corresponding angles mean that lines are parallel.

7. Interior angles on the same side of the transversal.

8. Supplementary interior angles on the same side of a transversal mean that lines are parallel.

9. $\angle 2$ and $\angle 4$.

10. Yes. $\angle 1$ and $\angle 2$ (or $\angle 3$ and $\angle 4$) are a linear pair; so they are supplementary. If $\angle 1$ and $\angle 4$ are supplementary, then $\angle 2 = \angle 4$ (or $\angle 1 = \angle 3$) because supplements of the same angle are equal. It follows that the wipers are parallel because they form equal corresponding angles with the transversal.

Exactly One.

11. That a cat has at least one flea.

•12. At least, no more than, exactly.

13. At least, no more than, exactly.

•14. No more than.

15. No more than.

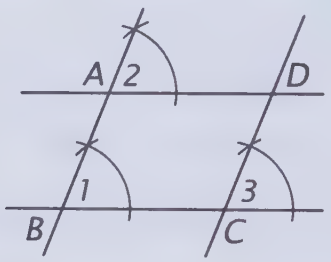
16. At least.

17. At least, no more than, exactly.

18. At least, no more than, exactly.

Parallelogram.

19.



•20. They form equal corresponding angles with a transversal.

21. They seem to be equal.

•22. They seem to be supplementary.

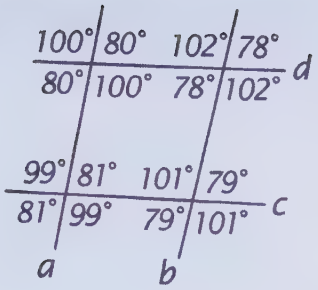
23. They seem to be equal.

24. No. It can't be folded so that the two halves coincide.

Euclid's Assumption.

•25. All four of them.

26.



•27. Down, below the figure.

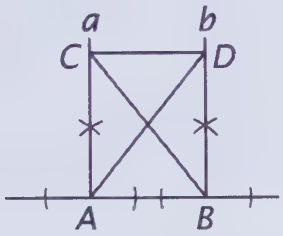
28. To the left.

Set II (pages 228–229)

The restriction to the plane in Theorem 18 is, of course, unnecessary, because the theorem is also true in space. (The proof in solid geometry is based on theorems on perpendicular lines and planes.)

Gateway.

29.



•30. One.

31. One.

32. They are parallel.

•33. In a plane, two lines perpendicular to a third line are parallel.

34. They seem to be parallel.

35. One.

•36. Through a point not on a line, there is exactly one line parallel to the line.

•37. Yes. (SAS; AC = BD, ∠CAB = ∠DBA, AB = AB.)

38. Yes. (Corresponding parts of congruent triangles are equal.)

39. Yes. (SSS; AC = BD, AD = BC, CD = CD.)

40. Yes. (Corresponding parts of congruent triangles are equal.)

41. No. (It will be possible to prove these triangle congruent by methods learned in future lessons, however.)

Theorem 18.

•42. a and b are not parallel.

•43. Given.

44. Through a point not on a line, there is exactly one line parallel to the line.

45. $a \parallel b$.

Proclus's Claim.

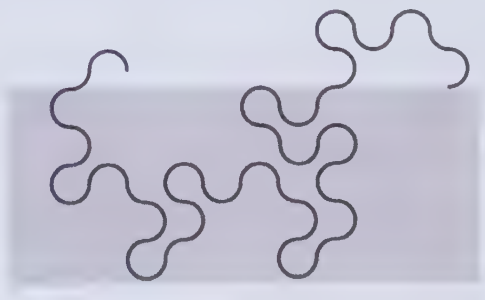
46. Yes.

47. If c doesn't intersect b, then c is parallel to b. This would mean that, through P, lines c and a are both parallel to b, which contradicts the Parallel Postulate.

Set III (page 229)

The "dragon curve" was discovered by NASA physicist John Heighway. Martin Gardner explains in the chapter titled "The Dragon Curve and Other Problems" in his book titled *Mathematical Magic Show* (Knopf, 1977) that the curve gets its name from the fact that, when it is drawn with each right angle rounded, it "vaguely resembles a sea dragon paddling to the left with clawed feet, his curved snout and coiled tail just above an imaginary waterline."

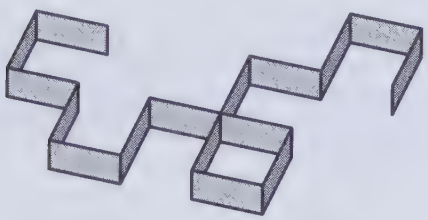
The curve can be generated in various ways including paper folding, by geometric construction, and from a sequence of binary digits. The version in the text is called an order-4 dragon because it is produced by four folds. The order-6 dragon shown below with its right angles rounded makes the origin of the name more evident.



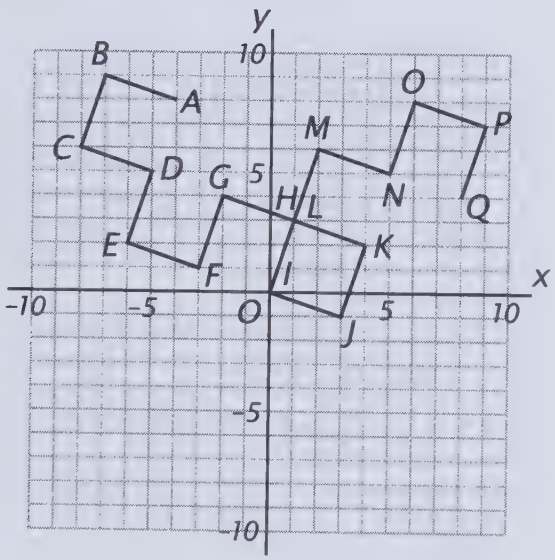
The length of the strip of paper suggested for folding the dragon was chosen so that, if a student uses graph paper of 4 units per inch, the unfolded dragon will fit nicely on the graph. Each time the paper is unfolded, the half produced is a 90° rotation. The original fold point has been placed at the origin so that someone studying the graph may discover the connection between the 90° clockwise turn and the coordinate transformation: $(a, b) \rightarrow (b, -a)$.

A Dragon Curve.

1. (The folded paper should look like the dragon curve shown in the text.)



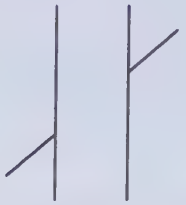
2.



3. It looks like the edge of the folded strip.
4. Point I.
5. Points E and M.
6. It has the same shape but is rotated 90° clockwise.
7. For each point with coordinates (a, b) in the first half, there is a corresponding point with coordinates $(b, -a)$ in the second half.

Chapter 6, Lesson 4

Set I (pages 232–234)



The Poggendorff Illusion

The fishpole puzzle is based on an illusion created by J. C. Poggendorff in 1860. The original illusion is illustrated above. Although the oblique segments are aligned, the upper one appears raised in relation to the lower one. The Poggendorff illusion and the theories developed to explain it are discussed in depth in *Perception,* by Irvin Rock (Scientific American Library, 1984).

Which Fishpole?

•1. They are equal.

•2. Parallel lines form equal corresponding angles.

3. Fishpole A.

4. It doesn't look like fishpole A.

Corollary Proofs.

•5. (1) Given.
(2) Parallel lines form equal corresponding angles.
(3) Vertical angles are equal.
(4) Substitution.

6. (1) Given.
(2) Parallel lines form equal corresponding angles.
(3) The angles in a linear pair are supplementary.
(4) The sum of two supplementary angles is 180°.
(5) Substitution.
(6) Two angles are supplementary if their sum is 180°.

7. (1) Given.
   (2) Perpendicular lines form right angles.
   (3) A right angle is 90°.
   (4) Given.
   (5) Parallel lines form equal corresponding angles.
   (6) Substitution.
   (7) A 90° angle is a right angle.
   (8) Lines that form a right angle are perpendicular.

Bent Pencil.

8. Parallel lines form equal corresponding angles.

•9. Betweenness of Rays Theorem.

10. Substitution.

11. Subtraction.

•12. Equal corresponding angles mean that lines are parallel.

Find the Angles.

•13. $\angle 1 = 115°$, $\angle 2 = 65°$, $\angle 3 = 115°$.

14. $\angle 1 = 90°$, $\angle 2 = 90°$, $\angle 3 = 90°$.

15. $\angle 1 = 70°$, $\angle 2 = 55°$.

16. $\angle 1 = 110°$, $\angle 2 = 70°$, $\angle 3 = 110°$.

City Streets.

•17. They must be perpendicular because, in a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

18. They must be parallel because, in a plane, two lines perpendicular to a third line are parallel.

19. 113°, 67°, and 113°.

•20. 67°, 113°, 67°, and 113°.

21. Parallel lines form equal corresponding angles.

T Puzzle.

•22. $\angle 1$ and $\angle 2$ are supplementary because parallel lines form supplementary interior angles on the same side of a transversal.

23. $\angle 2 = \angle 3$ because parallel lines form equal corresponding angles.

•24. $\angle 3 = \angle 4$ because parallel lines form equal alternate interior angles.

25. $\angle 2 = \angle 4$ by substitution.

26. $\angle 4$ and $\angle 5$ are complementary because they together with the right angle add up to 180°.

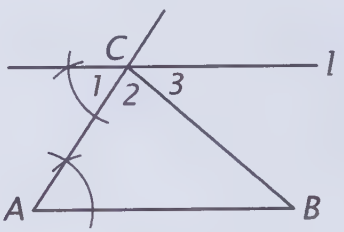
27. $\angle 5$ and $\angle 6$ are supplementary because parallel lines form supplementary interior angles on the same side of a transversal.

Set II (pages 234–235)

In his book titled *The Flying Circus of Physics* (Wiley, 1977), Jearl Walker reports: "Looking down on desert sand dunes from a high altitude airplane, one sees curious long, narrow dune belts running across the desert, roughly from north to south, in almost straight lines, as if one were viewing well-designed parallel streets. The dune belts are characteristic of every major desert in the world." Walker explains that the belts are caused by air currents and that the dominant winds in every desert of the world are north or south.

Parallel Construction.

28.



29. Equal alternate interior angles mean that lines are parallel.

•30. Through a point not on a line, there is exactly one line parallel to the line (the Parallel Postulate).

•31. Parallel lines form equal alternate interior angles.

32. 180°.

33. 180°.

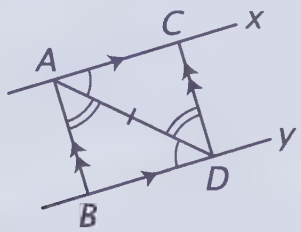
Overlapping Angles.

34. Yes. $\angle B = \angle DGC$ and $\angle DGC = \angle E$ because parallel lines form equal corresponding angles. So $\angle B = \angle E$ by substitution.

35. No. If one angle were turned with respect to the other, their sides would obviously not be parallel.

Sand Dune Lines.

36.



•37. Two points determine a line.

38. Parallel lines form equal alternate interior angles.

•39. In a plane, two lines perpendicular to a third line are parallel.

40. Parallel lines form equal alternate interior angles.

41. Reflexive.

42. ASA. ($\angle CAD = \angle BDA$, $AD = AD$, $\angle BAC = \angle CDA$.)

43. Corresponding parts of congruent triangles are equal.

•44. The length of a perpendicular segment between them.

SAT Problem.

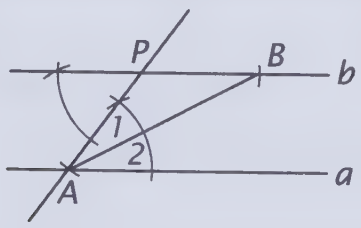
45. $x + (2x - 30) = 180$.

•46. $3x = 210$; so $x = 70$. The acute angle is 70°.

47. $2x - 30 = 2(70) - 30 = 110°$. The obtuse angle is 110°.

Construction Problem.

48.



49. $\angle 1$ and $\angle 2$ seem to be equal.

50. Because $a \parallel b$, $\angle PBA = \angle 2$ (parallel lines form equal alternate interior angles). Also, $PA = PB$, and so $\angle 1 = \angle PBA$ (angles opposite equal sides in a triangle are equal). So $\angle 1 = \angle 2$ by substitution.

Set III (page 235)

The context of Kepler's method for determining the shape of Earth's orbit is set forth by David Layzer in *Constructing the Universe* (Scientific American Library, 1984). The parallel lines are in the direction of an arbitrary star. The angle numbered 1 is measured when Mars is "at opposition," that is, when Earth is in alignment between Mars and the sun (M-E-S). The point E' is the position of Earth 1 Martian year later. Layzer explains: "By means of observations of Mars and the sun at intervals separated by one Martian year, Kepler calculated the shape of the earth's orbit and its dimension relative to the reference distance SM."

Plotting successive positions of Earth in this way led to Kepler's realization that the orbits of the planets are elliptical rather than circular and to his three laws of planetary motion to describe this motion.

1. $\angle 4 = \angle 1$ because parallel lines form equal corresponding angles.

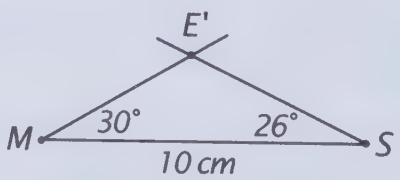
2. $\angle 5 = 180° - \angle 3$ because parallel lines form supplementary interior angles on the same side of a transversal.

3. $\angle 6 = \angle 4 - \angle 5$.

4. $\angle M$ is the third angle of $\Delta MSE'$; so $\angle M = 180° - \angle 2 - \angle 6$.

5. $\angle 4 = \angle 1 = 84°$.
   $\angle 5 = 180° - \angle 3 = 180° - 122° = 58°$.
   $\angle 6 = \angle 4 - \angle 5 = 84° - 58° = 26°$.
   $\angle M = 180° - \angle 2 - \angle 6 = 180° - 124° - 26° = 30°$.

6.



Chapter 6, Lesson 5

Set I (pages 238–239)

It is strange to think that a steel rod bent into the shape of a triangle would make a good musical instrument and to realize that geometry is useful in describing the sounds produced. They depend in part on where the triangle is struck. According to Thomas Rossing in *The Science of Sound* (Addison-Wesley, 1990), single strokes played on the base create vibrations in the plane of the triangle; tremolos are played between the sides of the vertex angle and create vibrations perpendicular to the plane of the triangle.

The primary appearance of geometry in economics is in graphs. The consumption function example is about the "break even" point. As described in *Economics* by Paul Samuelson and William Nordhaus (McGraw-Hill, 1989), "at any point on the 45° line, consumption exactly equals income . . . . The 45° line tells us immediately, therefore, whether consumption spending is equal to, greater than, or less than the level of income. The point on the consumption schedule where it intersects the 45° line shows us the level of disposable income at which households just break even."

Corollary Proofs.

•1. The sum of the angles of a triangle is 180°.

•2. Substitution.

•3. Substitution.

•4. Subtraction.

5. A right angle equals 90°.

6. Substitution.

7. Subtraction.

8. Two angles are complementary if their sum is 90°.

•9. An equilateral triangle is equiangular.

•10. An equiangular triangle has three equal angles.

11. Substitution (and algebra).

12. Division.

13. Substitution.

Exterior Angle Theorem.

•14. Through a point not on a line, there is exactly one line parallel to the line.

15. Parallel lines form equal alternate interior angles.

•16. Parallel lines form equal corresponding angles.

17. Betweenness of Rays Theorem.

18. Substitution.

UFO Angles.

•19. $\angle 2 > \angle 1$ and $\angle 2 > \angle A$.

20. $\angle 2 = \angle 1 + \angle A$.

21. 28°.

Right and Wrong.

22. Acute.

23. The acute angles of a right triangle are complementary; so their sum is a right angle.

Another Kind of Triangle.

•24. An equilateral triangle.

25. As a musical instrument.

26. 60°.

Economics Graph.

•27. It is a right triangle.

•28. They are complementary.

29. 45°.

30. If two angles of a triangle are equal, the sides opposite them are equal.

Set II (pages 240–241)

The direction of the North Star is treated in exercises 31 through 37 as if it coincides exactly with that of celestial north even though there is a slight discrepancy. Peculiar, a small town near the western border of Missouri, got its name by accident. When the residents applied for a post office, several names that they proposed were rejected because other towns in the state had already taken them. Finally the appointed post master for the town wrote to Washington D. C. saying, if the names previously submitted would

not do, then please assign some "peculiar" name to the town. The Post Office Department did exactly that.

The catch question about the height of the triangle was created by Leo Moser of the University of Alberta.

North Star and Latitude.

31. Parallel lines form equal alternate interior angles.

32. $\angle 4$ and $\angle 3$ are complementary.

33. The acute angles of a right triangle are complementary.

•34. $\angle 2$ and $\angle 3$ are complementary.

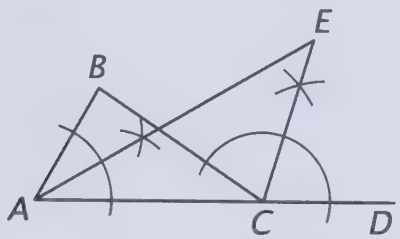
35. $\angle 4 = \angle 2$.

•36. Complements of the same angle are equal.

37. Substitution. [$\angle 1 = \angle 4$ (exercise 31) and $\angle 4 = \angle 2$ (exercise 35).]

Angle Bisectors.

38.



39. *(Student answer.)* ($\angle B$ seems to be twice as large as $\angle E$.)

•40. An exterior angle of a triangle is equal to the sum of the remote interior angles.

•41. Substitution.

42. Multiplication.

43. Substitution.

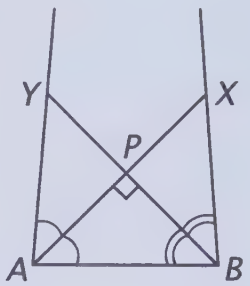
44. Subtraction.

45. *Proof.*
    (1) In ΔABC and ΔADE, $\angle ADE = \angle B$. (Given.)
    (2) $\angle A = \angle A$. (Reflexive.)
    (3) $\angle AED = \angle C$. (If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.)

46. *Proof.*
    (1) ΔABC is a right triangle with right $\angle ABC$; $\angle ABD$ and $\angle C$ are complementary. (Given.)
    (2) $\angle A$ and $\angle C$ are complementary. (The acute angles of a right triangle are complementary.)
    (3) $\angle ABD = \angle A$. (Complements of the same angle are equal.)
    (4) AD = BD. (If two angles of a triangle are equal, the sides opposite them are equal.)
    (5) ΔABD is isosceles. (A triangle is isosceles if it has at least two equal sides.)

Catch Question.

47.



48. Because AX ⊥ BY, ΔABP is a right triangle; so $\angle PAB$ and $\angle PBA$ are complementary. Therefore, $\angle PAB + \angle PBA = 90°$. Because AX and BY bisect two of the angles, $\angle YAP = \angle PAB$ and $\angle XBP = \angle PBA$. It therefore follows that $\angle YAB + \angle XBA = 180°$. If $\angle YAB$ and $\angle XBA$ are supplementary, however, then AY || BX (supplementary interior angles on the same side of a transversal mean that lines are parallel).

The problem is a "catch" question because AY and BX do not intersect to form a triangle.

Set III (page 241)

As this proof without words establishes, the sum of the corner angles of any five-pointed star, no matter how irregular, is 180°. A possible area of exploration would be to draw stars with other numbers of points and try to discover if anything comparable is true of them. (We will return to this problem in our study of circles. Using the fact that an inscribed angle is half the measure of its intercepted arc makes it easy to show for the special case of symmetric stars with an odd number of sides that the sum of the corner angles is 180°.)

1. The sum of the angles that are the corners of a five-pointed star is 180°.
2. A parallel line is drawn through one corner of the star to form angles with one of the sides extended. Applying the exterior angle theorem to two of the large overlapping triangles gives 1 + 3 and 2 + 4 in the smaller triangle on the right. Equal corresponding angles and equal alternate interior angles formed by the parallels reveal that $5 + (2 + 4) + (1 + 3) = 180°$.

Chapter 6, Lesson 6

Set I (pages 245–246)

The Rogallo glider is named after Francis Rogallo, a NASA engineer who invented its triangular-winged design in the late 1940s. According to James E. Mrazek in *Hang Gliding and Soaring* (St. Martin's Press, 1976), Rogallos have been tested by the U. S. Army at altitudes of 40 miles and speeds of 2,100 miles per hour and have successfully carried loads as great as three tons! NASA has even considered using radio-controlled Rogallos to bring 50-ton spacecraft boosters back to Earth for reuse.

The Roman alphabet is used almost without exception throughout the world to write the symbols of mathematics, even in such languages as Russian, Hebrew, and Chinese. No matter where you study geometry, the triangles are probably named ABC!

Hang Glider.

•1. ASA.

2. SSS.

3. SAS.

4. AAS.

•5. HL.

Congruent or Not?

•6. Parallel lines form equal corresponding angles.

7. Three.

8. No.

•9. Three.

10. No.

11. Three.

•12. Yes.

13. They are congruent by AAS.

Korean Theorem.

14. That $\angle C$ and $\angle C'$ are right angles.

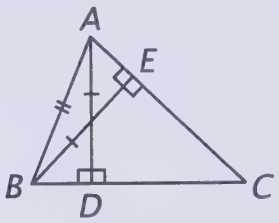
15. That ΔABC and $\Delta A'B'C'$ are congruent.

16. The HL Theorem. If the hypotenuse and a leg of one right triangle are equal to the corresponding parts of another right triangle, the triangles are congruent.

17. Right, hypotenuse, and side.

Triangle Problem.

18.



•19. HL.

20. Corresponding parts of congruent triangles are equal.

•21. If two angles of a triangle are equal, the sides opposite them are equal.

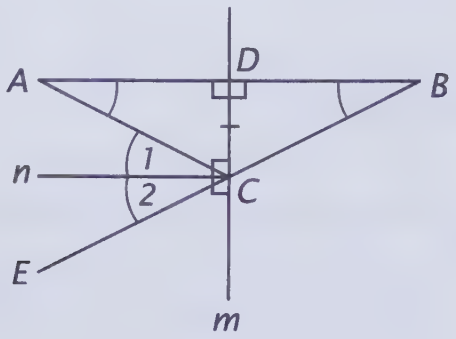
22. Isosceles.

Set II (pages 246–248)

Richard G. Pawley explored the question, "Under what conditions can two triangles have five parts of one congruent to five parts of the other, but *not* be congruent?" in an article in *The Mathematics Teacher* (May 1967). He called such triangles "5-con" and proved, among other things, that the sides of such triangles must be in geometric sequence. The angle measures for the two triangles in exercise 52 have been rounded to the nearest degree. Their exact and more precise values are: $\angle A = \text{Arccos}\ \frac{101}{108} \approx 20.74°$; $\angle B = \text{Arccos}\ \frac{-29}{48} \approx 127.17°$; $\angle C = \text{Arccos}\ \frac{61}{72} \approx 32.09°$. Calculating these values might make an amusing problem for trigonometry students to do.

Mirror Distances.

23.



•24. In a plane, two lines perpendicular to a third line are parallel.

25. Parallel lines form equal alternate interior angles.

26. Parallel lines form equal corresponding angles.

•27. Substitution.

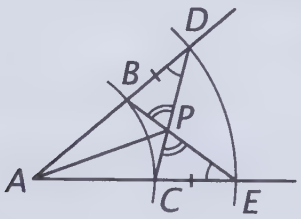
28. Reflexive.

•29. AAS.

30. Corresponding parts of congruent triangles are equal.

Angle Bisection.

31.



•32. AB = AC, AD = AE, and ∠A = ∠A.

33. SAS.

34. Corresponding parts of congruent triangles are equal.

35. BD = CE (because AD – AB = AE – AC), ∠ADC = ∠AEB, and ∠BPD = ∠CPE.

36. AAS.

37. Corresponding parts of congruent triangles are equal.

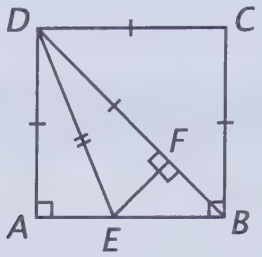
38. One set of parts: AD = AE, ∠ADC = ∠AEB, and DP = PE. (Another set of parts: AD = AE, DP = PE, AP = AP.)

39. SAS (or SSS).

•40. Because ∠DAP = ∠EAP (corresponding parts of congruent triangles are equal).

Origami Frog.

41.



•42. SSS.

43. Corresponding parts of congruent triangles are equal.

•44. 45°.

45. 45°.

46. If two angles of a triangle are equal, the sides opposite them are equal.

47. Right triangles.

48. HL.

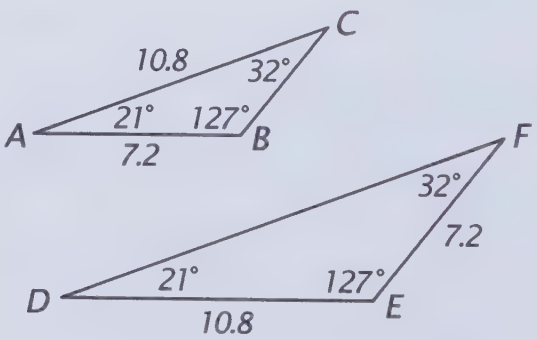
49. Corresponding parts of congruent triangles are equal.

•50. Substitution.

More on Equal Parts.

51. Two triangles that have these equal parts must be congruent by SSS.

52.



53. Yes. ΔABC and ΔDEF are an example.

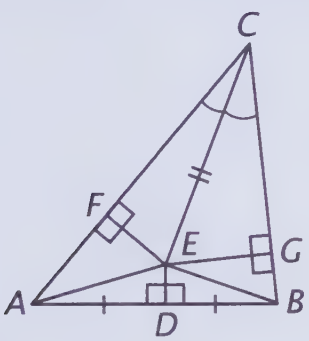
54. No. Two triangles that have six pairs of equal parts must be congruent by definition (or SSS).

Set III (page 248)

An entire "Mathematical Games" column by Martin Gardner in *Scientific American* is on the topic of geometrical fallacies. It is included in his *Wheels, Life and Other Mathematical Amusements* (W. H. Freeman and Company, 1983). Gardner reports that Euclid wrote a book on the subject. Called *Pseudaria*, the book is lost and there is no record of its content.

The proof that every triangle is isosceles appeared for the first time in W. W. Rouse Ball's *Mathematical Recreations and Essays*. Lewis Carroll included it in *The Lewis Carroll Picture Book* (1899). As Gardner explains: "The error is one of construction. E is always outside the triangle and at a point such that, when perpendiculars are drawn from E to sides CA and CB, one perpendicular will intersect one side of the triangle but the other will intersect an extension of the other side."

Is Every Triangle Isosceles?

1. 

2. AAS.
3. Corresponding parts of congruent triangles are equal.
4. SAS.
5. Corresponding parts of congruent triangles are equal.
6. HL. [EA = EB (exercise 5), EF = EG (exercise 3).]
7. Corresponding parts of congruent triangles are equal.
8. Addition. (FC = GC because $\Delta CEF \cong \Delta CEG$.)
9. Betweenness of Points Theorem.
10. Substitution.
11. The false statements appear at the end. Although it is true for this triangle that A-F-C and so AF + FC = AC, it is not true that B-G-C; so BG + GC ≠ BC. So the last equation, AC = BC, does not follow.

Chapter 6, Review

Set I (pages 250–253)

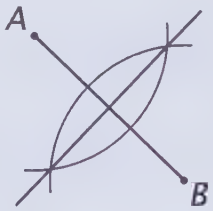
Exercises 20 through 22 are based on the rather remarkable dissection of an equilateral triangle into five isosceles triangles such that none, one, or two of the triangles are equilateral. Other such dissections are possible.

Flag Symmetries.

•1. Fold the figure along the proposed line of symmetry and see if the two halves coincide.

2. The Cuban flag. The star prevents it from having a horizontal line of symmetry.

3. The Algerian flag has a horizontal line of symmetry. The flag of Barbados has a vertical line of symmetry. The Jamaican flag has two lines of symmetry: one horizontal and the other vertical.

4. 

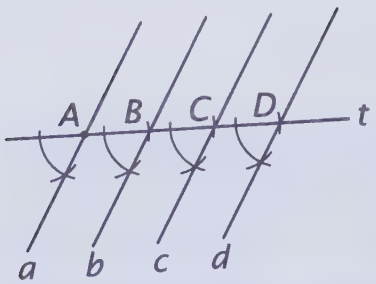
•5. It is the perpendicular bisector of AB.

Lots of Lines.

6. An overhead view of a parking lot.

•7. Equal corresponding angles mean that lines are parallel.

8.



Alphabet.

•9. To the sum of the remote interior angles.

10. Form equal corresponding angles.

11. Perpendicular to a third line are parallel.

•12. They are perpendicular.

13. Interior angles mean that lines are parallel.

•14. Are equal, their sides are perpendicular.

15. Are equal.

Ollie's Mistakes.

•16. In a plane, *two points each* equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

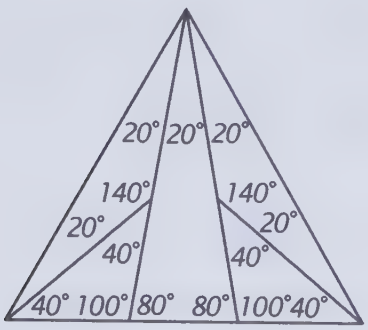
17. If two angles *and the side opposite one of them in one triangle* are equal to *the corresponding parts* of another triangle, the triangles are congruent.

18. If two *angles* of one triangle are equal to two *angles* of another triangle, the third *angles* are equal.

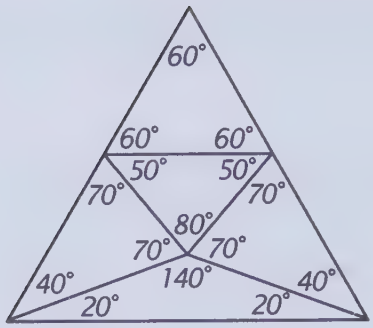
19. *In a plane,* a line perpendicular to one of two parallel lines is also perpendicular to the other.

Triangle Division.

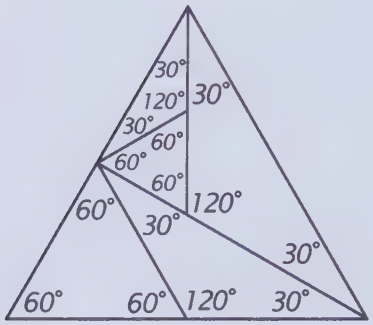
20.



21.



22.

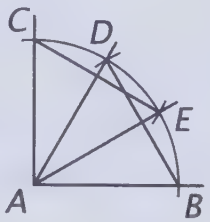


23. They are all isosceles.

24. First figure: no pieces. Second figure: one piece. Third figure: two pieces.

Angle Trisection.

25.



•26. Equilateral.

27. $\angle DAB = \angle EAC = 60°$. Each angle of an equilateral triangle is 60°.

28. $\angle EAB = 30°$.
$\angle EAB = \angle CAB - \angle DAB = 90° - 60° = 30°$.

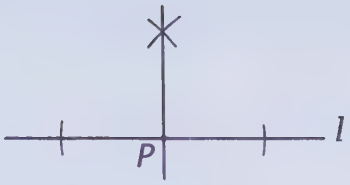
Surveyor's Triangle.

29. Exactly one.

•30. In a plane, two lines perpendicular to a third line are parallel.

31. It would bisect the angle.

32.



Set II (pages 253–254)

Magnifying Glass.

•33. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

34. All right angles are equal.

•35. Concurrent.

36. SAS. (BD = DF, ∠BDF = ∠FDP, DP = DP.)

37. Corresponding parts of congruent triangles are equal.

38. They are equal (by addition).

Measuring a Tree.

•39. ∠B = ∠CAD = 45°.

40. BA = AC and AC = CD.

41. If two angles of a triangle are equal, the sides opposite them are equal.

•42. BA = CD by substitution.

Quadrilateral Problem.

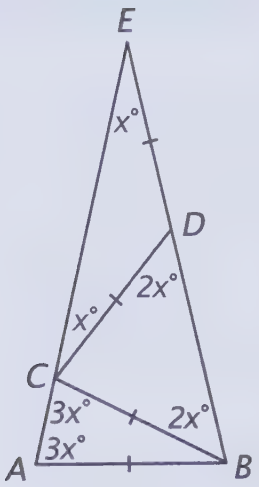
43. ∠A = ∠D. (Or ∠B = ∠C.)

44. Parallel lines form supplementary interior angles on the same side of a transversal.

45. No. (This answer may surprise some students. An example of why the answer is "no" is an isosceles trapezoid.)

Isosceles Triangle.

46.



47. ∠ECD = $x°$.

•48. ∠CDB = $2x°$.

49. ∠DBC = $2x°$.

•50. ∠ACB = $3x°$. (∠ACB is an exterior angle of ΔBCE; so ∠ACB = ∠E + ∠EBC = $x° + 2x°$.)

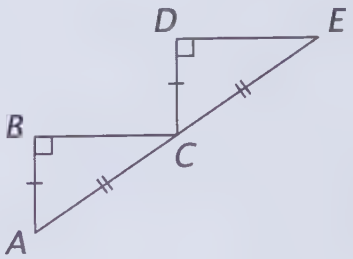
51. ∠A = $3x°$.

52. ∠ABE = $3x°$.

53. $\angle E + \angle A + \angle ABE = x + 3x + 3x = 180$; so $7x = 180$ and $x = \frac{180}{7} \approx 25.7$. ∠E ≈ 25.7°.

Stair Steps.

54.



ΔABC ≅ ΔCDE (HL); so ∠BCA = ∠E (corresponding parts of congruent triangles are equal). BC || DE because equal corresponding angles (with a transversal) mean that lines are parallel.

SAT Problem.

55. $q = 40$ because parallel lines form equal alternate interior angles. $p = 140$ because two angles in a linear pair are supplementary. So $p - q = 140 - 40 = 100$.

Algebra Review (page 256)

1. $\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$.

2. $\frac{5x}{20}-\frac{4x}{20}=\frac{x}{20}$.

3. $\frac{x+10}{2x}$.

4. $\frac{4x}{x^3}-\frac{3}{x^3}=\frac{4x-3}{x^3}$.

5. $\frac{24}{36}=\frac{2}{3}$.

6. $\frac{x^6}{8}$.

7. $\frac{1}{2}\cdot\frac{10}{7}=\frac{5}{7}$.

8. $\frac{x+2}{x}\cdot\frac{x}{2}=\frac{x+2}{2}$.

9. $\frac{12x}{16}=\frac{3x}{4}$.

10. $\frac{x+4-x-2}{10}=\frac{2}{10}=\frac{1}{5}$.

11. $\frac{1}{6}$.

12. $\frac{10}{x}\cdot\frac{1}{5}=\frac{10}{5x}=\frac{2}{x}$.

13. $\frac{4x}{8}+\frac{2y}{8}-\frac{z}{8}=\frac{4x+2y-z}{8}$.

14. $\frac{8\pi r^3}{6}=\frac{4\pi r^3}{3}$.

15. $\frac{x}{y}-\frac{y}{y}=\frac{x-y}{y}$.

16. $\frac{6(x-3)}{3(2x-3)}=\frac{2(x-3)}{2x-3}$ or $\frac{2x-6}{2x-3}$.

17. $\frac{3x}{(x+1)(x-1)}-\frac{2(x-1)}{(x+1)(x-1)}=$
$\frac{3x-2x+2}{(x+1)(x-1)}=\frac{x+2}{x^2-1}$.

18. $\frac{y}{y(x-y)}+\frac{x}{x-y}=\frac{1+x}{x-y}$.

19. $\frac{(x+3)(x+5)}{(x+3)(x-2)}\cdot\frac{x-2}{x+5}=1$.

20. $\frac{(x-1)(x^2+x+1)}{x-1}+\frac{1}{x-1}=\frac{x^3-1+1}{x-1}=\frac{x^3}{x-1}$.

Chapter 7, Lesson 1

Set I (pages 260–261)

The aerial photographer Georg Gerster, the author of *Below from Above* (Abbeville Press, 1986), wrote about his photograph of the freeway overpass covered by carpets:
"Years after having taken this photograph, I am still amazed at this super-scale Paul Klee . . . . In Europe, special mats, a Swiss development, have been adopted. They retain the heat generated as the concrete hardens and make it unnecessary to keep wetting the concrete. The new procedure is more economical, though less colorful . . . ."

Theorem 24.

•1. Two points determine a line.

2. The sum of the angles of a triangle is 180°.

•3. Addition.

4. Betweenness of Rays Theorem.

5. Substitution.

Corollary.

•6. The sum of the angles of a quadrilateral is 360°.

7. Division.

8. An angle whose measure is 90° is a right angle.

•9. A quadrilateral each of whose angles is a right angle is a rectangle.

•10. Each angle of a rectangle is a right angle.

11. All right angles are equal.

Carpets.

12. Rectangles.

13. Right.

•14. Lines that form right angles are perpendicular.

15. In a plane, two lines perpendicular to a third line are parallel.

Rectangles.

16. They are equal and they are right angles.

•17. They seem to be parallel and equal.

18. They seem to be equal (and they seem to bisect each other).

Rhombuses.

19. They seem to be equal (and the opposite sides seem to be parallel).

20. They seem to be perpendicular (and they seem to bisect each other).

•21. They are equiangular (or each of their angles is a right angle).

22. They are equilateral.

Parallelograms.

23. They seem to be parallel and equal.

24. They seem to be equal.

25. They seem to bisect each other.

SAT Problem.

•26. $2x + 2y = 180$. ($2x + 2y + 180 = 360$.)

27. $x + y = 90$.

Set II (pages 262–264)

Martin Gardner dedicated his book of mathematical recreations, *Penrose Tiles to Trapdoor Ciphers* (W. H. Freeman and Company, 1989) to Roger Penrose with these words:

"To Roger Penrose, for his beautiful, surprising discoveries in mathematics, physics, and cosmology; for his deep creative insights into how the universe operates; and for his humility in not supposing that he is exploring only the products of human minds."

Two chapters of Gardner's book are about Penrose tilings, a subject full of rich mathematical ideas. The two tiles in the exercises come from a rhombus with angles of 72° and 108°. John Horton Conway calls them "kites" and "darts." Amazingly, in *all* nonperiodic tilings of these tiles, the ratio of the number of kites to the number of darts is the golden ratio: $\frac{1+\sqrt{5}}{2} = 1.618\ldots$! There are a number of simple ways to prove that there are infinitely many ways to tile the plane with these two tiles. Even so, any finite region of one tiling appears infinitely many times in every other tiling; so no finite sample of a Penrose tiling can be used to determine which of the infinitely

many different tilings it is! Students interested in such mind-boggling ideas will enjoy reading Gardner's material and doing the experiments suggested in it.

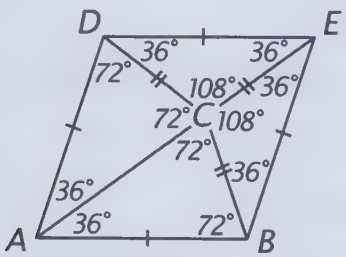
The exercises on the angles of polygons are intended to encourage students to realize that memorizing a result such as "each angle of an equiangular n-gon has a measure of $\frac{(n-2)180°}{n}$," is not important. What *is* important is for the student to feel comfortable and confident in being able to reconstruct such a result if it is needed.

Penrose Tiles.

28. The one labeled ABCD.

•29. 144°. (360° – 3·72°.)

30.



•31. SSS.

32. All are isosceles.

33. They seem to be collinear.

34. They are collinear because $\angle ACE = 72° + 108° = 180°$.

35. Yes. It appears that it could be folded along the line through A, C, and E so that the two halves would coincide.

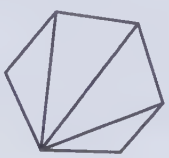
36. No. Quadrilateral ABED is equilateral but not equiangular.

Gemstone Pattern.

•37. Triangles, quadrilaterals, and an octagon.

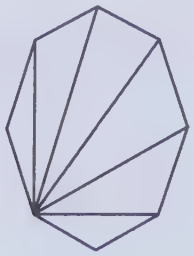
38. Five sides, two diagonals, and three triangles.

39. *Example figure:*



•40. Six sides, three diagonals, and four triangles.

41. *Example figure:*



42. Eight sides, five diagonals, and six triangles.

•43. $n - 3$.

44. $n - 2$.

45. A quadrilateral has $4 - 3 = 1$ diagonal which forms $4 - 2 = 2$ triangles.

•46. 720°. ($4 \times 180°$.)

•47. 120°. ($\frac{720°}{6}$.)

48. 1,080°. ($6 \times 180°$.)

49. 135°. ($\frac{1,080°}{8}$.)

•50. $(n - 2)180°$.

51. $\frac{(n-2)180°}{n}$.

Linkage Problems.

•52. Their sum is 360°.

53. Their sum is 180° if we think of the linkage as a triangle. (It is 360° if we think of the linkage as a quadrilateral, because $\angle BCD = 180°$.)

•54. They all appear to be smaller.

55. $\angle A + \angle B + \angle APB = 180°$, and $\angle C + \angle D + \angle CPD = 180°$; so $\angle A + \angle B + \angle APB = \angle C + \angle D + \angle CPD$. Because $\angle APB = \angle CPD$, it follows that $\angle A + \angle B = \angle C + \angle D$ by subtraction.

Quadrilateral Angle Sum.

56. Addition.

57. Substitution.

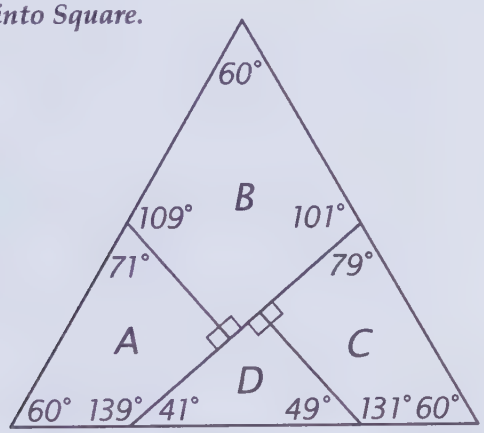
58. Subtraction.

59. $\angle 1$ and $\angle 2$ are a linear pair; so they are supplementary and their sum is 180°. Likewise for $\angle 3$ and $\angle 4$.
$(\angle 1 + \angle 2) + (\angle 3 + \angle 4) = S$; so
$180° + 180° = S$, and so $S = 360°$.

Set III (page 264)

This triangle-to-square puzzle is the puzzle of Henry Dudeney first introduced in the Set III exercise of Chapter 1, Lesson 3. It is perhaps surprising that, by carefully comparing the two figures and knowing that the angles of an equilateral triangle and a square are 60° and 90° respectively, we can find all of the remaining angles from just the one given. (By using the fact that the triangle and square have equal areas and applying some simple trigonometry to the triangle labeled D, we can find the exact values of *all* of the angles in the figure. The 41° angle, for example, is rounded from Arcsin $\frac{\sqrt[4]{3}}{2}$.)

Triangle into Square.



Chapter 7, Lesson 2

Set I (pages 267–268)

1. The first card (the card with the picture of the baseball player).

•2. Point.

•3. See if the figure looks exactly the same when it is turned upside down.

Optical Illusion.

•4. Three.

•5. Yes. The opposite sides of a parallelogram are equal.

6. BC looks longer than CF, but they are actually the same length.

7. The opposite angles of a parallelogram are equal.

•8. An exterior angle of a triangle is greater than either remote interior angle.

9. Substitution.

Theorem 26.

10. The opposite sides of a parallelogram are equal.

•11. The opposite sides of a parallelogram are parallel.

12. Parallel lines form equal alternate interior angles.

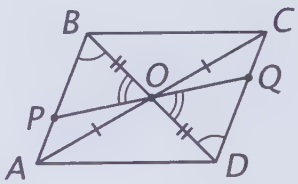
13. ASA.

14. Corresponding parts of congruent triangles are equal.

•15. A line segment is bisected if it is divided into two equal segments.

Point Symmetry.

16.



•17. The diagonals of a parallelogram bisect each other.

•18. Two points are symmetric with respect to a point if it is the midpoint of the line segment connecting them.

19. Parallel lines form equal alternate interior angles.

20. Vertical angles are equal.

21. ASA.

22. Corresponding parts of congruent triangles are equal.

23. Two points are symmetric with respect to a point if it is the midpoint of the line segment connecting them.

Set II (pages 268–269)

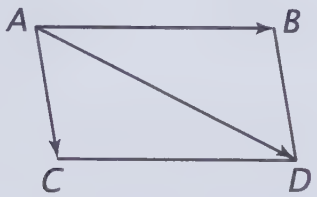
In encyclopedias and unabridged dictionaries, the entry for "parallelogram" is followed by an entry for "parallelogram law." The *Encyclopedia Britannica* describes it as the "rule for obtaining geometrically the sum of two vectors by constructing a parallelogram. A diagonal will give the requisite vector sum of two adjacent sides. The law is commonly used in physics to determine a resultant force or stress acting on a structure." The except from Newton's *Mathematical Principles of Natural Philosophy* is taken from the original Latin edition published in 1687. It immediately follows statements of his three laws of motion.

A specific measure for ∠BAD in the figure for exercises 47 through 50 was given to make the exercises a little easier. Actually, *any* measure other than 60° or 120° will work. A nice problem for better students would be to rework the exercises letting ∠BAD = x° and to explain what happens when x = 60 or 120.

Parallelogram Rule.

24. Corollary.

25.



•26. 10. (5 cm represents 50 lb; so 1 cm represents 10 lb.)

27. 25. (2.5 × 10 = 25.)

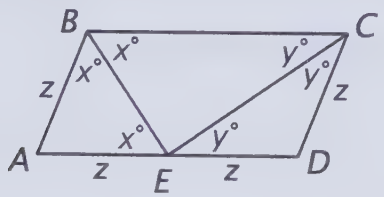
28. About 6 cm.

29. About 60. (6 × 10 = 60.)

•30. About 24°.

Angle Bisectors.

31.



•32. ∠AEB = x°; ∠CED = y°.

33. AB, AE, and ED.

34. Point E is the midpoint of AD.

•35. AD = 2DC.

36. They are supplementary. Parallel lines form supplementary interior angles on the same side of a transversal.

•37. 180.

38. 90.

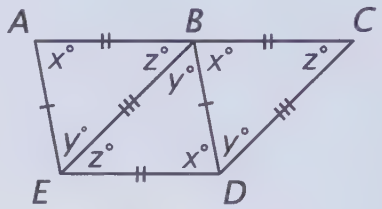
39. Isosceles.

40. Right. Because $x + y = 90$, ∠BEC = 90°.

41. BE ⊥ EC.

Two Parallelograms.

42.



43. Because x, y, and z are the measures of the angles of a triangle, $x + y + z = 180$. Therefore, ∠ABC is a straight angle; so A, B, and C are collinear.

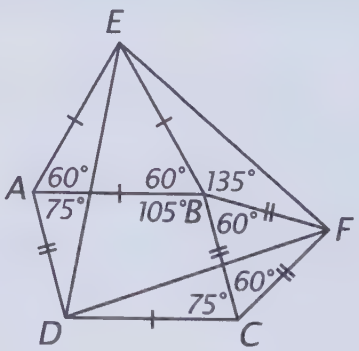
44. Because ABDE and BCDE are parallelograms, AB || ED and BC || ED. If A, B, and C are not collinear, then through B there are two lines parallel to ED. This contradicts the Parallel Postulate; so A, B, and C must be collinear.

•45. All three (ΔABE ≅ ΔDEB ≅ ΔBCD).

46. No.

Hidden Triangles.

47.



48. ΔADE, ΔCDF, and ΔEBF.

49. SAS.

50. Three. In addition to ΔABE and ΔBCF, ΔDEF is equilateral because its sides are the corresponding sides of the three congruent triangles.

Set III (page 269)

The four symbols used on modern playing cards were designed in 1392 by the court painter to King Charles VI of France. The four suits represented four classes of French society: the spades, soldiers, the clubs, farmers, the diamonds, artisans, and the hearts, the clergy. According to John Scarne in *Scarne's New Complete Guide to Gambling* (Simon & Schuster, 1974), playing cards first arrived in the New World in 1492 with Columbus and immediately found favor with the Native Americans, who made them of deerskin or sheepskin!

Playing Card Symmetries.

1. So that they look the same whether they are held "right side up" or "upside down."
2. No playing card has line symmetry.
3. The diamonds are more symmetric because the diamond symbol itself has point symmetry, whereas the symbols of the other suits do not. (They do have line symmetry, however.)
4. All of the diamond cards have point symmetry except for the 7. All face cards have point symmetry. In the club, spade, and heart suits, the 2s and 4s have point symmetry.
5. The 6s and 8s of clubs, spades, and hearts would have point symmetry if one of the symbols in the middle row were turned upside down. (It is impossible for the odd-numbered cards in these three suits to be redesigned so as to have point symmetry!)

Chapter 7, Lesson 3

Set I (pages 272–273)

The letter *p* is the only letter of the English alphabet that transforms into three other letters by reflections and rotations. Some students may enjoy thinking about other examples of letters that transform into at least one other letter in this way.

Pop-Up Parallelogram.

•1. A quadrilateral is a parallelogram if its opposite sides are equal.

2. The opposite sides of a parallelogram are parallel.

•3. Parallel lines form equal corresponding angles.

•4. In a plane, if a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.

Theorem 28.

5. The sum of the angles of a quadrilateral is 360°.
6. Division.
7. Two angles are supplementary if their sum is 180°.

•8. Supplementary interior angles on the same side of a transversal mean that lines are parallel.

9. A quadrilateral is a parallelogram if its opposite sides are parallel.

Theorem 29.

10. Two points determine a line.
11. Parallel lines form equal alternate interior angles.
12. Reflexive.

•13. SAS.

14. Corresponding parts of congruent triangles are equal.

•15. A quadrilateral is a parallelogram if its opposite sides are equal.

Theorem 30.

16. A line segment that is bisected is divided into two equal parts.
17. Vertical angles are equal.
18. SAS.
19. Corresponding parts of congruent triangles are equal.

•20. Equal alternate interior angles mean that lines are parallel.

•21. A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.

Letter Transformations.

•22. Line symmetry. The figure can be folded along a vertical line so that the two halves coincide.

23. Point symmetry. If the figure is turned upside down, it looks exactly the same.

24. Line symmetry. The figure can be folded along a horizontal line so that the two halves coincide.

Quadrilateral Symmetries.

25. Yes. Point symmetry.

•26. No. Line symmetry (one vertical line).

27. Yes. Point symmetry and line symmetry (two lines, one vertical and one horizontal).

28. Yes. Point symmetry and line symmetry (four lines, one vertical, one horizontal, and two at 45° angles with them).

Economics Graph.

•29. They seem to be parallel.

30. The quadrilateral formed by the four lines is a parallelogram (a quadrilateral is a parallelogram if two opposite sides are equal and parallel); so the lines C and C + I are parallel.

Set II (pages 273–275)

The method for constructing a parallel to a line through a given point not on it is another example of a "rusty compass" construction. Martin Gardner observes that, although the method has long been known, it is still being rediscovered and reported as new.

The grid-bracing illustrations are taken from the section titled "Bracing Structures" in Jay Kappraff's *Connections—The Geometric Bridge between Art and Science* (McGraw-Hill, 1991). Kappraff cites the article "How to Brace a One-Story Building" in *Environment and Planning* (1977) as the original source of this material, an important topic for engineers and architects. Examples of pertinent theorems dealing with this subject in graph theory are: "In any distorted grid all the elements of a row (column) are parallel" and "A bracing of an *n* by *m* grid is a minimum rigid bracing if and only if the bracing subgraph is a tree."

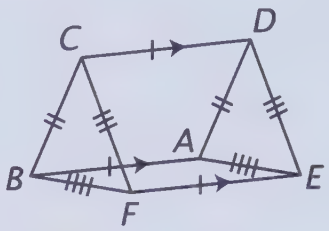
Parallel Rulers.

31. They are connected so that AB = CD and AC = BD. A quadrilateral is a parallelogram if its opposite sides are equal.

•32. The pairs of opposite angles.

Tent Geometry.

33.

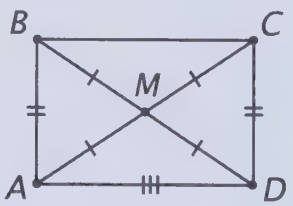


34. Yes. ABCD and CDEF are parallelograms; so AB = CD and CD = EF (the opposite sides of a parallelogram are equal) and AB || CD and CD || EF (the opposite sides of a parallelogram are parallel). It follows that AB = EF (substitution) and AB || EF (two lines parallel to the same line are parallel to each other). Therefore, ABFE is a parallelogram because two opposite sides are parallel and equal.

35. Yes. AD = BC, DE = CF, and AE = BF (the opposite sides of a parallelogram are equal). Therefore, $\Delta ADE \cong \Delta BCF$ (SSS).

Rope Trick.

36.



•37. A quadrilateral is a parallelogram if its diagonals bisect each other.

38. The opposite sides of a parallelogram are equal.

•39. SSS.

40. Corresponding parts of congruent triangles are equal.

41. The opposite angles of a parallelogram are equal.

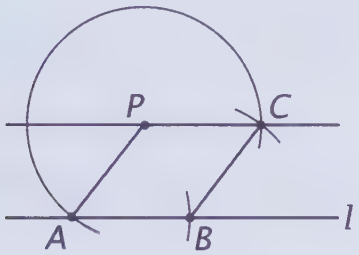
42. Substitution.

•43. A quadrilateral is a rectangle if all of its angles are equal.

Parallel Postulate.

44. Through point P, there is exactly one line parallel to line *l*.

45.



46. PC is parallel to line *l* because ABCP is a parallelogram (a quadrilateral is a parallelogram if its opposite sides are equal).

Flexible Grid.

•47. 12.

48. 4.

49. 12.

50. 4.

51. They remain unchanged because their shapes are determined by their sides (SSS) and their sides do not change.

52. They remain parallel because the opposite sides of the quadrilaterals remain equal; so they are always parallelograms.

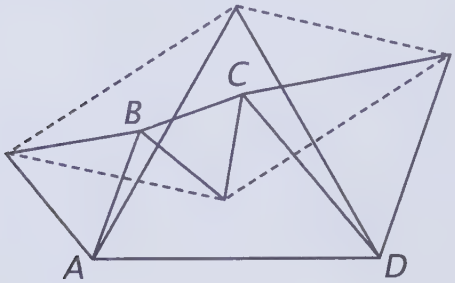
•53. They are the sides of the quadrilaterals above and below the braced square.

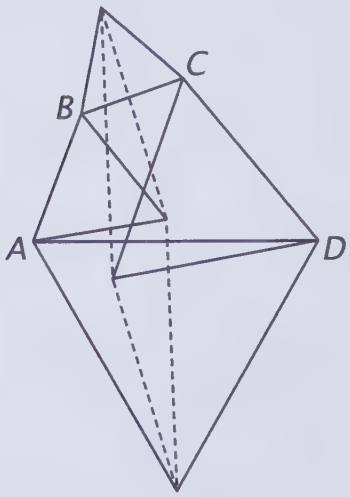
54. They are the sides of the quadrilaterals to the right of the braced square.

55. Because all of the quadrilaterals remain parallelograms, these sides remain parallel to the horizontal and vertical sides of the braced square.

Set III (page 275)

1. *(The figures below show two possible answers depending on which triangles are constructed inward and outward.)*





2. The quadrilateral formed seems to be a parallelogram.

Chapter 7, Lesson 4

Set I (pages 277–279)

In his definitive book on the subject, *Dissections—Plane and Fancy* (Cambridge University Press, 1997), Greg N. Frederickson (using radians, 2π radians = 360°) reports:

"We can think of a regular polygon as a union of various rhombuses and isosceles triangles (which are halves of rhombuses). Such a characterization is an *internal structure* of the corresponding polygon. If the regular polygon has an even number of sides, then the internal structure needs only rhombuses . . . . A regular polygon with $2q$ sides contains q each of $\frac{i\pi}{q}$-rhombuses, for each positive integer i that is less than $\frac{q}{2}$, plus $\frac{q}{2}$ squares if q is even. For example, the dodecagon contains

6 each of $\frac{\pi}{6}$-rhombuses and $\frac{\pi}{3}$-rhombuses, plus 3 squares."

[$\frac{\pi}{6}$ and $\frac{\pi}{3}$ indicate rhombuses having angles of 30° and 60°.] There are several ways in which the rhombuses can be arranged when the dodecagon is divided in this way.

The remark about a square being perfect because all of its angles are "just right" was made by Albert H. Beiler at the beginning of the chapter titled "On The Square" in his book titled *Recreations in the Theory of Numbers—The Queen of Mathematics Entertains* (Dover, 1966).

Regular Dodecagon.

•1. 12.

•2. Three.

3. A square.

4. Three.

•5. 15.

6. Three. (There are six narrow, six medium, and three square.)

Theorem 31.

•7. All of the angles of a rectangle are equal.

8. A quadrilateral is a parallelogram if its opposite angles are equal.

Theorem 32.

9. All of the sides of a rhombus are equal.

•10. A quadrilateral is a parallelogram if its opposite sides are equal.

Theorem 33.

11. All of the angles of a rectangle are equal.

•12. All rectangles are parallelograms.

13. The opposite sides of a parallelogram are equal.

14. Reflexive.

15. SAS.

16. Corresponding parts of congruent triangles are equal.

Theorem 34.

•17. All of the sides of a rhombus are equal.

18. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Which Parts?

•19. Two consecutive sides.

20. One side.

21. One side and one angle.

22. Two consecutive sides and one angle.

"Just Right."

23. If all of the angles of a triangle were right angles, the sum of their measures would be 270°, rather than 180°.

•24. No. It must be a rectangle.

25. *Shown below are two of many possible examples. (The ambiguity arises because the right angles may be on the inside or the outside of the polygon.)*



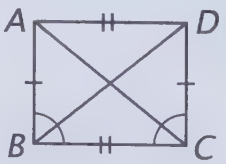
Set II (pages 279–280)

Checking a Wall.

26. No. If the opposite sides of a quadrilateral are equal, it must be a parallelogram but not necessarily a rectangle.

27. No. We know that the diagonals of a rectangle are equal but to conclude that, if the diagonals of a quadrilateral are equal, it is a rectangle would be to make the converse error.

28.



29. SSS.

30. Corresponding parts of congruent triangles are equal.

•31. A quadrilateral is a parallelogram if its opposite sides are equal.

32. The opposite angles of a parallelogram are equal.

33. Substitution.

•34. A quadrilateral all of whose angles are equal is a rectangle.

35. True.

•36. False.

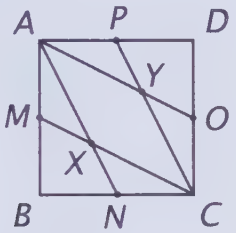
37. True.

38. False.

•39. True.

Square Problem.

40.



41. Parallelograms.

42. They must be parallelograms because two opposite sides are both parallel and equal. (For example, in AMCO, AM || CO because ABCD is a square and AM = CO because M and O are midpoints of the equal segments AB and CD.)

•43. Its opposite sides are parallel.

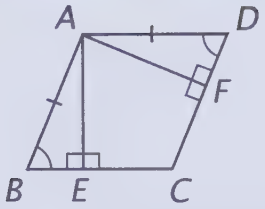
44. A rhombus.

45. They seem to be collinear.

46. It appears to have point symmetry and line symmetry (with respect to the lines of the diagonals of the square).

Rhombus Problem.

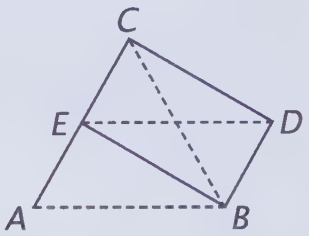
47.



48. Yes. Because ABCD is a rhombus, AB = AD. It is also a parallelogram; so $\angle B = \angle D$. $\Delta AEB \cong \Delta AFD$ (AAS); so AE = AF.

Triangle Problem.

49.



50. Because ABDE is a parallelogram, AB = ED (the opposite sides of a parallelogram are equal). Also, because BDCE is a rectangle, ED = CB (the diagonals of a rectangle are equal). So AB = CB (substitution). Therefore, ΔABC is isosceles. (Additional drawings using the conditions given reveal that it isn't necessarily equilateral.)

Set III (page 280)

Bill Leonard told the story of starting a lesson with a high-school class of remedial mathematics students with the problem of counting the squares in a 4 x 4 grid. He had hoped that the students would think about it for awhile, but it didn't take long for several students to start yelling out "16." Leonard then asked if they had counted squares such as the one shaded in the second figure. The class immediately yelled out in unison, "17!"

Counting Squares.

1. It contains 30 squares in all. There are 16 small squares, 9 2×2 squares, 4 3×3 squares, and 1 4×4 square.

2. It contains 15 squares in all. There are 7 small squares, 6 2×2 squares, 1 3×3 square, and 1 4×4 square.

Chapter 7, Lesson 5

Set I (pages 282–284)

Hal Morgan in his book titled *Symbols of America* (Viking, 1986), comments on the Chase Manhattan Bank trademark: "After World War II the representational images of the nineteenth century began to give way to abstract designs inspired by the spare, geometric forms of the Bauhaus. Some of these modern designs are quite striking in their clarity and simplicity. . . . The design firm of Chermayeff & Geismar Associates created the Chase Manhattan Bank's distinctive octagon logo in 1960. . . . The abstract, interlocking design seems to have become a sort of model for the designers of other bank and insurance logos." Morgan shows examples of several more-recent logos that are also point-symmetric designs of interlocking solid black congruent polygons.

Quadrilaterals in Perspective.

•1. A trapezoid.

•2. That it has exactly one pair of parallel sides.

3. Yes. Picnic blankets are usually rectangular. (The perspective alters the blanket's shape.)

•4. That all of its angles are right angles (or that they are equal).

•5. Yes. All rectangles are parallelograms.

6. Rhombuses and parallelograms.

7. Squares, rectangles, rhombuses, and parallelograms.

Geometric Trademark.

•8. Point.

9. It is a square because all of its sides and angles are equal.

•10. They seem to be congruent.

•11. Trapezoids.

12. CD and JE.

•13. No. The legs of a trapezoid must be equal for it to be isosceles.

14. No. One is a right angle and the other is acute.

15. An octagon.

Theorem 35.

•16. The bases of a trapezoid are parallel.

•17. Through a point not on a line, there is exactly one line parallel to the line.

18. A quadrilateral is a parallelogram if its opposite sides are parallel.

•19. The opposite sides of a parallelogram are equal.

20. The legs of an isosceles trapezoid are equal.

21. Substitution.

•22. If two sides of a triangle are equal, the angles opposite them are equal.

23. Parallel lines form equal corresponding angles.

24. Substitution.

•25. Parallel lines form supplementary interior angles with a transversal.

26. The sum of two supplementary angles is 180°.

27. Substitution.

28. Subtraction.

Theorem 36.

29. The legs of an isosceles trapezoid are equal.

•30. The base angles of an isosceles trapezoid are equal.

31. Reflexive.

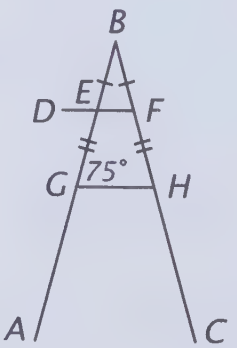
32. SAS.

33. Corresponding parts of congruent triangles are equal.

Set II (pages 284–285)

Stepladder.

34.



35. An isosceles trapezoid.

•36. 75°.

37. The base angles of an isosceles trapezoid are equal.

38. Isosceles.

39. 75°.

•40. Parallel lines form equal corresponding angles.

•41. 105°.

42. 30°.

Regular Pentagon.

43. Examples are ABCD, ABCE, ACDE, and ABDE. Their bases are parallel (they form supplementary interior angles on the same side of a transversal) and their legs are equal (given).

44. Examples are ABCH, ABGE, and AFDE. Their opposite sides are parallel (established in exercise 43).

45. Examples are ABCH, ABGE, and AFDE. They are rhombuses because all of their sides are equal. In regard to ABCH, for example, AB = BC (given), AB = HC, and AH = BC (the opposite sides of a parallelogram are equal); so HC = AH (substitution).

Trapezoid Diagonals.

•46. AC and DB bisect each other.

•47. A quadrilateral is a parallelogram if its diagonals bisect each other.

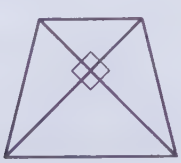
48. The opposite sides of a parallelogram are parallel.

•49. A trapezoid has only one pair of opposite sides parallel.

50. AC and DB bisect each other.

51. AC and DB do not bisect each other.

52. Yes. One example of such a trapezoid is shown below.



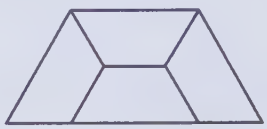
Set III (page 285)

The Moscow Puzzles is the English language edition of Boris Kordemsky's *Mathematical Know-how*, which, according to Martin Gardner, is the best and most popular puzzle book ever published in the Soviet Union. Kordemsky taught mathematics at a high school in Moscow for many years.

Gardner reported Andrew Miller's discovery of a second solution to Kordemsky's trapezoid dissection problem in the February 1979 issue of *Scientific American* and posed it as a new puzzle for the readers of that magazine.

Congruence Puzzle.

Kordemsky's solution is shown below.



The solution thought of by Andrew Miller is shown below. The three cuts intersect the upper base at points $\frac{1}{8}$, $\frac{3}{4}$, and $\frac{7}{8}$ of the way from its left endpoint.



Chapter 7, Lesson 6

Set I (pages 288–289)

The "state capital" quadrilateral was chosen partly for fun but primarily to show that the original quadrilateral can have any shape. It doesn't even have to be convex. Not only can the quadrilateral be concave, it can even be "self-intersecting" or skew!

Portuguese Theorem.

1. Midpoints.

•2. A midsegment.

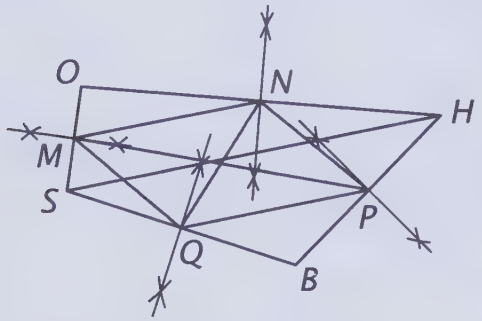
3. Other side.

4. A midsegment of a triangle is parallel to the third side and half as long.

•5. DE || AC and DE = $\frac{1}{2}$AC.

State Capitals Problem.

6.



7. MNPQ looks like a parallelogram.

•8. A midsegment of a triangle is parallel to the third side. (MN and QP are midsegments of ΔSOH and ΔSBH.)

•9. In a plane, two lines parallel to a third line are parallel to each other.

•10. A midsegment of a triangle is half as long as the third side.

11. Substitution.

12. That it is a parallelogram.

•13. MNPQ is a parallelogram because two opposite sides are both parallel and equal.

14. MP and NQ seem to bisect each other.

15. The diagonals of a parallelogram bisect each other.

Midpoint Quadrilateral.

16. A midsegment of a triangle is half as long as the third side.

•17. Addition.

•18. Substitution.

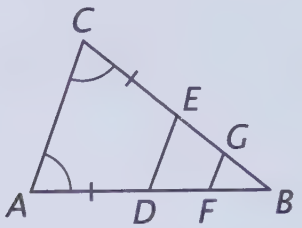
19. Its perimeter.

•20. The sum of its diagonals.

21. The perimeter of the midpoint quadrilateral is equal to the sum of the diagonals of the original quadrilateral.

Two Midsegments.

22.



23. ADEC is an isosceles trapezoid because DE || AC (a midsegment of a triangle is parallel to the third side) and AD = CE.

•24. ∠A = ∠C because the base angles of an isosceles trapezoid are equal.

25. ΔABC is isosceles because, if two angles of a triangle are equal, the sides opposite them are equal.
(Alternatively, AB = 2AD = 2CE = CB .)

26. FG = $\frac{1}{4}$AC because FG = $\frac{1}{2}$DE and DE = $\frac{1}{2}$AC [so FG = $\frac{1}{2}(\frac{1}{2}$AC) = $\frac{1}{4}$AC].

Also, FG || AC because FG || DE and DE || AC (in a plane, two lines parallel to a third line are parallel to each other).

Set II (pages 289–291)

Exercises 27 through 36 are intended to review some basics of coordinate geometry, including the distance formula, and to get students to thinking about how midpoints and the directions of lines might be treated.

The Midsegment Theorem is, in a way, a "special case" of the fact that the line segment connecting the midpoints of the legs of a trapezoid is parallel to the bases and half their sum.

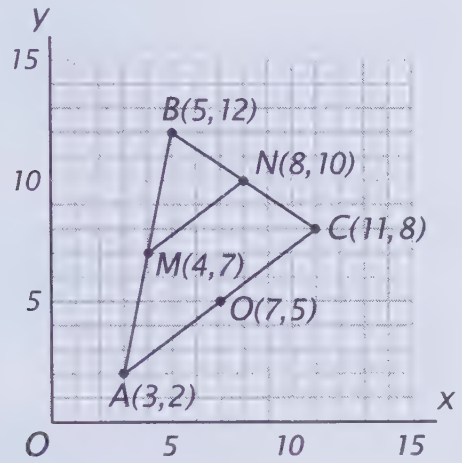
The edition of Euclid's *Elements* with the fold-up figures was its first translation into English, by Henry Billingsley. After a career in business, Billingsley became sheriff of London and was elected its mayor in 1596. An excerpt from *The Elements of Geometrie*, including an actual fold-up model, is reproduced in Edward R. Tufte's wonderful book *Envisioning Information* (Graphics Press, 1990).

In *Mathematical Gems II* (Mathematical Association of America, 1976), Ross Honsberger wrote: "Solid geometry pays as much attention to the tetrahedron as plane geometry does to the

triangle. Yet many elementary properties of the tetrahedron are not very well known." Exercises 37 through 41 establish that any triangle can be folded along its midsegments to form a tetrahedron. Unlike tetrahedra in general, however, a tetrahedron formed in this way is always isosceles; that is, each pair of opposite edges are equal. All the faces of an isosceles tetrahedron are congruent triangles and consequently have the same area; furthermore, the sum of the face angles at each vertex is 180°. More information on this geometric solid can be found in chapter 9 of Honsberger's book.

Midpoint Coordinates.

27.



•28. (4, 7).

•29. (8, 10).

30. (7, 5).

31. Yes. The coordinates of the midpoint are "midway between" the respective coordinates of the endpoints. (They are their averages.)

•32. $MN = \sqrt{(8-4)^2+(10-7)^2} = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5.$

33. $AC = \sqrt{(11-3)^2+(8-2)^2} = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10.$

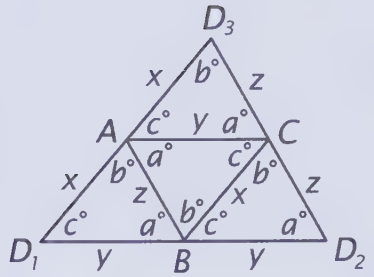
34. Yes. $MN = \frac{1}{2}AC$ as the Midsegment Theorem states.

•35. They should be parallel.

36. Yes. They seem to be parallel because they have the same direction; each line goes up 3 units as it goes 4 units to the right.

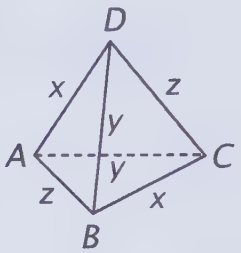
Tetrahedron.

37.



•38. They are congruent by SSS.

39.

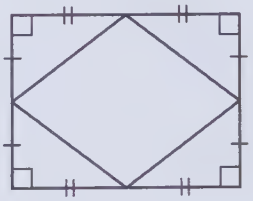


40. They are equal.

41. 180°. (The measures of the three angles that meet at each vertex are *a*, *b*, and *c*.)

More Midpoint Quadrilaterals.

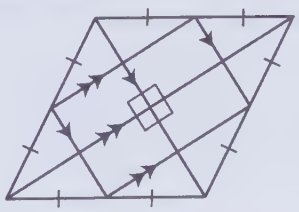
42. *Example figure:*



•43. A rhombus.

44. The line segments divide the rectangle into four right triangles and the quadrilateral. The triangles are congruent by SAS; so their hypotenuses are equal. Because the hypotenuses are the sides of the quadrilateral, it is equilateral and must be a rhombus.

45. *Example figure:*



46. A rectangle.

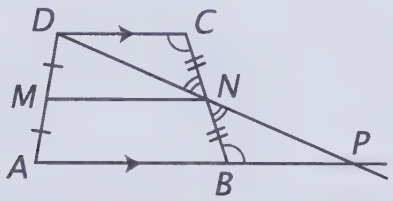
47. The sides of the new quadrilateral are parallel to the diagonals of the rhombus because of the Midsegment Theorem. Consequently, they form four small parallelograms. Each parallelogram has a right angle because the diagonals of a rhombus are perpendicular. Because the opposite angles of a parallelogram are equal, it follows that the four angles of the new quadrilateral are right angles; so it is a rectangle.

48. Another square.

Base Average.

•49. MN seems to be parallel to AB and DC, the bases of the trapezoid.

50.



51. DC || AP; so ∠C = ∠NBP (parallel lines form equal alternate interior angles). CN = NB because N is the midpoint of CB. ∠CND = ∠BNP (vertical angles are equal). So ΔDCN ≅ ΔPBN (ASA).

[An alternative proof could use ∠CDN = ∠BPN (equal alternate interior angles) in place of either pair of angles above, in which case the triangles would be congruent by AAS.]

52. DN = NP (corresponding parts of congruent triangles are equal); so N is the midpoint of DP. Therefore, MN is a midsegment of ΔDAP and so MN || AP.

53. MN || AP and DC || AP; so MN || DC. (In a plane, two lines parallel to a third line are parallel to each other.)

54. $MN = \frac{1}{2}AP$ by the Midsegment Theorem. Because AP = AB + BP and DC = BP (corresponding parts of congruent triangles are equal), $MN = \frac{1}{2}(AB + BP) = \frac{1}{2}(AB + DC)$ (substitution).

Set III (page 291)

The problem of the nested triangles is a preview of the idea of a limit, a topic considered later in the treatment of measuring the circle. The sum of the lengths of all of the sides of the triangles is the sum of the geometric series:

$$6 + \frac{1}{2}6 + (\frac{1}{2})^2 6 + (\frac{1}{2})^3 6 + \cdots = 2 \times 6$$

Students who tackle this problem will be interested to know that they will explore it further in second-year algebra.

Infinite Series.

The perimeter of the largest triangle is 6 in. From the Midsegment Theorem, we know that the sides of the next triangle are half as long; so its perimeter is 3 in. Reasoning in the same way, we find that the perimeters in inches of successive triangles are 1.5, 0.75, 0.375, 0.1875, and so on.

Starting with the first two triangles, we find that the successive sums of the sides are 9, 10.5, 11.25, 11.625, 11.8125, and so on. The sums seem to be getting closer and closer to 12 inches and (as students who study infinite series will learn to prove) the sum of *all* of the sides of the triangles in the figure is 1 foot.

Chapter 7, Review

Set I (pages 292–294)

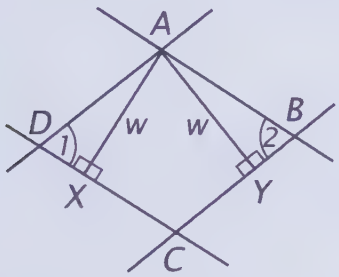
A question such as that in exercise 11 is intended to arouse the student's curiosity. According to Jearl Walker, who posed it in his book titled *The Flying Circus of Physics* (Wiley, 1977), "If the ratio of the bar's density to the fluid's density is close to 1 or 0, the bar floats in stable equilibrium as in the first figure. If the ratio is some intermediate value, then the bar floats in stable equilibrium with its sides at 45° to the fluid's surface. In each case, the orientation of stable equilibrium is determined by the position in which the potential energy of the system is least." This floating-bar problem is a good illustration of how scientists arrive at conclusions by using inductive reasoning (performing experiments with floating bars to see what happens) and how they then "explain" the results by using deductive reasoning (calculating the potential energy and its minimum value by using mathematics, including *geometry*).

The frequent appearance in this chapter of theorems whose converses also are theorems can easily mislead students into forgetting that a

statement and its converse are not logically equivalent. The intent of exercises 17 through 30 is to remind students of this fact and to encourage them to draw figures to test unfamiliar conclusions (or even to remind themselves of facts that they have proved). Knowledge of the properties of the various quadrilaterals should not be as much a matter of memorization as of looking at pictures and applying simple common sense.

Palm Strand Rhombus.

1.



•2. Both pairs of its opposite sides are parallel (because the strips have parallel sides).

3. The opposite angles of a parallelogram are equal.

•4. AAS.

5. Corresponding parts of congruent triangles are equal.

6. The opposite sides of a parallelogram are equal.

•7. Substitution.

8. All of its sides are equal.

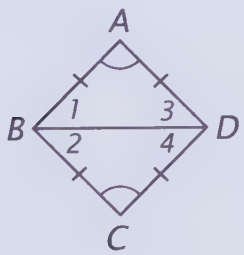
Floating Bar.

•9. Four.

10. Yes.

11. *(Student answer.)* (The bar can float either way depending on it and the liquid.)

12.



•13. All of the sides and angles of a square are equal.

14. SAS (or SSS).

15. Corresponding parts of congruent triangles are equal.

•16. If a line divides an angle into two equal parts, it bisects the angle.

Related Statements.

17. It is the converse of the first statement.

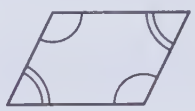
18. No. The first statement is true; the first figure shows that the triangles are congruent by SSS. The second figure shows that the second statement is false.

19. True.

20. True.

•21. True.

22. False.
Example of a quadrilateral that has two pairs of equal angles but is not an isosceles trapezoid:



23. False.
Example of a parallelogram that is not a rhombus:



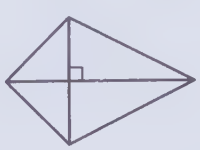
24. True.

25. True.

26. True.

•27. True.

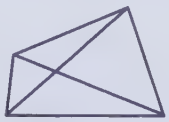
28. False.
Example of a quadrilateral with perpendicular diagonals that is not a rhombus:



29. False.
Example of a parallelogram whose diagonals are not equal:



30. False.
Example of a quadrilateral whose diagonals are equal that is not a parallelogram:



Jumping Frog.

•31. The sum of the angles of a quadrilateral is 360°.

32. They are rhombuses and all rhombuses are parallelograms. (Or, Both pairs of their opposite sides are equal.)

•33. The opposite angles of a parallelogram are equal.

34. They are parallelograms because their diagonals bisect each other.

35. The opposite sides of a parallelogram are parallel and, in a plane, two lines parallel to a third line are parallel to each other.

Set II (pages 294–295)

Only the right angles in the pieces of the puzzle of exercises 36 through 50 actually have measures that are integers. From ΔABG it is evident that

$\tan A = \frac{2c}{c} = 2$; so $\angle A = \text{Arctan } 2 \approx 63.43°$.

Although part of basic vocabulary in books of the past, words such as "trapezium" and "rhomboid" are now close to being obsolete. Strangely, in British usage, the word "trapezium" is used to mean "trapezoid."

Even though the quadrilaterals in exercises 54 through 57 are drawn to scale, it is almost impossible to recognize them merely by their appearance. It is tempting to conclude that, because quadrilateral ABCD is a trapezoid whose base angles are equal, it is isosceles, which, of course, is the converse error. To prove that it is isosceles would require adding an extra line to the figure to form a parallelogram and a triangle; the proof is comparable to that of Theorem 35, outlined in exercises 16 through 28 of Lesson 5. The additional fact that the base opposite the larger base angles of an isosceles trapezoid is longer than the other base is the basis for the magic trick of exercise 58. Like the question about the floating bar in Set I, the question about the magic trick is included for fun and not meant to be taken too seriously.

Dissection Puzzle.

•36. D, E, and F.

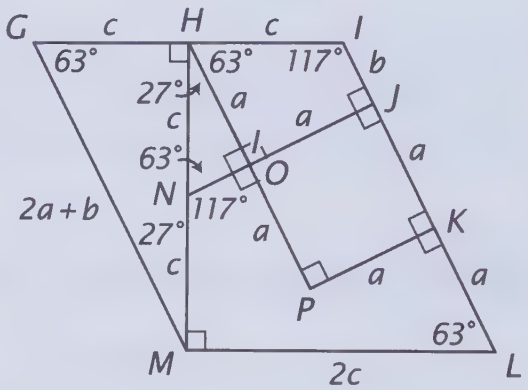
37. D.

38. D and F.

39. C and D.

40. A, B, D, E, and F.

41.



•42. GI ⊥ HM and ML ⊥ HM. In a plane, two lines perpendicular to a third line are parallel.

43. A parallelogram.

44. A quadrilateral is a parallelogram if its opposite sides are parallel.

•45. $2c$.

46. $2a + b$. [Some students may recognize that GM can also be expressed in terms of c by using the Pythagorean Theorem.
$GM^2 = c^2 + (2c)^2 = 5c^2$; so $GM = c\sqrt{5}$.]

47. A square.

48. All of its sides and angles are equal.

•49. A trapezoid.

50. It has exactly one pair of parallel sides. (HO || IJ because, in a plane, two lines perpendicular to a third line are parallel.)

Words from the Past.

•51. No, because a trapezoid has two parallel sides.

52. Yes, because, other than squares, rhombuses are parallelograms that have no right angles.

53. No, because a rectangle has four right angles.

Rectangles Not.

54. ABCD is a trapezoid because BC || AD (they form supplementary interior angles on the same side of a transversal). (It is also possible to show that ABCD is isosceles.)

55. EFGH is a parallelogram because both pairs of opposite angles are equal.

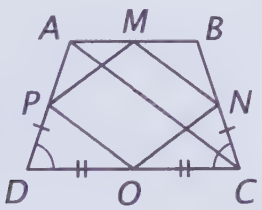
•56. IJKL is a trapezoid because IJ || LK.

57. From its angles we can conclude that MNOP is neither a parallelogram nor a trapezoid; so it is not any of the special types of quadrilaterals that we have studied.

58. While you look at the card, the magician rotates the pack 180° so that the bottom edges of the cards are at the top. When your chosen card is put back, it is easy to slide it up because it is now wider at the top than any of the others.

Another Midpoint Quadrilateral.

59.



60. MN || AC and PO || AC (a midsegment of a triangle is parallel to the third side); so MN || PO. $MN = \frac{1}{2}AC$ and $PO = \frac{1}{2}AC$ (a midsegment of a triangle is half as long as the third side); so MN = PO. MNOP is a parallelogram because two opposite sides are both parallel and equal.

61. It appears to be a rhombus. $\Delta PDO \cong \Delta NCO$ (SAS); so PO = NO. Because MN = PO and MP = NO, MN = MP; so all of the sides of MNOP are equal.

Algebra Review (page 296)

•1. 2.236.

•2. 22.36.

•3. 223.6.

•4. 7.07.

•5. 70.7.

•6. $\sqrt{500} = \sqrt{100 \cdot 5} = \sqrt{100}\sqrt{5} = 10\sqrt{5}$.

•7. $\sqrt{50{,}000} = \sqrt{100 \cdot 500} = \sqrt{100}\sqrt{500} = 10(10\sqrt{5}) = 100\sqrt{5}$.

•8. $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$.

•9. $\sqrt{5{,}000} = \sqrt{100 \cdot 50} = \sqrt{100}\sqrt{50} = 10(5\sqrt{2}) = 50\sqrt{2}$.

•10. $\sqrt{25x} = \sqrt{25}\sqrt{x} = 5\sqrt{x}$.

•11. $\sqrt{x^5} = \sqrt{x^4 \cdot x} = \sqrt{x^4}\sqrt{x} = x^2\sqrt{x}$.

•12. $\sqrt{\pi x^2} = \sqrt{\pi}\sqrt{x^2} = x\sqrt{\pi}$.

•13. $\sqrt{12x^{12}} = \sqrt{4 \cdot 3x^{12}} = \sqrt{4}\sqrt{3}\sqrt{x^{12}} = 2x^6\sqrt{3}$.

•14. $\sqrt{147} + \sqrt{3} = \sqrt{49 \cdot 3} + \sqrt{3} = \sqrt{49}\sqrt{3} + \sqrt{3} = 7\sqrt{3} + \sqrt{3} = 8\sqrt{3}$.

•15. $\sqrt{147 + 3} = \sqrt{150} = \sqrt{25 \cdot 6} = \sqrt{25}\sqrt{6} = 5\sqrt{6}$.

•16. $\sqrt{700} - \sqrt{175} = \sqrt{100 \cdot 7} - \sqrt{25 \cdot 7} = \sqrt{100}\sqrt{7} - \sqrt{25}\sqrt{7} = 10\sqrt{7} - 5\sqrt{7} = 5\sqrt{7}$.

•17. $\sqrt{700 - 175} = \sqrt{525} = \sqrt{25 \cdot 21} = \sqrt{25}\sqrt{21} = 5\sqrt{21}$.

•18. $\sqrt{20 + 20 + 20} = \sqrt{60} = \sqrt{4 \cdot 15} = \sqrt{4}\sqrt{15} = 2\sqrt{15}$.

•19. $\sqrt{20} + \sqrt{20} + \sqrt{20} = 3\sqrt{20} = 3\sqrt{4 \cdot 5} = 3\sqrt{4}\sqrt{5} = 6\sqrt{5}$.

•20. $\sqrt{6^2} + \sqrt{8^2} = 6 + 8 = 14$.

•21. $\sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$.

•22. $\sqrt{14}\sqrt{21} = \sqrt{14 \cdot 21} = \sqrt{2 \cdot 7 \cdot 3 \cdot 7} = 7\sqrt{6}$.

•23. $\dfrac{\sqrt{242}}{\sqrt{2}} = \sqrt{\dfrac{242}{2}} = \sqrt{121} = 11$.

•24. $(4\sqrt{5})(6\sqrt{2}) = 24\sqrt{10}$.

•25. $(3\sqrt{7})^2 = 9 \cdot 7 = 63$.

•26. $\dfrac{12\sqrt{15}}{4\sqrt{3}} = 3\sqrt{5}$.

•27. $(5 - \sqrt{2}) + (5 + \sqrt{2}) = 10$.

•28. $(5 - \sqrt{2})(5 + \sqrt{2}) = 25 - 2 = 23$.

•29. $\sqrt{3}(\sqrt{27} + 1) = \sqrt{81} + \sqrt{3} = 9 + \sqrt{3}$.

•30. $\sqrt{3} + (\sqrt{27} + 1) = \sqrt{3} + \sqrt{9 \cdot 3} + 1 =$
$\sqrt{3} + 3\sqrt{3} + 1 = 4\sqrt{3} + 1$.

•31. $(\sqrt{x} + \sqrt{y}) + (\sqrt{x} + \sqrt{y}) = 2\sqrt{x} + 2\sqrt{y}$.

•32. $(\sqrt{x} + \sqrt{y})^2 = (\sqrt{x})^2 + 2(\sqrt{x}\sqrt{y}) + (\sqrt{y})^2 =$
$x + 2\sqrt{xy} + y$.

Chapter 8, Lesson 1

Set I (pages 300–302)

In their book titled *Symmetry—A Unifying Concept* (Shelter Publications, 1994), István and Magdolna Hargittai point out that it was Louis Pasteur who first discovered that otherwise identical crystals can be mirror images of one another. Except for glycine, all amino acids can exist in opposite-handed forms, but only the left-handed version occurs naturally. The Hargittais write: "Many biologically important chemical compounds exist in left-handed and right-handed forms, and the biological activity of the two forms may be very different. . . . Humans metabolize only right-handed glucose. Left-handed glucose, although still sweet, passes through the system untouched." Martin Gardner's book titled *The New Ambidextrous Universe* (W. H. Freeman and Company, 1990) is a good source of further information on the subject.

At first glance, it might appear that Roger Shepard's figure (exercises 18 through 21) also illustrates rotations because the arrows point in opposite directions. The arrows pointing to the left, however, are not congruent to the arrows pointing to the right. The figure is featured on the cover of Al Seckel's excellent book titled *The Art of Optical Illusions* (Carlton Books, 2000).

Transformations in Art.

•1. A translation.

2. A rotation.

•3. A rotation.

4. No.

5. No.

Mirror Molecules.

•6. A reflection.

7. "Left-handed" and "right-handed."

Reflections.

8.



9.



10.



11.



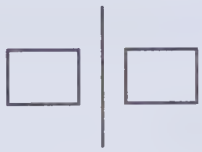
12.



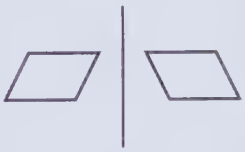
13.



14.



15.



•16. Exercises 9, 11, and 14.

•17. Each has a vertical line of symmetry.

Down the Stairs.

18. A translation.

19. A one-to-one correspondence between two sets of points.

•20. Yes.

21. A transformation that preserves distance and angle measure.

Peter Jones.

22. A rotation.

23. The image is produced from the original figure by rotating the figure 90° clockwise.

24. S.

Set II (pages 302–304)

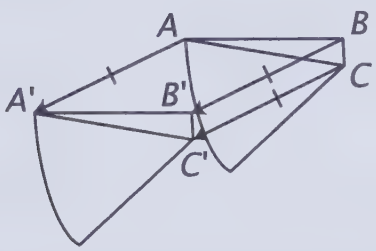
As the Adobe Illustrator User Guide explains, the program "defines objects mathematically as vector graphics." Exactly how vector graphics combines geometry with linear algebra and matrix theory is explained in detail in *The Geometry Toolbox for Graphics and Modeling* by Gerald E. Farin and Dianne Hansford (A. K. Peters, 1998).

Escalator Transformations.

•25. A translation.

26. A rotation.

27.



•28. They are parallelograms because they have two sides that are both parallel and equal.

29. The opposite sides of a parallelogram are parallel.

30. It is a parallelogram.

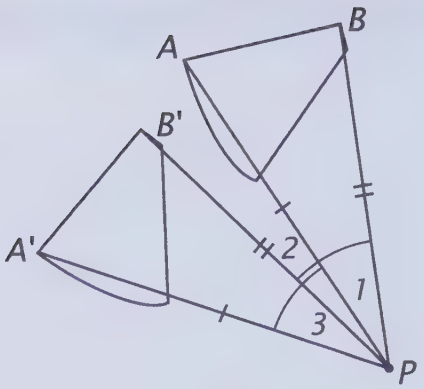
•31. The opposite sides of a parallelogram are equal.

32. SSS.

33. Corresponding parts of congruent triangles are equal.

34. A transformation that preserves distance and angle measure.

35.



•36. Betweenness of Rays Theorem.

•37. Substitution.

38. Subtraction.

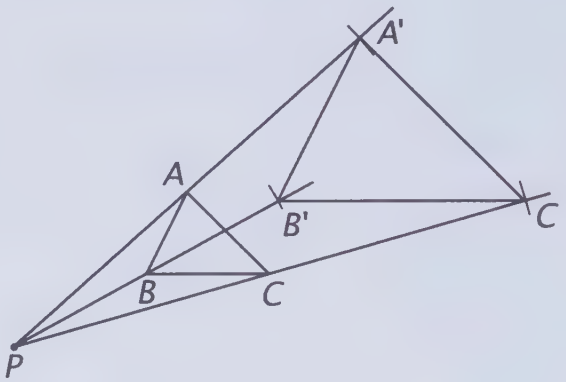
39. SAS.

40. Corresponding parts of congruent triangles are equal.

41. Distances.

Triangle Construction.

42.



•43. A dilation.

•44. They seem to be twice as long.

45. They seem to be equal.

46. No. It is not an isometry, because it doesn't preserve distance.

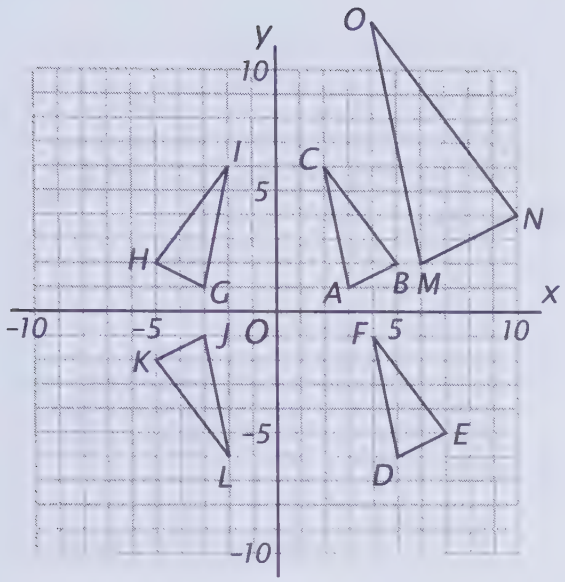
Computer Geometry.

47. A translation.

48. A rotation.

49. A dilation.

50.



51. A(3, 1) → D(3 + 2, 1 – 7), or D(5, –6).
B(5, 2) → E(5 + 2, 2 – 7), or E(7, –5).
C(2, 6) → F(2 + 2, 6 – 7), or F(4, –1).

•52. A translation.

53. A(3, 1) → G(–3, 1).
B(5, 2) → H(–5, 2).
C(2, 6) → I(–2, 6).

54. A reflection.

55. A(3, 1) → J(–3, –1).
B(5, 2) → K(–5, –2).
C(2, 6) → L(–2, –6).

•56. A rotation.

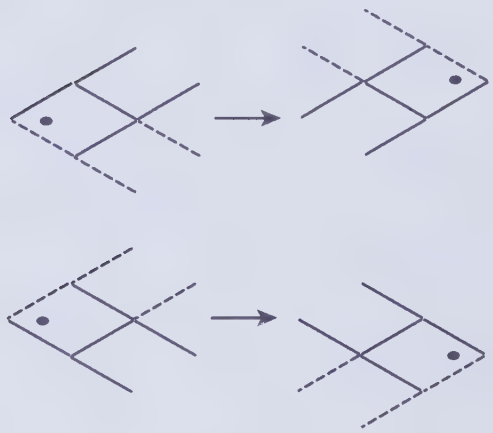
57. A(3, 1) → M(6, 2).
B(5, 2) → N(10, 4).
C(2, 6) → O(4, 12).

58. A dilation.

Set III (page 304)

Toothpick Puzzle.

1. (There are two possible ways to solve the puzzle, one of which is a reflection of the other. The three toothpicks moved are shown as dotted lines in the figures below.)



2. Yes. The reversed fish is a rotation image of the original fish. The center of rotation can be the midpoint of either toothpick forming the fish's back.

Chapter 8, Lesson 2

Set I (pages 308–309)

Capital letters in many typefaces do not have the simple symmetries suggested by exercises 10 through 13. For example, the horizontal bar of an H doesn't always connect the midpoints of the side bars. The upper part of an S is frequently smaller than the lower part.

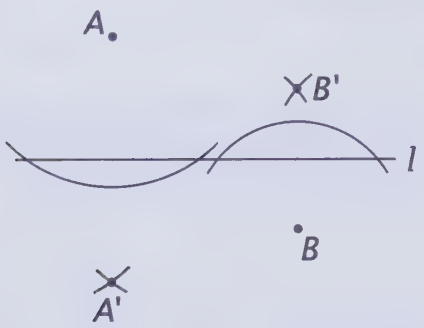
David Moser, the creator of the "China" transformation, has designed some other amazing figures of this sort. Two are included in the book of visual illusions titled *Can You Believe Your Eyes* by J. R. Block and Harold E. Yuker (Brunner/Mazel, 1992). One changes "England" from Chinese into English and the other does the same thing with "Tokyo"! Both word transformations are accomplished, like the example in the text, by a 90° rotation.

A Suspicious Cow.

1. The water from which the cow is drinking.

2. It is upside down. The cow and barn that we see "above" the water are actually reflections.

3.



•4. $2x$ units.

•5. It is the segment's perpendicular bisector.

Double Reflections.

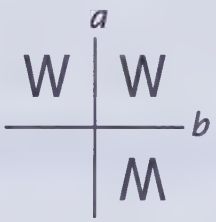
6. *b*.

•7. A rotation.

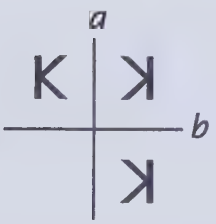
8. 90°.

•9. 180°.

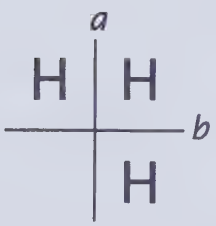
10.



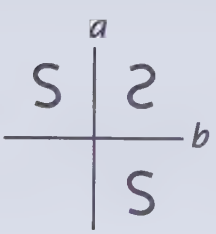
11.



12.



13.



•14. Its image looks the same.

15. Its image looks the same.

16. Its image looks the same.

17. In exercise 12. The figure has both a vertical and a horizontal line of symmetry.

SAT Problem.

•18. B.

19. A reflection.

20. Measure the distances from B and P to the dotted line and see if they are the same.

Can You Read Chinese?

21. A reflection.

•22. A reflection.

23. A rotation.

•24. Yes, because it is the composite of two reflections in intersecting lines.

25. It is a translation of the word "China" from Chinese into English.

Set II (pages 309–311)

Ethologist Niko Tinbergen carried out a number of interesting laboratory studies concerning animal vision, including one with the "goose-hawk" and an experiment with circular disks interpreted by newly hatched blackbirds as their "mother." These examples and others are included by cognitive scientist Donald D. Hoffman in *Visual Intelligence—How We Create What We See* (Norton, 1998).

Kaleidoscope Patterns.

26. B, D, and F.

•27. C and E.

•28. 120°.

29. Three. The three mirror lines. (The monkey faces are almost mirror symmetric; so the lines that bisect the angles formed by the mirrors almost look like lines of symmetry.)

•30. No. The figure does not look exactly the same upside down. (The monkey's left nostril is closer to its left eye than its right nostril is to its right eye; so it is possible to tell the difference.)

Scaring Chickens.

31. B.

•32. E.

33. D.

34. A.

•35. E.

36. A.

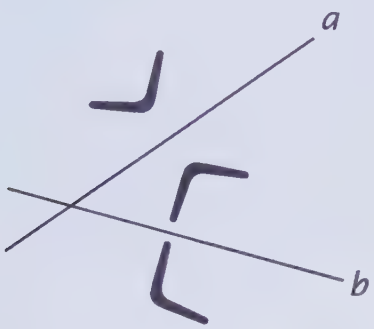
•37. A translation.

38. A translation is the composite of two reflections in parallel lines.

39. Seeing the bird at C flying to A.

Boomerang.

40.



•41. A rotation.

42. A rotation is the composite of two reflections in intersecting lines.

43. At the point in which the two lines intersect.

Triangle Reflections.

•44. If a point is reflected through a line, the line is the perpendicular bisector of the segment connecting the point and its image.

45. A translation.

•46. Its magnitude.

47. It is twice as long; AA" = 2XY.

•48. SAS.

49. Corresponding parts of congruent triangles are equal.

50. A rotation.

51. Its magnitude.

52. It is twice as large; $\angle AOA'' = 2\angle XOY$.

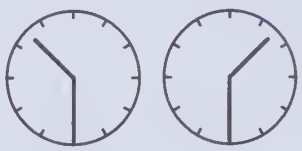
Set III (page 311)

A. E. W. Mason, the author of *The House of the Arrow* (1924), is best known for his novels *The Four Feathers* and *Fire Over England*. *The House of the Arrow*, which featured his detective Inspector Hanaud investigating the murder of a French widow, was made into a movie three times. Following the excerpt quoted in the Set III exercise, the story continues:

"It was exactly half-past one; the long minute hand pointing to six, the shorter hour hand on the right-hand side of the figure twelve, half-way between the one and the two. With a simultaneous movement they all turned again to the mirror; and the mystery was explained. The shorter hour-hand seen in the mirror was on the left-hand side of the figure twelve, and just where it would have been if the hour had been half-past ten and the clock actually where its reflection was. The figures on the dial were reversed and difficult at a first glance to read."

What Time Was It?

1.



2. Examples suggest that the reflection of a clock face in a vertical mirror always looks like an actual time. This is not true for the reflection of a clock face in a horizontal mirror. For example, a horizontal reflection of a clock face reading 10:30 would not look like an actual time because the hour hand would be in the wrong position when the minute hand is pointing to the top of the clock.



Chapter 8, Lesson 3

Set I (pages 314–315)

The number "eight" seems to have a special connection to sports in which synchronization is important. One of the definitions given in *The American Heritage Dictionary of the English Language* for the word "eight" is "an eight-oared racing shell." Team routines in synchronized swimming consist of eight swimmers.

Prevaricator.

1. He has "liar" written all over his face!
2. A translation.
•3. A reflection.
4. A rotation.
•5. A glide reflection.
6. Yes.
7. If two figures are congruent, there is an isometry such that one figure is the image of the other.

Synchronized Oars.

•8. ∠PAB and ∠PCD.
9. If the oars are assumed to be identical, AB = CD. A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.
•10. The opposite sides of a parallelogram are equal.
11. A translation.
12. A glide reflection.
13. Synchronized swimming.

Swing Isometries.

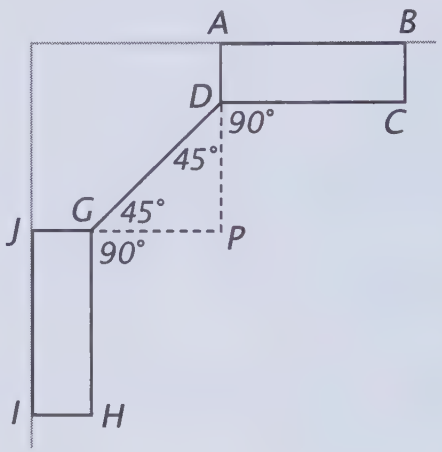
•14. A reflection.
15. A rotation.
16. Its center.
•17. The magnitude of the rotation.
•18. The lines bisect these angles.
19. They are the perpendicular bisectors of these line segments.
20. It is twice as large.

Quadrilateral Reflections.

21. Two points determine a line.
•22. They are the perpendicular bisectors of these line segments.
23. They appear to be parallel.
24. A translation.
•25. Two figures are congruent if there is an isometry such that one figure is the image of the other.

Set II (pages 315–316)

The symmetry of the illustration for the piano-moving problem suggests that the distances from A and J to the corner of the room are equal. If this is the case, then it is easy to prove that ΔDGP in the figure below is an isosceles right triangle. It follows that ∠CDG = ∠DGH = 135°. The piano, then, has been rotated 270° in all. If the symmetry doesn't exist, ΔDGP will still be a right triangle with complementary acute angles; so, even though the measures of ∠CDG and ∠DGH will change, their sum will remain the same.



Moving a Piano.

26. A rotation.
•27. D.
28. E.
•29. G.
30. D.
31. Another rotation about point G.
32. 270°.

Bulldogs.

33. Two figures are congruent if there is an isometry such that one figure is the image of the other.

34. A translation.

•35. No, because the two dogs are not mirror images of each other.

36. Yes. A translation is the composite of two reflections through parallel lines.

37. No, because, if there were three reflections, one dog would be a mirror image of the other.

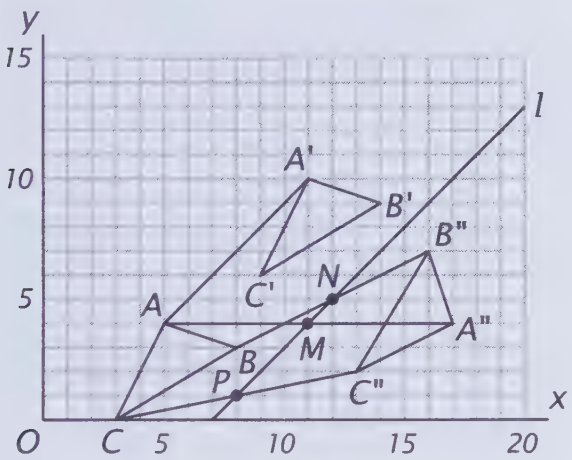
38. A glide reflection.

•39. A translation and a reflection.

40. Three reflections.

Grid Problem.

41.



•42. B′(14, 9), C′(9, 6).

•43. $6\sqrt{2}$. ($AA' = \sqrt{(11-5)^2+(10-4)^2} = \sqrt{36+36} = \sqrt{36\cdot 2} = 6\sqrt{2}$.)

44. AA′ appears to be parallel to line l.

45. A glide reflection, because it is the composite of a translation and a reflection in a line parallel to the direction of the translation.

•46. M(11, 4), N(12, 5), P(8, 1).

47. The y-coordinate is 7 less than the x-coordinate.

48. They lie on line l.

Set III (pages 317–318)

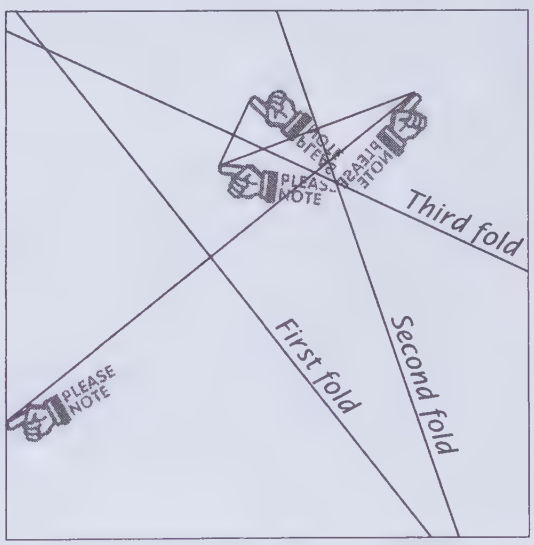
The irregular rubber stamp figure was chosen to try to impress upon the student's mind that the number of reflections needed to show that two figures are congruent has nothing to do with their complexity. David Henderson suggests a nice experiment in his book titled *Experiencing Geometry—In Euclidean, Spherical and Hyperbolic Spaces* (Prentice Hall, 2001). He says: "Cut a triangle out of an index card and use it to draw two congruent triangles in different orientations on a sheet of paper. . . . Now, can you move one triangle to the other by three (or fewer) reflections? You can use your cutout triangle for the intermediate steps." This leads to proving that "on the plane, spheres, or hyperbolic planes, every isometry is the composition of one, two, or three reflections." Chapter 11 of Henderson's book is a good resource for students (and teachers) wanting to learn more about isometries.

Stamp Tricks.

1. Ollie stamped one of the images on the back of the tracing paper.

2. One figure is now the mirror image of the other and two reflections (of an asymmetric figure) cannot produce a mirror image.

3. Yes. The figure below shows one of the many ways in which it can be done.



Chapter 8, Lesson 4

Set I (pages 321–322)

Two wonderful sources of ambigrams are Scott Kim's *Inversions* (Byte Books, 1981) and John Langdon's *Wordplay* (Harcourt Brace Jovanovich, 1992). In the introduction to his book, Langdon wrote:

"Ambigrams come in a number of forms, limited only by the ambigrammist's imagination, and usually involve some kind of symmetry. This book is made up of three types:

1. words with rotational symmetry . . .
2. words that have bilateral, or mirror-image symmetry . . .
3. chains. These are ambigrams that cannot stand alone as single words, but depend on being linked to the preceding and ensuing words."

In other words, ambigrams are based on the three basic types of symmetry in the plane: rotation, reflection, and translation.

The fact that games normally have two opponents or two opposing teams requires that almost all playing fields and courts have two lines of symmetry. Baseball and its related versions can use a field with just one line of symmetry because the teams regularly interchange their positions in the game.

It is interesting that, although Washington had the extra window painted solely to "complete the symmetry," the two windows on either side of the front door, with the windows above them, are not quite in the right places.

Ambigrams.

•1. Rotation (or point) symmetry.

2. See if it coincides with its rotation image. (Or, for point symmetry, see if it looks the same upside down.)

3. Reflection (line) symmetry.

4. See if it coincides with its reflection image or fold it to see if the two halves coincide.

Sport Symmetry.

5. Baseball.

•6. It has reflection (line) symmetry with respect to a line through home plate and second base.

7. Basketball.

8. It has reflection (line) and rotation (point) symmetry. It has two lines of symmetry and 2-fold rotation symmetry.

9. The same type as the basketball court.

10. *(Student answer.)* (It is so that each team has the same view of the other side.)

Mount Vernon.

11. To make his house look more symmetric.

Symmetries of Basic Figures.

•12. The point itself. (Not its "center," because a point does not have a center!)

13. Yes. A point is symmetric with respect to every line that contains it (because a point on a reflection line is its own image.)

•14. A line looks the same if it is rotated 180°.

15. Any point on the line can be chosen as its center of symmetry.

•16. A line has reflection symmetry because it can be reflected (folded) onto itself.

17. Infinitely many. The line itself and every line that is perpendicular to it.

18. Yes. A line can be translated any distance along itself and still look the same.

•19. That rays OA and OD are opposite rays and that rays OB and OC are opposite rays.

20. If $\angle AOB$ is rotated 180° about point O, it coincides with $\angle COD$.

21. Vertical angles are equal.

•22. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

23. A and C.

24. If two sides of a triangle are equal, the angles opposite them are equal.

•25. They bisect each other.

26. CD.

27. The opposite sides of a parallelogram are equal.

28. No. BD cannot be a rotation image of AC because they do not have the same length.

Set II (pages 322–324)

For the piano keyboard, it is interesting to note that all of the translation images of a given key have the same letter name. The key that is the first translation image to the right of a given key is one octave higher and has a frequency twice as great. Without the translation pattern of the black keys, pianists would have trouble keeping their place!

In his book titled *Reality's Mirror* (Wiley, 1989), Bryan Bunch explains the connection of odd and even wave functions to such topics as cold fusion and the Pauli Exclusion Principle. Bunch writes: "It would be only a slight exaggeration to say that symmetry accounts for all the observable behavior of the material world."

Piano Keyboard.

29. *Example answer:*



•30. Translation.

31. No.

Water Wheel.

32. Rotation and point symmetry.

•33. 22.5°. ($\frac{360°}{16}$.)

•34. No. ($\frac{125°}{22.5°}$ is not an integer.)

35. Yes. ($\frac{225°}{22.5°} = 10$.)

•36. 16.

37. No.

Wave Functions.

•38. The y-axis.

39. Reflection (line).

40. The origin.

41. Rotation (or point).

•42. Even.

43. Neither.

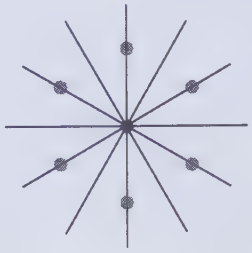
44. Odd.

Cherry Orchard.

•45. Because it can be translated (in various directions) and look exactly the same.

46. The distance between any pair of neighboring trees.

47.



48. Because it can be reflected (in various lines) and look exactly the same.

49. Because it can be rotated (about various points) and look exactly the same.

50. Any tree. Also, any point centered between three neighboring trees. (Also, the midpoints of the segments between neighboring trees are centers of 2-fold rotation symmetry.)

51. 60°.

Set III (page 324)

Short Story.

1. *in the summer, sis goes to the beach and suns herself, adds up sums of numbers, does stunts and collects tin cans. The ride home is hilly, and that makes her head feel fizzy.*

2. The figure can be read as either Scott's first name or his last name.

Chapter 8, Review

Set I (pages 325–327)

In *Visual Intelligence—How We Create What We See* (Norton, 1998), Donald Hoffman reports that perhaps the earliest version of the "vase-faces" illusion appeared in a picture puzzle in 1795. Perceptual psychologists first began to study the illusion in 1915. Of the many variations that have appeared since, surely the most clever is its use in the vase pictured in the text.

In 1941–42, Escher filled a large notebook with notes and drawings that he titled Regular Division of the Plane with Asymmetrical Congruent Polygons. This notebook is reproduced in its entirety in *Visions of Symmetry—Notebooks, Periodic Drawings, and Related Work of M. C. Escher*, by Doris Schattschneider (W. H. Freeman and Company, 1990). She remarks about this work: "It is impossible to look at this notebook and not conclude that in this work, Escher was a research mathematician."

Remarkably, the first book to be published on Escher's work was written by Caroline H. Macgillavry, a professor of chemical crystallography. In *Symmetry Aspects of M. C. Escher's Periodic Drawings* (The International Union of Crystallography, 1965), Macgillavry wrote that Escher's notebook had been a "revelation" to her, commenting: "It is no wonder that x-ray crystallographers, confronted with the ways in which nature solves the same problem of packing identical objects in periodic patterns, are interested in Escher's work."

In lecturing on his work, Escher explicitly discussed his use of the four isometries of the plane in creating his mosaics. For more on this, see the section titled "The geometric rules" (p. 31ff) in *Visions of Symmetry*.

Amazing Vase.

1. No. The left and right sides of the upper part of the vase do not quite look like mirror images of each other.

2. The profiles of Prince Philip and Queen Elizabeth can be seen in silhouette.

Double Meanings.

3. A one-to-one correspondence between two sets of points.

4. A transformation that preserves distance and angle measure.

5. A transformation in which the image is an enlargement or reduction of the original.

•6. No.

Monkey Rug.

7. A translation.

•8. A glide reflection.

•9. A rotation.

10. A reflection.

Clover Leafs.

•11. That it can be rotated so that it looks the same in three positions.

12. 120°.

13. No. It doesn't look the same upside down.

14.



15.



16. 90°.

•17. Yes.

18. Good.

Musical Transformations.

19. A translation.

•20. A reflection.

21. A translation.

22. A rotation.

Fish Design.

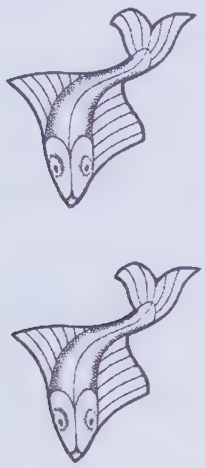
23. No. There seem to be two shapes of fish in the mosaic. The differences in their noses and tails are the most obvious.

•24. Two figures are congruent if there is an isometry such that one figure is the image of the other.

25. A glide reflection.

26. A rotation.

27. Yes. Translations.
Example answer:



Set II (pages 327–329)

Keep Your Eye on the Ball—The Science and Folklore of Baseball, by Robert G. Watts and A. Terry Bahill (W. H. Freeman and Company, 1990) is a great source of examples for use in illustrating the application of algebra, geometry, and physics to the analysis of baseball. The authors describe the figure used for exercises 28 through 32: "The swing of a baseball bat exhibits two types of motion: translational and rotational. . . . To move the bat from position A to position C, one can first *rotate* the bat about the center of mass and then *translate* the center of mass."

John Langdon wrote that his "past" and "future" ambigrams "are intended to represent the interminable conveyor belt of time, and the individual letters and words are the experiences, the 'life bites' of our existence."

Batter's Swing.

28. A rotation.

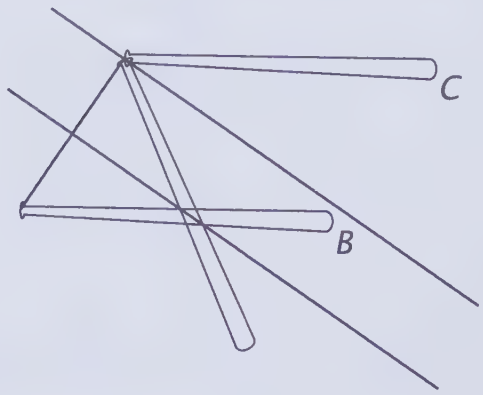
•29. A translation.

30. For a rotation, the two reflection lines intersect. For a translation, they are parallel.

•31. About 47°.

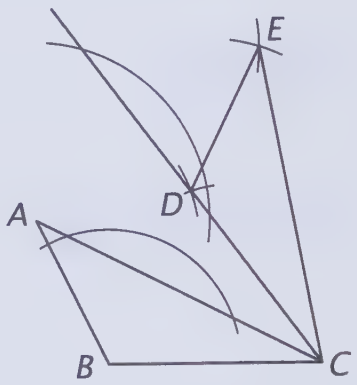
32. About 24 in. (In the figure, the length of the bat is 42 mm and the magnitude of the translation is about 24 mm.)

33. *Example figure:*



Construction Problem.

34.



35. The image of AB is AD and the image of BC is DC.

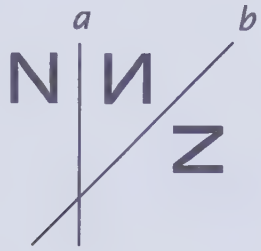
•36. Because ΔADC is the reflection image of ΔABC.

•37. It has reflection (line) symmetry with respect to line AC.

38. A rotation.

From N to Z.

39.



40. A rotation.

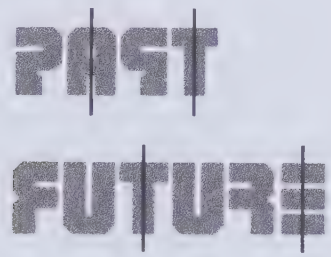
•41. 45°.

42. 90°.

43. The image of H looks like I. The image of W looks somewhat like E. (And, the image of Z looks like N.)

Past and Future.

44. *Example answers:*



45. Translation.

46. So that the E's look "correct" in both directions.

•47. No.

48. The letters A and T in PAST.
The letters T, U, and E in FUTURE.

49. Yes. The letter E.

Dice Symmetries.

50. *Group 1:* 1, 4, and 5.
These faces have 4 lines of symmetry and 4-fold rotation (as well as point) symmetry.
Group 2: 2, 3, and 6.
These faces have 2 lines of symmetry and 2-fold rotation (as well as point) symmetry.
In regard to 2 and 3, the symmetry lines contain the diagonals of the face; in regard to 6, they are parallel to the edges and midway between them.

Midterm Review

Set I (pages 330–333)

Chill Factor.

1. A conditional statement.
2. If you don't put your coat on.

Losers, Sleepers.

3. An Euler diagram.
4. If you snooze, you lose.
5. You lose.

Marine Logic.

6. A syllogism.
7. If one of its premises were false.

Finding Truth.

8. Theorems.
9. Postulates.

Only in Geometry.

10. Points that lie on the same line.
11. The side opposite the right angle in a right triangle.
12. A triangle or trapezoid that has two equal legs.
13. A quadrilateral all of whose sides are equal.

Why Three?

14. Three noncollinear points determine a plane.
15. Point, plane.

What Follows?

16. A line.
17. 180°.
18. Bisects it.
19. Are equal.
20. Equiangular.
21. Either remote interior angle.
22. The perpendicular bisector of the line segment.

23. Are parallel.
24. Are parallel.
25. Also perpendicular to the other.
26. The remote interior angles.
27. Equal.
28. Are equal.

Formulas.

29. The area of a circle is π times the square of its radius.
30. The area of a rectangle is the product of its length and width.
31. This is the Distance Formula. The distance between two points is the square root of the sum of the squares of the differences of their x-coordinates and their y-coordinates.
32. The perimeter of a triangle is the sum of the lengths of its sides.
33. The perimeter of a rectangle is the sum of twice its length and twice its width.

Protractor Problems.

34. 84°. (123 – 39.)
35. 42°. ($\frac{1}{2}$84°.)
36. 81. (39 + 42 or 123 – 42.)
37. 48°. (90° – 42°.)
38. 171. (123 + 48 = 171.)
39. Yes. OD-OC-OB because 171 > 81 > 39.

Metric Angles.

40. Two angles are supplementary iff their sum is 200 grades.
41. An angle is obtuse if it is greater than 100 grades but less than 200 grades.
42. The sum of the angles of a quadrilateral is 400 grades.
43. Each angle of an equilateral triangle is $66\frac{2}{3}$ grades.

Linear Pair.

44. That the figure contains two opposite rays.
45. No. If the angles are equal, they would be right angles.
46. They are supplementary.
47. Each angle is a right angle.

Polygons.

48. $\sqrt{1125}$ or $15\sqrt{5}$.
    ($c^2 = 15^2 + 30^2 = 225 + 900 = 1125$,
    $c = \sqrt{1125} = 15\sqrt{5}$.)
49. 30. (It can't be 15, because 15 + 15 = 30.)
50. 30°, 150°, and 150°. (A rhombus is a parallelogram. The opposite angles of a parallelogram are equal and the consecutive angles are supplementary.)
51. 30°, 150°, and 150°. (The base angles of an isosceles trapezoid are equal. The parallel bases form supplementary angles with the legs.)

Bent Pyramid.

52. 126°. (180° – 54°.)
53. 94°. [180° – 2(43°).]
54. 54°. ($\angle E = \angle A$.)
55. 169°. (126° + 43°.)

Six Triangles.

56. ΔAFP and ΔDEP.
57. By SAS. (ED = AP because AB = ED and AB = AP.)
58. By HL. (AF = PB because AF = CD and CD = PD.)

Italian Theorem.

59. $\angle B > \angle A$.
60. If two sides of a triangle are unequal, the angle opposite the greater side is greater than that opposite the smaller side. (If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.)
61. Major and minor.

Impossibly Obtuse.

62. $\angle A + \angle B + \angle C > 270°$.

63. The fact that the sum of the angles of a triangle is 180°.

64. It shows that what we supposed is false.

65. Indirect.

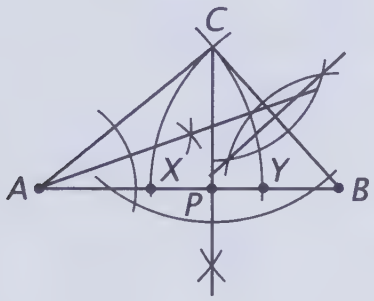
Converses.

66. If two angles are complementary, then they are the acute angles of a right triangle. False.

67. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. True.

Construction Exercises.

68.



69. No.

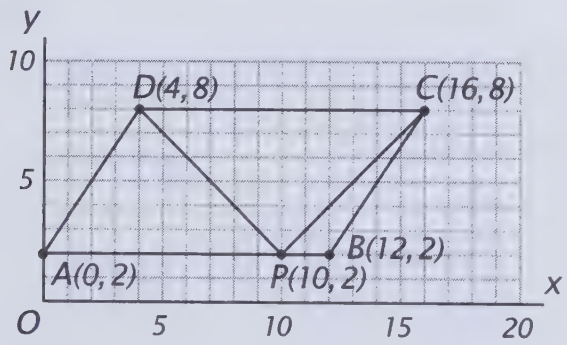
70. Point P.

71. CA, CP, and CB.

Set II (pages 333–336)

Grid Exercise.

72.



73. A parallelogram.

74. AB = DC = 12; AD = BC = $\sqrt{80}$ or $4\sqrt{5}$.
$(AD = \sqrt{(4-0)^2+(8-2)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$.
$BC = \sqrt{(16-12)^2+(8-2)^2} = \sqrt{16+64} = \sqrt{80} = 4\sqrt{5}$.

75. *(Student answer.)* (PD looks longer to many people.)

76. $PD = \sqrt{(4-10)^2+(8-2)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$.
$PC = \sqrt{(16-10)^2+(8-2)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$.

Roof Truss.

77. $\angle A = \angle L$. $\angle A$ and $\angle L$ are corresponding parts of ΔAGF and ΔLGF, which are congruent by SSS.

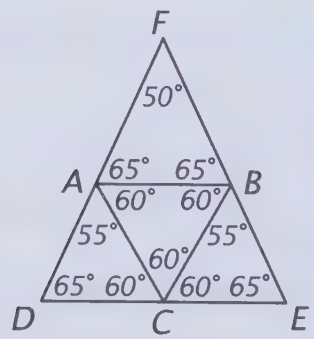
78. BC || FG. BC || DE and DE || FG because equal alternate interior angles mean that lines are parallel. BC || FG because, in a plane, two lines parallel to a third line are parallel to each other.

79. Yes. CD || EF because they form equal alternate interior angles with DE.

80. No. If $\angle AGF$ and $\angle GFA$ were supplementary, GA would be parallel to FA (supplementary angles on the same side of a transversal mean that lines are parallel). GA and FA intersect at A.

Angles Problem 1.

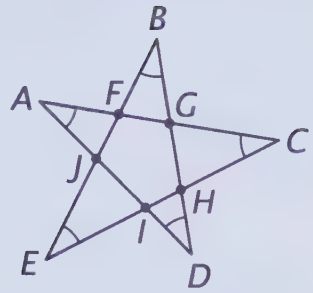
81.



82. $\Delta ACD \cong \Delta BCE$ by ASA. AC = BC because ΔABC is equilateral, $\angle DAC = 55° = \angle EBC$, and $\angle DCA = 60° = \angle ECB$.

Irregular Star.

83.



84. True. ΔEFC is isosceles because EF = FC because ∠C = ∠E.

85. True. FGHIJ is equiangular because each of its angles is an angle of one of the five overlapping isosceles triangles, whose base angles are the equal angles at the corners of the star. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.

86. False. (This is evident from the figure.)

87. True. There are 10 isosceles triangles in all. The base angles of the triangles that are acute (ΔBFG is an example) are supplementary to the equal angles of pentagon FGHIJ; so they are equal.

Angle Trisector.

88. These triangles are congruent by SSS.

89. ∠1 = ∠2 and ∠2 = ∠3 because they are corresponding parts of the congruent triangles; so ∠1 = ∠3.

Quadrilateral Problem.

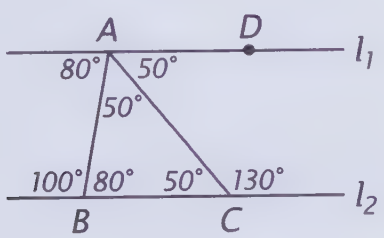
90. $x + 2x + 3x + 4x = 360$, $10x = 360$, $x = 36$.

91. 144°. [4(36°).]

92. ∠B = 72°, ∠C = 108°, AB || DC, ABCD is a trapezoid.

Angles Problem 2.

93.



94. ΔABC is isosceles. ∠BAC = 50° = ∠ACB; so AB = BC.

A, B, C.

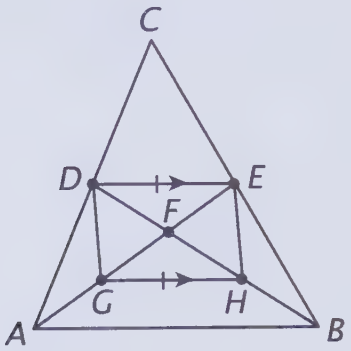
95. $a^2 + b^2 = c^2$.

96. $a + b > c$, $a + c > b$, $b + c > a$.

97. No. The sum of two of its sides would equal the third side, which would contradict the Triangle Inequality Theorem.

Midsegments.

98.



99. Applying the Midsegment Theorem to ΔABC gives $DE = \frac{1}{2}AB$ and applying it to ΔABF gives $GH = \frac{1}{2}AB$; so DE = GH by substitution.

100. Again, by the Midsegment Theorem, DE || AB and GH || AB; so DE || GH. In a plane, two lines parallel to a third line are parallel to each other.

101. DEHG is a parallelogram because two opposite sides, DE and GH, are both parallel and equal.

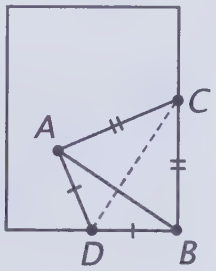
102. AG = GF and BH = HF because G and H are the midpoints of AF and BF. GF = FE and HF = FD because the diagonals of a parallelogram bisect each other. So AG = GF = FE and BH = HF = FD.

On the Level.

103. The design of the swing is based on a parallelogram. The supports of the plank are equal, and the part of the plank between the supports is equal to the distance between the supports at the top. The top remains level with the ground. The plank, the opposite side of the parallelogram, is always parallel to the top; so it also stays level with the ground.

Folding Experiment.

104. *Example figure (the student's figure will depend on the point and corner chosen):*



105. It appears to be the perpendicular bisector of AB.

106. When B falls on A, CA = CB and DA = DB. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Earth Measurement.

107. 20,520 mi. (360 × 57.)

108. About 3,300 mi. ($c = 2\pi r$; so $r = \frac{c}{2\pi} = \frac{20{,}520}{2\pi} \approx 3{,}300$.)

Not a Square.

109. Yes. ABCD could be a rhombus if all of its sides were equal.

110. Yes. If ABCD is a rhombus, its diagonals will be perpendicular.

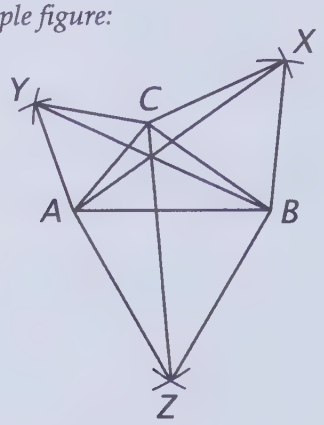
111. Yes. ABCD is a parallelogram and the diagonals of a parallelogram bisect each other.

112. No. If ABCD were a rectangle, ∠DAB would be equal to ∠ABC.

113. No. If AC = DB, then ΔDAB ≅ ΔCBA by SSS (DA = CB because ABCD is a parallelogram.) If ΔDAB ≅ ΔCBA, then ∠DAB = ∠ABC but ∠DAB ≠ ∠ABC.

Construction Exercise.

114. *Example figure:*



115. They seem to be concurrent.

SAT Problem.

116. A rotation.

117. A translation.

118. A reflection.

119. A reflection.

120. Figure E.

Quilt Patterns.

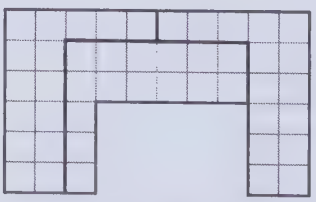
121. Rotation (point) symmetry (4-fold).

122. Rotation (point) symmetry and reflection (line) symmetry (4 lines).

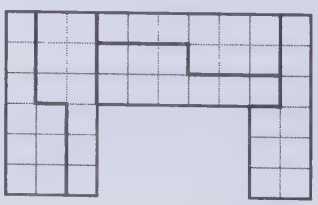
123. Reflection (line) symmetry (1 line).

Dividing a Lot.

124.



125.



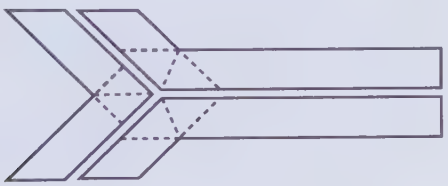
Chapter 9, Lesson 1

Set I (pages 340–341)

The Amtrak trademark first appeared in 1971 with the beginning of the government corporation in charge of the nation's passenger trains. According to Hal Morgan in *Symbols of America* (Viking, 1986), "the arrow design was chosen to convey the image of speed and purpose of direction."

The geometry of drums is not well known and rather surprising. In 1911, Hermann Weyl proved that the overtones of a drum determine its area. In 1936, another mathematician proved that a drum's overtones also determine its perimeter! In 1966, Mark Kac published a paper titled "Can One Hear the Shape of a Drum?" In 1991, Carolyn Gordon gave a lecture at Duke University about drum geometry. She and her husband David Webb have subsequently discovered a number of pairs of sound-alike drums, including the pair pictured in the exercises. The drums in each pair are polygonal in shape, have at least eight sides, and the same perimeter and area. All of this information is from the chapter titled "Different Drums" in Ivars Peterson's *The Jungles of Randomness* (Wiley, 1998). The book also includes a page of color photographs showing the standing waves of the first few normal modes of these drums.

Train Logo.

1. 

   (The dotted lines in this figure illustrate the possible choices of ways to divide the figure into six quadrilaterals.)

2. Trapezoids and parallelograms.

•3. Its area is equal to the sum of the areas of its nonoverlapping parts.

Star in Square.

4. The star because of the Triangle Inequality Theorem.

5. The square because of the Area Postulate. (The area of the square is equal to the area of the "star" plus the positive areas of the four white triangular regions.)

6. No.

Area Relations.

•7. True, because $\Delta ADC \cong \Delta ABC$.

8. True, because $\Delta 1 \cong \Delta 4$ and $\Delta 2 \cong \Delta 3$.

•9. True, because $\alpha EPHD = \alpha\Delta ADC - \alpha\Delta 1 - \alpha\Delta 2 = \alpha\Delta ABC - \alpha\Delta 4 - \alpha\Delta 3 = \alpha GBFP$.

10. True, because $\alpha AGHD = \alpha EPHD + \alpha AGPE = \alpha GBFP + \alpha AGPE = \alpha ABFE$.

11. True, because $\alpha EFCD = \alpha EPHD + \alpha PFCH = \alpha GBFP + \alpha PFCH = \alpha GBCH$.

12. False. $\alpha AGPE > \alpha PFCH$.

Flag Geometry.

13. x square units.

•14. $2x$ square units.

15. $6x$ square units.

•16. $(x + y)$ square units.

17. $(2x - 2y)$ square units.

18. $(x + y)$ square units.

19. Yes. $(x + y) + (2x - 2y) + (x + y) = 4x$.

Drum Polygons.

•20. They are both concave octagons.

21. They have equal areas. ($3.5b^2$.)

22. They have equal perimeters.

•23. $3a + 6b$.

24. No.

Set II (pages 341–342)

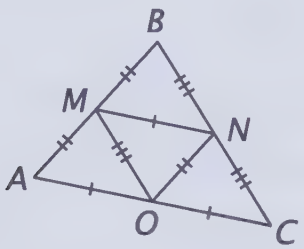
Exercises 35 through 39 are a nice example of how some areas can be easily compared. The areas of the regular hexagons circumscribed about a circle and inscribed in it are related in a surprisingly simple way: their ratio is 4 to 3.

Plato's method for doubling the square is based on the same simple idea of counting triangles. It nicely circumvents the fact that, if the side of the original square is 1 unit, the side of the doubled square is an irrational number.

According to Sir Thomas Heath in his commentary on Euclid's *Elements*, Proclus stated

that Euclid coined the word "parallelogram." Euclid first mentions parallelograms in Proposition 34 of Book I. It is the next theorem, Proposition 35, that is demonstrated in exercises 46 through 55. (The figure is the one that Euclid used; it illustrates just one of three possible cases relating the upper sides of the parallelograms.) Euclid does not actually mention area, stating the theorem in the form "Parallelograms which are on the same base and in the same parallels are equal to one other." Heath remarks: "No *definition* of equality is anywhere given by Euclid; we are left to infer its meaning from the few *axioms* about 'equal things.'" Previously to the theorem, Euclid had used equality to mean congruence exclusively.

Midsegment Triangle.

25. 

•26. They are congruent (SSS) and therefore equal in area.

27. $\alpha\Delta MNO = \frac{1}{4}\alpha\Delta ABC$.

28. They are parallelograms.

29. Their opposite sides are parallel (or, their opposite sides are equal). Both follow from the Midsegment Theorem.

•30. No.

•31. Yes. Each one contains two of the four triangles, all of whose areas are equal.

32. Trapezoids.

33. No.

34. Yes. Each one contains three of the four triangles, all of whose areas are equal.

Circle Area.

•35. $\frac{1}{6}$ square unit. ($\frac{3}{18}$.)

36. $\frac{1}{2}$ square unit. [$3(\frac{1}{6})$.]

•37. 2 square units. [$12(\frac{1}{6})$.]

38. 4 square units. [$24(\frac{1}{6})$.]

39. Roughly 3.5 square units. (The area of the circle appears to be about halfway between the areas of the two hexagons. Using methods that the students won't know until later, we can show that the area of the circle is $\frac{2\pi}{\sqrt{3}} \approx 3.63\ldots$)

Doubling a Square.

40. SAS.

41. Congruent triangles have equal areas.

•42. It is equilateral and equiangular. (Both follow from the fact that corresponding parts of congruent triangles are equal.)

•43. BFED contains four of the congruent triangles, and ABCD contains two of them.

SAT Problem.

•44. 4 square units. ($\frac{100-84}{4} = 4$.)

45. 116 square units. [100 + 4(4) = 116.]

Comparing Parallelograms.

46. The opposite sides of a parallelogram are equal.

47. The opposite sides of a parallelogram are parallel.

48. Parallel lines form equal corresponding angles.

•49. They are both equal to AB.

•50. Because DF = DC + CF = CF + FE = CE.

51. SAS.

52. Congruent triangles have equal areas.

53. Substitution. ($\alpha\Delta ADF = \alpha 1 + \alpha 3$ and $\alpha\Delta BCE = \alpha 3 + \alpha 4$.)

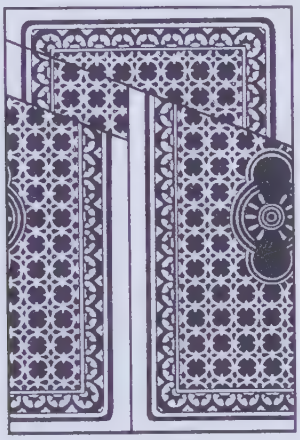
54. Subtraction.

55. $\alpha 1 + \alpha 2 = \alpha 2 + \alpha 4$ (addition); so $\alpha ABCD = \alpha ABEF$ (substitution).

Set III (page 343)

According to Jerry Slocum and Jack Botermans (*New Book of Puzzles*, W. H. Freeman and Company, 1992), the magic playing card puzzle first appeared in a version known as "The Geometric Money" in *Rational Recreations*, by William Hooper, published in 1794.

Unfortunately, it isn't possible to make this puzzle out of an ordinary playing card, because the back of the card, before it is cut up, would have to be redesigned for the purpose of the trick. The figure below shows what it would look like for the card shown in the text.



Magic Playing Card.

1. No. If they were, the two would have equal areas. Including the "hole," the second rectangle has a greater area.
2. The widths are obviously the same; so the lengths must be different.
3. The second figure is slightly longer than the first. The extra area is equal to the area of the "hole."

Chapter 9, Lesson 2

Set I (pages 345–347)

The "Abstract Painting, 1960–61" pictured in the text is minimal artist Ad Reinhardt's most famous work. It is actually composed of nine congruent squares, each painted in a slightly different shade of black. At the time of this writing, there is a site on the Internet where another of his pictures, "Black Painting, 1960–66" can be "viewed". The site, which appropriately has the word "piffle" in its name, is . . . black.

In Book II of the *Elements*, Euclid used geometric methods to prove a series of algebraic identities. Proposition 1 deals with the distributive property. Proposition 4, which concerns the square of a binomial, is stated as: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments." All this to say: $(a + b)^2 = a^2 + b^2 + 2ab$. In his *History of Mathematics* (Allyn & Bacon, 1985), David Burton wrote: "By Euclid's time, Greek geometric algebra had reached a stage of development where it could be used to solve simple equations." The equations, both linear and quadratic, were solved by making geometric constructions.

Abstract Art.

•1. 20 ft.

2. 240 in.

3. 25 ft^2.

•4. 3,600 in^2.

Tile Pattern.

5. *(Student answer.)*

6. Blue region: 36 units; green region: 28 units; yellow region: 36 units. ($6^2 = 36$; $8^2 - 6^2 = 28$; $10^2 - 8^2 = 36$.)

Fishing Nets.

7. The sides of the small squares are $\frac{1}{2}$ inch long.

•8. Four.

9. 16.

10. To allow the smaller, younger fish to escape.

Distributive Property.

•11. $a(b + c)$.

•12. ab and ac.

13. $a(b + c) = ab + ac$.

Binomial Square.

•14. $(a + b)^2$.

15. a^2, ab, ab, and b^2.

16. $(a + b)^2 = a^2 + 2ab + b^2$.

Difference of Two Squares.

17. $a^2 - b^2 = (a + b)(a - b)$.

Area Connection.

18. $A = ab$.

•19. Two congruent triangles.

20. The Area Postulate: Congruent triangles have equal areas.

21. $A = \frac{1}{2}ab$.

Two Squares.

•22. $\sqrt{26}$.

23. No. ($\sqrt{26} = 5.0990195\ldots$)

24. 25.999801.

•25. No. (5.0990195^2 is a long decimal ending in 5, not the integer 26.)

26. No.

27. Irrational. (As students may recall from algebra, an irrational number is a number that cannot be written as the quotient of two integers. If an integer is not the square of an integer, its square roots are irrational.)

Set II (pages 347–349)

In his book titled *The Cosmological Milkshake* (Rutgers University Press, 1994), physics professor Robert Ehrlich remarks that "lying on a bed of nails need be no more painful than lying on your own bed, provided that the nail spacing is small enough, so that the fraction of your weight supported by any one nail is not too large." He also provides a simple explanation of why a spacing of one nail per square inch is practical.

Lumber measurements are interesting in that the three dimensions of a board are not all given in the same units: a 10-foot "2-by-4", for example, has one dimension given in feet and the other two in inches (a "2-by-4" was originally just that but is now actually 1.5-by-3.5). Another example of combining different units is the "board foot." A board foot is a unit of cubic measure that is the volume of a piece of wood measuring 1 *foot* by 1 *foot* by 1 *inch*. (Because the grading of a piece of lumber is based on cutting it into pieces of the same thickness as the lumber, the calculations can be done in terms of area rather than volume.)

Bed of Nails.

•28. 2,592 nails. ($6 \times 12 \times 3 \times 12$.)

29. *(Student answer.)* (Standing on it would hurt the most.)

Surveyor's Chain.

•30. 66 ft. ($\frac{100 \times 7.92}{12}$.)

31. One chain by ten chains.

32. Ten square chains.

33. 5,280 ft. (80×66.)

•34. 6,400 square chains. (80^2.)

35. 640. ($\frac{6,400}{10}$.)

Map Reading.

36. 480 cm^2.

•37. 84 cm^2.

•38. 17.5%. ($\frac{84}{480} = 0.175$.)

39. $33\frac{1}{3}\%$. (The area of the awkward region is now 160 cm^2; $\frac{160}{480} = \frac{1}{3}$.)

Wallpaper Geometry.

•40. 21.6 ft long. ($36\ ft^2 = 5,184\ in^2$; $\frac{5,184\ in^2}{20\ in} = 259.2\ in = 21.6\ ft$.)

41. 16 ft long. ($\frac{5,184\ in^2}{27\ in} = 192\ in = 16\ ft$.)

42. The wallpaper strips have to be placed so that the pattern matches, which means that some paper will be wasted.

•43. The total surface area of the walls.

44. In rounding to the nearest number, if we round down, we may not have enough paper. (For example, if the result is 21.4, 21 rolls may not be enough.)

45. It is about taking into account the area of windows and doors.

46. It would reduce it. (The basic rule is to *subtract 1 roll for every 50 square feet of door or window opening.*)

Cutting a Board.

•47. 1,728 in^2. (12 in × 144 in.)

•48. 189 in^2. (3.5 in × 54 in.)

49. 243 in^2. (4.5 in × 54 in.)

50. 459 in^2. (8.5 in × 54 in.)

51. 408 in^2. (6 in × 68 in.)

52. Yes. The total area of the four cuttings is 1,299 in^2 and $\frac{1,299}{1,729} \approx 75\%$.

The Number 17.

53.

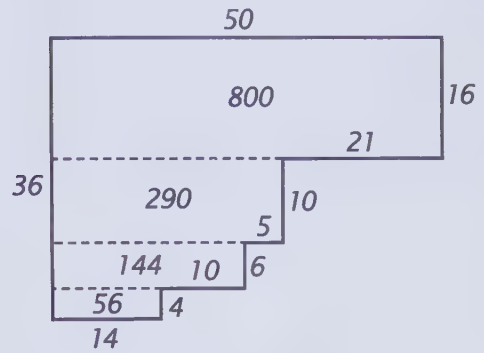


54. The number of linear units in the perimeter of each figure is equal to the number of square units in its area.

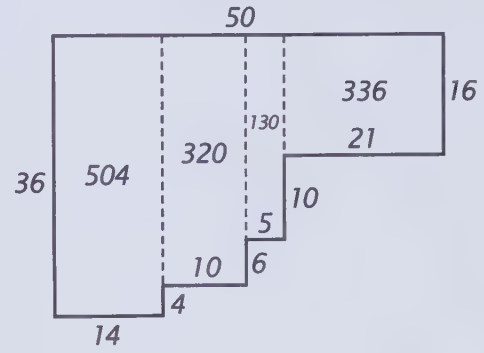
55. [This is a challenging problem. Letting x and y be the dimensions of such a rectangle, $2x + 2y = xy$. Finding solutions to this equation is easiest if we first solve for one variable in terms of the other. $2x = xy - 2y$; so $y(x - 2) = 2x$, and so $y = \frac{2x}{x-2}$. Letting $x = 5$ gives $y = \frac{10}{3}$; so another rectangle is a 5 by $3\frac{1}{3}$ rectangle. Of course, x does not have to be an integer. Any value of $x > 2$ will do.]

Total Living Area.

56. 1,290 ft^2.
Example figure (answers will vary):



57. *Example figure (answers will vary):*



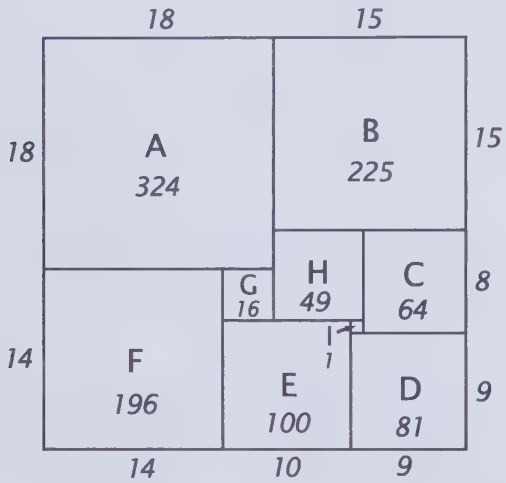
Set III (page 350)

According to David Wells *(The Penguin Dictionary of Curious and Interesting Geometry,* Penguin Books, 1991), a Russian mathematician in the early part of the twentieth century claimed that the "squaring the square" problem couldn't be done. In 1939, Roland Sprague discovered a solution using 55 squares and, in 1978, Dutch mathematician A. J. W. Duijvestijn found the simplest solution, consisting of 21 squares. Martin Gardner's "Mathematical Games" column in the November 1958 issue of *Scientific American* contained William T. Tutte's account of how he and his fellow Cambridge students solved the problem of squaring the square, which was the reason for the topic being featured on the cover of the magazine. The figure on the cover was discovered by Z. Morón in 1925, and it was proved in 1940 that it is the simplest possible way in which to square a rectangle. Tutte's account is charming and very instructive. It shows what fun and progress can be made if you are persistent, systematic, and lucky. Mrs. Brooks, the mother of R. L. Brooks, one of the four Cambridge students

who did this work, put together a jigsaw puzzle that her son had made of a squared rectangle but, to everybody's astonishment, she got a rectangle of different dimensions. Such a thing had never been seen before. It was a key observation and unexpected demonstration of the fact that making a real model and playing with it can lead to real progress. More recent discoveries are reported by Gardner in chapter 11 of his book titled *Fractal Music, Hypercards and More* (W. H. Freeman and Company, 1992).

Dividing into Squares.

1.



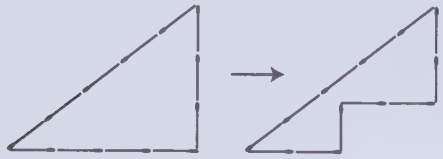
2. A, 324; B, 225; C, 64; D, 81; E, 100; F, 196; G, 16; H, 49; I, 1.

3. No. Its base is 33 and its altitude is 32.

Chapter 9, Lesson 3

Set I (pages 353–355)

Matchstick puzzles such as the one in exercises 1 through 3 first became popular in the nineteenth century, when matches were universally used to light lamps and stoves. As a followup to these exercises, it might be amusing to place 12 matches in the 3-4-5 right triangle arrangement on an overhead projector, observe that they enclose an area of 6 square units, and consider how they might be rearranged to get areas of 5 and 4 square units (the obvious solutions are "corollaries" to the solution for getting an area of 3 square units). Even better would be to pose the puzzle of what areas can be enclosed by 12 matches as a problem for interested students to explore. In one of his early "Mathematical Games" columns for *Scientific American*, Martin Gardner posed the "triangle to area of 4" version of the problem. Several readers of the magazine pointed out that, if 12 matches are arranged to form a six-pointed star, the widths





of the star's points can be adjusted to produce any area between 0 and 3 cot 15° (11.196 . . .), the area of a regular dodecagon (*The Scientific American Book of Mathematical Puzzles and Diversions*, Martin Gardner, Simon & Schuster, 1959).

In the days before students had calculators, the problem of finding the area of the four-sided field required a lot more work. The value of knowing how to factor is evident if the problem of calculating

$$\frac{1}{2}(14.36)(8.17) + \frac{1}{2}(14.36)(5.74)$$

is compared with the problem of calculating

$$\frac{1}{2}(14.36)(8.17 + 5.74),$$

as it would have been done more than a century ago.

Match Puzzle.

•1. 6 square units.

2. Yes. 3 square units have been removed; so 3 square units are left.

3. They are the same: 12 units.

Isosceles Right Triangle.

4. $\frac{1}{2}a^2$.

•5. $\frac{1}{4}c^2$.

Theorem 39.

•6. The area of a right triangle is half the product of its legs.

•7. The area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts (the Area Postulate).

8. Substitution.

•9. Substitution.

10. The area of a right triangle is half the product of its legs.

11. The area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts (the Area Postulate).

12. Subtraction.

13. Substitution.

14. Substitution.

Enlargement.

•15. They seem to be equal.

16. They are twice as long.

•17. ΔABC, 40 units; ΔDEF, 80 units.

18. It is twice as large.

19. ΔABC, 60 square units; ΔDEF, 240 square units.

20. It is four times as large.

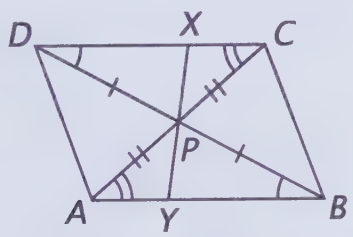
•21. They seem to stay the same.

22. It is doubled.

23. It is multiplied by four.

Chinese Parallelogram.

24.



25. Yes. The triangles bounding them are congruent. (The diagonals of a parallelogram bisect each other, parallel lines form equal alternate interior angles, and the pairs of vertical angles at P are equal.)

26. They have equal areas. (They are also congruent.)

Four-Sided Field.

•27. $\alpha ABCD \approx 99.87$ square chains. [$\alpha\Delta ABD + \alpha\Delta CBD = \frac{1}{2}BD \cdot AE + \frac{1}{2}BD \cdot CF = \frac{1}{2}BD(AE + CF) = 7.18(8.17 + 5.74) \approx 99.87$.]

28. About 9.987 acres.

Set II (pages 355–357)

The key to the puzzle of the surfer is Viviani's Theorem, named after an obscure Italian mathematician (1622–1703): In an equilateral triangle, the sum of the perpendiculars from any point P to the sides is equal to the altitude of the triangle. The proof outlined in exercises 29 through 34 works for the sum of the perpendiculars from any point P in the interior of a convex equilateral *n*-gon to its *n* sides. The pentagon case appeared as the "Clairvoyance Test" in Chapter 6, Lesson 1 (page 216, exercises 30 through 32).

A practical application of Viviani's Theorem is in "trilinear charts." A trilinear chart is useful in presenting data belonging to three sets of numbers, each set of which sums to 100%. A. S. Levens describes them in his book titled *Graphics* (Wiley, 1962): "These charts are in the shape of an equilateral triangle. It can be shown that the sum of the perpendiculars, drawn from a point within the triangle, to the sides of the triangle is equal to the altitude of the triangle. Now, when we graduate each side of the triangle in equal divisions ranging from 0% to 100% and regard each side as representing a variable, we have a means for determining the percentages of each of the three variables that will make their sum equal 100%."

The kite problem lends itself to further exploration. For what other quadrilaterals would taking half the product of the lengths of the diagonals give the area? Do the diagonals have to be perpendicular? Does one diagonal have to bisect the other? Does the diagonal formula work for squares? Rhombuses? Rectangles? Are there any trapezoids for which it works? And so on.

In his book titled *Mathematical Encounters of the Second Kind* (Birkhäuser, 1997), Philip J. Davis relates an amusing story about a phone call from a man who identified himself as a roofer and who

had a roof whose area he wanted to estimate. As Davis tells it: "He recalled that there were mathematical ways of doing this and had known them in high school, but he had forgotten them and would I help him. It seemed to me that any roofer worthy of his trade could at a single glance estimate the area of a roof to within two packages of shingles, whereas any professional mathematician would surely make an awful hash of it." After finally getting out of the man the fact that the roof consisted of four triangles, Davis asked: "'These four triangles, what shape are they? Are they equilateral triangles like one sees in some high pitched garage roofs or are they isosceles triangles? I've got to know.' No reply from my caller. 'Well, let's look at it another way: You've got four triangles. Find the area of each triangle and add them up.' 'How do you do that?' 'The area of a triangle is one half the base times its altitude.' 'My triangles don't have an altitude. . . .' "

Surfer Puzzle.

•29. The area of a triangle is half the product of any base and corresponding altitude.

30. The area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts (the Area Postulate).

•31. Substitution.

32. Substitution.

•33. Division.

34. It proves that the sum of the lengths of the three paths is equal to the altitude of the triangle.

Three Equal Triangles.

•35. 84 square units.

(α right $\Delta DEF = \frac{1}{2}7 \cdot 24 = 84$.)

•36. 11.2 units. ($\frac{1}{2}15h = 84$.)

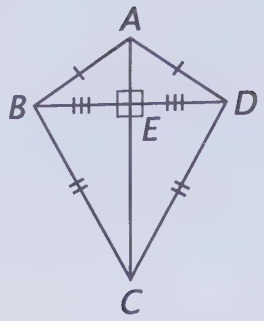
37. 6.72 units. ($\frac{1}{2}25h = 84$.)

38. 24 units. (EF = 24.)

39. 16.8 units. ($\frac{1}{2}10h = 84$.)

Kite Geometry.

40.



•41. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

•42. Betweenness of Points Theorem.

43. The area of a triangle is half the product of any base and corresponding altitude.

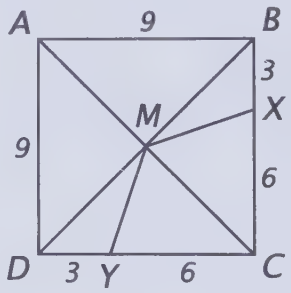
44. Area Postulate.

45. Substitution.

Dividing a Cake.

46. Because it has less icing on its sides.

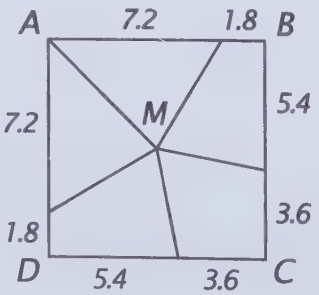
47.



•48. $\alpha\Delta AMB = \alpha\Delta AMD = 20.25$ in^2;
$\alpha\Delta BMX = \alpha\Delta DMY = 6.75$ in^2;
$\alpha\Delta CMX = \alpha\Delta CMY = 13.5$ in^2.

49. Yes. The area of the top of each piece is 27 in^2 and each piece has the same amount of icing.

50.



Divide the perimeter of the cake, 36, by 5 to get 7.2. Starting at A, mark off points around the square at intervals of 7.2 in. Cut from the center of the cake through these points.

Shingle Roof.

51. On the scale drawing, the lengths of the base and corresponding altitude are 2.5 in and 1.5 in. Because 1 in represents 8 ft, these lengths on the roof are 20 ft and 12 ft. Therefore, the area of the section is 120 ft^2.

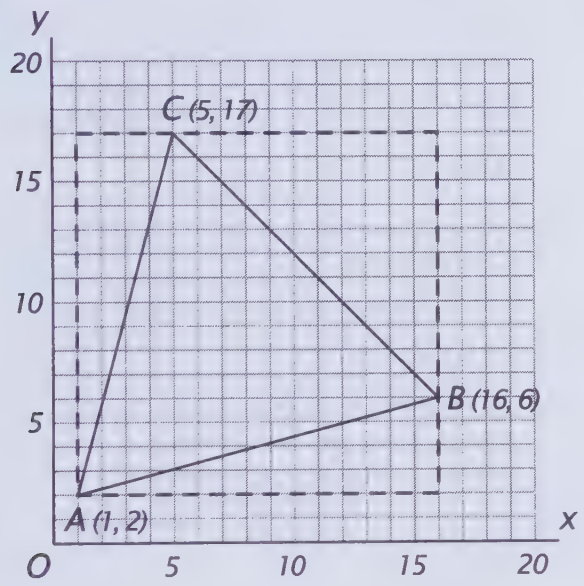
•52. About 1,200 shingles.

53. 4.8, or about 5 bundles. (Because 4 bundles contain 1,000 shingles, 1 bundle contains 250 shingles. $\frac{1{,}200}{250} = 4.8$.)

54. 3.6 pounds. ($\frac{120}{100} \times 3 = 3.6$.)

Grid Exercise.

55.



56. It is isosceles. By the Distance Formula,

$AB = \sqrt{15^2 + 4^2} = \sqrt{241} \approx 15.52$;

$BC = \sqrt{11^2 + 11^2} = \sqrt{242} \approx 15.56$;

$AC = \sqrt{4^2 + 15^2} = \sqrt{241} \approx 15.52$.

57. The exact area is 104.5 square units. ($15^2 - 2 \cdot \frac{1}{2} \cdot 4 \cdot 15 - \frac{1}{2} 11^2$.)

58. Irrational.

59. Rational.

Set III (page 357)

Although Heron's name is attached to the method for finding the area of a triangle from the lengths of its sides, Archimedes is thought to have actually discovered it. Heron's proof of the method is included in his book titled *Metrica*, which was lost until a manuscript of it was discovered in Constantinople in 1896. J. L. Heilbron remarks: "A good example of the difference in approach between the geometer and the algebraist may be drawn from proofs of the famous formula of Heron for the area of a triangle." Heilbron includes Heron's (or Archimedes's) geometric proof based on similar triangles, as well as an elegant trigonometric proof, in the "Tough Knots" chapter of his *Geometry Civilized* (Clarendon Press, 1998). An algebraic proof is included in the preceding editions of my *Geometry* (1974 and 1987). The algebra can be checked by symbol manipulation programs but is otherwise more challenging than instructive.

Heron's Theorem.

1. *(Student answer.)* ("Triangle 3" with sides 5, 5, and 10 seems "the most obvious.")

2. Triangle 1: perimeter 16; so $s = 8$.
   Area $= \sqrt{8(8-5)(8-5)(8-6)} = \sqrt{8 \cdot 3 \cdot 3 \cdot 2} = \sqrt{144} = 12$.
   Triangle 2: perimeter = 18; so $s = 9$.
   Area $= \sqrt{9(9-5)(9-5)(9-8)} = \sqrt{9 \cdot 4 \cdot 4 \cdot 1} = \sqrt{144} = 12$.
   Triangle 3: perimeter = 20; so $s = 10$.
   Area $= \sqrt{10(10-5)(10-5)(10-10)} = \sqrt{10 \cdot 5 \cdot 5 \cdot 0} = \sqrt{0} = 0$.

3. "Triangle 3" can't be a triangle, because, if it were, the lengths of its sides would contradict the Triangle Inequality Theorem.

4. Triangle 4 because "triangle 5" can't be a triangle.

5. Perimeter = 18; so $s = 9$.
   Area $= \sqrt{9(9-4)(9-6)(9-8)} = \sqrt{9 \cdot 5 \cdot 3 \cdot 1} = \sqrt{135}$, or $3\sqrt{15}$, or approximately 11.6.

Chapter 9, Lesson 4

Set I (pages 360–362)

The two arrangements of the automobiles in exercises 1 through 5 have something in common with D'Arcy Thompson's fish transformation. Both are examples of a "shear" transformation, a transformation not usually discussed in geometry but one that is a basic tool of computer graphics. For example, some computer programs use shear transformations to generate slanted italic fonts from standard ones. Although these transformations are not isometries, they do preserve area.

An interesting story of the construction and problems of the Hancock Tower, headquarters of the John Hancock Mutual Life Insurance Company in Boston, is told by Mattys Levi and Mario Salvadori in their book titled *Why Buildings Fall Down* (Norton, 1992). Although now altered to be more structurally sound, the tower at the beginning had problems with the wind because of its unusual nonrectangular cross section. Among other things, all of the more than 10,000 panels of reflective glass covering its walls had to be replaced with stronger panels!

From Above.

•1. That they are equal.

•2. 2,800 ft². [$x = (5)(14) = 70$, $y = (5)(8) = 40$; $xy = 2{,}800$.]

3. 220 ft. [$2(x + y) = 2(70 + 40) = 220$.]

4. The perimeter of the second figure is greater.

5. No.

Tax Assessor Formula.

•6. Area $= \frac{1}{4}(s + s)(s + s) = \frac{1}{4}(2s)(2s) = s^2$.

7. Yes. Area $= \frac{1}{4}(b + b)(h + h) = \frac{1}{4}(2b)(2h) = bh$.

•8. $\frac{1}{4}(20 + 20)(13 + 13) = \frac{1}{4}(40)(26) =$ 260 square units.

9. No. The area of a parallelogram is the product of any base and corresponding altitude; so the area $= (20)(12) =$ 240 square units.

10. $\frac{1}{4}(3 + 24)(10 + 17) = \frac{1}{4}(27)(27) =$ 182.25 square units.

11. No. The area of a trapezoid is half the product of its altitude and the sum of its bases; so the area $= \frac{1}{2}(8)(3 + 24) = 4(27) =$ 108 square units.

12. $\frac{1}{4}(20 + 24)(7 + 15) = \frac{1}{4}(44)(22) =$ 242 square units.

13. No. The area of each right triangle is half the product of its legs; so the area $= \frac{1}{2}(7)(24) + \frac{1}{2}(20)(15) = 84 + 150 =$ 234 square units.

•14. Area $= \frac{1}{4}(37 + 37)(37 + 37) = \frac{1}{4}(74)(74) =$ 1,369 square units.

15. No. The diagonals of a rhombus are perpendicular and bisect each other; so the area of this rhombus is $4(\frac{1}{2})(12)(35) = 4(210) =$ 840 square units.

16. Because the formula seems to always give an area either equal to or larger than the correct answer.

Pegboard Quadrilaterals.

•17. A parallelogram. (Two of its opposite sides are both parallel and equal.)

•18. 1 unit. $A = bh = (1)(1) = 1$.

19. A parallelogram.

20. 1 unit.

21. A parallelogram.

22. 1 unit. It can be divided by a horizontal diagonal into two triangles, each with a base and altitude of 1; $2(\frac{1}{2})(1)(1) = 1$.

Skyscraper Design.

•23. 30,794 ft². [(300)(104) – 2($\frac{1}{2}$)(14)(29) = 31,200 – 406 = 30,794.]

24. 237,000 ft². [(300)(790) = 237,000.]

Shuffleboard Court.

25. 3 ft². (Triangle: $\frac{1}{2}(2)(3) = 3$.)

•26. 4.5 ft². (Trapezoid: $\frac{1}{2}(3)(1 + 2) = 4.5$.)

27. 7.5 ft². (Trapezoid: $\frac{1}{2}(3)(2 + 3) = 7.5$.)

•28. 8.25 ft². (Trapezoid: $\frac{1}{2}(1.5)(5 + 6) = 8.25$.)

29. 35.25 ft².

Set II (pages 362–364)

Students who do exercises 47 through 49 in which the trapezoidal rule is used to find the approximate area of the region between a parabola and the *x*-axis may be intrigued to know that the actual area is represented by the expression

$$\int_0^3 \frac{x^2}{2}\,dx$$

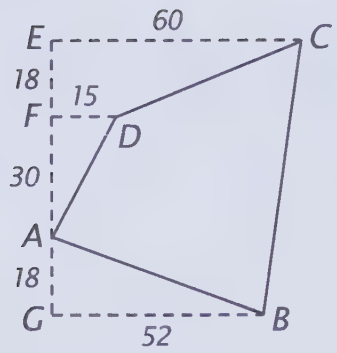
and that those who take calculus will learn how to find its exact value, 4.5.

The SAT problem is a good example of how important it is to "look before you leap." The method of subtracting the shaded area from the rectangle to get the unshaded region is tempting but, even if the areas of all of the regions could be found, would be more time consuming than would simply adding the areas of the two unshaded regions. Time taken to consider alternative approaches to solving a problem is often time well spent.

Inaccessible Field.

30. The entire figure (ECBG) is a trapezoid. Subtract the areas of trapezoid ECDF, ΔFDA, and ΔABG from its area.

31.



•32. 675 m². [$\frac{1}{2}(18)(60 + 15) = 675$.]

33. 225 m². [$\frac{1}{2}(15)(30) = 225$.]

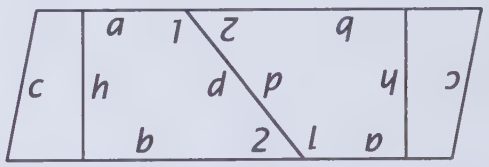
34. 468 m². [$\frac{1}{2}(18)(52) = 468$.]

•35. 3,696 m². [$\frac{1}{2}(60 + 52)(18 + 30 + 18) = 3{,}696$.]

36. 2,328 m². (3,696 – 675 – 225 – 468 = 2,328.)

A Fold-and-Cut Experiment.

37.



38. The bases line up because ∠1 and ∠2 are supplementary (parallel lines form supplementary interior angles on the same side of a transversal); together ∠1 and ∠2 form a straight angle.

39. A parallelogram.

40. A quadrilateral is a parallelogram if its opposite sides are equal. (Alternatively, a quadrilateral is a parallelogram if two opposite sides are both parallel and equal.)

•41. Its area is $(a + b)h$.

42. The parallelogram consists of two congruent trapezoids; so the area of each is $\frac{1}{2}(a + b)h$, or $\frac{1}{2}h(a + b)$.

Trapezoidal Rule.

•43. In a plane, two lines perpendicular to a third line are parallel.

44. $\alpha ABGH = \frac{1}{2}x(y_1 + y_2)$. $\alpha BCFG = \frac{1}{2}x(y_2 + y_3)$. $\alpha CDEF = \frac{1}{2}x(y_3 + y_4)$.

45. $\alpha ABGH + \alpha BCFG + \alpha CDEF =$
$\frac{1}{2}x(y_1 + y_2) + \frac{1}{2}x(y_2 + y_3) + \frac{1}{2}x(y_3 + y_4) =$
$\frac{1}{2}x(y_1 + y_2 + y_2 + y_3 + y_3 + y_4) =$
$\frac{1}{2}x(y_1 + 2y_2 + 2y_3 + y_4)$.

•46. 165 square units.
$[\frac{1}{2}(5)(8 + 2(15) + 2(11) + 6) = \frac{1}{2}(5)(66) = 165.]$

47. B(2, 2), C(3, 4.5).

48. 4.75. $[\frac{1}{2}(1)(0 + 2(0.5) + 2(2) + 4.5).]$

49. It is larger.

SAT Problem.

•50. ΔEBH, FHCG (and FHCD).

51. We don't know exactly where point H is; so we don't know the lengths of BH and HC.

52. 9 square units.

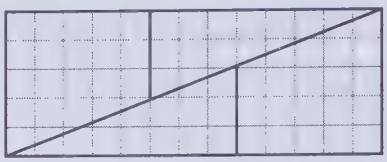
53. The unshaded region consists of two triangles, ΔEFD and ΔEFH, with a common base of 2. The corresponding altitudes are 5 and 4; so $\alpha EDFH = \frac{1}{2}(2)(5) + \frac{1}{2}(2)(4) = 5 + 4 = 9$.

Set III (page 364)

The chessboard puzzle was discovered by a German mathematician named Schlömilch and published in 1868 in an article titled "A Geometric Paradox." Good references on it and similar puzzles can be found in the second chapter on "Geometrical Vanishes" in Martin Gardner's *Mathematics, Magic and Mystery* (Dover, 1956) and the chapter titled "Cheated, Bamboozled, and Hornswoggled" in *Dissections—Plane and Fancy,* by Greg N. Frederickson (Cambridge University Press, 1997). A generalization of the chessboard paradox was made by Lewis Carroll; an algebraic analysis of it can be found in J. L. Heilbron's *Geometry Civilized* (Clarendon Press, 1998).

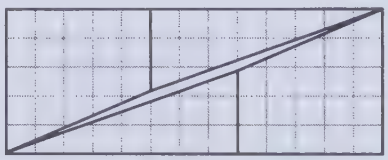
Chessboard Mystery.

1. The area of each right triangle is $\frac{1}{2}(3)(8) = 12$ units. The area of each trapezoid is $\frac{1}{2}(5)(3 + 5) = 20$ units.
The four areas add up to 64 units, as we would expect for a square board consisting of 64 unit squares.

2. 

3. 65 units.

4. According to the Area Postulate, the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts. But $65 \neq 64$.
(The explanation is that the pieces don't quite fill the rectangle. The exaggerated drawing below shows where the extra unit comes from.)



Chapter 9, Lesson 5

Set I (pages 367–368)

J. L. Heilbron in *Geometry Civilized* (Clarendon Press, 1998), wrote: "The Pythagorean Theorem and its converse bring the first book of the *Elements* to a close. They therefore mark a climax, the high point toward which all the apparatus of definitions, postulates, and propositions was aimed. The diagram of the windmill [the figure used by Euclid to prove this theorem] may be taken as a symbol of geometry, and hence of Greek thought, which generations . . . have regarded as the bedrock of their culture."

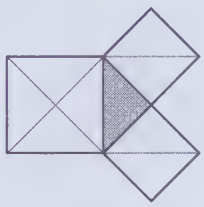
Note that in exercises 1 through 3, dealing with the special case of an isosceles right triangle, the areas of the squares on its sides are expressed in terms of the area of the triangle rather than the length of one of its sides.

The proof of the converse of the Pythagorean Theorem is, in essence, that of Euclid. The only difference is that Euclid chose to construct the second triangle so that it shared one of its shorter sides with the original triangle.

In thinking about exercise 32, it is helpful to consider the squares representing a^2 and b^2 as being hinged together at one of their vertices. If the angle of the hinge is adjusted to 90°, we have the Pythagorean Theorem with $c^2 = a^2 + b^2$. As the angle is made smaller, the distance c shrinks and c^2 gets smaller; so, for acute triangles, $c^2 < a^2 + b^2$. If instead, the angle is made larger, the distance c increases and c^2 grows; so, for obtuse triangles, $c^2 > a^2 + b^2$.

Batik Design.

1. *Example figure:*



•2. 2, 2, and 4 units.

3. $2 + 2 = 4$. (The square on the hypotenuse of a right triangle is equal to the sum of the squares on its legs.)

Theorem 43.

•4. The Ruler Postulate.

•5. The Protractor Postulate.

6. The Ruler Postulate.

7. Two points determine a line.

•8. The square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs.

•9. Substitution.

10. SSS.

11. Corresponding parts of congruent triangles are equal.

•12. Substitution.

13. A 90° angle is a right angle.

14. A triangle with a right angle is a right triangle.

Squares on the Sides.

•15. 12.

•16. 1,225.

17. 1,369.

•18. 37.

19. 10.

20. 26.

21. 576.

22. 24.

23. 484.

24. 16.

25. 15.

•26. No. $256 + 225 = 481 \neq 484$.

27. 2,304.

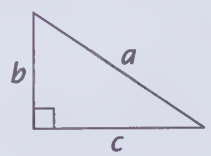
28. 3,025.

29. 73.

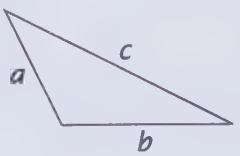
30. Yes. $2{,}304 + 3{,}025 = 5{,}329$. (If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.)

Ollie's Triangles.

31. *Example figure:*



32. *Example figure:*



Set II (pages 368–370)

Frank J. Swetz and T. I. Kao, the authors of *Was Pythagoras Chinese? An Examination of Right Triangle Theory in Ancient China* (Pennsylvania State University Press, 1977), report that, although estimates of the date of origin of the *Chou Pei Suan Ching* date as far back as 1100 B.C., much of the material in it seems to have been written at the time of Confucius in the sixth century B.C. It is interesting to observe that, although there is no proof of the Pythagorean Theorem given in the *Chou Pei*, the figure, based on the 3-4-5 right triangle, may have been the inspiration for the figure on which the proof in the text of Lesson 5 is based.

In *The Mathematical Universe* (Wiley, 1994), William Dunham tells the story of James A. Garfield and his proof of the Pythagorean Theorem. After completing his college education, Garfield began teaching mathematics at Hiram College in Ohio. Within a few years, he was elected to the Ohio Senate and then joined the Union Army; he served for 17 years in the House of Representatives before being elected president in 1880. At the time of its publication in *The New England Journal of Education* in 1876, it was reported that Garfield had discovered his proof of the Pythagorean Theorem during "some mathematical amusements and discussions with other congressmen."

The figure illustrating exercises 50 and 51 is another nice example of a "proof without words." The fact that the bisector of the right angle of a right triangle bisects the square on the hypotenuse is clearly connected to the second figure's rotation symmetry.

Was Pythagoras Chinese?

•33. 6 units. $[\frac{1}{2}(3)(4).]$

34. 24 units. $(4 \times 6.)$

•35. 25 units. $(24 + 1.)$

36. $\frac{1}{2}ab$.

•37. $2ab$. $[4(\frac{1}{2}ab).]$

•38. $(b - a)^2$.

39. $\alpha ABCD = 2ab + (b - a)^2$ and $\alpha ABCD = c^2$.

40. $2ab + (b - a)^2 = c^2$; so $2ab + b^2 - 2ab + a^2 = c^2$, and so $a^2 + b^2 = c^2$.

Garfield's Proof.

41. SAS.

42. Corresponding parts of congruent triangles are equal.

•43. The acute angles of a right triangle are complementary (so the sum of these angles is 90°).

44. A right angle.

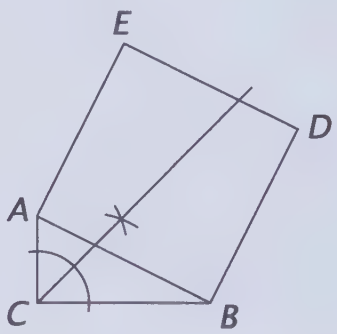
•45. A trapezoid.

46. $\alpha ABDE = \frac{1}{2}(a + b)(a + b)$ and $\alpha ABDE = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$.

47. $\frac{1}{2}(a + b)^2 = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$; so $(a + b)^2 = 2ab + c^2$; so $a^2 + 2ab + b^2 = 2ab + c^2$, and so $a^2 + b^2 = c^2$.

Angle-Bisector Surprise.

48.



•49. It appears to bisect ABDE (that is, divide it into two congruent parts having equal areas).

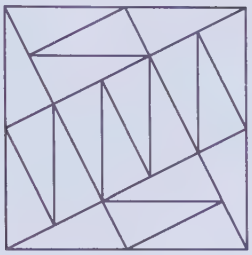
50. It bisects $\angle ACB$. $\Delta FCG \cong \Delta HCG$; so $\angle FCG = \angle HCG$.

51. It bisects ABDE.

Set III (page 370)

Twenty Triangles.

1. *Example figure:*



[The areas of the three squares are 4, 16, and 20 square inches; so the sides of the large square are $\sqrt{20} \approx 4.5$" (or exactly, $\sqrt{20} = 2\sqrt{5}$"). We can't get this length by using the legs of the triangles; so we have to use their hypotenuses: $1^2 + 2^2 = c^2$, and so $c = \sqrt{5} \approx 2.2$. Two hypotenuses are needed to form each side of the large square.]

2. It is tempting to try to use the right angles of the triangles to form the right angles of the third square. This won't work, however, because the legs of the triangles have to be *inside* the square.

Chapter 9, Review

Set I (pages 371–373)

In the section on Babylonian mathematics in *The History of Mathematics* (Allyn & Bacon, 1985), David Burton wrote: "In the ancient world, the error was widespread that the area of a plane figure depended entirely on its perimeter; people believed that the same perimeter always confined the same area. Army commanders estimated the number of enemy soldiers according to the perimeter of their camp, and sailors the size of an island according to the time for its circumnavigation. The Greek historian Polybius tells us that in his time unscrupulous members of communal societies cheated their fellow members by giving them land of greater perimeter (but less area) than what they chose for themselves; in this way they earned reputations for unselfishness and generosity, while they really made excessive profits."

An excellent source of information on tangrams are the two chapters on them by Martin Gardner in *Time Travel and Other Mathematical Bewilderments* (W. H. Freeman and Company, 1988). Although many references report that tangrams date back about 4,000 years, historians now believe that they actually originated in China in about 1800. An ornate set of carved ivory tangrams once owned by Edgar Allan Poe is now in the New York Public Library.

Olympic Pools.

•1. 125 m^2. (50×2.5.)

2. No. There is room left on the sides. The area of the pool is 1,050 m^2 and the total area of the 8 lanes is 1,000 m^2 (the width of the pool is 21 m and the total width of the 8 lanes is 20 m.)

•3. 525. ($\frac{1,050}{2}$.)

Enemy Camps.

•4. Camp A, 68 paces; camp B, 64 paces; camp C, 60 paces.

5. Camp A, 208 square paces; camp B, 220 square paces; camp C, 216 square paces.

6. Camp A because it has the greatest perimeter.

7. No. Camp B has the greatest area.

Tangrams.

•8. 16 units.

9. 4 units.

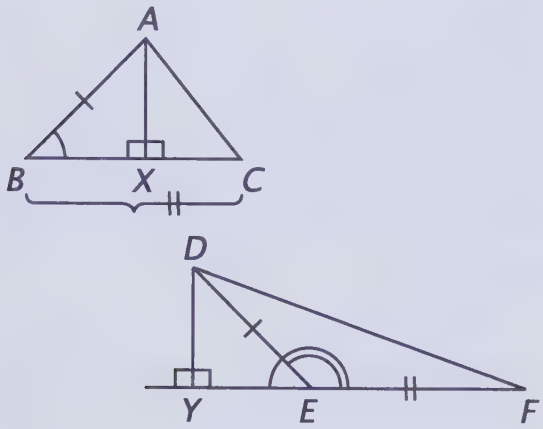
10. The other triangle, 8 units; the square, 8 units; the parallelogram, 8 units.

11. 64 units.

•12. The area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts.

Altitudes and Triangles.

13.



•14. ∠DEY.

15. ΔABX ≅ ΔDEY.

•16. AAS.

17. Congruent triangles have equal areas.

•18. Corresponding parts of congruent triangles are equal.

19. ΔABC and ΔDEF.

•20. Triangles with equal bases and equal altitudes have equal areas.

Suriname Stamp.

21. The square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs.

22. If the square on one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.

•23. 11.

24. 49.

25. 169.

26. No. $121 + 49 = 170 \neq 169$.

Moroccan Mosaic.

•27. $4a$.

28. a^2.

•29. $4a + 4b$.

•30. $a(a + 2b)$ or $a^2 + 2ab$.

31. $4a + 2b$.

32. $b(a + b)$ or $ab + b^2$. $[\frac{1}{2}b(a + a + 2b.]$

Set II (pages 373–375)

The Daedalus project was named after the character in Greek mythology who constructed wings from feathers and wax to escape from King Minos. The planes built in the project were designed and built at M.I.T. Daedalus 88, the plane shown in the photograph, was flown 74 miles from Crete to the island of Santorini in the Mediterranean in 3 hours, 54 minutes, all by human power alone! According to an article in the October 1988 issue of *Technology Review*, the plane weighed 70 pounds when empty. Its wing span and area are worked out in the exercises. The plane had a flight speed ranging from 14 to 17 miles per hour and its human "engine" power was about a quarter of a horsepower!

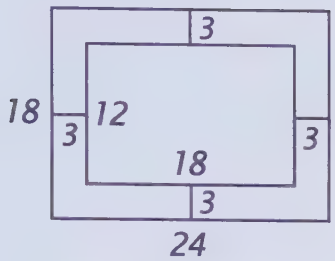
The problems about a crack forming in a stretched bar are based on information in *The New Science of Strong Materials* (Princeton University Press, 1976), by J. E. Gordon, who writes: "When a crack appears in a strained material it will open up a little so that the two faces of the crack are separated. This implies that the material immediately behind the crack is relaxed and the strain energy in that part of the material is released. If we now think about a crack proceeding inwards from the surface of a stressed material we should expect the area of material in which the strain is relaxed to correspond roughly to the two shaded triangles. Now the area of these triangles is roughly l^2, where l is the length of the crack. The relief of strain energy would thus be expected to be proportional to the square of the crack length, or rather depth, and in fact this rough guess is confirmed by calculation. Thus a crack two microns deep releases four times as much strain energy as one one micron deep and so on."

The exercises on the area of an irregular tract reveal that the trapezoidal rule is used in surveying as well as in calculus.

George Biddle Airy was the astronomer royal at the Greenwich Observatory from 1836 to 1881. A nice treatment of dissection proofs of the Pythagorean Theorem and their connection to superposing tessellations is included by Greg Frederickson in chapter 4 of *Dissections—Plane and Fancy* (Cambridge University Press, 1997).

Courtyard Design.

33.



•34. 216 ft^2. (12×18.)

35. 216 ft^2. ($18 \times 24 - 216$.)

36. Yes.

Daedalus Wing.

37. 112 ft.

•38. 332.5 ft^2. $[2 \cdot \frac{1}{2}(14)(1.25 + 2.5) + 2 \cdot \frac{1}{2}(28)(2.5 + 3.75) + 28(3.75) = 52.5 + 175 + 105 = 332.5.]$

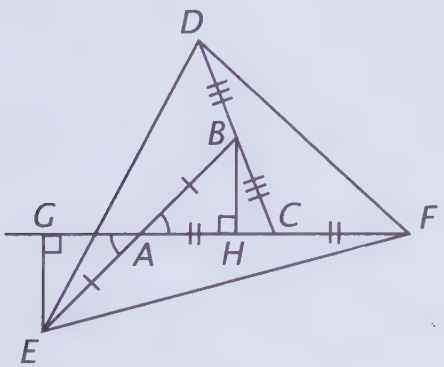
Crack Formation.

39. $a + b = 2l$. The colored area is $\frac{1}{2}al + \frac{1}{2}bl = l^2$; so $\frac{1}{2}a + \frac{1}{2}b = l$, and so $a + b = 2l$.

•40. It would be four times as great [because $(2l)^2 = 4l^2$].

Triangle Comparisons.

41.



42. AAS.

43. Corresponding parts of congruent triangles are equal.

•44. $\alpha\Delta AEF = 2x$ (because $AF = 2AC$ and $BH = EG$).

45. $\alpha\Delta DEF = 7\alpha\Delta ABC$. (By the same reasoning, $\alpha\Delta BDE = 2x$ and $\alpha\Delta CFD = 2x$. Because $\alpha\Delta DEF = \alpha\Delta ABC + \alpha\Delta AEF + \alpha\Delta BDE + \alpha\Delta CFD$, $\alpha\Delta DEF = x + 2x + 2x + 2x = 7x = 7\alpha\Delta ABC$.)

SAT Problem.

•46. $4x^2 - xy$.

•47. $4x - y$.

48. $8x$. [$4(2x)$.]

49. $10x - 2y$. $\{2[x + (4x - y)] = 2(5x - y)$.$\}$

Surveying Rule.

50. $\frac{1}{2}x(a + b) + \frac{1}{2}x(b + c) + \frac{1}{2}x(c + d) + \frac{1}{2}x(d + e)$ or $\frac{1}{2}x(a + 2b + 2c + 2d + e)$.

•51. It would be rectangular.

52. $\frac{1}{2}x(a + 2a + 2a + 2a + a) = \frac{1}{2}x(8a) = 4xa$.

Pythagorean Proof.

53. In the first arrangement, the three pieces form the squares on the legs of the right triangle and their total area is $a^2 + b^2$. In the second arrangement, the three pieces form the square on the hypotenuse of the right triangle and their total area is c^2; so $c^2 = a^2 + b^2$.

54. It refers to the area of the red piece. [In the first figure, the area of the red piece is clearly the sum of the areas of the two smaller squares minus the areas of the two triangles: $a^2 + b^2 - 2(\frac{1}{2}ab)$. When the two triangles "stand on" the red piece, the square on the hypotenuse is formed and, when the red piece "stands on" the two triangles, the squares on the legs are formed.]

Chapter 9, Algebra Review (page 376)

•1. 500. [$x = 10(50) = 500$.]

•2. $\frac{1}{5}$ or 0.2. ($3 = 15x$, $x = \frac{3}{15} = \frac{1}{5} = 0.2$.)

•3. 2.5. [$6 = 4(x - 1)$, $6 = 4x - 4$, $10 = 4x$, $x = \frac{10}{4} = \frac{5}{2} = 2.5$.]

•4. ±10. ($x^2 = 100$, $x = \pm 10$.)

•5. –3. ($11 + 2x = 5$, $2x = -6$, $x = -3$.)

•6. 42. ($2x = 3x - 42$, $x = 42$.)

•7. 15. ($x - 12 = 3$, $x = 15$.)

•8. 20. ($5x = 4x + 20$, $x = 20$.)

•9. 41. [$7(x + 5) = 2(4x - 3)$, $7x + 35 = 8x - 6$, $41 = x$.]

•10. 3.5. [$\frac{4}{x-3} = 8$, $4 = 8(x - 3)$, $4 = 8x - 24$, $28 = 8x$, $x = \frac{28}{8} = \frac{7}{2} = 3.5$.]

•11. 0. [$15 - 2(x + 3) = 3(x + 3)$, $15 - 2x - 6 = 3x + 9$, $9 - 2x = 3x + 9$, $5x = 0$, $x = 0$.]

•12. 8. ($\frac{8x}{x} = x$, $x = 8$.)

•13. 5. [$3(15) + 1(x + 4) = 2(3)(x + 4)$, $45 + x + 4 = 6x + 24$, $49 + x = 6x + 24$, $25 = 5x$, $x = 5$.]

•14. 5. [$5(3x + 1) - 4x = 20(x - 2)$, $15x + 5 - 4x = 20x - 40$, $11x + 5 = 20x - 40$, $45 = 9x$, $x = 5$.]

•15. –36. [$4(x + 8) - (x - 4) = 2x$, $4x + 32 - x + 4 = 2x$, $3x + 36 = 2x$, $x = -36$.]

Chapter 10, Lesson 1

Set I (pages 380–382)

According to Benno Artmann in *Euclid—The Creation of Mathematics* (Springer, 1999), the architects of the Parthenon, Ictinos and Callikrates, based its dimensions on the square numbers 4 and 9. Starting with a height of 4^2, they chose $4 \cdot 9$ for the width and 9^2 for the length, so that the width was the geometric mean between the height and the length.

Euclid developed the theory of proportion with a series of 25 theorems in Book V of the *Elements*. His definition of *ratio* as "a sort of relation in respect of size between two magnitudes of the same kind" is, as one commentator remarked, unusual for Euclid because it is so vague as to be of no practical use.

The numbers connected with the *Turtles Forever* picture illustrate the fact that any three consecutive terms of a geometric sequence are proportional (as are *any* three evenly spaced terms). The term in the middle is consequently the geometric mean between the other two terms.

Parthenon Architecture.

•1. 2.25.

2. 2.25.

3. $81 \times 16 = 1{,}296$ and $36 \times 36 = 1{,}296$.

•4. The geometric mean.

•5. $\frac{9}{6} = \frac{6}{4}$.

6. $1.5 = 1.5$.

7. $101\frac{1}{3}$ ft.

•8. $h \approx 45$ ft. (*Example proportion:* $\frac{228}{h} = \frac{81}{16}$.)

•9. $\frac{2}{7}$ (or 0.285714 . . .). ($\frac{2d}{7d}$.)

10. $\frac{4}{5}$ (or 0.8). ($\frac{8d}{10d}$.)

11. $\frac{10}{1}$ (or 10). ($\frac{10d}{d}$.)

Turtles Forever.

12. $\frac{16}{32} = \frac{32}{64}$.

13. $\frac{2}{16} = \frac{16}{128}$.

14. 4 and 16, 2 and 32, 1 and 64.

Ratio.

•15. Division.

16. The second.

17. Euclid referred to "magnitudes of the same kind," which suggests that numbers of the same kind of units are being compared.

Four Rectangles.

18. Rectangle 1, ab; rectangle 2, ad; rectangle 3, cd; rectangle 4, bc.

•19. Rectangles 2 and 4. ($ad = bc$ because $\frac{b}{a} = \frac{d}{c}$.)

20. Rectangles 1 and 3. ($\frac{a}{c} = \frac{b}{d}$ because $ad = bc$.)

21. Rectangles 1 and 3.

Enlargement.

•22. 30.

23. 36.

24. Correct. (Both ratios are equal to $\frac{13}{14}$.)

25. Correct. ($\frac{126}{84} = \frac{42}{28} = \frac{3}{2} = 1.5$.)

26. Correct. [$\frac{\frac{1}{2} \cdot 42 \cdot 36}{\frac{1}{2} \cdot 28 \cdot 24} = 2.25$ and $(\frac{39}{26})^2 = 2.25$.]

Set II (pages 382–384)

The official design of the current United States flag was signed into law by President Eisenhower in 1959, soon after Alaska became the 50th state. The first chapter of *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics* (Princeton University Press, 1999), by Robert Banks, deals with the geometry of the flag.

Concerning the ratio $\frac{19}{10}$ Banks writes: "Why the relative length of the flag is precisely 1.9—or indeed why the relative length of the union is exactly 0.76 (could it be 1776?)—is not known. It just is."

As Thomas Rossing remarks in *The Science of Sound* (Addison Wesley, 1990), "the word *scale* is derived from a Latin word (*scala*) meaning *ladder* or *staircase*. A musical scale is a succession of notes arranged in ascending or descending order." In chapter 9 of his book, Rossing includes complete discussions of the scales based on just intonation and equal temperament as well as the Pythagorean scale on which exercises 41 through 47 are based. Another good reference on the structure of musical scales is *Connections—The Geometric Bridge between Art and Science*, by Jay Kappraff (McGraw-Hill, 1991). It is interesting to see from the relative lengths of the strings producing the notes of the Pythagorean scale that they are all derived from the primes 2 and 3:

$$1 \quad \frac{2^3}{3^2} \quad \frac{2^6}{3^4} \quad \frac{3}{2^2} \quad \frac{2}{3} \quad \frac{2^4}{3^3} \quad \frac{2^7}{3^5} \quad \frac{1}{2}$$

In the currently used even-tempered scale, the ratios $\frac{C}{D}$, $\frac{D}{E}$, $\frac{F}{G}$, $\frac{G}{A}$, and $\frac{A}{B}$ are $\left(\sqrt[12]{2}\right)^2 \approx 1.122$, in contrast with the Pythagorean ratio of $\frac{9}{8} = 1.125$. The "even tempered" ratios $\frac{E}{F}$ and $\frac{B}{C'}$ are $\sqrt[12]{2} \approx 1.059$ in contrast with the Pythagorean ratio, $\frac{256}{243} \approx 1.053$.

The fact that the means of a proportion can be interchanged is proved in Book V of the *Elements* as Proposition 16: "If four magnitudes be proportional, they will also be proportional alternately." The addition property proved in exercises 51 and 52 appears in the *Elements* as Proposition 18 of the same book.

Arm Spans.

27. No. The units of length being compared are not the same; so the result is meaningless.

•28. 36 in; $\frac{36}{15} = 2.4$.

29. 1.25 ft; $\frac{3}{1.25} = 2.4$.

30. No.

United States Flag.

•31. $\frac{19}{10}$.

32. $\frac{7}{13}$.

33. $\frac{7}{6}$.

34. The length of the flag.

35. 304 feet.

•36. The width of one stripe.

•37. Approximately 12.3 ft. ($\frac{160}{13} \approx 12.3$.)

38. Approximately 86 ft. ($7 \times 12.3 \approx 86$.)

•39. Approximately 122 ft. ($\frac{b}{160} = 0.76$, $b = 121.6$.)

40. No. $\frac{a}{b} \approx \frac{86}{121.6} \approx 0.7$ and $\frac{w}{l} = \frac{10}{19} \approx 0.53$.

Pythagorean Tuning.

•41. $\frac{D}{E} = \frac{\frac{8}{9}}{\frac{64}{81}} = \frac{8}{9} \cdot \frac{81}{64} = \frac{9}{8}$.

•42. $\frac{E}{F} = \frac{\frac{64}{81}}{\frac{3}{4}} = \frac{64}{81} \cdot \frac{4}{3} = \frac{256}{243}$.

43. $\frac{F}{G} = \frac{\frac{3}{4}}{\frac{2}{3}} = \frac{3}{4} \cdot \frac{3}{2} = \frac{9}{8}$.

44. $\frac{G}{A} = \frac{\frac{2}{3}}{\frac{16}{27}} = \frac{2}{3} \cdot \frac{27}{16} = \frac{9}{8}$.

45. $\frac{A}{B} = \frac{\frac{16}{27}}{\frac{128}{243}} = \frac{16}{27} \cdot \frac{243}{128} = \frac{9}{8}$.

46. $\frac{B}{C'} = \frac{\frac{128}{243}}{\frac{1}{2}} = \frac{128}{243} \cdot \frac{2}{1} = \frac{256}{243}$.

47. G is the geometric mean between F and A; A is the geometric mean between G and B. ($\frac{F}{G} = \frac{G}{A} = \frac{9}{8}$ and $\frac{G}{A} = \frac{A}{B} = \frac{9}{8}$.)

Geometric Proportions.

•48. The means are interchanged.

49. Multiplication (or, in a proportion, the product of the means is equal to the product of the extremes).

•50. Division.

51. Addition.

52. Substitution (and algebra).

Sinker or Floater.

53. Alice is a floater because her density is $\frac{120 \text{ pounds}}{1.95 \text{ cubic feet}} \approx 61.5$ pounds per cubic foot.

54. Let x = Ollie's weight in pounds. His density is $\frac{x \text{ pounds}}{2.3 \text{ cubic feet}}$. Because he is a sinker, $\frac{x}{2.3} > 62.4$ and $x > 143.52$. Ollie weighs more than 143 pounds.

Set III (page 384)

The White Horse of Uffington was created by removing sod from the chalky rock lying underneath. A book written as far back as the fourteenth century ranked it second only to Stonehenge among the tourist attractions of Britain. The horse is in excellent condition owing to its being maintained by the National Trust.

The White Horse.

1. *(Student answer.)* (The length of the horse in the photograph is about 95 mm and its height is about 24 mm: $\frac{l}{h} = \frac{x \text{ m}}{28 \text{ m}} = \frac{95 \text{ mm}}{24 \text{ mm}}$, $x \approx 110.8$. The horse is about 110 meters long.)

2. *(Student answer.)* (The scale is about $\frac{1}{1{,}200}$. $\frac{24 \text{ mm}}{28{,}000 \text{ mm}} \approx \frac{1}{1{,}167}$.)

Chapter 10, Lesson 2

Set I (pages 387–389)

According to Robert Bauval and Adrian Gilbert (*The Orion Mystery*, Crown, 1994), the apparent relative positions of the three pyramids at Giza not only match those of the three stars in Orion's belt but their orientation with respect to the Nile matches Orion's apparent orientation with respect to the Milky Way! The arrangement of the pyramids and their relative sizes (seemingly based on the apparent sizes of the stars) seem to indicate that the Egyptians were trying to build a replica of heaven on Earth.

A "plane table" is a drawing board and ruler mounted on a tripod, used by surveyors in the field to sight and map data. More details on how it works can be found in J. L. Heilbron's *Geometry Civilized* (Clarendon Press, 1998).

According to Robert C. Yates in *The Trisection Problem* (Franklin Press, 1942; N.C.T.M., 1971), Alfred Kempe was "one of the cleverest amateur mathematicians of the [eighteenth] century." Kempe gave a historic lecture on linkages to a group of science teachers in London in the summer of 1876. This lecture, titled "How to Draw a Straight Line," and published in 1877, was reprinted by N.C.T.M. in 1977. The linkage invented by Kempe for trisecting angles is also discussed by Yates in the chapter on mechanical trisectors in *The Trisection Problem* and by Martin Gardner in the chapter titled "How to Trisect an Angle" in *Mathematical Carnival* (Knopf, 1975). As Kempe himself observed, his idea can be extended to produce a linkage that can divide an angle into any number of equal parts.

The Pyramids and Orion.

•1. They are proportional.

•2. They are equal.

Plane Table.

•3. $\angle DAC = \angle FBC$.

4. $\angle AEC = \angle BGC$.

5. $\angle GCB = \angle ECA$.

•6. $\frac{AD}{BF} = \frac{DC}{FC}$.

7. $\frac{GC}{EC} = \frac{BC}{AC}$.

8. $\frac{BC}{AC} = \frac{BG}{AE}$.

Kempe's Linkage.

•9. OE = 4 and OG = 8. ($\frac{1}{2} = \frac{2}{OE} = \frac{OE}{OG}$.)

10. They are always equal. Corresponding angles of similar triangles are equal.

11. It trisects ∠AOG.

Billboards.

12. Their corresponding angles are equal because billboards are rectangular and all right angles are equal. Their corresponding sides, however, are not proportional, because $\frac{5}{11} \neq \frac{11}{23}$. As a calculator shows, $\frac{5}{11} \approx 0.45$, whereas $\frac{11}{23} = 0.47826\ldots$.

•13. 24.2. ($\frac{5}{11} = \frac{11}{x}$, $5x = 121$, $x = 24.2$.)

14. $\frac{121}{23}$ (or $5\frac{6}{23}$ or approximately 5.26). ($\frac{x}{11} = \frac{11}{23}$, $23x = 121$, $x = \frac{121}{23} = 5\frac{6}{23}$.)

Dilation Problem.

•15. The center of the dilation.

16. AB = 5, BC = 4, AC = 6, A′B′ = 2.5, B′C′ = 2, and A′C′ = 3.

17. They are half as long.

•18. $\frac{1}{2}$.

19. They are their midpoints.

Quadrilateral Conclusions.

•20. Corresponding sides of similar polygons are proportional.

21. Multiplication. (In a proportion, the product of the means is equal to the product of the extremes.)

22. They have equal areas.

23. They also have equal areas. αADGH = αAEFH + αEDGF and αEDCI = αEDGF + αFGCI. Because αADGH = αEDCI (exercise 22), αAEFH + αEDGF = αEDGF + αFGCI (substitution); so αAEFH = αFGCI (subtraction).

Set II (pages 389–391)

A good article on the instruments of the violin family is "The Physics of Violins" by Carleen Maley Hutchins, published in the November 1962 issue of *Scientific American* and reprinted in the *Scientific American Resource Library* (W. H. Freeman and Company, 1978).

The International Standard of paper sizes is now in common use in Europe. As the exercises reveal, they are based on a sheet with an area of 1 square meter that can be cut in half to produce a second sheet similar in shape. Further sizes are produced in the same way. As the exercises also reveal, this pattern requires that all sizes have the ratio $\frac{\sqrt{2}}{1}$. The sequence beginning with size A0 continues to size A10 (a tiny 26 mm by 37 mm sheet). Computer programs usually offer sizes A4 and A5 in their menus of choices for printing formats.

Similar-Triangles Proof.

•24. A midsegment of a triangle is parallel to the third side.

25. Parallel lines form equal corresponding angles.

26. Reflexive.

27. A midsegment of a triangle is half as long as the third side.

•28. Division.

29. The midpoint of a line segment divides it into two equal segments.

•30. Betweenness of Points Theorem.

31. Substitution.

32. Division.

33. Substitution (they are all equal to $\frac{1}{2}$).

•34. Two triangles are similar if their corresponding angles are equal and their corresponding sides are proportional.

Violin Family.

•35. OC = 56 mm. $56 = r_2 28$; so $r_2 = \frac{56}{28} = 2.0$.

36. OD = 87 mm. $87 = r_3 28$; so $r_3 = \frac{87}{28} \approx 3.1$.

•37. BF = 24 mm. $\frac{BF}{AE} = \frac{24}{21} \approx 1.1$.

38. CG = 42 mm. $\frac{CG}{AE} = \frac{42}{21} = 2.0$.

39. DH = 65 mm. $\frac{DH}{AE} = \frac{65}{21} \approx 3.1$.

40. Viola: $1.1 \times 24 \approx 26$ inches.
Cello: $2.0 \times 24 = 48$ inches.
Bass: $3.1 \times 24 \approx 74$ inches.

Paper Sizes.

41. $\frac{1188}{x} = \frac{x}{594}$.

•42. Approximately 840 mm. ($x^2 = 1188 \times 594$, $x \approx 840$.)

43. The geometric mean.

•44. Approximately 1 m^2.
(1188 mm × 840 mm = 997,920 $mm^2 \approx 1\ m^2$.)

45. Approximately 420 mm. ($\frac{x}{2} \approx \frac{840}{2} = 420$).

46. $\frac{r}{1} = \frac{1}{\frac{r}{2}}$.

•47. $\frac{r^2}{2} = 1$, $r^2 = 2$, $r = \sqrt{2}$; so the exact ratio of the length to width is $\sqrt{2}$.

Picture Frames.

48. $\frac{28+2w}{28} = \frac{20+2w}{20}$.

•49. $\frac{14+w}{14} = \frac{10+w}{10}$, $140 + 10w = 140 + 14w$, $0 = 4w$, $w = 0$.

50. $\frac{a+2w}{a} = \frac{b+2w}{b}$, $ab + 2bw = ab + 2aw$, $2bw = 2aw$, $bw = aw$. If $w \neq 0$, then $a = b$.

51. They mean that it is only possible for a rectangular picture to be surrounded by a frame of constant width whose outer edge is similar to it if the picture is square.

Set III (page 391)

The Villa Foscari, also called "La Malcontenta," is located on the Brenta River near Venice. It was commissioned by the brothers Nicolo and Alvise Foscari and is still in the possession of the Foscari family.

Similar Rectangles.

1. *Ratio 1:2.*

   Dimensions 1 × 2:
   HION, JKQP, NOUT, PQWV.

   Dimensions 1.5 × 3:
   ACIG, BDJH, CEKI, DFLJ.

   Dimensions 2 × 4:
   GHTS, IJVU, KLXW, HKQN, NQWT.

   Dimensions 4 × 8:
   GLXS.

2. *Ratio 1:4.*

   Dimensions 1 × 4:
   HIUT, JKWV.

   Dimensions 1.5 × 6:
   AEKG, BFLH.

   Dimensions 2 × 8:
   GLRM, MRXS.

3. *Ratio 2:3.*

   Dimensions 1 × 1.5:
   BCIH, DEKJ.

   Dimensions 2 × 3:
   GIOM, HJPN, IKQO, JLRP, MOUS, NPVT, OQWU, PRXV.

   Dimensions 4 × 6:
   GKWS, HLXT.

4. *Ratio 3:4.*

   Dimensions 1.5 × 2:
   ABHG, BDJI, CEKI, DFLJ.

   Dimensions 3 × 4:
   GIUS, HJVT, IKWU, JLXV.

Chapter 10, Lesson 3

Set I (pages 394–396)

Mathematics is truly a universal language. Although the words in the Turkish version of the Side-Splitter Theorem are incomprehensible to someone who does not know Turkish, the figure and symbolic statements about it are recognizable everywhere.

Descartes's method for multiplying two numbers geometrically appeared in Book I, "Problems the Construction of Which Requires Only Straight Lines and Circles," of his *Géométrie*, published as Appendix I to his *Discours de la méthode* (1637). As Descartes described the construction: "Taking one line which I shall call the unit in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to the unit (which is the same as multiplication)." More on this subject can be found in *A Source Book in Mathematics, 1200–1800*, edited by D. J. Struik (Harvard University Press, 1969).

In his book titled *Perspective in Perspective* (Routledge & Kegan Paul, 1983), Lawrence Wright describes the various ways in which artists have drawn a cube, ranging from a mere square to "spherical perspective." Although the "two-point perspective" version used for exercises 22 through 25 was the basis for perspective drawings in the Renaissance, Wright observes: "None of the three faces shown is square, the overall shape is not even partly square, none of the twelve angles seen is a right angle, and of the three sets of parallels only the verticals are shown as such."

Turkish Theorem.

1. "Paralel" and "hipotez" obviously mean "parallel" and "hypothesis." "Distan" seems to mean "distance." ("Hüküm" evidently means "conclusion," but it is unrecognizable as such.)

•2. If a line parallel to one side of a triangle intersects the other two sides in different points.

3. It divides the sides in the same ratio.

Errors of Omission.

4. *Example answer:*



5. *Example answer:*



6. *Example answer:*



Picturing Products.

•7. The area of the rectangle.

•8. As lengths: AC = 3, AD = 2, and AE = 6.

9. $\frac{1}{a} = \frac{b}{c}$.

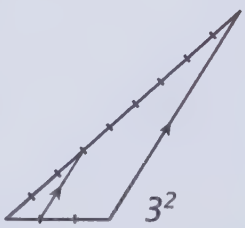
10. $c = ab$.

11.



3^2

12.



Supply and Demand.

•13. $\frac{AC}{CO} = \frac{AP}{PB}$.

14. $\frac{CO}{AO} = \frac{PB}{AB}$.

15. $\frac{OD}{DB} = \frac{AP}{PB}$.

16. $\frac{OD}{OB} = \frac{AP}{AB}$.

•17. Substitution. (Both are equal to $\frac{AP}{PB}$.)

Side-Splitter Practice.

•18. $x = 2.4$. ($\frac{x}{4} = \frac{3}{5}$, $5x = 12$, $x = 2.4$.)

19. $x = 8$. ($\frac{x}{4} = \frac{16}{x}$, $x^2 = 64$, $x = 8$.)

20. $x = 7.5$. ($\frac{3}{x} = \frac{4}{10}$, $4x = 30$, $x = 7.5$.)

21. $x = 10$. ($\frac{x}{24} = \frac{15}{36}$, $36x = 360$, $x = 10$.)

Two-Point Perspective.

•22. Reasonable because BE || CF in ΔOCF (the Side-Splitter Theorem).

23. Not reasonable, because lines AD and BC intersect in O.

24. Reasonable because of division (BC = CD and EF = FG).

25. Reasonable because DG || CF in ΔPCF (the corollary to the Side-Splitter Theorem).

Set II (pages 396–398)

The Dames Point Bridge is the longest cable-stayed bridge in the United States and the only one to feature the "harp," or parallel-stay, arrangement on two vertical planes. Completed in 1989, it is 2 miles long.

Parallelogram Exercise.

•26. It is a parallelogram because DF || EB and DF = EB.

27. XE || YB in ΔAYB and FY || DX in ΔCDX. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

28. AX = XY = YC. Because AE = EB, $\frac{AE}{EB} = 1$; so $\frac{AX}{XY} = 1$ and AX = XY. Because DF = FC, $\frac{DF}{FC} = 1$; so $\frac{XY}{YC} = 1$ and XY = YC.

Corollary to Theorem 44.

•29. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

30. Multiplication. (In a proportion, the product of the means is equal to the product of the extremes.)

31. Addition.

32. Substitution (factoring).

33. Division (and substitution).

Parallels Path.

34. If a line parallel to one side of a triangle intersects the other two sides in different points, it cuts off segments proportional to the sides.

35. $\frac{AM}{MN} = \frac{BM}{MO} = \frac{CN}{NO} = \frac{DN}{MN} = \frac{EO}{MO} = \frac{FO}{NO} = \frac{GM}{MN}$.

•36. EO. (Because $\frac{BM}{MO} = \frac{EO}{MO}$.)

37. FO. (Because $\frac{CN}{NO} = \frac{FO}{NO}$.)

38. Because $\frac{AM}{MN} = \frac{GM}{MN}$, AM = GM by multiplication.

•39. Point G is the same point as point A.

40. It would retrace itself.

Bridge Cables.

•41. EF = FG; so $\frac{EF}{FG} = 1$ by division.

42. $\frac{EH}{HI} = 1$. In ΔIEG, HF || IG; so, by the Side-Splitter Theorem, $\frac{EH}{HI} = \frac{EF}{FG}$.

•43. In a plane, two lines perpendicular to a third line are parallel.

44. ABHI and BCEH are parallelograms (both pairs of their opposite sides are parallel).

•45. The opposite sides of a parallelogram are equal.

46. $\frac{BC}{AB} = 1$. Because $\frac{EH}{HI} = 1$, BC = EH and AB = HI, $\frac{BC}{AB} = 1$ by substitution.

47. Because $\frac{BC}{AB} = 1$, AB = BC by multiplication.

Original Proofs.

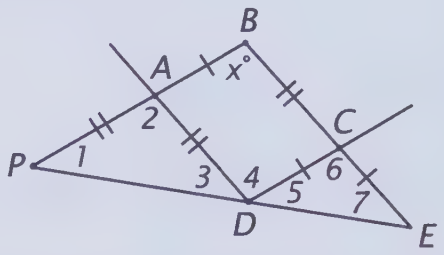
48. *Proof. (One possibility.)*
(1) In ΔABC, AD bisects ∠BAC. (Given.)
(2) ∠BAD = ∠DAC. (If an angle is bisected, it is divided into two equal angles.)
(3) AE = ED. (Given.)
(4) ∠EDA = ∠DAC. (If two sides of a triangle are equal, the angles opposite them are equal.)
(5) ∠BAD = ∠EDA. (Substitution.)
(6) ED || AB. (Equal alternate interior angles mean that lines are parallel.)
(7) $\frac{AE}{EC} = \frac{BD}{DC}$. (If a line parallel to one side of a triangle intersects the other two sides in different points, it divides them in the same ratio.)

49. *Proof.*
(1) In ΔABC, AB = AC and DE || BC. (Given.)
(2) $\frac{AD}{AB} = \frac{AE}{AC}$. (If a line parallel to one side of a triangle intersects the other two sides in different points, it cuts off segments proportional to the sides.)
(3) $\frac{AD}{AB} = \frac{AE}{AB}$. (Substitution.)
(4) AD = AE. (Multiplication.)

Set III (page 398)

According to an article on the reproducing pantograph by Jack W. Jacobsen (www.carousels.com), "the reproducing pantograph first saw use as an early copying machine, making exact duplicates of written documents. Artists soon adopted its use to duplicate drawings. It is known that da Vinci used one to make duplicates of his drawings and possibly to duplicate those drawings onto canvas. . . . It was not long before sculptors and carvers adapted the pantograph's use for tracing drawings onto blocks of marble or wood. They would then use the reproduced lines as guidelines for carving. True advancement in the pantograph design came about late in the 18th century with the advent of typeset printing. A pantograph was used to cut out the typeset letters. . . ."

Pantograph.

1. 

2. ABCD is always a parallelogram because both pairs of its opposite sides are equal.

3. ∠2, ∠4, and ∠6.

4. $\angle 1 = \angle 3 = \angle 5 = \angle 7 = (\frac{180 - x}{2})°$ or $(90 - \frac{x}{2})°$.

5. 180°.

6. 180°. (∠4 = ∠2 and ∠5 = ∠1; so ∠3 + ∠4 + ∠5 = ∠3 + ∠2 + ∠1 by substitution).

7. It is always a straight angle.

8. They are always collinear.

9. $\frac{PD}{DE}$ always stays the same because $\frac{PD}{DE} = \frac{PA}{AB} = \frac{BC}{CE}$ and PA, AB, BC, and CE are fixed lengths.

10. $\frac{PE}{PD} = \frac{PB}{PA} = \frac{35\text{ cm}}{20\text{ cm}} = 1.75$.

Chapter 10, Lesson 4

Set I (pages 401–403)

It is a remarkable thing that, if two similar triangles are in *any* position but have the same orientation, the "average" triangle formed by taking the midpoints of the line segments

connecting their corresponding vertices is similar to them. David Wells points out in *The Penguin Dictionary of Curious and Interesting Geometry* (Penguin Books, 1991) that "the same is true of polygons in general. It is also true if, instead of taking the midpoints of the lines, they are just divided in the same ratio."

The apple on the stamp honoring Sir Isaac Newton symbolizes the story that an apple falling from a tree led Newton to the idea of universal gravitation by getting him to thinking that, if the earth pulls on an apple, it also pulls on more distant objects such as the moon. (The moon doesn't fall as does the apple, because it falls *around* the earth. Its sideways motion makes it miss.) The geometric figure on the stamp is from Newton's *Principia Mathematica*. As William Dunham remarks in *The Mathematical Universe* (Wiley, 1994), "This work presented Newtonian mechanics in a precise, careful, and mathematical fashion. In it he introduced the laws of motion and the principle of universal gravitation and deduced, mathematically, everything from tidal flows to planetary orbits. *Principia Mathematica* is regarded by many as the greatest scientific book ever written."

The figure for exercises 9 through 11 is more remarkable than it might seem. As recently as 1972, C. Stanley Ogilvy observed in *Tomorrow's Math* (Oxford University Press) that, although a way to dissect a right triangle into five triangles similar to itself was known, a way to do this for nonright triangles was not. The division of the 30°-30°-120° triangle on which exercises 9 through 11 are based was discovered soon afterward and has been proved to be the only such solution. Zalman Usiskin and S. G. Wayment reported this result in an article published in *Mathematics Magazine* titled "Partitioning a Triangle into Five Triangles Similar to It." All of this is reported by Martin Gardner in *Wheels, Life and Other Mathematical Amusements* (W. H. Freeman and Company, 1983).

The chessboard puzzle was first introduced in the Set III exercises of Chapter 9, Lesson 4.

Two entire chapters of J. V. Field's *The Invention of Infinity* (Oxford University Press, 1997) deal with Piero della Francesca's mathematics and his treatise on perspective, a work strongly influenced by Euclid.

Thales' Method.

•1. Parallel lines form equal corresponding angles.

2. All right angles are equal.

•3. AA.

4. Corresponding sides of similar triangles are proportional.

Triangle Average.

5. They appear to be their midpoints.

•6. Yes. Two triangles similar to a third triangle are similar to each other.

Newton's Figure.

7. $\Delta PRI \sim \Delta PDF$ and $\Delta PQI \sim \Delta PES$. (Also, $\Delta PDF \sim \Delta SEF$, from which it follows that $\Delta PRI \sim \Delta SEF$.)

8. AA. (All these triangles are right triangles. The triangles in the first two pairs share acute angles at P. The triangles in the third pair have equal vertical angles at F. The fourth pair follows from the fact that two triangles similar to a third triangle are similar to each other.)

Nine Triangles.

9. ΔABC, ΔADC, ΔBCE, ΔCDF, ΔCEF, ΔDEF.

•10. 30°, 30°, and 120°. (It is easiest to see this by starting with the three equal angles at F.)

11. ΔCDE is equiangular (and equilateral). ΔACE and ΔBCD are right triangles with acute angles of 30° and 60°.

Chessboard Puzzle.

12. The square has an area of 64 units and the other arrangement seems to have an area of 65 units.

13. AA. [$\angle CAE = \angle DAF$ (reflexive), and $\angle AEC = \angle AFD$ (all right angles are equal.]

•14. Corresponding sides of similar triangles are proportional.

15. EC = 3 and FD = 5.

16. No. $\frac{8}{13} \neq \frac{3}{5}$.

•17. They are not collinear.

18. Indirect.

Piero's Theorem.

•19. That they are parallel.

20. That lines contain the same point.

•21. $\Delta AHK \sim \Delta ABD$ and $\Delta AKL \sim \Delta ADE$.

22. $\frac{HK}{BD} = \frac{AK}{AD}$ and $\frac{AK}{AD} = \frac{KL}{DE}$.

•23. $\frac{HK}{BD} = \frac{KL}{DE}$.

24. $\frac{LM}{EF} = \frac{MN}{FG} = \frac{NI}{GC}$.

Set II (pages 403–405)

Students who have solved "work problems" in algebra may be interested in knowing that the figure illustrating two electrical resistances in parallel also illustrates problems with two workers working simultaneously. The equation $R = \frac{R_1R_2}{R_1+R_2}$ is equivalent to the equation $\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{R}$, a form more familiar in the work-problem context: R_1 and R_2 become the times needed for the two workers to complete the job individually and R the time for them to complete it working together.

Exercises 31 through 36 are based on an interesting fact that is not well known. Although the angles of a 3-4-5 triangle are not related in any interesting way, the largest angle of a 4-5-6 triangle is exactly twice as large as its smallest angle.

Mascheroni's work *Geometria del Compasso*, in which he proved that every straightedge and compass construction can be done with an adjustable compass alone, was published in 1797. For more on Mascheroni constructions, see chapter 17 of Martin Gardner's *Mathematical Circus* (Knopf, 1979).

Electrician's Formula.

25. AA.

26. Corresponding sides of similar triangles are proportional.

•27. Addition.

•28. Multiplication.

29. Substitution (factoring).

30. Division.

Sides and Angles.

•31. $\Delta ABD \sim \Delta CBA$.

32. $\frac{AB}{CB} = \frac{BD}{BA} = \frac{AD}{CA}$.

33. $\frac{12}{18} = \frac{x}{12} = \frac{y}{15}$.

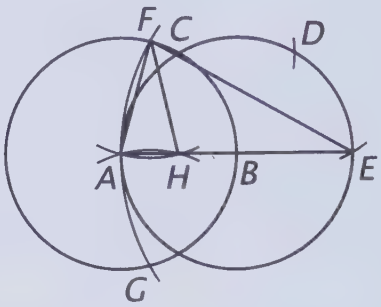
•34. $x = 8$, $y = 10$, and $z = 10$. ($z = 18 - x$.)

35. $\angle 2 = \angle C$. ($y = 10 = z$.)

36. $\angle BAC = 2\angle C$. ($\angle BAC = \angle 1 + \angle 2$, $\angle 1 = \angle C$, $\angle 2 = \angle C$; so $\angle BAC = \angle C + \angle C$.)

Mascheroni Construction.

37.



38. Isosceles.

•39. AA ($\angle A = \angle EFA$ and $\angle FHA = \angle A$).

40. Corresponding sides of similar triangles are proportional.

41. It is equal to $\frac{1}{2}$. $FH = FA = AB = \frac{1}{2}AE$; so $\frac{FH}{AE} = \frac{1}{2}$.

42. $\frac{AH}{FA} = \frac{FH}{EA} = \frac{1}{2}$ (exercises 40 and 41); so $\frac{AH}{FA} = \frac{1}{2}$ and $AH = \frac{1}{2}FA$. Because $FA = AB$, $AH = \frac{1}{2}AB$ by substitution.

Deja Vu.

43. AA ($\angle A = \angle A$ and $\angle C = \angle ADE$).

44. $\frac{BC}{ED} = \frac{AC}{AD}$.

•45. $\frac{a}{s} = \frac{b}{b-s}$.

46. $ab - as = bs$; so $ab = as + bs$; so $s(a + b) = ab$; so $s = \frac{ab}{a+b}$.

47. It represents a single resistance to which resistances a and b are equivalent.

Original Proofs.

48. *Proof.*
    (1) ΔACD with $BE \parallel CD$. (Given.)
    (2) $\angle ABE = \angle C$ and $\angle AEB = \angle D$. (Parallel lines form equal corresponding angles.)
    (3) $\Delta ABE \sim \Delta ACD$. (AA.)

49. *Proof.*
    (1) Trapezoid ABCD with bases AB and DC and diagonals AC and BD meeting at E. (Given.)
    (2) $AB \parallel DC$. (The bases of a trapezoid are parallel.)
    (3) $\angle BAC = \angle ACD$ and $\angle ABD = \angle BDC$. (Parallel lines form equal alternate interior angles.)
    (4) $\Delta AEB \sim \Delta CED$. (AA.)
    (5) $\frac{AE}{CE} = \frac{EB}{ED}$. (Corresponding sides of similar triangles are proportional.)
    (6) $AE \times ED = BE \times EC$. (Multiplication.)

50. *Proof.*
    (1) ΔABC with midsegments MN, MO, and NO. (Given.)
    (2) $MN \parallel AB$, $NO \parallel BC$, and $MO \parallel AC$. (A midsegment of a triangle is parallel to the third side.)
    (3) ANMO and BMNO are parallelograms. (A quadrilateral is a parallelogram if both pairs of opposite sides are parallel.)
    (4) $\angle OMN = \angle A$ and $\angle ONM = \angle B$. (The opposite angles of a parallelogram are equal.)
    (5) $\Delta MNO \sim \Delta ABC$. (AA.)

Set III (page 406)

Dividing Line.

1. 4.5 square units each.

2. The area of the purple region is $\frac{1}{2}(4 \times 2) + 1 = 5$ square units; so the area of the yellow region is $9 - 5 = 4$ square units.

3. AA ($\angle DBQ = \angle CBA$ and $\angle QDB = \angle C$).

4. Corresponding sides of similar triangles are proportional.

5. $QD = \frac{1}{4}$. Because $\frac{QD}{AC} = \frac{DB}{CB}$, $\frac{QD}{1} = \frac{1}{4}$.

6. The area of the purple region is $\frac{1}{2}(4 \times 1.75) + 1 = 4.5$ square units.

7. Yes.

Chapter 10, Lesson 5

Set I (pages 409–410)

Exercises 25 through 30 are a good basis for further exploration. After confirming the answers to exercises 29 and 30, it might be fun to experiment with other linear transformations of the coordinates; that is, $(a, b) \rightarrow (ca + e, db + f)$, to see what choices of c, d, e, and f produce dilations and what sorts of transformations are produced when $c \neq d$.

Fish Story.

•1. CG and CF.

2. In a plane, two lines perpendicular to a third line are parallel.

3. $\Delta CDE \sim \Delta CAB$ by AA ($\angle CDE = \angle A$ and $\angle CED = \angle B$ because parallel lines form equal corresponding angles.)

•4. Corresponding altitudes.

5. $\frac{CG}{CF} = \frac{DE}{AB}$.

6. Corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides.

•7. CG, the distance of the fish from the camera.

8. The fish is 1 foot long. ($\frac{2}{12} = \frac{DE}{6}$, $DE = 1$.)

9. The center.

•10. 7.5 cm.

11. 3 cm.

•12. 2.5. ($r_1 = \frac{BE}{DE} = \frac{7.5}{3}$.)

13. It indicates that the larger fish is 2.5 times as long as the smaller fish.

14. AB = 5 cm. CD = 2 cm. $\frac{AB}{CD} = \frac{5}{2} = 2.5$.

•15. 0.4. ($r_2 = \frac{DE}{BE} = \frac{3}{7.5}$.)

16. It indicates that the smaller fish is 0.4 times as long as the larger fish.

17. $\frac{CD}{AB} = \frac{2}{5} = 0.4$.

•18. r_2 is the reciprocal of r_1.

Drawing Conclusions.

19. No. ΔAGB appears to be larger than ΔDGE. Also, we do not know that any sides are equal.

20. Yes. The triangles are similar by AA. (The right angles at G and the alternate interior angles formed by AB and ED.)

•21. Yes. Corresponding sides of similar triangles are proportional.

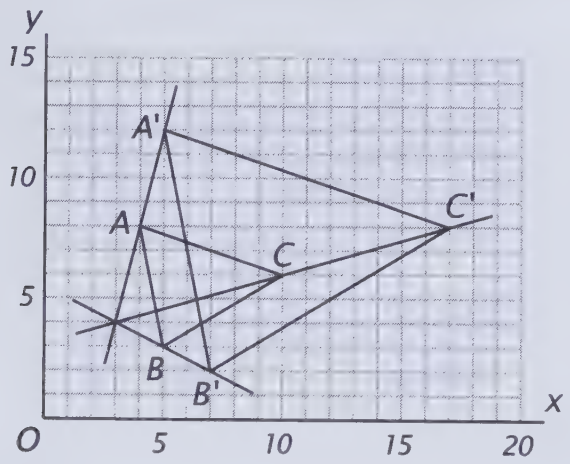
22. Yes. Because $\frac{AG}{DG} = \frac{GB}{GE}$, AG × GE = GB × DG.

•23. No. These areas cannot be equal if ΔAGB is larger.

24. Yes. Adding αΔDGE to each side of αAGEF = αBGDC gives αADEF = αBEDC.

Grid Exercise.

25.



•26. B(5, 3) → B′(2 · 5 − 3, 2 · 3 − 4), or B′(7, 2). C(10, 6) → C′(2 · 10 − 3, 2 · 6 − 4), or C′(17, 8).

•27. ΔA′B′C′ appears to be an enlargement of ΔABC. (Also, the corresponding sides of the two triangles appear to be parallel.)

28. A dilation.

29. (3, 4).

30. 2.

Set II (pages 410–413)

The story of the Washington Monument is told by former engineering professor Robert Banks in *Slicing Pizzas, Racing Turtles, and Further Adventures in Applied Mathematics* (Princeton University Press, 1999). A summary of the story follows: The monument, 555 feet 5 inches in height, is made of marble and granite. The cornerstone for its construction was laid on July 4, 1848, when James Polk was president. As originally planned, the tower was not tapered and would have had a star rather than a pyramid at its top. After 5 years, it had risen to a height of 152 feet, but then the money ran out and construction didn't start up again until 1880. It was completed in 1884. As Banks points out, if the monument were not capped with the pyramid but extended until the tapered sides of the main column met at a point, it would be more than twice as tall, taller than the Empire State Building.

The rest of the exercises in Set II consider the SAS and SSS similarity theorems, which are nice to know but have been omitted from the main sequence of theorems. In most applications of similar triangles, it is angles that we know to be equal rather than sides being proportional. It is unfortunate that the proofs of these theorems are so cumbersome with so many algebraic steps. As is often the case with proofs of this nature, after the stage has been set with the extra line forming the third triangle, it is probably easier to construct the proofs rather than to read them.

Washington Monument.

•31. 555.4 ft.

32. ΔAGF ~ ΔEHF, ΔBGF ~ ΔCHF, and ΔABF ~ ΔECF.

33. $\frac{FG}{FH} = \frac{AB}{EC}$, $\frac{x}{x - 500.4} = \frac{55}{34.4}$,
$34.4x = 55x - 27{,}522$, $20.6x = 27{,}522$,
$x \approx 1{,}336$ ft.

•34. Corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides.

Congruence and Similarity.

•35. ASA.

•36. AA.

37. AAS.

38. AA.

39. Yes. If $\frac{a}{b} = \frac{c}{d} = 1$, then $a = b$ and $c = d$. The triangles are congruent by SAS.

SAS Similarity Theorem.

•40. Ruler Postulate.

•41. Through a point not on a line, there is exactly one line parallel to the given line.

42. A line parallel to one side of a triangle cuts off segments on the other two sides proportional to the sides.

43. Substitution.

44. Multiplication.

45. SAS.

46. Corresponding parts of congruent triangles are equal.

•47. Parallel lines form equal corresponding angles.

48. Substitution.

•49. AA.

50. Yes. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = 1$, then $a = b$, $c = d$, and $e = f$. The triangles are congruent by SSS.

SSS Similarity Theorem.

51. Ruler Postulate.

52. Two points determine a line.

•53. Reflexive.

54. If an angle of one triangle is equal to an angle of another triangle and the sides including these angles are proportional, then the triangles are similar.

•55. Corresponding sides of similar triangles are proportional.

56. Substitution.

57. Multiplication.

•58. SSS Congruence Theorem.

59. Corresponding parts of congruent triangles are equal.

60. AA Similarity Theorem.

61. Two triangles similar to a third triangle are similar to each other.

Set III (page 413)

Camera Experiment.

1. It gets larger.

2. If $\Delta AHB \sim \Delta CHD$, $\frac{x}{y} = \frac{AB}{CD}$. So $x\,CD = y\,AB$, and $CD = \frac{y}{x} AB$.

3. It gets bigger.

4. It gets smaller.

5. It gets bigger.

6. It gets bigger.

Chapter 10, Lesson 6

Set I (pages 416–417)

Judo, which means "gentle way" in Japanese, didn't become an Olympic sport until 1972. The three regions of its competition area, although square in shape, are covered with "tatami," rectangular mats 1 meter wide and 2 meters long. (Tatami are the same mats used by Japanese architects in creating floor plans.) A consequence of the use of these mats is that, when the judo competition area was first designed, its inner space, the "contest area," had to have sides that were an even number of meters. Interested students might enjoy exploring this requirement and explaining why.

Lewis Carroll included whimsical references to maps in two of his books. In *The Hunting of the Snark* (1876), he wrote: "He had brought a large map representing the sea, Without the least vestige of land: And the crew were much pleased

when they found it to be A map they could all understand." Martin Gardner remarks in *Lewis Carroll's The Hunting of the Snark—Centennial Edition* (William Kaufman, Inc., 1981): "In contrast, a map in Carroll's *Sylvie and Bruno Concluded*, Chapter 11, has *everything* on it. The German Professor explains how his country's cartographers experimented with larger and larger maps until they finally made one with a scale of a mile to the mile."

Triangle Ratios.

•1. $\frac{1}{2}$.

•2. $\frac{1}{4}$.

3. $\frac{1}{3}$.

4. $\frac{1}{9}$.

5. $\frac{2}{3}$.

6. $\frac{4}{9}$.

7. $\frac{1}{2}$. ($\frac{2}{4}$.)

8. $\frac{1}{4}$. ($\frac{4}{16}$.)

9. The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides.

Judo Mat.

•10. side of 1 = 8 m.

•11. $\rho 1 = 32$ m.

•12. $\alpha 1 = 64$ m^2.

13. side of 2 = 10 m.

14. $\rho 2 = 40$ m.

15. $\alpha 2 = 100$ m^2.

16. side of 16 m.

17. $\rho 3 = 64$ m.

18. $\alpha 3 = 256$ m^2.

•19. $\frac{4}{5}$. ($\frac{\text{side of 1}}{\text{side of 2}} = \frac{8}{10}$.)

20. $\frac{4}{5}$. ($\frac{\rho 1}{\rho 2} = \frac{32}{40}$.)

•21. $\frac{16}{25}$. ($\frac{\alpha 1}{\alpha 2} = \frac{64}{100}$.)

22. $\frac{5}{8}$. ($\frac{\text{side of 2}}{\text{side of 3}} = \frac{10}{16}$.)

23. $\frac{5}{8}$. ($\frac{\rho 2}{\rho 3} = \frac{40}{64}$.)

24. $\frac{25}{64}$. ($\frac{\alpha 2}{\alpha 3} = \frac{100}{256}$.)

Map Scaling.

•25. They must be proportional.

26. They must be equal.

27. They must be proportional to the square of the ratio of the corresponding distances.

•28. 440. (The scale is $\frac{1}{5{,}280}$, and so 1 inch represents 5,280 inches; $\frac{5{,}280 \text{ in}}{12} = 440$ ft.)

29. 193,600. [(440 ft)^2 = 193,600 ft^2.]

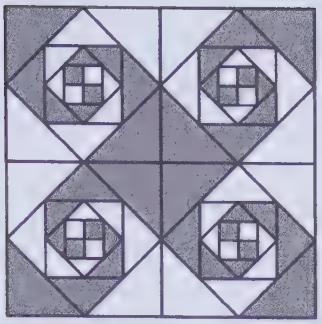
•30. $\frac{1}{63{,}360}$. ($\frac{1 \text{ inch}}{5{,}280 \times 12 \text{ inches}}$.)

31. $\frac{1}{1}$ or 1.

Set II (pages 417–419)

The question asked in the SAT problem of exercises 32 through 36 is the area of ΔADE, for which four numerical choices were listed, followed by the choice, "It cannot be determined from the information given." This choice, though incorrect, is somewhat tempting in that there is no way to know the lengths of ΔADE's sides.

Cranberries are grown commercially in level bog areas that can be drained. When the berries are ready to harvest, the bog is flooded and they are shaken loose from the vines by tractors. They then float to the surface where they are corralled as shown in the photograph before being transported.



The pattern used by Barbara Dean in designing her quilt is called the "Virginia Reel." Copies of it are turned through 90°, 180°, and 270° to form the four-fold rotation-symmetric design shown above. Jinny Beyer presents this design and others in Chapter 9, "Creating Geometric Tessellations," of her beautiful book titled *Designing Tessellations* (Contemporary Books, 1998).

SAT Problem.

•32. Yes. $\Delta ADE \sim \Delta ABC$ by AA.

33. Yes. Corresponding sides of similar triangles are proportional.

•34. No.

35. No.

36. Yes. The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides; so

$$\frac{\alpha\Delta ADE}{\alpha\Delta ABC} = (\frac{1}{3})^2.$$

$$\alpha\Delta ADE = \frac{1}{9}\alpha\Delta ABC = \frac{1}{9}(54) = 6.$$

Quilt Pattern.

•37. Six.

38. $\frac{1}{4}$.

39. $\frac{1}{2}$.

•40. $\frac{1}{2}$.

41. $\frac{2}{1}$ or 2.

42. $\sqrt{\frac{2}{1}} = \frac{\sqrt{2}}{1}$ or $\sqrt{2}$.

43. $\frac{\sqrt{2}}{1}$ or $\sqrt{2}$.

Cranberry Circles.

•44. Because all circles have the same shape.

45. 45 ft and 60 ft.

•46. $90\pi \approx 283$ ft and $120\pi \approx 377$ ft.

47. $2{,}025\pi \approx 6{,}362$ ft^2 and $3{,}600\pi \approx 11{,}310$ ft^2.

•48. $\frac{3}{4} = 0.75$. ($\frac{90}{120}$.)

49. $\frac{3}{4} = 0.75$. ($\frac{45}{60}$.)

50. $\frac{3}{4} = 0.75$. ($\frac{90\pi}{120\pi}$.)

51. $\frac{9}{16} = 0.5625$. [$\frac{2{,}025\pi}{3{,}600\pi}$ or $(\frac{3}{4})^2$.]

Pentagon Measurements.

52. 108°.

•53. 921 ft. (The scale is $\frac{1}{11{,}052}$; so 1 inch represents 11,052 in; $\frac{11{,}052 \text{ in}}{12} = 921$ ft.)

54. 4,605 ft. (5×921.)

55. Approximately 29 acres.

[$\frac{1.5 \text{ in}^2}{x \text{ in}^2} = (\frac{1}{11{,}052})^2$, $x = 183{,}220{,}056$ in^2,

$\frac{183{,}220{,}056 \text{ in}^2}{144} = 1{,}272{,}361.5$ ft^2,

$\frac{1{,}272{,}361.5 \text{ ft}^2}{43{,}560} \approx 29$ acres.]

Set III (page 419)

This exercise is based on Proposition 31 of Book VI of the *Elements*. Euclid stated it as follows: "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle."

Pythagoras on the Sides.

1. The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides.

2. Adding $\frac{\alpha\,\mathrm{Py}.a}{\alpha\,\mathrm{Py}.c} = (\frac{a}{c})^2$ and $\frac{\alpha\,\mathrm{Py}.b}{\alpha\,\mathrm{Py}.c} = (\frac{b}{c})^2$ gives $\frac{\alpha\,\mathrm{Py}.a}{\alpha\,\mathrm{Py}.c} + \frac{\alpha\,\mathrm{Py}.b}{\alpha\,\mathrm{Py}.c} = (\frac{a}{c})^2 + (\frac{b}{c})^2$, or $\frac{\alpha\,\mathrm{Py}.a + \alpha\,\mathrm{Py}.b}{\alpha\,\mathrm{Py}.c} = \frac{a^2 + b^2}{c^2}$. Because a, b, and c are the lengths of the sides of the right triangle, $a^2 + b^2 = c^2$; so $\frac{\alpha\,\mathrm{Py}.a + \alpha\,\mathrm{Py}.b}{\alpha\,\mathrm{Py}.c} = 1$. Multiplying, $\alpha\mathrm{Py}.a + \alpha\mathrm{Py}.b = \alpha\mathrm{Py}.c$.

Chapter 10, Review

Set I (pages 421–423)

J. L. Heilbron describes Galileo's figure for objects in free fall in *Geometry Civilized* (Clarendon Press, 1998): "After many false starts, he [Galileo] discovered that the fundamental characteristic of free fall is constant acceleration, which means that the velocity with which the body moves at any time is proportional to the elapsed time. . . . Galileo derived his rule relating distance and time using geometry. . . . The angle EOF is drawn so that AC = gOA [where g is a constant number that measures the pull of gravity]. . . . In a moment of great insight, he identified the *areas* of the triangles AOC, BOD, with the *distances* covered by the falling body in the times t_A, t_B." Heilbron goes on to explain the rest of Galileo's argument. By associating distances covered with areas of triangles, Galileo reasoned that, because $\mathrm{OA} = t_A$ and $\mathrm{AC} = g\mathrm{OA} = gt_A$,

$$d_A = \alpha\Delta\mathrm{AOC} = \frac{1}{2}\mathrm{OA} \cdot \mathrm{AC} = \frac{1}{2}t_A(gt_A) = \frac{1}{2}gt_A^2.$$

Because A is an arbitary constant, it follows that $d = \frac{1}{2}gt^2$.

Body Ratios.

•1. $\frac{1}{2}$. ($\frac{2}{4}$.)

2. $\frac{1}{5}$.

3. $\frac{1}{3}$. ($\frac{2}{6}$.)

4. $\frac{5}{12}$. ($\frac{2.5}{6}$.)

Perspective Law.

5. If a line parallel to one side of a triangle intersects the other two sides in different points, it cuts off segments proportional to the sides.

6. AA. ($\angle A = \angle A$ and the corresponding angles formed by BC and DE with AD and AE are equal.)

7. AC = 2 cm, AE = 5 cm, DE = 4 cm.

8. $\frac{\mathrm{AC}}{\mathrm{AE}} = \frac{\mathrm{BC}}{\mathrm{DE}}$; so $\frac{2}{5} = \frac{\mathrm{BC}}{4}$, 5BC = 8, BC = 1.6.

•9. 2.5. ($\frac{\mathrm{DE}}{\mathrm{BC}} = \frac{4\text{ cm}}{1.6\text{ cm}} = 2.5$.)

•10. It is 6.25 times as great. ($2.5^2 = 6.25$.)

Proportion Practice.

11. 8. ($\frac{x}{12} = \frac{10}{15}$, $15x = 120$, $x = 8$.)

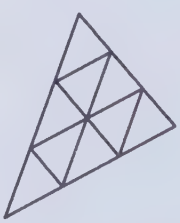
•12. 6.4. ($\frac{x}{4} = \frac{8}{5}$, $5x = 32$, $x = 6.4$.)

13. 5.6. ($\frac{x}{8} = \frac{7}{10}$, $10x = 56$, $x = 5.6$.)

•14. 12.8. ($\frac{x}{8} = \frac{8}{5}$, $5x = 64$, $x = 12.8$.)

Similar Triangles.

15. Nine.

16. 

Reptiles.

•17. Concave pentagons.

•18. ABHIG ≅ EKJGI.

19. AFEDC ~ GABHI.

•20. GFEI is a parallelogram. Both pairs of its opposite sides are equal because GFEKJ ≅ EIGJK.

21. αAFEDC = 4αABHIG.

22. ρAFEDC = 2ρABHIG.

23. $\frac{2}{1}$, or 2.

Free Fall.

24. In a plane, two lines perpendicular to a third line are parallel.

•25. It increases. (For example, BD > AC.)

26. AA. ($\angle O = \angle O$ and $\angle OAC$ and $\angle OBD$ are right angles.)

27. Corresponding sides of similar triangles are proportional.

28. The ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding sides.

Type Transformations.

29. Dilations.

•30. 2. ($\frac{48}{24}$.)

31. $\frac{3}{4}$ or 0.75. ($\frac{72}{96}$.)

•32. They are multiplied by 3. ($\frac{72}{24}$.)

33. They do not change.

Set II (pages 423–424)

The crossed-ladders problem on which exercises 45 through 50 are based is a famous puzzle in recreational mathematics. The puzzle concerns two crossed ladders of unequal length that lean against two buildings as shown in the figure: given the lengths of the ladders and the height of their crossing point, the problem is to find the distance between the two buildings. The puzzle is famous because the solution, based on similar triangles, leads to a difficult quartic equation. Different numbers appear in various versions of the puzzle and the solution is generally not an integer. The simplest set of numbers for which all of the distances in the figure are integers was reported in the *American Mathematical Monthly* in 1941, and these numbers are used in the version of the puzzle presented in the exercises. More on the crossed-ladders problem can be found in Chapter 5 of Martin Gardner's *Mathematical Circus* (Knopf, 1979).

Trick Card.

•34. $\frac{10}{x} = \frac{x}{5}$, $x^2 = 50$, $x = \sqrt{50} = 5\sqrt{2} \approx 7.1$.
The width of the original card is approximately 7.1 cm.

35. $50\sqrt{2} \approx 71$ cm^2.

•36. $\frac{\sqrt{50}}{10} = \frac{5\sqrt{2}}{10} = \frac{\sqrt{2}}{2} \approx 0.71$.

37. $(\frac{\sqrt{50}}{10})^2 = \frac{50}{100} = \frac{1}{2}$.

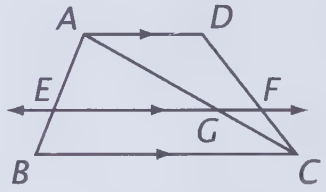
38. Yes. The second card is formed by folding the first card in half.

Catapult.

39. AB = 108. ($\Delta CFD \sim \Delta CAB$ by AA; so $\frac{CF}{CA} = \frac{FD}{AB}$. $\frac{60}{162} = \frac{40}{AB}$, 60AB = 6,480, AB = 108.)

Leg Splitter.

40.



•41. In a plane, two lines parallel to a third line are parallel to each other. (EF || BC and AD || BC.)

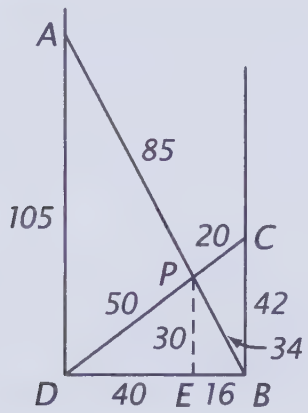
42. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

43. Substitution.

44. They prove that the line divides the legs of the trapezoid in the same ratio.

Crossed Ladders.

45.



•46. 30 ft. ($\Delta PED \sim \Delta CBD$; so $\frac{PE}{42} = \frac{40}{56}$.)

47. 105 ft. ($\Delta ADB \sim \Delta PEB$; so $\frac{AD}{30} = \frac{56}{16}$.)

•48. 70 ft. ($CD^2 = 56^2 + 42^2$; so $CD = \sqrt{4{,}900} = 70$.)

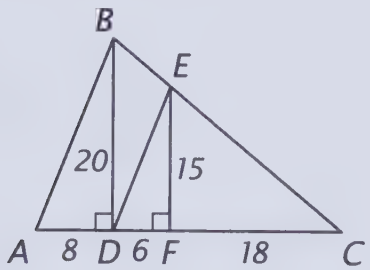
49. 119 ft. ($AB^2 = 105^2 + 56^2$; so $AB = \sqrt{14{,}161} = 119$.)

50. PA = 85, PB = 34, PC = 20, PD = 50.
($PD^2 = 30^2 + 40^2$; so $PD = \sqrt{2{,}500} = 50$.
$PB^2 = 30^2 + 16^2$; so $PB = \sqrt{1{,}156} = 34$.
In ΔADB, PE || AD; so $\frac{PA}{34} = \frac{40}{16}$,
16PA = 1,360, PA = 85.
In ΔCBD, PE || CB; so $\frac{PC}{50} = \frac{16}{40}$, 40PC = 800,
PC = 20.)

Seven Triangles.

51. $\Delta ABD \sim \Delta DEF$, $\Delta ABC \sim \Delta DEC$,
$\Delta BDC \sim \Delta EFC$.

52.



•53. 15. ($\Delta EFD \sim \Delta BDA$; so $\frac{EF}{20} = \frac{6}{8}$, 8EF = 120,
EF = 15.)

54. 18. ($\Delta EFC \sim \Delta BDC$; so $\frac{FC}{6+FC} = \frac{15}{20}$,
20FC = 90 + 15FC, 5FC = 90, FC = 18.)

•55. $\frac{4}{3}$. ($\Delta ABD \sim \Delta DEF$; so $\frac{20}{15} = \frac{8}{6} = \frac{4}{3}$.)

•56. 80. [$\alpha\Delta ABD = \frac{1}{2}(8 \cdot 20)$.]

57. 45. [$\alpha\Delta DEF = \frac{1}{2}(6 \cdot 15)$.]

•58. $\frac{16}{9}$. ($\frac{80}{45}$.)

59. $\frac{1}{3}$. (In Δ BCD, EF || BD; so $\frac{BE}{EC} = \frac{6}{18} = \frac{1}{3}$.)

60. 60. [$\alpha\Delta BED = \frac{1}{2}(20 \cdot 6)$.]

61. 180. [$\alpha\Delta DEC = \frac{1}{2}(24 \cdot 15)$.]

62. $\frac{1}{3}$. ($\frac{60}{180}$.)

Algebra Review 1 (page 425)

•1. $(x + 7)(x - 4) = 0$,
$x + 7 = 0$ or $x - 4 = 0$,
$x = -7$ or $x = 4$,
4 and –7.

•2. $(5x + 1)(x - 1) = 0$,
$5x + 1 = 0$ or $x - 1 = 0$,
$x = -\frac{1}{5}$ or $x = 1$,
1 and $-\frac{1}{5}$.

•3. $x^2 - 8x + 5 = 0$,
$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(5)}}{2(1)} = \frac{8 \pm \sqrt{64 - 20}}{2} =$
$\frac{8 \pm \sqrt{44}}{2} = \frac{8 \pm 2\sqrt{11}}{2} = 4 \pm \sqrt{11}$,
$4 + \sqrt{11}$ and $4 - \sqrt{11}$.

•4. $x^2 - 49 = 0$,
$(x + 7)(x - 7) = 0$,
$x + 7 = 0$ or $x - 7 = 0$,
$x = -7$ or $x = 7$.
7 and –7.

•5. $x^2 + 4x - 20 = 0$,

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-20)}}{2(1)} = \frac{-4 \pm \sqrt{16 - (-80)}}{2}$$

$$= \frac{-4 \pm \sqrt{96}}{2} = \frac{-4 \pm 4\sqrt{6}}{2} = -2 \pm 2\sqrt{6}.$$

$-2 + 2\sqrt{6}$ and $-2 - 2\sqrt{6}$.

•6. $(3x - 1)^2 = 0$,
$3x - 1 = 0$,

$$x = \frac{1}{3}.$$

$\frac{1}{3}$.

•7. $3x^2 + 5x - 12 = 0$,
$(3x - 4)(x + 3) = 0$,
$3x - 4 = 0$ or $x + 3 = 0$,

$$x = \frac{4}{3} \text{ or } x = -3.$$

$\frac{4}{3}$ and –3.

•8. $x^2 - 7x + 12 = 2$,
$x^2 - 7x + 10 = 0$,
$(x - 2)(x - 5) = 0$,
$x - 2 = 0$ or $x - 5 = 0$,
$x = 2$ or $x = 5$.
2 and 5.

•9. $x^2 + 10x + 25 = x + 17$,
$x^2 + 9x + 8 = 0$,
$(x + 1)(x + 8) = 0$,
$x + 1 = 0$ or $x + 8 = 0$,
$x = -1$ or $x = -8$.
–1 and –8.

•10. $5x^2 - 5 = 2x$,
$5x^2 - 2x - 5 = 0$,

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(-5)}}{2(5)} =$$

$$\frac{2 \pm \sqrt{4 - (-100)}}{10} = \frac{2 \pm \sqrt{104}}{10} = \frac{2 \pm 2\sqrt{26}}{10} =$$

$$\frac{1 \pm \sqrt{26}}{5}.$$

$\frac{1 + \sqrt{26}}{5}$ and $\frac{1 - \sqrt{26}}{5}$.

Algebra Review 2 (page 426)

•1. $r = \frac{c}{2\pi}$.

•2. $p_1 = \frac{p_2 v_2}{v_1}$.

•3. $v_2 = \frac{p_1 v_1}{p_2}$.

•4. $c = \sqrt{a^2 + b^2}$.

•5. $c^2 - b^2 = a^2$,
$a = \sqrt{c^2 - b^2}$.

•6. $2A = bh$,

$$h = \frac{2A}{b}.$$

•7. $V^2 = 2gh$,

$$h = \frac{V^2}{2g}.$$

•8. $\frac{A}{4\pi} = r^2$,

$$r = \sqrt{\frac{A}{4\pi}} \text{ or } r = \frac{1}{2}\sqrt{\frac{A}{\pi}}.$$

•9. $IR = E$,

$$R = \frac{E}{I}.$$

•10. $3V = \pi r^2 h$,

$$h = \frac{3V}{\pi r^2}.$$

•11. $S(1-r)=a$,
$a=S(1-r)$ or $a=S-Sr$.

•12. $S(1-r)=a$, or $S(1-r)=a$,
$S-Sr=a$,
$S-a=Sr$, $1-r=\frac{a}{S}$,
$r=\frac{S-a}{S}$ $r=1-\frac{a}{S}$.

•13. $8M=wL^2$,
$w=\frac{8M}{L^2}$.

•14. $W^2=w_1w_2$,
$w_1=\frac{W^2}{w_2}$.

•15. $r=\frac{R}{A-150}$.

•16. $R=rA-150r$, or $\frac{R}{r}=A-150$,
$rA=R+150r$,
$A=\frac{R+150r}{r}$ $A=\frac{R}{r}+150$.

•17. $Ed^2=I$,
$d^2=\frac{I}{E}$,
$d=\sqrt{\frac{I}{E}}$.

•18. $\frac{T}{2\pi}=\sqrt{\frac{l}{g}}$,
$\frac{l}{g}=\frac{T^2}{4\pi^2}$,
$l=\frac{gT^2}{4\pi^2}$.

•19. $I_n-I_x=Ad^2$,
$A=\frac{I_n-I_x}{d^2}$.

•20. $I_n-I_x=Ad^2$,
$\frac{I_n-I_x}{A}=d^2$,
$d=\sqrt{\frac{I_n-I_x}{A}}$.

•21. $r_1r_2=Rr_2+Rr_1$, or $R=\frac{1}{\frac{1}{r_1}+\frac{1}{r_2}}$,
$R(r_1+r_2)=r_1r_2$,
$R=\frac{r_1r_2}{r_1+r_2}$ $R=\frac{r_1r_2}{r_1+r_2}$.

•22. $r_1r_2=Rr_2+Rr_1$, or $\frac{1}{r_1}=\frac{1}{R}-\frac{1}{r_2}$,
$r_1r_2-Rr_1=Rr_2$,
$r_1(r_2-R)=Rr_2$, $r_1=\frac{1}{\frac{1}{R}-\frac{1}{r_2}}$,
$r_1=\frac{Rr_2}{r_2-R}$ $r_1=\frac{Rr_2}{r_2-R}$.

•23. $nur+vr=vu(n-1)$,
$nur+vr=vun-vu$,
$nur-vun=-vr-vu$,
$n(ur-vu)=-(vr+vu)$,
$n=-\frac{vr+vu}{ur-vu}$ or $n=-\frac{v(r+u)}{u(r-v)}$ or
$n=\frac{vr+vu}{vu-ur}$ or $n=\frac{v(r+u)}{u(v-r)}$.

•24. $F-32=\frac{9}{5}C$, or $F-32=\frac{9}{5}C$,
$C=\frac{5}{9}(F-32)$ $9C=5(F-32)$,
$C=\frac{5}{9}(F-32)$.

•25. $\frac{f_0}{f_s}=\sqrt{\frac{c+v}{c-v}}$,
$\frac{f_0^2}{f_s^2}=\frac{c+v}{c-v}$,
$f_0^2(c-v)=f_s^2(c+v)$,
$f_0^2c-f_0^2v=f_s^2c+f_s^2v$,
$f_0^2c-f_s^2c=f_0^2v+f_s^2v$,
$c(f_0^2-f_s^2)=v(f_0^2+f_s^2)$,
$v=c\frac{f_0^2-f_s^2}{f_0^2+f_s^2}$.

Chapter 11, Lesson 1

Set I (pages 430–431)

The chameleon's coiled tail is a pretty good approximation of a logarithmic spiral different from that of the chambered nautilus. Its graph can be obtained from the polar equation, $r = 1.1167^{\theta}$, where θ is the measure of the rotation in radians and $-2\pi \le \theta \le 5\pi$.

Exercises 20 through 30 show a special case of the fact that, in any right triangle, the altitude to the hypotenuse divides the hypotenuse into two segments that have the same ratio as the square of the ratio of the legs. The general case will be considered in a later exercise.

Exercise 31 appears in the *Elements* as Proposition 17 of Book VI: "If three straight lines be proportional, the rectangle contained by the extremes is equal to the square on the mean." (Euclid includes its converse in the same proposition.)

Chameleon Tail.

1. A spiral.

•2. The altitude to the hypotenuse.

•3. AP and PC.

4. The altitude to the hypotenuse of a right triangle is the geometric mean between the segments into which it divides the hypotenuse.

•5. EP.

•6. EP and EG.

7. Each leg of a right triangle is the geometric mean between the hypotenuse and its projection on the hypotenuse.

Shadows.

8. AD.

9. DB.

10. The legs.

•11. Their projections on the hypotenuse.

•12. AC.

13. CD.

14. CB.

Find the Lengths.

•15. 4. ($\frac{8}{x} = \frac{x}{2}$, $x^2 = 16$, $x = 4$.)

16. 6. ($\frac{9}{x} = \frac{x}{4}$, $x^2 = 36$, $x = 6$.)

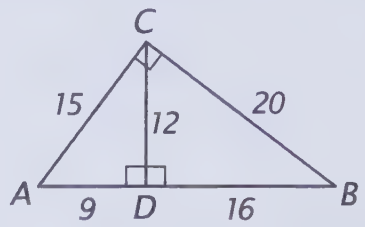
•17. $2\sqrt{10} \approx 6.32$. ($\frac{8}{x} = \frac{x}{5}$, $x^2 = 40$, $x = \sqrt{40} = 2\sqrt{10} \approx 6.32$.)

•18. 4.05. ($\frac{20}{9} = \frac{9}{x}$, $20x = 81$, $x = 4.05$.)

19. 10. ($\frac{x+8}{12} = \frac{12}{8}$, $8x + 64 = 144$, $8x = 80$, $x = 10$.)

With and Without Pythagoras.

20.



•21. 16. ($\frac{9}{12} = \frac{12}{DB}$, $9DB = 144$, $DB = 16$.)

22. 15. ($\frac{25}{AC} = \frac{AC}{9}$, $AC^2 = 225$, $AC = 15$.)

23. 15. ($AC^2 = 9^2 + 12^2 = 81 + 144 = 225$, $AC = 15$.)

24. 20. ($\frac{25}{BC} = \frac{BC}{16}$, $BC^2 = 400$, $BC = 20$.)

25. 20. ($BC^2 = 12^2 + 16^2 = 144 + 256 = 400$, $BC = 20$.)

•26. $\frac{9}{16}$.

27. $\frac{3}{4}$. ($\frac{15}{20}$.)

28. $\frac{9}{16}$.

•29. False.

30. True.

Rectangle and Square.

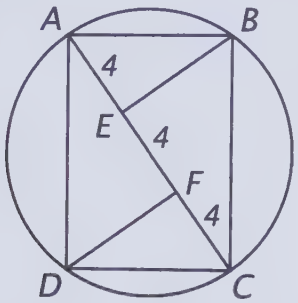
31. They are equal because $\frac{a}{b} = \frac{b}{c}$; so $ac = b^2$.)

Set II (pages 431–432)

Exercises 38-45 illustrate in theory an easy way to construct, given a segment of unit length, the successive integral powers of a number greater than 1. The process can be reversed to construct the corresponding powers of a number less than 1.

Strongest Beam.

32.



•33. $4\sqrt{6}$. ($\frac{12}{BC} = \frac{BC}{8}$, $BC^2 = 96$, $BC = \sqrt{96} = 4\sqrt{6}$.)

34. $4\sqrt{3}$. ($\frac{12}{AB} = \frac{AB}{4}$, $AB^2 = 48$, $AB = \sqrt{48} = 4\sqrt{3}$.)

•35. $\frac{\sqrt{2}}{1}$ (or $\sqrt{2}$.) ($\frac{4\sqrt{6}}{4\sqrt{3}}$.)

36. 68 in^2. [$(4\sqrt{6})(4\sqrt{3}) = 16\sqrt{18} \approx 68$.]

•37. 113 in^2. ($\pi 6^2 = 36\pi \approx 113$.)

Zigzag.

•38. $\frac{OA}{OB} = \frac{OB}{OC}$.

39. Each leg of a right triangle is the geometric mean between the hypotenuse and its projection on the hypotenuse.

40. $\frac{OB}{OC} = \frac{OC}{OD}$ and $\frac{OC}{OD} = \frac{OD}{OE}$.

•41. r^2. ($\frac{1}{r} = \frac{r}{OC}$, $OC = r^2$.)

•42. r^3. ($\frac{r}{r^2} = \frac{r^2}{OD}$, $r\,OD = r^4$, $OD = r^3$.)

43. r^4. ($\frac{r^2}{r^3} = \frac{r^3}{OE}$, $r^2\,OE = r^6$, $OE = r^4$.)

44. $OF = r^5$ and $OG = r^6$.

45. 1,000,000 inches (almost 16 miles!). (10^6.)

Plato's Proportions.

46. Yes. $\frac{OA}{OB} = \frac{OB}{OC}$ and $\frac{OB}{OC} = \frac{OC}{OD}$; so $\frac{OA}{OB} = \frac{OC}{OD}$ by substitution.)

47. Yes. $\Delta ABC \sim \Delta BCD$ by AA ($\angle A$ is complementary to $\angle ABO$ and $\angle ABO$ is complementary to $\angle DBC$; so $\angle A = \angle DBC$, and so $\angle ABC = \angle BCD$. $\frac{AB}{BC} = \frac{BC}{CD}$ because corresponding sides of similar triangles are proportional.)

SAT Problem.

•48. 5. ($AC^2 = 4^2 + 3^2 = 25$, $AC = 5$.)

49. 3.2. ($\frac{5}{4} = \frac{4}{x}$, $5x = 16$, $x = 3.2$.)

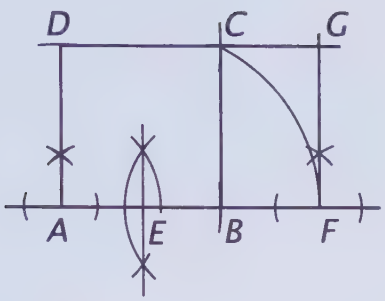
50. 1.8.

51. 2.4. ($\frac{3.2}{z} = \frac{z}{1.8}$, $z^2 = 5.76$, $z = 2.4$.)

Set III (page 433)

Martin Gardner's column on the golden ratio can be found in *The 2nd Scientific American Book of Mathematical Puzzles and Diversions* (Simon & Schuster, 1961). As Gardner points out, a lot of material that has been written on the golden rectangle and golden ratio is crankish in nature.

The Golden Rectangle.

1. 

2. a) $AD = 2$.
   b) $EB = 1$.
   c) $EC = \sqrt{5}$.
   d) $EF = \sqrt{5}$.
   e) $AF = 1 + \sqrt{5}$.
   f) $\frac{AF}{AD} = \frac{1+\sqrt{5}}{2}$.
3. 1.618033989 . . . .
4. a) $GF = 2$.
   b) $BF = \sqrt{5} - 1$. [$BF = AF - AB = (1 + \sqrt{5}) - 2$.]
   c) $\frac{GF}{BF} = \frac{2}{\sqrt{5}-1}$.
5. 1.618033989 . . . . [It is easy to show that $\frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2} : \frac{2}{\sqrt{5}-1} = \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = \frac{2(\sqrt{5}+1)}{5-1} = \frac{\sqrt{5}+1}{2}$.]
6. That it is also a golden rectangle (also that it is similar to rectangle AFGD.)
7. 0.618033989 . . . .
8. If r represents the golden ratio, it appears that $\frac{1}{r} = r - 1$. [This equality can be proved by solving the following equation for r:
   $1 = r(r-1)$, $r^2 - r = 1$, $r^2 - r - 1 = 0$,
   $r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$, $r = \frac{1 \pm \sqrt{1+4}}{2}$;
   $r > 0$, and so $r = \frac{1+\sqrt{5}}{2}$.]

Chapter 11, Lesson 2

Set I (pages 436–437)

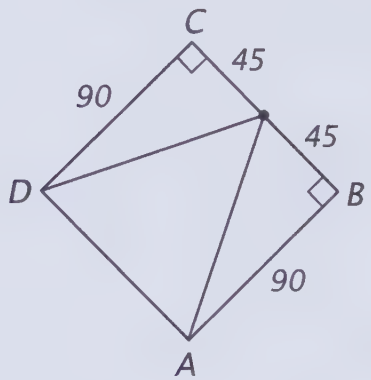
Pythagorean triples have been a popular subject of recreational mathematics since ancient times. Martin Gardner includes many interesting facts about these numbers in the *Sixth Book of Mathematical Games from Scientific American* (W. H. Freeman and Company, 1971). Some of your students might enjoy trying to discover some of them for themselves. For example, one number must be a multiple of 3 and one of the other two numbers a multiple of 5. Another great source of information on this subject is the chapter titled "The Eternal Triangle" in *Recreations in the Theory of Numbers—The Queen of Mathematics Entertains,* by Albert H. Beiler (Dover, 1966). In *Geometry and Algebra in Ancient Civilizations* (Springer Verlag, 1983), B. L. van der Waerden points out that, although Pythagorean triples exist in the early mathematics of Egypt, India, Greece, and China, there is no mention of them in the work of Archimedes or Euclid.

Baseball Distances.

•1. 30 ft.

•2. About 95 ft. ($\sqrt{30^2 + 90^2} = \sqrt{9{,}000} \approx 95$.)

3. About 108 ft. ($\sqrt{60^2 + 90^2} = \sqrt{11{,}700} \approx 108$.)

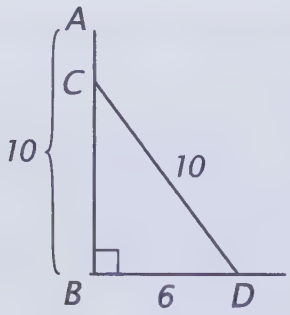
4. 

•5. About 101 ft. ($\sqrt{45^2 + 90^2} = \sqrt{10{,}125} \approx 101$.)

6. About 101 ft (by symmetry).

Pole Problem.

7.

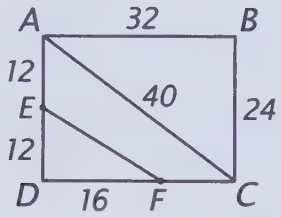


•8. 8 cubits. ($BC = \sqrt{10^2 - 6^2} = \sqrt{64} = 8$.)

9. 2 cubits.

Emergency Exits.

10.

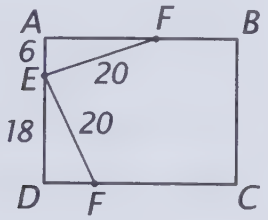


11. 40 ft. ($AC = \sqrt{32^2 + 24^2} = \sqrt{1{,}600} = 40$.)

•12. 20 ft. ($EF \geq \frac{1}{2}AC$.)

13. Yes. If F is midway between D and C, $DF = FC = 16$. $EF = \sqrt{12^2 + 16^2} = \sqrt{400} = 20$.

14.



•15. About 8.7 ft. ($DF = \sqrt{20^2 - 18^2} = \sqrt{76} \approx 8.7$.)

16. About 19 ft. ($AF = \sqrt{20^2 - 6^2} = \sqrt{364} \approx 19$.)

17. Yes. It could be placed anywhere on wall BC. (Every point on BC is more than 20 ft from point E.)

Pythagorean Triples.

•18. $7^2 + 24^2 = 49 + 576 = 625 = 25^2$.

19. Yes. (The list contains many examples.)

20. No. The square of an odd number is odd. The sum of two odd numbers is even.

•21. Each number in 14-48-50 is twice the corresponding number in 7-24-25.

22. A dilation.

•23. 6-8-10 and 9-12-15.

24. 12-16-20; 15-20-25; 18-24-30; 21-28-35; 24-32-40; 27-36-45; 30-40-50.

25. 10-24-26 and 15-36-39.

Not Quite Right.

26. In ΔABC, $15^2 + 10^2 = 325$ and $18^2 = 324$. In ΔDEF, $19^2 + 9^2 = 442$ and $21^2 = 441$.

27. Acute. The longest side of each triangle is slightly less than what it would be if the triangle were a right triangle.

Distance Formula.

•28. AB.

•29. $y_2 - y_1$.

30. Because ΔABC is a right triangle, $AC^2 = AB^2 + BC^2$; so $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ and

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Set II (pages 438–440)

The solution of the oil-well puzzle is based on a fact whose generality is quite surprising. As is demonstrated in exercises 47 through 49, given a point inside a rectangle, the sum of the squares of its distances from two opposite corners is equal to the sum of the squares of its distances from the other two corners. This is also true if the point is outside the rectangle or even on it. In fact, the point doesn't even have to lie in the plane of the rectangle. It is true for *any* point whatsoever!

Matthys Levy and Mario Salvadori point out in *Why Buildings Fall Down* (Norton, 1992) that "the earliest suspension bridges in the world, in China and South America, were 'vine bridges' made of vegetable fiber ropes that served as cables, hangers, and roadway."

In *Einstein's Legacy—The Unity of Space and Time* (Scientific American Library, 1986), Julian Schwinger reports that Einstein at age 12 "encountered and was deeply moved by the wonders of Euclidean geometry. An uncle had told him about the theorem of Pythagoras. . . . About this he later said, 'After much effort I succeeded in "proving" this theorem on the basis of the similarity of triangles.'" It is interesting that some of the basic equations of relativity come from the Pythagorean Theorem. Exercises 56 and 57 provide an example.

TV Screens.

•31. $d = 5x$. [$d^2 = (4x)^2 + (3x)^2$, $d^2 = 25x^2$, $d = 5x$.]

32. $x = \frac{1}{5}d$.

•33. 8.4 in by 11.2 in. [If $d = 14$, $x = \frac{1}{5}(14) = 2.8$; so $3x = 8.4$ and $4x = 11.2$.]

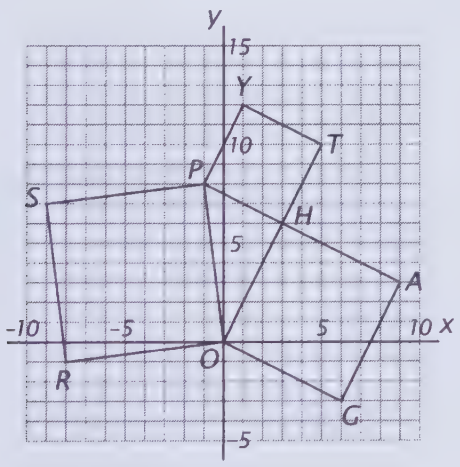
34. 16.2 in by 21.6 in. [If $d = 27$, $x = \frac{1}{5}(27) = 5.4$; so $3x = 16.2$ and $4x = 21.6$.]

•35. $\frac{27}{14}$ or about 1.9.

36. About 3.7 times as big. [$(\frac{27}{14})^2 = \frac{729}{196} \approx 3.7$.]

Grid Exercise.

37.



•38. $OH = \sqrt{(3-0)^2 + (6-0)^2} = \sqrt{9+36} = \sqrt{45}$ or $3\sqrt{5}$.
$OP = \sqrt{(8-0)^2 + (-1-0)^2} = \sqrt{64+1} = \sqrt{65}$.
$PH = \sqrt{(3-(-1))^2 + (6-8)^2} = \sqrt{16+4} = \sqrt{20}$ or $2\sqrt{5}$.

39. ΔOHP is a right triangle because $OH^2 + PH^2 = OP^2$. [$(\sqrt{45})^2 + (\sqrt{20})^2 = (\sqrt{65})^2$, $45 + 20 = 65$.]

40. They appear to be squares.

41. 20, 45, and 65 square units.

Rope Surveying.

•42. The length of AP.

•43. $x^2 + 20^2 = (40 - x)^2$,
$x^2 + 400 = 1{,}600 - 80x + x^2$, $80x = 1{,}200$,
$x = 15$.

44. $32 - y$.

45. $y^2 + 16^2 = (32 - y)^2$,
$y^2 + 256 = 1{,}024 - 64y + y^2$, $64y = 768$, $y = 12$.

46. The converse of the Pythagorean Theorem.

Oil Well.

47. $b^2 = x^2 + y^2$, $c^2 = x^2 + z^2$, $d^2 = w^2 + z^2$.

48. $a^2 + c^2 = w^2 + y^2 + x^2 + z^2$,
$b^2 + d^2 = x^2 + y^2 + w^2 + z^2$.

49. That $a^2 + c^2 = b^2 + d^2$.

50. 27,000 ft. ($21{,}000^2 + 18{,}000^2 = 6{,}000^2 + d^2$, $d^2 = 129{,}000{,}000$, $d = 27{,}000$.)

Suspension Bridge.

51.



52. About 7 ft. ($\sqrt{50.5^2 - 50^2} = \sqrt{50.25} \approx 7.1$.)

Turning Radius.

•53. $D = R - \sqrt{R^2 - B^2}$.

54. About 2.8 ft.
$(27 - \sqrt{27^2 - 12^2} = 27 - \sqrt{585} \approx 2.8.)$

55. It becomes about half as large.
$(54 - \sqrt{54^2 - 12^2} = 54 - \sqrt{2{,}772} \approx 1.4.)$

Rocket Car.

56. Because $A^2 + B^2 = C^2$, $\frac{A^2}{C^2} + \frac{B^2}{C^2} = 1$; so $\frac{A^2}{C^2} = 1 - \frac{B^2}{C^2}$. Because $\left(\frac{A}{C}\right)^2 = 1 - \left(\frac{B}{C}\right)^2$, $\frac{A}{C} = \sqrt{1 - \left(\frac{B}{C}\right)^2}$.

57. About 0.44. $[\sqrt{1 - (0.90)^2} = \sqrt{0.19} \approx 0.44.]$

Set III (page 440)

Fermat's Last Theorem—Unlocking the Secret of an Ancient Mathematical Problem, by Amir D. Aczel (Four Walls Eight Windows, 1996) is an appropriate book for students who are curious about the subject and its geometric aspects. As *Kirkus Reviews* describes the book, it is "an excellent short history of mathematics, viewed through the lens of one of its great problems—and achievements."

Fermat's Last Theorem.

1. Infinitely many.
2. Infinitely many. There are infinitely many Pythagorean triples.
3. None.
4. The reporter observed that all positive integer powers of a number whose last digit is 4 end in either 4 or 6 and that all positive integer powers of a number whose last digit is 1 end in 1. The last digits of the three numbers in Krieger's equation, then, are either 4, 1, and 1 or 6, 1, and 1. But 4 + 1 = 5, not 1, and 6 + 1 = 7, not 1; so regardless of the value of n chosen, the equation cannot be true.

Chapter 11, Lesson 3

Set I (pages 443–445)

Isosceles Right Triangles.

•1. Yes. They are similar by AA.

•2. $\frac{a}{1} = \frac{c}{\sqrt{2}}$.

•3. $c = a\sqrt{2}$.

•4. $a = \frac{c}{\sqrt{2}}$.

•5. In an isosceles right triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

30°-60° Right Triangles.

6. Yes.

7. $\frac{a}{1} = \frac{b}{\sqrt{3}} = \frac{c}{2}$.

8. $b = a\sqrt{3}$.

9. $c = 2a$.

10. $a = \frac{b}{\sqrt{3}}$.

•11. $c = \frac{2b}{\sqrt{3}}$.

12. $a = \frac{c}{2}$.

13. $b = \frac{c\sqrt{3}}{2}$.

14. In a 30°-60° right triangle, the hypotenuse is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.

Length Problems.

•15. $8\sqrt{2}$.

•16. $5\sqrt{2}$. $\left(x = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}.\right)$

•17. $x = 12, y = 6\sqrt{3}$.

18. $x = 2, y = 2\sqrt{3}$.

•19. $x = 5\sqrt{3}, y = 10\sqrt{3}$. ($x = \frac{15}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$.)

20. $12\sqrt{2}$.

21. 14.

22. $10\sqrt{2}$. ($x = \frac{20}{\sqrt{2}} = \frac{20\sqrt{2}}{2} = 10\sqrt{2}$.)

Equilateral Triangles.

•23. $\sqrt{3}$. [$\frac{\sqrt{3}}{4}(2)^2$.]

•24. $4\sqrt{3}$. [$\frac{\sqrt{3}}{4}(4)^2$.]

25. $9\sqrt{3}$. [$\frac{\sqrt{3}}{4}(6)^2$.]

26. ΔADE contains 4 triangles congruent to ΔABC; ΔAFG contains 9 triangles congruent to ΔABC.

Line of Sight.

•27. $90\sqrt{2} \approx 127$ ft.

28. 150 ft. (A quick method: this triangle is a "3-4-5" right triangle with sides 3×30, 4×30, and 5×30.)

29. $120\sqrt{2} \approx 170$ ft.

Cathedral Design.

•30. $75\sqrt{3} \approx 130$ ft. ($\frac{\sqrt{3}}{2}150$.)

•31. About 23 in. ($20 = \frac{\sqrt{3}}{2}s$, $s = \frac{40}{\sqrt{3}} \approx 23$.)

Olympic Course.

32. About 2.3 nautical miles. ($\frac{3.25}{\sqrt{2}} \approx 2.3$.)

33. About 7.8 nautical miles.
[$3.25 + 2(\frac{3.25}{\sqrt{2}}) \approx 7.8$.]

Set II (pages 445–447)

Exercises 38 through 41 deal with a special case of the navigator's rule called "doubling the angle on the bow." Another special case is the one in which the angle is doubled from 22.5° to 45°. It is called the "$\frac{7}{10}$ rule," on the basis of the fact that $\frac{\sqrt{2}}{2} \approx 0.7 = \frac{7}{10}$.

Exercises 42 through 47 provide a preview to students who continue their study of mathematics of the prominence of the 30°-60° and isosceles right triangle theorems in quick-calculation trigonometry.

Exercises 48 through 52 are based on a special case of "rep-tiles," polygons that can be put together or subdivided to make larger or smaller copies of themselves. According to Solomon Golomb, Professor of Electrical Engineering at the University of Southern California, a rep-tile of order k is one that can be divided into k copies similar to it and congruent to one another. As Martin Gardner explains in the chapter on rep-tiles in *The Unexpecting Hanging and Other Mathematical Diversions* (Simon & Schuster, 1969), *every* triangle is of order 4. The 30°-60° right triangle is unusual (unique?) in that it is also of order 3, as the construction demonstrates.

Not Quite Right.

•34. 21 square units. [$\frac{3}{7}(7)^2 = 3(7) = 21$.]

35. $\frac{\sqrt{3}}{4}s^2$.

36. 21.2 square units. [$\frac{\sqrt{3}}{4}(7)^2 = \frac{49\sqrt{3}}{4} \approx 21.2$.]

37. If $\sqrt{3} = \frac{12}{7}$, $\frac{\sqrt{3}}{4}s^2 = \frac{\frac{12}{7}}{4}s^2 = \frac{3}{7}s^2$.

Navigator's Rule.

38. AB = BD. $\angle ABD = 120°$; so $\angle ADB = 30° = \angle A$. Because two angles of ΔABD are equal, the sides opposite them are equal.

•39. $\frac{x}{2}$. (Because AB = BD = x, in 30°-60° right ΔBCD, BC = $\frac{x}{2}$.)

40. $\frac{x}{2}\sqrt{3}$.

41. $AB = x$ and $CD = \frac{x}{2}\sqrt{3} = \frac{\sqrt{3}}{2}x$.

$\frac{\sqrt{3}}{2} \approx 0.866 \approx 0.875 = \frac{7}{8}$.

Trigonometric Angles.

42. $\angle A = 90°$, $\angle O = 60°$, and $\angle P = 30°$.

•43. $OA = \frac{1}{2}$, $AP = \frac{1}{2}\sqrt{3}$.

•44. $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$.

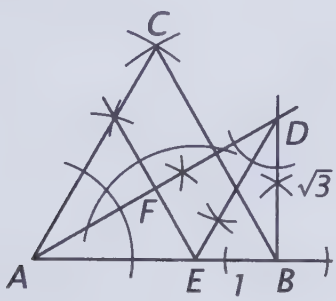
45. $\angle A = 90°$, $\angle O = 45°$, and $\angle P = 45°$.

46. $OA = AP = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ or $\frac{1}{2}\sqrt{2}$.

47. $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}.)$

Construction Exercise.

48.



49. ΔABD, ΔAEF, ΔDEF, and ΔBDE.

50. The following lengths are in inches: $DE = 2$, $DB = \sqrt{3}$, $DF = \sqrt{3}$, $EF = 1$, $AE = 2$, $AF = \sqrt{3}$.

51. *(Student answer.)* (Measurements of DB, DF, and AF suggest that they are $1\frac{3}{4}$ inches long, and $\sqrt{3} \approx 1\frac{3}{4}$.)

52. Yes. Because $DB = \sqrt{3}$, it follows that $AD = 2\sqrt{3} = \sqrt{3} + \sqrt{3} = AF + FD$, and $AB = \sqrt{3}\sqrt{3} = 3$.

Area Problem.

•53. $\frac{1}{2}$ or 0.5. $(\frac{1}{2}1^2 = \frac{1}{2}.)$

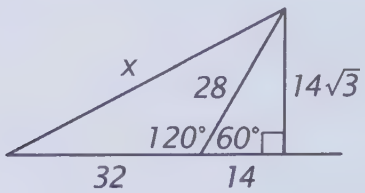
•54. $\frac{\sqrt{3}}{4}$. $(\frac{\sqrt{3}}{4}1^2 = \frac{\sqrt{3}}{4}.)$

55. $6 + 3\sqrt{3}$. $[12(\frac{1}{2}) + 12(\frac{\sqrt{3}}{4}).]$

56. 11.2 square units.

Moscow Challenge.

57.



$x = 52$. [Drawing the altitude to the side labeled 32 produces a 30°-60° right triangle with hypotenuse 28 and legs 14 and $14\sqrt{3}$. By the Pythagorean Theorem, $x^2 = 46^2 + (14\sqrt{3})^2 = 2{,}116 + 588 = 2{,}704$; so $x = 52$.]

Set III (page 447)

A Big Mistake.

1. A 3-4-5 triangle is a right triangle because $3^2 + 4^2 = 5^2$. If one acute angle of a right triangle has a measure of 45°, so must the other. But, if two angles of a triangle are equal, the sides opposite them are equal. This proves that a 3-4-5 right triangle cannot have a 45° angle.

2. If the object is to find how far the tree will reach when it falls, an isosceles right triangle should be used.

Chapter 11, Lesson 4

Set I (pages 450–451)

Students with graphing calculators will find it of interest to look at the graph of $y = \tan\theta$ over the interval $0° \le \theta < 90°$, especially in relation to exercises 10 through 16. The graph of this function climbs from 0 to 1 in the interval 0° to 45° and then from 1 to ∞ in the interval 45° to 90°. Exercises 27 and 28 suggest that, if two angles are complementary, their tangents are reciprocals of each other. This is nicely illustrated by comparing the graphs of $y = \tan\theta$ and $y = \frac{1}{\tan(90-\theta)}$.

Tangent Practice.

•1. $\frac{c}{b}$.

2. $\frac{d}{f}$.

•3. $\frac{f}{d}$.

•4. 0.445.

5. 0.017.

•6. 57.290.

•7. 5.711°.

8. 87.614°.

9. 89.943°.

Isosceles Right Triangle.

10. $\angle A = 45°$.

•11. $\tan A = \frac{a}{b} = \frac{a}{a} = 1$.

12. $\tan 45° = 1$.

•13. $\angle A > 45°$.

•14. $\tan A > 1$.

15. $\angle B < 45°$.

16. $\tan B < 1$.

30°-60° Right Triangle.

17. $AB = 2b$, $BC = b\sqrt{3}$.

•18. $\tan 60° = \frac{BC}{AC} = \frac{b\sqrt{3}}{b} = \sqrt{3}$.

19. $\tan 60° \approx 1.7320508$ and $\sqrt{3} \approx 1.7320508$.

20. $\tan 30° = \frac{AC}{BC} = \frac{b}{b\sqrt{3}} = \frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$.

21. $\tan 30° \approx 0.5773502$ and $\frac{\sqrt{3}}{3} \approx 0.5773502$.

3-4-5 Triangle.

•22. A right triangle because $3^2 + 4^2 = 5^2$.

23. $\tan A = \frac{3}{4}$.

•24. $\tan B = \frac{4}{3}$.

•25. $\angle A \approx 37°$.

26. $\angle B \approx 53°$.

•27. $\angle A < \angle B$; $\angle A$ and $\angle B$ are complementary.

28. $\tan A < \tan B$; $\tan A$ is the reciprocal of $\tan B$.

Finding Lengths.

•29. 16.5. ($\tan 70° = \frac{x}{6}$, $x = 6 \tan 70° \approx 16.5$.)

30. 10.4. ($\tan 46° = \frac{x}{10}$, $x = 10 \tan 46° \approx 10.4$.)

31. 20.0. ($\tan 55° = \frac{x}{14}$, $x = 14 \tan 55° \approx 20.0$.)

Finding Angles.

•32. 20°. ($\tan x = \frac{4}{11}$, $x \approx 20°$.)

33. 36°. ($\tan x = \frac{18}{25}$, $x \approx 36°$.)

34. 57°. ($\tan x = \frac{20}{13}$, $x \approx 57°$.)

Set II (pages 451–453)

The lighthouse at Alexandria stood at the entrance to the most famous harbor of the ancient world. It was begun by the first Ptolemy and completed by his son in about 285 B.C. It survived storms and earthquakes until being destroyed by a particularly violent earthquake in 1303. Estimates of its height vary. The drawing of the lighthouse and the measurements of exercises 39 through 43 are based on material in Herman Thiersch's book *Der Pharos von Alexandrien* (1909), included in B. L. van der Waerden's *Science Awakening* (Oxford University Press, 1961).

In an article for the *Encyclopedia Britannica* 1970 Yearbook of *Science and the Future*, Jay Holmes wrote: "Because of the immense distances to be traversed, all present and contemplated space flights follow curved paths. . . . The distances proportionately magnify the effect of the slightest navigation error. A miscalculation of only one degree in the course from the moon back to the earth could result in missing the earth by more than 4,000 miles. And unlike sailors of old, modern astronauts could not turn around if they should find themselves so far off course."

Fly Ball.

•35. 58°. ($\tan e = \frac{80}{50}$, $e \approx 58$.)

36. 67°. ($\tan e = \frac{70}{30}$, $e \approx 67$.)

37. 72°. ($\tan e = \frac{60}{20}$, $e \approx 72$.)

38. It increases.

Lighthouse.

•39. 197 ft. ($\tan 33.3° = \frac{AB}{300}$, $AB = 300 \tan 33.3° \approx 197$.)

40. 295 ft. ($\tan 44.5° = \frac{AC}{300}$, $AC = 300 \tan 44.5° \approx 295$.)

41. 370 ft. ($\tan 51.0° = \frac{AD}{300}$, $AD = 300 \tan 51.0° \approx 370$.)

42. 392 ft. ($\tan 52.6° = \frac{AE}{300}$, $AE = 300 \tan 52.6° \approx 392$.)

•43. About 22 ft.
($DE = AE - AD \approx 392 - 370 = 22$.)

Binoculars.

44. 0.13. ($\frac{AC}{BD} = \frac{130}{1{,}000}$.)

45. 65 m. ($\frac{130}{2}$.)

•46. 3.7°. ($\tan \angle ABD = \frac{AD}{BD} = \frac{65}{1{,}000}$, $\angle ABD \approx 3.72$.)

47. 7.4°. ($\angle ABC = 2\angle ABD$.)

Grid Triangle.

•48. 26.565051°. ($\tan \angle ABD = \frac{1}{2}$.)

49. 18.434949°. ($\tan \angle EBC = \frac{1}{3}$.)

50. 45°. ($\angle ABC = \angle ABD + \angle EBC$.)

•51. $AB = \sqrt{5}$, $BC = \sqrt{10}$, and $AC = \sqrt{5}$.
($AB^2 = 1^2 + 2^2$, $AB = \sqrt{5}$; $BC^2 = 1^2 + 3^2$, $BC = \sqrt{10}$; $AC^2 = 1^2 + 2^2$, $AC = \sqrt{5}$.)

52. ΔABC is an isosceles right triangle because it has two equal sides and the sum of the squares of these sides is equal to the square of the third side.

Diamond Cut.

•53. 40.8°. ($\tan \angle 1 = \frac{0.431x}{0.5x} = 0.862$; $\angle 1 \approx 40.8°$.)

54. 34.6°. [$\tan \angle 2 = \frac{0.162x}{0.5(x - 0.53x)} = \frac{0.162}{0.235} \approx 0.689$; $\angle 2 \approx 34.6°$.]

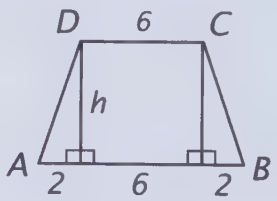
Space-Flight Errors.

•55. About 4,172 mi. ($\tan 1° = \frac{PE}{239{,}000}$, $PE = 239{,}000 \tan 1° \approx 4{,}172$.)

56. About 14,254 mi. ($\tan \frac{1}{60}° = \frac{PE}{49{,}000{,}000}$, $PE = 49{,}000{,}000 \tan \frac{1}{60}° \approx 14{,}254$.)

Exam Problem.

57.



$\alpha ABCD = \frac{1}{2}h(10 + 6) = 8h$. $\tan A = \frac{h}{2}$; so $h = 2 \tan A$.
So $\alpha ABCD = 8(2 \tan A) = 16 \tan A$.

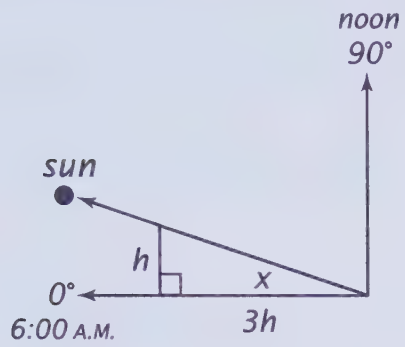
58. About 68°. ($40 = 16 \tan A$, $\tan A = \frac{40}{16} = 2.5$, $\angle A \approx 68°$.)

Set III (page 453)

The overhead photograph of the camel caravan was taken in the Kerman Province of Iran by aerial photographer Georg Gerster.

What Time is It?

To find out what time it is, we can find the angle of elevation of the sun as shown in this figure:



$\tan x = \frac{h}{3h} = \frac{1}{3}$; so $x \approx 18.435°$.

The sun's angle of elevation changes at a constant rate because the earth rotates at a constant rate. Let t = the number of hours elapsed since 6:00 A.M.: $\frac{18.435°}{90°} = \frac{t}{6}$,

$t \approx 1.229$ hours $\approx$ 1 hour, 14 minutes.

The time was about 7:14 A.M.

Chapter 11, Lesson 5

Set I (pages 456–457)

In addition to providing practice in using a calculator to find sines and cosines and their inverses, exercises 7 through 12 and the two examples illustrate the fact that a function of an angle is the cofunction of its complement. Exercises 13 through 22 reinforce this idea. The notion of the cosine of an angle developed long after the sine, and the name "cosine" was not introduced until the seventeenth century, the prefix "co-" referring to the word "complement."

More information about Bhaskara's calculation of sines can be found in B. L. van der Waerden's *Geometry and Algebra in Ancient Civilizations* (Springer Verlag, 1983). According to van der Waerden, Bhaskara divided an arc of 90° in a circle first into 6, then 12, and finally 24 equal parts. He computed the values of the sines rather than finding them by measurement.

Sine and Cosine Practice.

•1. $\frac{h}{a}$ or $\frac{b}{x+y}$.

•2. $\frac{h}{y}$ or $\frac{a}{b}$.

•3. $\frac{y}{b}$ or $\frac{b}{x+y}$.

4. $\frac{x}{a}$ or $\frac{a}{x+y}$.

5. $\frac{x}{a}$.

6. $\frac{h}{b}$.

•7. 0.996.

8. 0.719.

9. 0.719.

•10. 82.935°.

•11. 45.000°.

12. 45.000°.

Pythagorean Triangle.

13. A right triangle because $5^2 + 12^2 = 13^2$.

•14. $\sin A = \frac{12}{13}$.

•15. $\angle A \approx 67°$.

16. $\cos B = \frac{12}{13}$.

17. $\angle B \approx 23°$.

18. $\angle A > \angle B$; $\angle A$ and $\angle B$ are complementary.

19. $\sin A = \cos B$.

•20. $\cos A = \frac{5}{13}$.

21. $\sin B = \frac{5}{13}$.

22. $\cos A = \sin B$.

Finding Lengths and Angles.

•23. 5.3. ($\sin 32° = \frac{x}{10}$, $x = 10 \sin 32° \approx 5.3$.)

24. 18.3. ($\cos 66° = \frac{x}{45}$, $x = 45 \cos 66° \approx 18.3$.)

•25. 61°. ($\tan x = \frac{16}{9}$, $x \approx 61°$.)

•26. 36.5. ($\cos 70° = \frac{12.5}{x}$, $x \cos 70° = 12.5$, $x = \frac{12.5}{\cos 70°} \approx 36.5$.)

27. 155.6. ($\sin 13° = \frac{35}{x}$, $x \sin 13° = 35$, $x = \frac{35}{\sin 13°} \approx 155.6$.)

28. 60°. ($\cos x = \frac{27}{54} = 0.5$, $x = 60°$.)

Sines from a Circle.

•29. BH = 2.6 cm, CI = 5.0 cm, DJ = 7.1 cm, EK = 8.7 cm, and FL = 9.7 cm.

30. $\sin 30° \approx \frac{5.0}{10} = 0.50$,

$\sin 45° \approx \frac{7.1}{10} = 0.71$,

$\sin 60° \approx \frac{8.7}{10} = 0.87$,

$\sin 75° \approx \frac{9.7}{10} = 0.97$.

31. $\sin 30° = 0.5$,
$\sin 45° \approx 0.7071067 \approx 0.71$,
$\sin 60° \approx 0.8660254 \approx 0.87$,
$\sin 75° \approx 0.9659258 \approx 0.97$.

32. It gets larger.

33. A number very close to zero.

34. A number very close to one.

35. According to a calculator, $\sin 0° = 0$ and $\sin 90° = 1$.

Set II (pages 458–460)

A wealth of information on flying kites to great altitudes can be found on Richard P. Synergy's Website, www.total.net/~kite/world.html. The line for the kite shown in the photograph was more than 4 miles long and was spooled from a winch. At times it exerted a force of 100 pounds! The calculation of the kite's altitude was more complicated than that implied in the exercises because the line for the kite was not actually straight; its weight made it sag considerably. The altitude of the kite was established through altimeter readings.

As the title of Robert B. Banks' *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics* (Princeton University Press, 1998) implies, he includes a good discussion of the mathematics and physics of toppling dominoes. The *Guinness Book of World Records* always has an entry on the latest record in this sport. In 1988, some students from three universities in the Netherlands spent a month setting up 1,500,000 dominoes. A single push caused 1,382,101 of them to fall!

Kite Flying.

•36. The sine of an acute angle of a right triangle is the ratio of the length of the opposite leg to the length of the hypotenuse.

37. Multiplication.

•38. In a 30°-60° right triangle, the side opposite the 30° angle is half the hypotenuse.

39. In ΔABC, $\sin 30° = \frac{h}{l}$. Because $\sin 30° = 0.5$, $\frac{h}{l} = 0.5$, so $h = 0.5l$.

•40. Approximately 14,500 ft, or more than 2.7 mi. ($\sin 42° = \frac{h}{21{,}600}$, $h = 21{,}600 \sin 42° \approx 14{,}453$; $\frac{14{,}453}{5{,}280} \approx 2.7$.)

Force Components.

41. $h = \frac{1}{2}(27{,}000) = 13{,}500$ pounds,
$v = 13{,}500\sqrt{3} \approx 23{,}383$ pounds.

42. $\sin 60° = \frac{v}{27{,}000}$,
$v = 27{,}000 \sin 60° \approx 23{,}383$ pounds.
$\cos 60° = \frac{h}{27{,}000}$,
$h = 27{,}000 \cos 60° = 13{,}500$ pounds.

Flight Path.

•43. About 806 ft. ($\sin 3.3° = \frac{CB}{14{,}000}$, $CB = 14{,}000 \sin 3.3° \approx 806$.)

44. About 13.6°. ($\sin A = \frac{3{,}300}{14{,}000}$, $\angle A \approx 13.6°$.)

Leaning Tower.

45. About 10 ft. ($\sin 3° = \frac{x}{193}$, $x = 193 \sin 3° \approx 10.1$.)

•46. About 4°. ($\sin A = \frac{13.5}{193}$, $\angle A \approx 4°$.)

47. About 18.5 ft. ($\sin 5.5° = \frac{x}{193}$, $x = 193 \sin 5.5° \approx 18.5$.)

Fiber Optics.

•48. Parallel lines form equal alternate interior angles.

•49. AAS.

50. Corresponding parts of congruent triangles are equal.

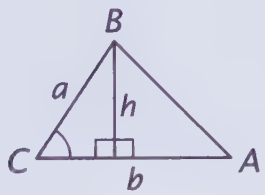
•51. $5a$.

52. $5d$.

53. Because $\frac{5a}{5d} = \frac{a}{d}$ and the cosine of each numbered angle is $\frac{a}{d}$.

SAS Area.

54.



55. $\sin C = \frac{h}{a}$; so $h = a \sin C$.

$\alpha\Delta ABC = \frac{1}{2}bh = \frac{1}{2}b(a \sin C) = \frac{1}{2}ab \sin C$.

Dominoes.

56. Parallel lines form equal alternate interior angles.

•57. $\sin \angle 2 = \frac{d-w}{h}$.

58. 30°. (If $d - w = \frac{1}{2}h$, $\sin \angle 2 = \frac{1}{2}$; so $\angle 2 = 30° = \angle 1$.)

59. 90°. (If $d - w = h$, the domino has room to turn 90° without hitting the next one. Also, if $d - w = h$, $\sin \angle 2 = 1$; so $\angle 2 = 90° = \angle 1$.)

60. The next domino might not fall over.

Set III (page 460)

The first distance to a star beyond the sun was determined by the German astronomer Friedrich Bessel in 1838. The star, 61 Cygni, was chosen because its position against the distant sky changes more than those of most stars. A Scottish astronomer, Thomas Henderson, found the distance to Alpha Centauri (and its faint companion Proxima Centauri) a year later. The measure of the angle $\angle ACS$ in the figure is called the "parallax" of the star. The unit now used to express distances in space is the "parsec," defined as the distance of a star whose parallax is 1 second. More details on the process of determining astronomical distances by parallax can be found in Eli Maor's book titled *Trigonometric Delights* (Princeton University Press, 1998). Another excellent reference source is *Parallax—The Race to Measure the Cosmos*, by Alan W. Hirshfeld (W. H. Freeman and Company, 2001).

Star Distance.

1. About 25,000,000,000,000 mi.

$(0.76'' = \frac{0.76°}{3{,}600}$, $\sin \frac{0.76°}{3{,}600} = \frac{93{,}000{,}000}{AC}$,

$AC = \frac{93{,}000{,}000}{\sin \frac{0.76°}{3{,}600}} \approx 2.5 \times 10^{13}$.)

2. About 4.3 light-years. $(\frac{2.5 \times 10^{13}}{5.88 \times 10^{12}} \approx 4.3)$

Chapter 11, Lesson 6

Set I (pages 463–465)

About 90% of all avalanches occur on slopes between 30° and 45°, although very wet snow can avalanche on slopes of only 10° to 25°. Skiers sometimes measure the slope angle with a clinometer and sometimes merely estimate it with a ski pole.

According to Cleo Baldon and Ib Melchior in *Steps and Stairways* (Rizzoli, 1989), the eighteenth-century French architect Jacques-François Blondel, "concluded that the stride or pace of walking was 24 inches and that when negotiating a stairway this pace should be decreased in

proportion to the height of the riser. His theory was that two times the height of the riser plus the depth of the tread should equal the 24 inches of the pace." Blondel's formula is still used today, but the pace number has been increased to 26.

Slope Practice.

•1. h.

2. c.

•3. The x-axis.

4. The y-axis.

•5. e.

•6. j, $\frac{1}{4}$; e, -4.

7. g.

8. g, 2; b, $-\frac{1}{2}$.

9. Two nonvertical lines are perpendicular iff the product of their slopes is –1.

10. No. Two lines with the same slope would be parallel. All of the lines named with letters intersect at the origin.

Snow Avalanches.

•11. The slope is $\tan 30° \approx 0.58$.

12. $\tan 45° = 1$.

Hydraulic Gradient.

•13. –60 m.

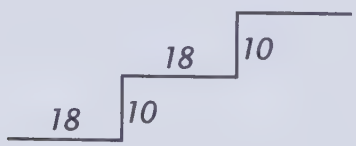
14. 1,000 m.

15. –0.06. ($\frac{-60}{1{,}000}$.)

•16. About 3.4°. ($\tan \angle 1 = 0.06$; $\angle 1 \approx 3.4°$.)

Stair Design.

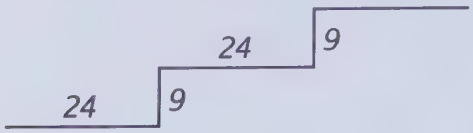
17.



•18. $\frac{5}{9}$ (or about 0.56). ($\frac{10}{18}$.)

19. About 29°. ($\tan A = \frac{5}{9}$, $\angle A \approx 29°$.)

20.



21. $\frac{3}{8}$ (or 0.375). ($\frac{9}{24}$.)

22. About 21°. ($\tan A = \frac{3}{8}$, $\angle A \approx 20.6°$.)

Quadrilateral Problem.

•23. $\frac{3}{11}$.

24. $\frac{9}{7}$.

25. $\frac{3}{11}$.

26. $\frac{2}{3}$.

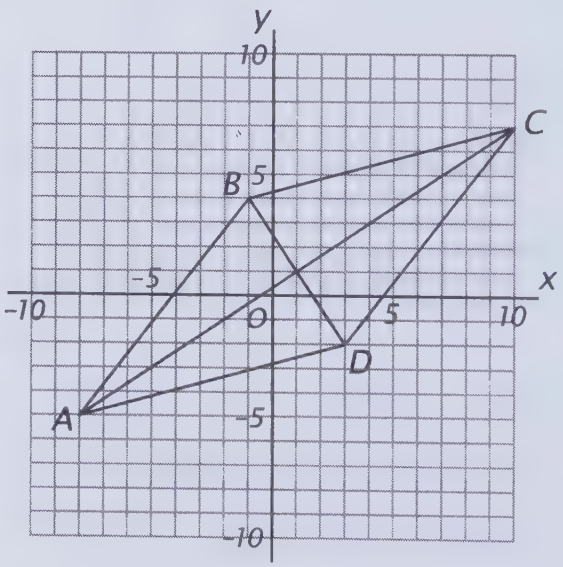
•27. $-\frac{3}{2}$.

28. $\sqrt{130}$.

29. $\sqrt{130}$.

30. $\sqrt{130}$.

31.



32. It is a rhombus because all of its sides are equal. (Or, it is a parallelogram because both pairs of opposite sides are parallel.)

33. They are perpendicular because the diagonals of a rhombus are perpendicular.

34. They are equal.

35. Their product is –1. (Or, one slope is the opposite of the reciprocal of the other.)

Set II (pages 465–467)

The two theorems of this lesson about the slopes of parallel and perpendicular lines are only partly proved in exercises 52 through 65. Exercises 52 through 59 establish that, if two nonvertical lines are parallel, their slopes are equal. The converse of this theorem can be proved by showing that $\frac{AC}{BC} = \frac{DF}{EF}$; so $\Delta ABC \sim \Delta DEF$ by SAS similarity, from which it follows that $\angle 1 = \angle 2$, and so $l_1 \parallel l_2$.

Exercises 60 through 65 establish that, if two nonvertical lines are perpendicular, the product of their slopes is –1. The converse theorem can be proved by showing that, if $\frac{AD}{BD} \cdot \frac{-DC}{BD} = -1$, then $\frac{AD}{BD} = \frac{BD}{DC}$. From this it follows that $\Delta ADB \sim \Delta BDC$ by SAS similarity. Because ΔADB and ΔBDC are right triangles, the facts that corresponding angles of similar triangles are equal and the acute angles of a right triangle are complementary can be used to show that $\angle ABC$ is a right angle; so $l_1 \perp l_2$.

Slope and x-Intercept.

•36. Its run is 550 m and its rise is –10 m.

37. 290 m above the ground. (300 – 10.)

38. Its run is x m and its rise is $-\frac{1}{55}x$ (or $-\frac{x}{55}$) m.

•39. $(300 - \frac{x}{55})$ m above the ground.

•40. 16,500 m. ($300 - \frac{1}{55}x = 0$, $\frac{1}{55}x = 300$, $x = 16{,}500$.)

41. Yes. (16,500 m = 16.5 km > 15 km.)

Point–Slope Equation of a Line.

•42. Run $= x - x_1$; rise $= y - y_1$.

•43. $y = m(x - x_1) + y_1$. [$m = \frac{y - y_1}{x - x_1}$, $m(x - x_1) = y - y_1$, $y = m(x - x_1) + y_1$.]

44. $y = -\frac{1}{55}x + 300$.

45. *(Student answer.)* (It is hoped, yes.)

46. The points (5,500, 200) and (1,100, 280) are on the path. The point (2,200, 250) is not.

[$y = -\frac{1}{55}(5{,}500) + 300 = 200$;

$y = -\frac{1}{55}(1{,}100) + 300 = 280$;

$y = -\frac{1}{55}(2{,}200) + 300 = 260 \neq 250$.]

100-Meter Dash.

•47. From the start to about 2.5 seconds.

48. From about 5 to 8 seconds.

49. From about 2.5 to 5 seconds.

50. The runner's speed is decreasing.

51. The runner's speed is not changing.

Theorem 52.

•52. $\frac{AC}{BC}$.

53. $\frac{DF}{EF}$.

54. Parallel lines form equal corresponding angles.

55. AA.

•56. Corresponding sides of similar triangles are proportional.

57. Multiplication.

•58. Division.

59. Substitution.

Theorem 53.

•60. l_2.

61. $\frac{AD}{BD}$.

•62. The altitude to the hypotenuse of a right triangle is the geometric mean between the segments of the hypotenuse.

63. $-\frac{1}{m_2}$.

64. Substitution.

65. Multiplication.

Fire Speed.

66. First figure, $\tan 25° \approx 0.47$; second figure, $\tan 50° \approx 1.19$.

67. About 6.5 times as fast. The squares of the two slopes are approximately $(0.47)^2 \approx 0.22$ and $(1.19)^2 \approx 1.42$; $\frac{1.42}{0.22} \approx 6.5$.

Set III (page 467)

The second-century Greek physician Galen first suggested a temperature scale based on ice and boiling water, but not until the seventeenth century were thermometers invented. In 1714, German physicist Gabriel Fahrenheit chose the temperature of a freezing mixture of ammonium chloride, ice, and water for the zero on his scale and the temperature of the human body as 96. The number 96 has many more divisors than 100 does, and Fahrenheit thought it would be more convenient for calculations! Swedish astronomer Anders Celsius introduced his scale, originally called the centigrade scale, in 1742. Strangely, Celsius chose 100° for the *freezing* point of water and 0° for its *boiling* point.

Celsius and Fahrenheit.

1. $\frac{9}{5}$ or 1.8.

   $(m = \frac{\text{rise}}{\text{run}} = \frac{212-32}{100-0} = \frac{180}{100} = \frac{9}{5} = 1.8.)$

2. $\frac{F-32}{C}$.

3. $\frac{F-32}{C} = \frac{9}{5}$.

4. $C = \frac{5}{9}(F-32)$.

   $[9C = 5(F-32),\ C = \frac{5}{9}(F-32).]$

5. $F = \frac{9}{5}C + 32$. $(F - 32 = \frac{9}{5}C,\ F = \frac{9}{5}C + 32.)$

6. Yes. The temperature is –40°. If F = C, then $C = \frac{9}{5}C + 32$; so $5C = 9C + 160$, $-4C = 160$, $C = -40$.

Chapter 11, Lesson 7

Set I (pages 471–472)

According to David Eugene Smith in his *History of Mathematics* (Ginn, 1925; Dover, 1958), the Law of Sines "was known to Ptolemy (c.150) in substance, although he expressed it by means of chords." About the Law of Cosines, Smith wrote that it is "essentially a geometric theorem of Euclid" (Book II, Propositions 12 and 13) and that "in that form it was known to all medieval mathematicians."

Greek Equations.

1. Sine.

•2. They are opposite each other.

3. $\frac{\gamma}{\eta\mu\Gamma}$.

•4. The Law of Sines.

5. Cosine.

6. ∠A is opposite side α.

•7. They include ∠A.

8. ∠B is opposite side β.

9. They include ∠B.

10. $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\,\sigma\upsilon\nu\Gamma$.

11. The Law of Cosines.

Law of Sines.

•12. 40°. $(\frac{\sin x}{50} = \frac{\sin 75°}{75}$,

$\sin x = \frac{50 \sin 75°}{75} \approx 0.644,\ x \approx 40°.)$

•13. 17.0. $(\frac{x}{\sin 50°} = \frac{17}{\sin 50°}$, $x = 17$. This is consistent with the fact that the triangle is isosceles.)

14. 75°. ($\frac{\sin x}{4} = \frac{\sin 14°}{1}$, $\sin x = 4 \sin 14° \approx 0.968$, $x \approx 75°$.)

Law of Cosines.

•15. 60°. [$19^2 = 16^2 + 21^2 - 2(16)(21) \cos x$, $361 = 256 + 441 - 672 \cos x$, $672 \cos x = 336$, $\cos x = \frac{336}{672} = 0.5$, $x = 60°$.]

•16. 32.0. [$x^2 = 30^2 + 40^2 - 2(30)(40) \cos 52°$, $x^2 = 900 + 1600 - 2400 \cos 52°$, $x^2 \approx 1022$, $x \approx 32.0$.]

17. 15.0. [$x^2 = 26^2 + 30^2 - 2(26)(30)\cos 30°$, $x^2 = 676 + 900 - 1560 \cos 30°$, $x^2 \approx 225$, $x \approx 15.0$.]

Sines of Supplementary Angles.

•18. $\frac{\sin \angle 1}{y} = \frac{\sin \angle 4}{x}$.

19. $\frac{\sin \angle 4}{x} = \frac{\sin \angle 3}{y}$.

20. They imply that $\sin \angle 1 = \sin \angle 3$. Substituting for $\frac{\sin \angle 4}{x}$, we get $\frac{\sin \angle 1}{y} = \frac{\sin \angle 3}{y}$; multiplying by y gives $\sin \angle 1 = \sin \angle 3$.

•21. 108°. ($\angle 3 = 180° - \angle 2 = 180° - \angle 1$.)

22. $\sin \angle 1 = \sin 72° \approx 0.9510565 \ldots$

23. $\sin \angle 3 = \sin 108° \approx 0.9510565 \ldots$

24. They suggest that the sines of supplementary angles are equal.

Set II (pages 472–474)

In a caption to a figure on which the figure for exercises 25 through 30 is based, John Keay, the author of *The Great Arc—The Dramatic Tale of How India Was Mapped and Everest Was Named* (Harper Collins, 2000), describes the triangulation process: "Given the distance between points A and B (the base-line), the distance from each to point C is calculated by trigonometry using the AB base-line measurement plus the angles CAB and ABC as measured by sighting with a theodolite. The distance between C and B having now been established, it may be used as the base-line for another triangle to plot the position of D. CD may then be used as the base for a third triangle, and so on." Keay points out that the accuracy of the Survey of India was important not only in measuring a subcontinent but also in determining the curvature of the earth.

As it travels in its orbit around the earth, the moon's distance from the earth varies between 221,463 miles and 252,710 miles. In his fascinating book titled *What If the Moon Didn't Exist?* (Harper Collins, 1993), astronomer Neil F. Comins observes that, ever since the Apollo 12 astronauts left the laser reflector on the moon in 1969, the distance at any given time can be determined to within a few inches. These measurements reveal that the moon is moving away from the earth at a rate of about 2 inches per year.

Triangulation.

•25. $\angle 3 = 56°$. ($\angle 3 = 180° - \angle A - \angle 1 = 180° - 73° - 51° = 56°$.)

•26. CB ≈ 27.7 mi. ($\frac{CB}{\sin 73°} = \frac{24.0}{\sin 56°}$, $CB = \frac{24.0 \sin 73°}{\sin 56°} \approx 27.7$.)

27. $\angle D = 53°$. ($\angle D = 180° - \angle 2 - \angle 4 = 180° - 42° - 85° = 53°$.)

•28. DB ≈ 34.6 mi. ($\frac{DB}{\sin 85°} = \frac{27.7}{\sin 53°}$, $DB = \frac{27.7 \sin 85°}{\sin 53°} \approx 34.6$.)

29. CD ≈ 23.2 mi. ($\frac{CD}{\sin 42°} = \frac{27.7}{\sin 53°}$, $CD = \frac{27.7 \sin 42°}{\sin 53°} \approx 23.2$.)

30. $CD^2 = CB^2 + DB^2 - 2(CB)(DB)\cos 42°$, $CD^2 = 27.7^2 + 34.6^2 - 2(27.7)(34.6) \cos 42°$, $CD^2 \approx 539.96$, $CD \approx 23.2$.

Distance to the Moon.

31. $\angle M = 1.4°$.

•32. 238,000 mi. ($\frac{AM}{\sin 89.2°} = \frac{5{,}820}{\sin 1.4°}$, $AM = \frac{5{,}820 \sin 89.2°}{\sin 1.4°} \approx 238{,}187$.)

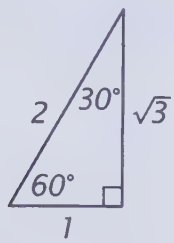
33. 238,000 mi. ($\frac{BM}{\sin 89.4°} = \frac{5{,}820}{\sin 1.4°}$,
$BM = \frac{5{,}820 \sin 89.4°}{\sin 1.4°} \approx 238{,}197$.)

The Case of the Equilateral Triangle.

•34. $a^2 = b^2 + c^2 - 2bc \cos A$.

•35. $a^2 = a^2 + a^2 - 2aa \cos A$, $a^2 = 2a^2 - 2a^2 \cos A$,
$2a^2 \cos A = a^2$, $\cos A = \frac{1}{2}$.

36. *Example figure:*



$\cos 60° = \frac{1}{2}$.

The Case of the Right Triangle.

37. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

38. 1.

39. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{1}{c}$.

•40. By multiplication.

41. The definitions of the sine and cosine based on a right triangle.

42. $c^2 = a^2 + b^2 - 2ab \cos C$.

43. 0.

•44. $c^2 = a^2 + b^2$.

45. The Pythagorean Theorem.

The Case of the "Flattened" Triangle.

46. $c^2 = a^2 + b^2 - 2ab \cos C$.

47. –1.

48. $c^2 = a^2 + b^2 + 2ab$.

49. Yes. Because $c^2 = a^2 + 2ab + b^2 = (a + b)^2$,
$c = a + b$.

Euclid's Law of Cosines.

50. The area of the rectangle is cx. In right ΔADC, $\cos A = \frac{x}{b}$; so $x = b \cos A$.
Substituting for x, the area of the rectangle is $c(b \cos A) = bc \cos A$.

Set III (page 474)

The idea for this exercise came from Carl Eckart's book titled *Our Modern Idol—Mathematical Science* (Scripps Institution of Oceanography, University of California, 1984). The 780-1351-1560 triangle is a remarkably good approximation of a 30°-60° right triangle because $\frac{1351}{780}$ matches $\sqrt{3}$ to five decimal places ($\frac{1351}{780} = 1.7320512\ldots$ and $\sqrt{3} = 1.7320508\ldots$) and $\frac{1560}{780} = 2$. Using *Mathematica* reveals that, to four decimal places, ΔABC is a 30.0000°-60.0000°-90.0000° triangle. To five decimal places, it is a 30.00000°-60.00003°-89.99997° triangle. To eleven decimal places, $\angle A = 30.00000000000°$ but, to twelve decimal places, $\angle A = 29.999999999996°$! Eckart remarks that "such a triangle differs from a right triangle by less than one-tenth of a second" and observes that it cannot be drawn on paper, because "paper is too flexible, and even printed straight lines are too coarse."

A Possible Case of Mistaken Identity.

1. About 30.000000°.
[$780^2 = 1351^2 + 1560^2 - 2(1351)(1560) \cos A$,
$\angle A \approx 30.000000°$.]

2. About 60.000027°. [By the Law of Sines, $\frac{\sin B}{1351} = \frac{\sin 30°}{780}$, $\sin B = \frac{1351 \sin 30°}{780}$,
$\angle B \approx 60.000027°$ or, by the Law of Cosines,
$1351^2 = 780^2 + 1560^2 - 2(780)(1560) \cos B$,
$\angle B \approx 60.000027°$.]

3. ΔABC is an acute triangle. From the measures of $\angle A$ and $\angle B$, it follows that $\angle C \approx 89.999973°$. Also,
$1351^2 + 780^2 = 2{,}433{,}601$, but
$1560^2 = 2{,}433{,}600$; so $a^2 + b^2 > c^2$. (The triangle is a very good imitation, however, of a 30°-60° right triangle!)

Chapter 11, Review

Set I (pages 476–478)

Exercises 23 through 28 provide another example of a pair of triangles that have five pairs of equal parts and yet are not congruent. Letting AD = 1 and AC = x and using the fact that each leg of a right triangle is the geometric mean between the hypotenuse and its projection on the hypotenuse, we have $\frac{x+1}{x} = \frac{x}{1}$. Solving for x, $x^2 = x + 1$, $x^2 - x - 1 = 0$, $x = \frac{1+\sqrt{5}}{2}$, the golden ratio! In ΔACD, $\cos A = \frac{1}{x}$; so $\cos A = \frac{2}{1+\sqrt{5}}$, $\angle A = 51.827292\ldots°$ and $\angle B = 38.172707\ldots°$

According to J. V. Field in *The Invention of Infinity—Mathematics and Art in the Renaissance* (Oxford University Press, 1997), "the great invention of the military engineers [of the sixteenth century] is the polygonal fort. . . . The commonest choice is the pentagon. A construction for a regular pentagon can be found in *Elements*, Book 4, but it is hardly the sort of construction one would wish to use in laying out the ground plan of a fortress. . . . The practical method used by the Venetians, preserved in a file in the State Archives, is shown in [the figure used in the exercises]. We are given the sides of a right-angled triangle. The triangle can be constructed from rope, or by linking together the chains used by surveyors, and repetitions of it will then allow the complete pentagon to be laid out. The person who actually lays out the ground plan using this method does not need to be very good at mathematics, but it is clear that some mathematical expertise was needed to devise the method, and to calculate the lengths of the sides of the triangle."

Greek Cross.

•1. 2.

2. $-\frac{1}{2}$.

3. 2.

4. $\frac{3}{4}$.

•5. $-\frac{4}{3}$.

6. AC ⊥ CD.

7. AC || DE.

•8. HF ⊥ BG.

9. They are equal.

10. Their product is –1. (Or, one slope is the opposite of the reciprocal of the other.)

Doubled Square.

•11. $d = s\sqrt{2}$.

12. s^2.

13. d^2.

14. $2s^2$. [$d^2 = (s\sqrt{2})^2 = 2s^2$.]

Escalator Design.

•15. 28 ft. [$c = 2a = 2(14)$.]

•16. About 24.2 ft. ($b = a\sqrt{3} = 14\sqrt{3} \approx 24.2$.)

17. 20 ft. ($40 = 2a$.)

18. About 34.6 ft. ($b = a\sqrt{3} = 20\sqrt{3} \approx 34.6$.)

•19. About 195 ft. ($b = 338$; so $a = \frac{338}{\sqrt{3}} \approx 195$.)

20. About 390 ft. [$c = 2a \approx 2(195)$.]

Basketball Angles.

•21. About 5.7°.
($\tan B = \frac{h}{D} = \frac{10-7.5}{25} = \frac{2.5}{25} = 0.1$, $\angle B \approx 5.7°$.)

22. About 14°. ($\tan B = \frac{2.5}{10} = 0.25$, $\angle B \approx 14°$.)

Equal Parts.

23. Three.

24. The altitude to the hypotenuse of a right triangle forms two triangles similar to it and to each other.

25. Three.

•26. Two.

27. Five.

28. No. The longest sides of the two triangles are not equal.

Pentagon.

29. $\angle BOC = 36°$. $\left(\frac{360°}{10}.\right)$

•30. BC = 100 ft. $\left(\frac{1{,}000}{10}.\right)$

•31. OB ≈ 137.6 ft. $\left(\tan 36° = \frac{100}{OB},\right.$
$\left.OB = \frac{100}{\tan 36°} \approx 137.6.\right)$

32. OC ≈ 170.1 ft. $\left(\sin 36° = \frac{100}{OC},\right.$
$\left.OC = \frac{100}{\sin 36°} \approx 170.1.\right)$

33. $OB^2 + BC^2 \approx 137.6^2 + 100^2 \approx 28{,}934$ and $OC^2 \approx 170.1^2 \approx 28{,}934$.

•34. About 408 feet.
(ρΔOBC ≈ 137.6 + 100 + 170.1 ≈ 408.)

Three Ratios.

•35. 0.87. (Letting the sides of the triangle and square be 1 unit, $CF = \frac{\sqrt{3}}{2}$ and EA = 1; so $\frac{CF}{EA} = \frac{\sqrt{3}}{2} \approx 0.87$.)

36. 0.75. (ρΔABC = 3 and ρABDE = 4; so $\frac{\rho\Delta ABC}{\rho ABDE} = \frac{3}{4} = 0.75$.)

37. 0.43. $\left(\alpha\Delta ABC = \frac{\sqrt{3}}{4}(1)^2\right.$ and $\left.\alpha ABDE = 1^2\right.$; so $\frac{\alpha\Delta ABC}{\alpha ABDE} = \frac{\sqrt{3}}{4} \approx 0.43$.)

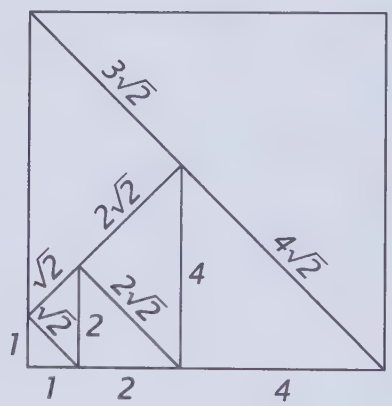
Set II (pages 478–480)

A discussion of various Steiner problems can be found in chapter 3, "Shortest and Quickest Connections," of *The Parsimonious Universe—Shape and Form in the Natural World*, by Stefan Hildebrandt and Anthony Tromba (Copernicus, 1996). Given a general set of points, the shortest network of segments connecting them is called their "minimal Steiner tree." These trees are important in the design of computer chips and have even been used by biologists studying the relationships of species. A rather amazing natural solution to finding Steiner trees comes from dipping models made from glass or plastic plates into soap solutions!

The Babylonian clay tablet in exercises 65 and 66 was deciphered in 1945. David M. Burton remarks in his *History of Mathematics* (Allyn & Bacon, 1985): "This tablet is written in Old Babylonian script, which dates it somewhere between 1900 B.C. and 1600 B.C. The analysis of this extraordinary group of figures establishes beyond any doubt that the so-called Pythagorean theorem was known to Babylonian mathematicians more than a thousand years before Pythagoras was born." Burton observes that the 541–481 error was probably that of a scribe because, in the sexagesimal notation of the Babylonians, 541 is written as 9,1 and 481 as 8,1. Eleanor Robson shows in "Words and Pictures—New Light on Plimpton 322" (*American Mathematical Monthly*, February 2002) that it is likely that these Pythagorean triples actually arose from computations concerning reciprocals. Even so, "Pythagorean triples" were of interest long before Pythagoras.

Isosceles Right Triangles.

38.



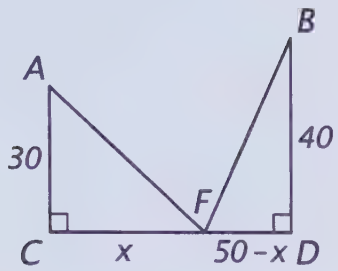
39. $\sqrt{2}$, 2, $2\sqrt{2}$, 4, $3\sqrt{2}$, and 7 units.

•40. 7 units.

41. $\frac{1}{2} + 1 + 2 + 4 + 8 + 9 + 24\frac{1}{2} = 49$.

Two Birds.

42.



•43. AF = BF.

44. $50 - x$.

45. $30^2 + x^2 = 40^2 + (50 - x)^2$,
$900 + x^2 = 1{,}600 + 2{,}500 - 100x + x^2$,
$100x = 3{,}200$, $x = 32$.

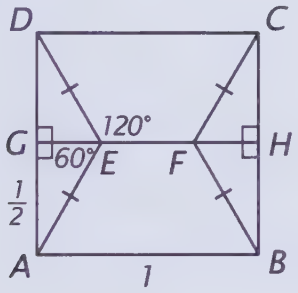
46. 32 ft and 18 ft.

•47. About 44 ft. ($AF^2 = 30^2 + 32^2 = 1{,}924$,
$AF \approx 43.9$, $BF^2 = 40^2 + 18^2 = 1{,}924$, $BF \approx 43.9$.)

Road Systems.

48. 3 mi.

49.



Approximately 2.7 mi.
In 30°-60° right ΔAGE, the longer leg

$AG = \frac{1}{2}$; so the shorter leg $GE = \frac{\frac{1}{2}}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$

and the hypotenuse $AE = 2GE = \frac{1}{\sqrt{3}}$.

$EF = 1 - 2(\frac{1}{2\sqrt{3}}) = 1 - \frac{1}{\sqrt{3}}$. The sum of the five segments is

$(1 - \frac{1}{\sqrt{3}}) + 4(\frac{1}{\sqrt{3}}) = 1 + \frac{3}{\sqrt{3}} = 1 + \sqrt{3} \approx 2.732$.

50. The second solution is better.

Sun and Moon.

51. In right ΔOMS, $\cos \angle O = \frac{m}{s}$. Because $\cos 87° \approx 0.052 \approx \frac{1}{19}$, $\frac{m}{s} = \frac{1}{19}$.

52. Because the product of the means is equal to the product of the extremes (or multiplication), $s = 19m$.

•53. About 380 times as far. ($\cos 89.85° = \frac{m}{s}$; so $\frac{s}{m} = \frac{1}{\cos 89.85°} \approx 382$.)

Eye Chart.

54. 20 ft = 240 in. $\tan P = \frac{3.75}{240} = 0.015625$, $\angle P \approx 0.895°$; so $\angle P \approx 0.895° \times 60$ minutes per degree ≈ 54 minutes.

•55. Approximately $\frac{1}{3}$ in.
[4.8 minutes $= (\frac{4.8}{60})° = 0.08°$.
$\tan 0.08° = \frac{x}{240}$, $x = 240 \tan 0.08° \approx 0.34$ in.]

UFO Altitude.

56. 75°. (180° – 60° – 45° = 75°.)

•57. About 600 ft. ($\frac{AC}{\sin 45°} = \frac{820}{\sin 75°}$, $AC = \frac{820 \sin 45°}{\sin 75°} \approx 600$.)

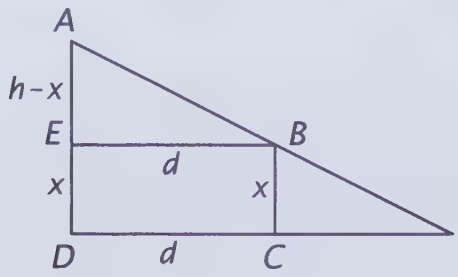
58. About 520 ft. ($\sin 60° \approx \frac{CD}{600}$, $CD \approx 600 \sin 60° \approx 520$.)

59. About 735 ft. ($\frac{BC}{\sin 60°} = \frac{820}{\sin 75°}$, $BC = \frac{820 \sin 60°}{\sin 75°} \approx 735$.)

60. About 520 ft; so it checks. ($\sin 45° \approx \frac{CD}{735}$, $CD \approx 735 \sin 45° \approx 520$.)

Outdoor Lighting.

61.



•62. $\tan \angle ABE = \dfrac{h-x}{d}$.

63. $x = h - (\tan \angle ABE)d$. ($d \tan \angle ABE = h - x$.)

64. They came from the equation for exercise 63 (and the fact that $\angle ABE = 90° - \angle A$.)

Number Mystery.

65. $\sqrt{75^2 - 45^2} = 60$

$\sqrt{97^2 - 65^2} = 72$

$\sqrt{481^2 - 319^2} = 360$

$\sqrt{769^2 - 541^2} = 546.516\ldots$

$\sqrt{1249^2 - 799^2} = 960$

$\sqrt{2929^2 - 1679^2} = 2400$

With one exception, each pair of numbers is part of a Pythagorean triple; specifically, the lengths of one leg and the hypotenuse.

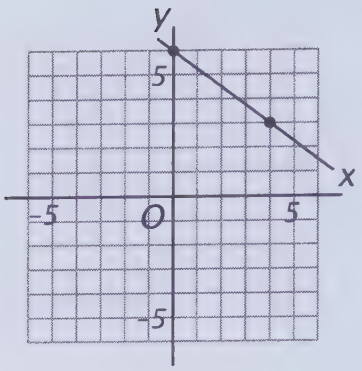
66. As the list for exercise 65 shows, the mistake is in the pair of numbers 541, 769. (541 should be 481.)

Algebra Review (page 482)

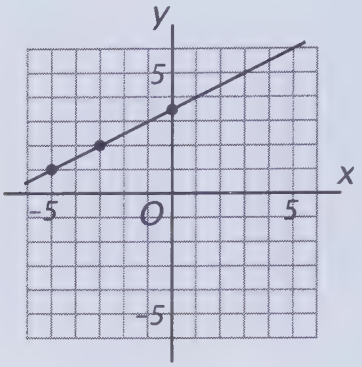
•1. $3x + 4y = 24$,
$4y = -3x + 24$,
$y = -\dfrac{3}{4}x + 6$.

•2. $m = -\dfrac{3}{4}$; $b = 6$.

•3.



•4.



•5. $y - 1 = \dfrac{1}{2}(x - (-5))$,

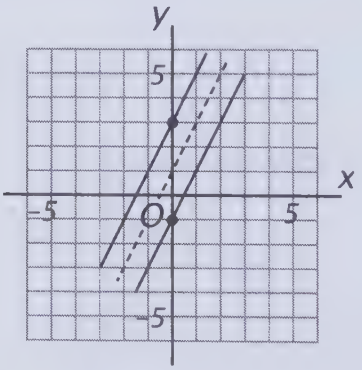
$y - 1 = \dfrac{1}{2}(x + 5)$.

•6. $y = \dfrac{1}{2}x + \dfrac{5}{2} + 1$,

$y = \dfrac{1}{2}x + \dfrac{7}{2}$.

•7. $b = \dfrac{7}{2}$ (or 3.5).

•8.

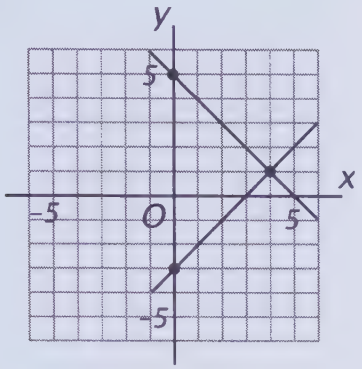


•9. They have the same slope, 2.

•10. $y = 2x + 1$.

•11. $x + y = 5$,
$y = -x + 5$;
$5x - 5y = 15$,
$-5y = -5x + 15$,
$y = x - 3$.

•12.



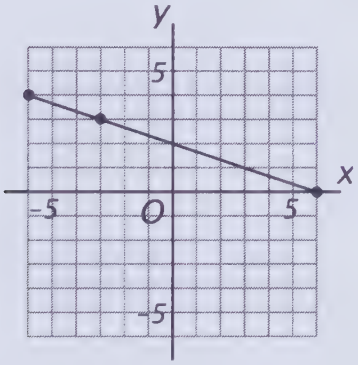
•13. –1 and 1.

•14. The product of their slopes is –1.

•15. $-\frac{1}{3}$.

•16. –6. [$4 - 4 = -\frac{1}{3}(x + 6)$, $0 = -\frac{1}{3}(x + 6)$,
$x + 6 = 0$, $x = -6$.]

•17.



•18. $y - 4 = -\frac{1}{3}(x + 6)$,

$y = -\frac{1}{3}x - 2 + 4$,

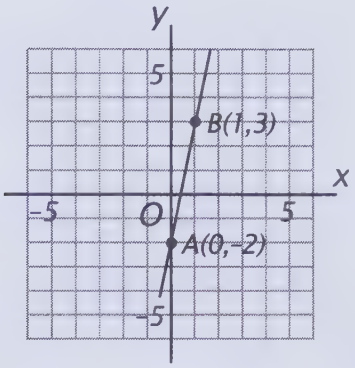
$y = -\frac{1}{3}x + 2$.

•19. x-intercept, 6; y-intercept, 2.

•20. $y = -\frac{1}{3}x + 2$,

$\frac{1}{3}x + y = 2$,
$x + 3y = 6$.

•21.



•22. 5.

•23. $y - 3 = 5(x - 1)$.

•24. $y = 5x - 5 + 3$,
$y = 5x - 2$.

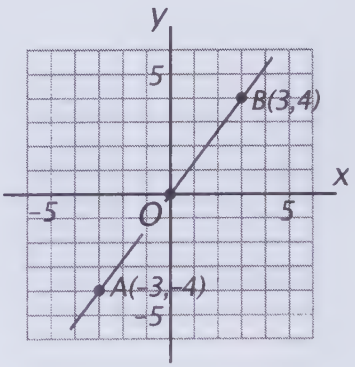
•25. $-5x + y = -2$,
$5x - y = 2$.

•26. $4x - 3y = 0$,
$-3y = -4x$,

$y = \frac{4}{3}x$ (or $y = \frac{4}{3}x + 0$).

•27. $m = \frac{4}{3}$; $b = 0$.

•28.



•29. 0.

•30. $AB = \sqrt{(3-(-3))^2 + (4-(-4))^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.

Chapter 12, Lesson 1

Set I (pages 486–487)

According to Matthys Levy and Mario Salvadori in their book titled *Why Buildings Fall Down* (Norton, 1992), the Romans built semicircular arches of the type illustrated in exercises 1 through 5 that spanned as much as 100 feet. These arches were used in the bridges of 50,000 miles of roads that the Romans built all the way from Baghdad to London!

The conclusions of exercises 13 through 15 are supported by the fact that a diameter of a circle is its longest chord and by symmetry considerations. As the second chord rotates away from the diameter in either direction about the point of intersection, it becomes progressively shorter.

Roman Arch.

1. Radii.

•2. Chords.

•3. A diameter.

4. Isosceles. Each has two equal sides because the radii of a circle are equal.

5. They are congruent by SSS.

Old Figure.

•6. Five.

7. One.

8. Four.

•9. Six.

Butterfly Spots.

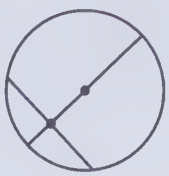
•10. Concentric.

11. It means that a set of circles lie in the same plane and have the same center.

12. *(Student answer.)* (The spots look like big eyes; so they might scare some predators away.)

Chord Problem.

13.



•14. They are perpendicular.

15. The longest chord is always a diameter; so its length doesn't depend on the location of the point. The length of the shortest chord depends on how far away from the center the point is; the greater the distance, the shorter the chord.

Theorem 56.

16. Perpendicular lines form right angles.

•17. Two points determine a line.

•18. All radii of a circle are equal.

19. Reflexive.

20. HL.

21. Corresponding parts of congruent triangles are equal.

•22. If a line divides a line segment into two equal parts, it bisects the segment.

Theorem 57.

23. If a line bisects a line segment, it divides the segment into two equal parts.

24. Two points determine a line.

25. All radii of a circle are equal.

•26. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Theorem 58.

•27. If a line through the center of a circle bisects a chord that is not a diameter, it is also perpendicular to the chord.

•28. Through a point on a line, there is exactly one line perpendicular to the line.

29. CD contains O.

Set II (pages 488–490)

The SAT problem of exercises 30 and 31 is something of an oddity in that it is a rather well known "trick question." Martin Gardner included it in his "Mathematical Games" column in the November 1957 issue of *Scientific American*—many years before it appeared on the SAT!

The CBS eye, a creation of designer William Golden, first appeared on television in 1951. Hal

Morgan observes in his book titled *Symbols of America* (Viking, 1986) that, "because it is essentially a picture of an eye, it has proved a challenge to the police efforts of the CBS trademark attorneys. As one company executive admitted, 'Bill Golden, after all, didn't invent the eye. God thought of it first.' "

The story of the use of Jupiter to determine the speed of light is told by William Dunham in *The Mathematical Universe* (Wiley, 1994). After the invention of the telescope in the seventeenth century and the discovery of the moons of Jupiter, astronomers were puzzled by the delay in the eclipses of the moon Io when the earth and Jupiter were farthest apart. Ole Roemer reasoned that the time discrepancy was due to the extra distance that the light from Jupiter and its moon had to travel to get to the earth. Surprisingly, it was not the speed of light that especially interested Roemer but rather the conclusion that light does not travel instantaneously from one place to another, something that had not been known previously.

The definition of the roundness of a grain of sand was devised by Haakon Wadell in about 1940. He defined it to be (for cross sections of the grain) the "ratio of average radius of curvature of the corners to radius of maximum inscribed circle." From this definition, it follows that roundness can vary from 0 to 1, with perfect roundness being 1.

Exercises 43 through 47 are based on a simple theorem discovered in 1916 by the American geometer Roger Johnson: "If three circles having the same radius pass through a point, the circle through their other three points of intersection also has the same radius." The appearance of the cube in the explanation and the role played by two of its opposite vertices is quite unexpected. For more information on this elegant theorem, see chapter 10 of George Polya's *Mathematical Discovery*, vol. 2 (Wiley, 1965), and the fourth of the "Four Minor Gems from Geometry" in Ross Honsberger's *Mathematical Gems II* (Mathematical Association of America, 1976).

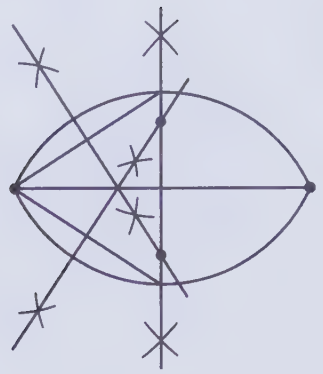
SAT Problem.

30. 10.

31. The diagonals of a rectangle are equal. The other diagonal of the rectangle is a radius of the circle.

CBS Eye.

32. *Example answer:*



33. The perpendicular bisector of a chord of a circle contains the center of the circle.

Red-Spot Puzzle.

•34. An isosceles right triangle.

•35. About 46 mm. ($\frac{65}{\sqrt{2}} \approx 45.96$.)

36. Very difficult because the disks are barely big enough to cover the spot.

Speed of Light.

37. 186,000,000 mi. ($2 \times 93{,}000{,}000$.)

•38. 1,000 seconds. ($16 \times 60 + 40$.)

39. 186,000 miles per second. ($\frac{186{,}000{,}000}{1{,}000}$.)

Roundness.

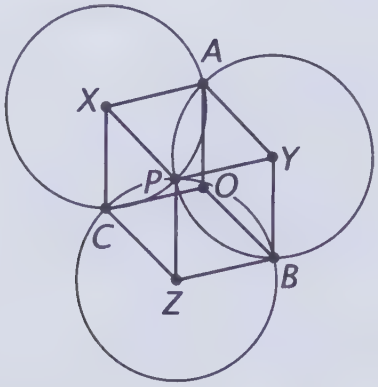
•40. $\frac{10}{27}$, or about 0.37. [$\frac{2+3+5}{3(9)}$.]

41. $\frac{20}{54} = \frac{10}{27}$, or about 0.37. [$\frac{4+6+10}{3(18)}$.]

42. No. The figure would be larger but still have the same shape.

Three Circles.

43.



44. They are each 1 unit.

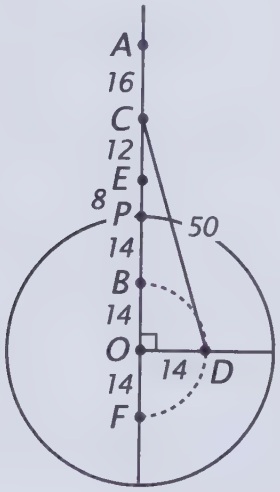
•45. They are rhombuses because all of their sides are equal.

46. They are equal. The opposite sides of a parallelogram are equal and the other segments all equal 1 unit.

47. The circle would have its center at O because OA = OB = OC = 1.

Steam Engine.

48.



49. EF = 50 in. [EP + PF = 8 + 3(14).]

50. CD = 50 in. (CD = EF.)

51. CO = 48 in. ($CO^2 + 14^2 = 50^2$; so $CO = \sqrt{50^2 - 14^2} = \sqrt{2{,}304} = 48$.)

52. CE = 12 in. (CE = CO – EO = 48 – 36.)

53. AC = 16 in. (AC = AB – CB = 50 – 34.)

Set III (page 490)

When this puzzle appeared in Martin Gardner's "Mathematical Games" column in *Scientific American*, William B. Friedman suggested tying the rope to two places high on the tree at B so that the person could walk along the lower part while holding onto the upper part and not get wet.

Island Puzzle.

Tie one end of the rope to the tree at B. Holding the other end of the rope, walk around the lake so that the rope wraps around the tree at A. Pull the rope tight and tie it at the other end to the tree at B. Hold onto the rope to get to the island.

Chapter 12, Lesson 2

Set I (pages 493–494)

Exercises 15 through 18 reveal how the tangent ratio got its name. As Howard Eves remarks in *An Introduction to the History of Mathematics* (Saunders, 1990), "the meanings of the present names of the trigonometric functions, with the exception of sine, are clear from their geometrical interpretations when the angle is placed at the center of a circle of unit radius. . . . The functions tangent, cotangent, secant, and cosecant have been known by various other names, but these particular names appeared as late as the end of the 16th century."

Band Saw.

1. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

•2. In a plane, two lines perpendicular to a third line are parallel to each other.

3. The opposite sides of a parallelogram are equal.

Concentric Circles.

•4. OB ⊥ AC.

•5. If a line through the center of a circle bisects a chord that is not a diameter, it is also perpendicular to the chord.

6. AC is tangent to the smaller circle.

7. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

Grid Exercise.

8. 5.

•9. $\frac{4}{3}$.

•10. $-\frac{3}{4}$.

11. $-\frac{3}{4}$.

12. Lines l and m are perpendicular to line AB.

13. Lines l and m are parallel.

14. The slopes of l and m are equal; the slope of AB is the opposite of the reciprocal of their slope.

Double Meaning.

•15. AB is tangent to circle O.

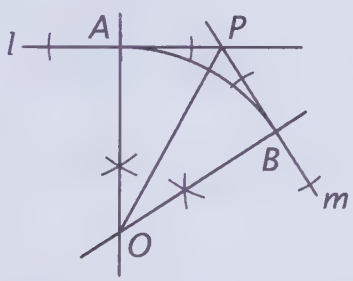
•16. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

17. The tangent of an acute angle of a right triangle is the ratio of the length of the opposite leg to the length of the adjacent leg.

•18. $\frac{AB}{1}$, or AB.

Railroad Curve.

19.



20. $\Delta APO \cong \Delta BPO$ by HL (they are right triangles with PO = PO and PA = PB).

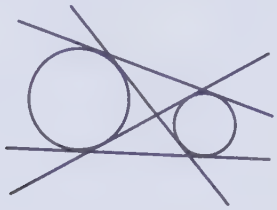
21. Corresponding parts of congruent triangles are equal.

22. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

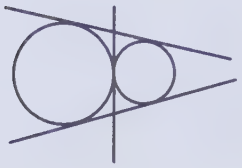
23. No.

Tangents to Two Circles.

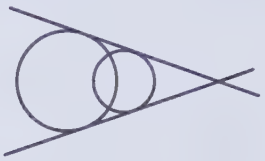
24.



25.



26. *Example figure:*



27. *Example figure:*



28. Yes. No lines can be drawn tangent to both circles if one circle is inside the other and the circles have no point in common.

Set II (pages 494–496)

Euclid's Book III, Proposition 17 is the following construction: "From a given point to draw a straight line touching a given circle." A construction based on the fact that an angle inscribed in a semicircle is a right angle is introduced in Lesson 6. In *Euclid—The Creation of Mathematics* (Springer, 1999), Benno Artmann says of Euclid's construction: "The unexpected way in which the tangent l, which at first sight has nothing to do with the problem, is used to produce the wanted tangent AB is a nice surprise and makes the construction very elegant."

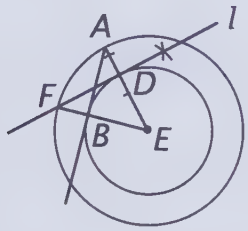
Although the angle trisector called the "tomahawk" has been included in many twentieth-century textbooks, no one knows who invented it. The tomahawk is one of many methods for

solving the trisection problem by going beyond the restrictions of classical constructions. According to Robert C. Yates in *The Trisection Problem* (Franklin Press, 1942; N.C.T.M., 1971), the tomahawk was mentioned in *Geométrie Appliquée a l'Industrie* published in 1835.

J. L. Heilbron in his *Geometry Civilized—History, Culture, and Technique* (Clarendon Press, 1998) writes that the Gothic arch of exercises 39 through 45 can be seen everywhere in churches built in the later middle ages and that it was often filled with other geometric shapes such as circles, squares, and smaller arches. The exercises reveal the surprising appearance of the 3-4-5 right triangle in the construction of an inscribed circle in a Gothic arch.

Construction Problem.

29.



30. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

31. SAS. (EF = EA, ∠E = ∠E, and ED = EB.)

•32. Because ∠FDE is a right angle and corresponding parts of congruent triangles are equal.

33. It is tangent to the smaller circle.

•34. Line AB.

Tomahawk.

35. SAS. (AB = BC, ∠ABY = ∠CBY, and BY = BY.)

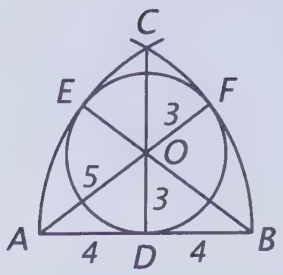
•36. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

37. HL. (ΔCBY and ΔCPY are right triangles with CY = CY and BC = CD.)

38. ∠AYB = ∠BYC and ∠BYC = ∠CYP because corresponding parts of congruent triangles are equal. Because ∠XYZ has been divided into three equal angles, it is trisected.

Gothic Window.

39.



40. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

•41. AD = 4 cm and AF = AB = 8 cm.

•42. $(8 - r)$ cm. (AO = AF − OF = $8 - r$.)

43. $(8 - r)^2 = 4^2 + r^2$ by the Pythagorean Theorem. $64 - 16r + r^2 = 16 + r^2$, $16r = 48$, $r = 3$.

•44. A 3-4-5 right triangle.

45. $4\sqrt{3} \approx 6.9$ cm. [If AC and BC were drawn, CD would be an altitude of equilateral ΔABC; so CD = $\frac{\sqrt{3}}{2}(8)$.]

Tangent Puzzle.

46. DC is perpendicular to AD and BC. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

47. They are parallel. In a plane, two lines perpendicular to a third line are parallel to each other.

•48. A parallelogram (also, a rectangle).

49. A right triangle.

50. By the Pythagorean Theorem: In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

51. AB = $y + z$, AE = $y - z$, EB = x.

52. $(y + z)^2 = (y - z)^2 + x^2$.

53. $y^2 + 2yz + z^2 = y^2 - 2yz + z^2 + x^2$, $4yz = x^2$, $x^2 = 4yz$.

Set III (page 496)

On a trip from the earth to the moon, it is obvious that at the halfway point the earth would look

bigger because it *is* bigger. What makes the earth and moon appear to have the same size as seen from point P is that the vertical angles formed by their common internal tangents are equal. When we observe an object in space, it is this angle, called its "angular size," rather than its true size, by which we judge it.

Trip to the Moon.

EP is approximately 184,000 miles. $\Delta AEP \sim \Delta BMP$ by AA ($\angle A$ and $\angle B$ are equal right angles and $\angle APE$ and $\angle BPM$ are equal vertical angles). $\frac{EP}{MP} = \frac{EA}{MB}$ (corresponding sides of similar triangles are proportional). Letting $EP = x$, $\frac{x}{234{,}000 - x} = \frac{3{,}960}{1{,}080}$, $1{,}080x = 926{,}640{,}000 - 3{,}960x$, $5{,}040x = 926{,}640{,}000$, $x \approx 183{,}857$.

Chapter 12, Lesson 3

Set I (pages 500–501)

In a section of Chapter 2 of *Geometry Civilized—History, Culture, and Technique* (Clarendon Press, 1998), J. L. Heilbron treats the subject of the size of the earth. Columbus used a number for the circumference of the earth that was about 75% of what it actually is. Heilbron says that this mistake "had more profound consequences than any other geometrical error ever committed." To try to convince others of the practicality of his plan, Columbus took several steps to minimize the length of the trip. Fortunately for him, he encountered the New World almost exactly where he had expected to find Japan.

The "three-body problem," to understand the mutual effects of gravity on three objects in space, frustrated scientists from the time of Newton to the end of the nineteenth century, when it was proved that mathematical solutions of the type that they had been seeking do not exist. The only situation for which there is an exact and stable solution is the case in which the three objects lie at the vertices of an equilateral triangle: this case exists in our solar system with the two equilateral triangles formed by the sun, Jupiter, and the two sets of asteroids called the Trojans. It has been suggested that this arrangement might also be appropriate for establishing the locations of the earth's first colonies in space. The two triangles would be formed by the earth, the moon, and the two colonies.

Routes to Japan.

•1. Central angles.

•2. $\overset{\frown}{PJ}$.

•3. $\overset{\frown}{PCJ}$.

4. 30°.

•5. 283°. (253 + 30.)

6. 77°. (360 – 283.)

•7. 77°.

Theorem 62.

•8. Two points determine a line.

9. All radii of a circle are equal.

•10. An arc is equal in measure to its central angle.

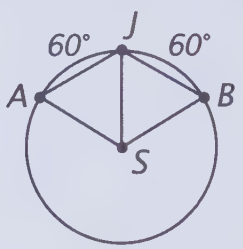
11. Substitution.

12. SAS.

13. Corresponding parts of congruent triangles are equal.

Jupiter's Asteroids.

14.



15. An arc is equal in measure to its central angle.

•16. The Arc Addition Postulate.

17. SAS. (SA = SJ = SB and $\angle ASJ = \angle BSJ$.)

18. Equilateral.

Arc Illusion.

•19. A semicircle.

20. Minor.

21. *(Student answer.)* (Many people assume that, because EF seems straightest, it comes from the largest circle.)

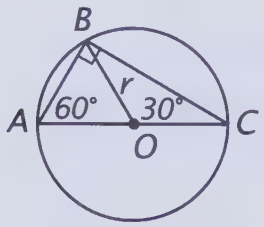
22. The three arcs seem to be part of congruent circles.

Chords and Arcs.

23. The Arc Addition Postulate.

•24. No. The sum of any two sides of a triangle is greater than the third side.

25.



26. $m\overset{\frown}{AB} = 60°$, $m\overset{\frown}{BC} = 120°$, $m\overset{\frown}{ABC} = 180°$.

•27. $AB = r$, $BC = r\sqrt{3}$, $AC = 2r$. (ΔABC is a 30°-60° right triangle.)

28. $m\overset{\frown}{ABC} = 3m\overset{\frown}{AB}$.

29. $BC = \sqrt{3}\,AB$.

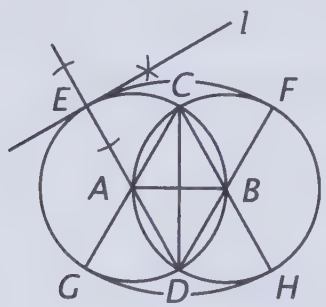
Set II (pages 501–503)

J. L. Heilbron observes that ovals formed from pairs of equal interlaced or tangent circles were commonly used in designing plazas in sixteenth-century Italy. St. Peter's Square in Rome is the most famous example.

Olympiodorus taught in Alexandria in the sixth century A.D. He wrote extensively on the rainbow, even attempting, as outlined in exercises 37 through 39, to determine the distance from the observer's eye to the center of the rainbow when the sun was on the horizon. The methods fails because, if SO were equal to OR, the rainbow would be well outside the atmosphere, which it is not.

The Square That Isn't.

30.



•31. ΔABC and ΔABD are equilateral and congruent.

32. A rhombus (and hence a parallelogram).

•33. Line l is tangent to circle A. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

34. Line l is also perpendicular to radius DE of the circle D of which $\overset{\frown}{EF}$ is an arc.

35. $m\overset{\frown}{GE} = 120°$, $m\overset{\frown}{EF} = 60°$, $m\overset{\frown}{FH} = 120°$, $m\overset{\frown}{HG} = 60°$.

36. 360°.

Rainbow Distance.

37. $x^2 + y^2 = 1$.

38. $(x + 1)^2 + y^2 = z^2$.

39. From the first equation, $y^2 = 1 - x^2$ and, from the second equation, $y^2 = z^2 - (x + 1)^2$.
So $1 - x^2 = z^2 - (x + 1)^2$,
$1 - x^2 = z^2 - (x^2 + 2x + 1)$,
$1 - x^2 = z^2 - x^2 - 2x - 1$,
$2x = z^2 - 2$,
$x = \dfrac{z^2 - 2}{2}$.

Night Sky.

40. The photograph is a time exposure. The stars seem to travel in circles about a point directly above the earth's axis because of the earth's rotation on its axis.

•41. All of the arcs have the same measure. (They all have equal central angles—namely, the angle through which the earth rotated during the time exposure.)

42. *(Student answer.)* (The earth rotates 360° in 24 hours, or 180° in 12 hours.The arcs appear to be somewhat smaller than semicircles; so a guess of about 10 or 11 hours is reasonable.)

Original Proofs.

43. *Proof.*
(1) Vertical angles $\angle AOB$ and $\angle COD$ are central angles of both circles. (Given.)
(2) $\angle AOB = \angle COD$. (Vertical angles are equal.)
(3) $m\overset{\frown}{AB} = \angle AOB$ and $m\overset{\frown}{CD} = \angle COD$. (An arc is equal in measure to its central angle.)
(4) $m\overset{\frown}{AB} = m\overset{\frown}{CD}$. (Substitution.)

44. *Proof.*
(1) The vertices of trapezoid ABCD (AD || BC) lie on a circle and $m\overset{\frown}{AB} = m\overset{\frown}{CD}$. (Given.)
(2) AB = CD. (In a circle, equal arcs have equal chords.)
(3) ABCD is isosceles. (A trapezoid is isosceles if its legs are equal.)
(4) $\angle B = \angle C$. (The base angles of an isosceles trapezoid are equal.)

45. *Proof.*
(1) AB = DE. (Given.)
(2) $m\overset{\frown}{AB} = m\overset{\frown}{DE}$. (In a circle, equal chords have equal arcs.)
(3) $m\overset{\frown}{AB} + m\overset{\frown}{BD} = m\overset{\frown}{BD} + m\overset{\frown}{DE}$. (Addition.)
(4) $m\overset{\frown}{AB} + m\overset{\frown}{BD} = m\overset{\frown}{AD}$ and $m\overset{\frown}{BD} + m\overset{\frown}{DE} = m\overset{\frown}{BE}$. (Arc Addition Postulate.)
(5) $m\overset{\frown}{AD} = m\overset{\frown}{BE}$. (Substitution.)
(6) AD = BE. (In a circle, equal arcs have equal chords.)

46. *Proof.*
(1) $m\overset{\frown}{AC} = m\overset{\frown}{AD}$ and $m\overset{\frown}{CB} = m\overset{\frown}{BD}$. (Given.)
(2) AC = AD and CB = BD. (In a circle, equal arcs have equal chords.)
(3) AB is the perpendicular bisector of CD. (In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.)
(4) AB contains the center of the circle. (The perpendicular bisector of a chord of a circle contains the center of the circle.)

47. *Proof.*
(1) AD is a diameter of circle O, and AB || OC. (Given.)
(2) $\angle A = \angle 2$. (Parallel lines form equal corresponding angles.)
(3) $\angle B = \angle 1$. (Parallel lines form equal alternate interior angles.)
(4) OA = OB. (All radii of a circle are equal.)
(5) $\angle B = \angle A$. (If two sides of a triangle are equal, the angles opposite them are equal.)
(6) $\angle 1 = \angle 2$. (Substitution.)
(7) $m\overset{\frown}{BC} = \angle 1$ and $m\overset{\frown}{CD} = \angle 2$. (An arc is equal in measure to its central angle.)
(8) $m\overset{\frown}{BC} = m\overset{\frown}{CD}$. (Substitution.)

Set III (page 503)

Obtuse Ollie's curve is an example of a curve of constant width. Such curves can be constructed by starting with any number of mutually intersecting lines. The radius of the initial arc is arbitrary; each arc is centered on the point of intersection of the two lines that bound it. Interested students may enjoy proving that such a curve actually closes and that it has a constant width. (The "constant width" in regard to Ollie's curve is DG = EH = FI.) The proofs are based on the fact that all radii of a circle are equal and some simple algebra. Martin Gardner discussed curves of constant width in his "Mathematical Games" column in *Scientific American* (February 1963). This column is included in *The Unexpected Hanging and Other Mathematical Diversions* (Simon & Schuster, 1969).

Ollie is correct. The measures of the six arcs are equal to the measures of their central angles. Three of the central angles are the angles of ΔABC; so their sum is 180°. The other three central angles are vertical to the three angles of ΔABC; so their sum also is 180°.

Chapter 12, Lesson 4

Set I (pages 506–507)

Skeet shooting is a sport in which clay targets are thrown into the air from two locations, a "high house" on the left and a "low house" on the right. The shooters advance from station to station, starting at A and ending at H. Although shooting was included in the first modern Olympics in 1896, skeet shooting did not become an Olympic event until 1968.

Skeet Range.

1. 16.5°. (In ΔAOG, OA = OG; so $\angle G = \angle A$.)

•2. 147°. (In ΔAOG, $\angle AOG = 180 - \angle A - \angle G = 180 - 16.5 - 16.5 = 147$.)

•3. 24.5°. ($m\overset{\frown}{AG} = \angle AOG$; $\frac{147}{6} = 24.5$.)

•4. 33°. [$m\overset{\frown}{GX} = 180 - m\overset{\frown}{AG} = 180 - 147 = 33$, or $\angle A = \frac{1}{2}m\overset{\frown}{GX}$, $m\overset{\frown}{GX} = 2\angle A = 2(16.5) = 33$.]

5. 147°. ($m\overset{\frown}{YX} = \angle YOX = \angle AOG = 147$.)

Theorem 63.

6. Two points determine a line.
•7. All radii of a circle are equal.
8. If two sides of a triangle are equal, the angles opposite them are equal.
•9. An exterior angle of a triangle is equal to the sum of the remote interior angles.
10. Substitution.
11. Division (or multiplication).
•12. A central angle is equal in measure to its intercepted arc.
13. Substitution.
14. Two points determine a line.
•15. Betweenness of Rays Theorem.
16. Substitution.
•17. Arc Addition Postulate.
18. Substitution.
19. The center of the circle lies outside the angle. (The proof is similar to case 2 but uses subtraction.)

Corollary 1.

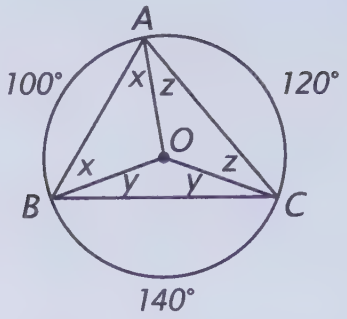
•20. An inscribed angle is equal in measure of half its intercepted arc.
21. Substitution.

Corollary 2.

•22. The measure of a semicircle is 180°.
23. An inscribed angle is equal in measure to half its intercepted arc.
24. Substitution.
•25. A 90° angle is a right angle.

Inscribed Triangle.

26.



27. Central angles.
•28. Inscribed angles.
29. $\angle AOB = 100°$, $\angle BOC = 140°$, $\angle COA = 120°$.
•30. $x = 40°$, $y = 20°$, and $z = 30°$.
31. $\angle ABC = 60°$, $\angle BCA = 50°$, $\angle CAB = 70°$. ($\angle ABC = x + y = 40° + 20° = 60°$, $\angle BCA = y + z = 20° + 30° = 50°$, $\angle CAB = x + z = 40° + 30° = 70°$.)
32. $\angle ABC = \frac{1}{2}m\overset{\frown}{AC} = \frac{1}{2}120° = 60°$;

$\angle BCA = \frac{1}{2}m\overset{\frown}{BA} = \frac{1}{2}100° = 50°$;

$\angle CAB = \frac{1}{2}m\overset{\frown}{CB} = \frac{1}{2}140° = 70°$.

Set II (pages 508–509)

The figure for exercises 33 and 34 appears in a book on the number π because early methods for determining π were based on approximating the circumference of a circle with the perimeters of regular polygons. In the figure, 40 equal chords form a regular polygon having 40 sides that looks very much like a circle. More on this subject will be considered in Chapter 14.

Exercises 35 through 41 establish the theorem that an angle formed by a tangent and a chord is equal in measure to half its intercepted arc (Book III, Proposition 32, of Euclid's *Elements*).

Exercises 43 through 47 show that, if two similar triangles are inscribed in a circle, they must also be congruent.

The method for constructing the square root of a number by inscribing a right triangle in a semicircle appears in the *Elements* as Proposition 13 of Book VI, stated as follows: "To two given straight lines to find a mean proportional."

Exercises 55 through 58 are connected to the following problem: Find the longest line segment that can be drawn through one of the intersection points of two intersecting circles such that its endpoints lie on the circles. This problem is discussed by Mogens Esrom Larsen in an article in *The Lighter Side of Mathematics*, Richard K. Guy and Robert E. Woodrow, editors (Mathematical Association of America, 1994). Larsen appropriately refers to it as a good example illustrating the role of insight in problem solving, "the experience of seeing right through the problem as can happen in geometry."

Forty Angles.

•33. 9°. ($\frac{360}{40}$.)

34. 63°. [$\frac{1}{2}(14 \times 9)$.]

Tangents, Chords, and Intercepted Arcs.

35. ∠BAT is a right angle. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

•36. They are complementary.

•37. It is a right angle. An angle inscribed in a semicircle is a right angle.

38. They are complementary. The acute angles of a right triangle are complementary.

39. They are equal. Complements of the same angle are equal.

40. An inscribed angle is equal in measure to half its intercepted arc.

41. Substitution.

42. They prove that the measure of an angle formed by a tangent and a chord is equal to half the measure of its intercepted arc.

Similar Inscribed Triangles.

•43. $n°$. ($\Delta ABC \sim \Delta DEF$; so $\angle F = \angle C$.)

•44. $2n°$.

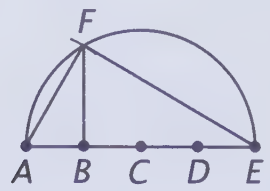
45. $2n°$.

46. AB = DE. In a circle, equal arcs have equal chords.

47. $\Delta ABC \cong \Delta DEF$ by ASA. ($\angle A = \angle D$ and $\angle B = \angle E$ because corresponding angles of similar triangles are equal and AB = DE.)

Square-Root Construction.

48.



49. ∠AFE is a right angle because an angle inscribed in a semicircle is a right angle.

•50. It is an altitude (to the hypotenuse) of the triangle.

51. $\frac{AB}{BF} = \frac{BF}{BE}$.

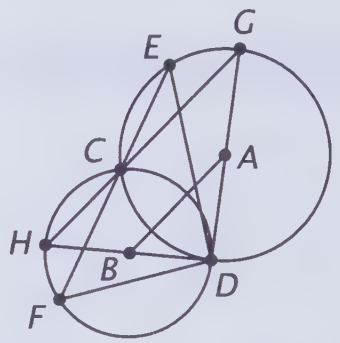
•52. BF = $\sqrt{3}$ in. ($\frac{1}{BF} = \frac{BF}{3}$, $BF^2 = 3$, BF = $\sqrt{3}$.)

53. 1.73 in.

54. Yes. ($BF \approx 1\frac{3}{4} = 1.75$ in.)

Intersecting Triangles and Circles.

55.



56. $\Delta DEF \sim \Delta DGH$ by AA. ($\angle E = \angle G$ and $\angle F = \angle H$ because inscribed angles that intercept the same arc are equal.)

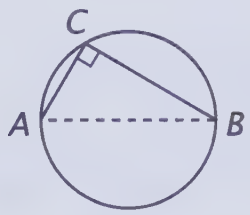
57. ΔDGH is larger because diameter DG > DE and diameter DH > DF.

58. AB = $\frac{1}{2}$GH and AB || GH. AB is a midsegment of ΔDGH, and a midsegment of a triangle is parallel to the third side and half as long.

Set III (page 509)

Finding the Center.

Example figure:



The carpenter should place the corner of the right angle of the set square on the edge of the wheel and mark the two points in which its sides intersect the wheel's edge. These two points must be the endpoints of a diameter of the wheel. (Because the right angle is inscribed in the circle, its measure is half its intercepted arc. $90° = \frac{1}{2}180°$; so the arc is a semicircle and its chord is a diameter.) Doing this a second time gives a second diameter and the diameters intersect in the center of the wheel.

(Alternatively, the carpenter could place the wheel *inside* the angle of the set square and mark successive quarter circles.)

Chapter 12, Lesson 5

Set I (pages 512–513)

A detailed discussion of danger angles is included by Elbert S. Maloney in *Dutton's Navigation and Piloting* (Naval Institute Press, 1985). A danger circle (not shown in the figure for exercises 17 through 21) is constructed with the middle point of the danger as its center and the distance from that center at which it is desired to pass as radius. A second circle is then constructed through points A and B tangent to the offshore side of the danger circle. Any inscribed angle such as $\angle D$ that intercepts AB in this second circle provides the measure of the danger angle. The navigator of a passing ship keeps track of the measure of $\angle S$, knowing that the ship is safe if $\angle S < \angle D$ and in danger if $\angle S > \angle D$.

Exercises 27 through 38 suggest that the sum of the corner angles of a symmetric star with an odd number of sides is 180°. The proof that this is true for any five-pointed star, no matter how irregular, was considered in the Set III exercise of Chapter 6, Lesson 5.

Theorem 64.

1. Two points determine a line.
•2. An exterior angle of a triangle is equal to the sum of the remote interior angles.
•3. An inscribed angle is equal in measure to half its intercepted arc.
4. Substitution.
•5. 36. ($x + 3x + 2x + 4x = 360$, $10x = 360$, $x = 36$.)
6. $m\overset{\frown}{AC} = 36°$, $m\overset{\frown}{CB} = 108°$, $m\overset{\frown}{BD} = 72°$, $m\overset{\frown}{DA} = 144°$.
•7. 54° and 126°. [$\frac{1}{2}(36 + 72) = \frac{1}{2}(108) = 54$; $180 - 54 = 126$.]

Theorem 65.

8. Two points determine a line.
9. An exterior angle of a triangle is equal to the sum of the remote interior angles.
•10. Subtraction.
11. An inscribed angle is equal in measure to half its intercepted arc.
12. Substitution.
13. 30°. ($360 - 100 - 110 - 120 = 30$.)
•14. 40°. [$\angle A = \frac{1}{2}(m\overset{\frown}{BC} - m\overset{\frown}{DE}) = \frac{1}{2}(110 - 30) = \frac{1}{2}(80) = 40$.]
•15. 75°. [$\angle B = \frac{1}{2}m\overset{\frown}{DEC} = \frac{1}{2}(30 + 120) = \frac{1}{2}(150) = 75$.]
16. 65°.

Danger Angle.

17. An inscribed angle.
•18. 55°.
•19. A secant angle.
•20. It is less than 55°. [$\angle S = \frac{1}{2}(m\overset{\frown}{AB}$ – a positive quantity); $\angle D = \frac{1}{2}m\overset{\frown}{AB}$.]

21. It would be more than 55°.

Perpendicular Chords.

22. $\angle AED$ is a right angle (or, $\angle AED = 90°$).

•23. $m\overset{\frown}{AD} + m\overset{\frown}{CB} = 180°$.

$[AED = \frac{1}{2}(m\overset{\frown}{AD} + m\overset{\frown}{CB}).]$

24. $m\overset{\frown}{AC} + m\overset{\frown}{DB} = 180°$.

25. 70°. (180 – 110.)

26. 100°. (180 – 80.)

Star Problem 1.

•27. 72°. $(\frac{360}{5}.)$

28. 144°. $(2 \times 72.)$

•29. 36°. $[\frac{1}{2}(144 - 72).]$

30. 216°. $(3 \times 72.)$

31. 108°. $(\frac{1}{2}216.)$

32. 180°. $(5 \times 36.)$

Star Problem 2.

33. 40°. $(\frac{360}{9}.)$

•34. 20°. $(\frac{1}{2}40.)$

35. 40°. $[\frac{1}{2}(40 + 40).]$

36. 60°. $[\frac{1}{2}(80 + 40).]$

37. 100°. $[\frac{1}{2}(80 + 120).]$

38. 180°. $(9 \times 20.)$

Set II (pages 514–515)

The angle of view of a camera lens determines the width of the scene in front of the camera that will fit in the frame of the film. It is inversely related to the focal length of the lens. Wide-angle lenses consequently have shorter-than-normal focal lengths, whereas the focal lengths of telephoto lenses with very small angles of view are longer than normal.

The model of the motion of the sun created by the second-century Greek astronomer Hipparchus and used by Ptolemy is described in *The Cambridge Illustrated History of Astronomy*, Michael Hoskin, editor (Cambridge University Press, 1997): "Hipparchus's elegant and simple model of the motion of the sun worked well enough for Ptolemy to retain it unaltered, and so it has been handed down to us. In the model, the sun orbited the earth on a circle, moving about the center of the circle with a uniform angular speed that took it round in 365 1/4 days. But Hipparchus knew that the seasons are of different lengths: in particular, as seen from earth, the sun moved the 90° from spring equinox to summer solstice in 94 1/2 days, and the 90° from summer solstice to autumn equinox in 92 1/2 days. Since both these intervals were longer than one-quarter of 365 1/4 days, the sun appeared to move across the sky with a speed that was not uniform but varied.

The earth therefore could not be located at the center of the circle, but must be 'eccentric'. By how far, and in which direction? To generate these longer intervals, the earth had to be displaced from the center of the circle in the opposite direction, so that the corresponding 90° arcs seen from earth were each more than one-quarter of the circle, and it therefore took the sun intervals of more than one-quarter of 365 1/4 days to traverse them. Hipparchus's calculations showed that the eccentricity needed to be 1/24th of the radius of the circle, and that the line from earth to center had to make an angle of 65.5° with the spring equinox.

Granted these parameters, the model was completely defined. Luckily it reproduced the remaining two seasons well enough, and so this single eccentric circle was sufficient to reproduce the motion of the sun."

The figure for exercises 56 through 59 is related to "The Tantalus Problem" introduced in the Set III exercise of Chapter 3, Lesson 3. The original problem was, given isosceles ΔABC with $\angle A = 20°$, $\angle DBC = 60°$, and $\angle HCB = 50°$, to find the measures of $\angle DHC$ and $\angle HDB$. Without the circle, the problem is notorious for being extremely difficult. With the circle, it is easily solved. For more on this, see pages 123–125 and pages 227–229 of *Tracking the Automatic Ant*, by David Gale (Springer, 1998), and pages 292–295 of *Geometry Civilized*, by J. L. Heilbron (Clarendon Press, 1998).

Angle of View.

39. 100°. ($\angle ACB = \frac{1}{2}m\overset{\frown}{AB}$.)

•40. If a line through the center of a circle bisects a chord that is not a diameter, it is also perpendicular to the chord.

41. SAS.

•42. 17 ft. ($\tan 25° = \frac{8}{CD}$, $CD \tan 25° = 8$, $CD = \frac{8}{\tan 25°} \approx 17$.)

43. To other points on circle O.

44. The photograph could not include the people at both ends of the row, because $\angle ACB$ would become larger than the camera's angle of view.

The Seasons.

•45. Each angle is 90°.
[$\angle AEB = \frac{1}{2}(93.14° + 86.86°) = \frac{1}{2}(180°) = 90°$, etc.]

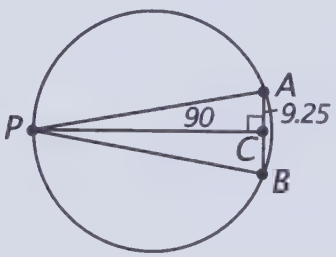
46. No. If E were the center of the circle, the angles at E would be central angles and be equal in measure to their intercepted arcs.

47. Spring.

48. About 92.5 days. ($\frac{91.17}{360} \times 365.25 \approx 92.5$.)

Place Kicker.

49.



50. C is the midpoint of AB. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

•51. 6°. ($\tan \angle APC = \frac{9.25}{90} \approx 0.103$, $\angle APC \approx 5.9°$.)

52. 12°. (2×6.)

53. 24°. ($\angle APB = \frac{1}{2}m\overset{\frown}{AB}$.)

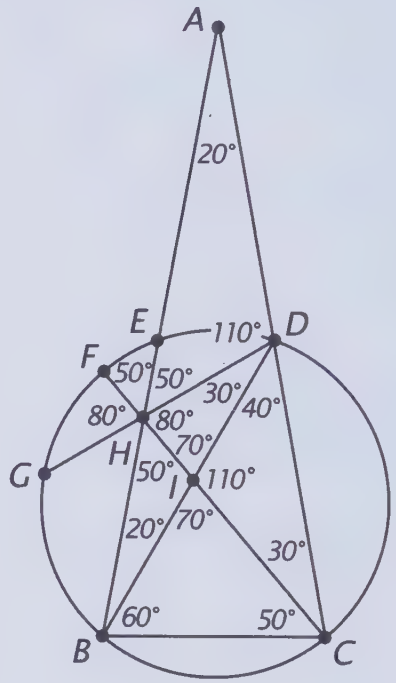
54. At other positions on the circle.

55. It would get smaller.

Angles Problem.

56. 20°. ($\frac{360}{18}$.)

57.



58. ΔABC, ΔABD, and ΔBCH.

59. ΔADH and ΔBIH, or ΔDHC and ΔDHI, or ΔBDH and ΔAHC.

Set III (page 515)

This problem was first posed by the fifteenth-century mathematician known as Regiomontanus in the following form: "At what point on the ground does a perpendicularly suspended rod appear largest (that is, subtend the greatest visual angle)?" Eli Maor, who includes the story of Regiomontanus as well as a discussion of the problem in his book titled *Trigonometric Delights* (Princeton University Press, 1998), notes that "it has been claimed that this was the first extreme problem in the history of mathematics since antiquity."

As the king walks toward the statue, the angle becomes larger until he gets to point B. Beyond

this point, the angle becomes smaller again.

The angle is at its largest at B because it is an inscribed angle that intercepts just the arc of the circle corresponding to the statue. All other viewing angles are secant angles whose vertices are outside the circle; therefore, each is measured by half the difference of the statue arc and another arc of the circle.

Chapter 12, Lesson 6

Set I (pages 517–519)

Exercises 18 through 21 demonstrate the method usually taught for constructing the tangents to a circle from an external point. Euclid's method for doing so (Book III, Proposition 17) was considered in exercises 29 through 33 of Lesson 2.

The radius of a circle inscribed in a triangle is not ordinarily determined by the lengths of the triangle's sides. For a right triangle, however, $r = \frac{1}{2}(a + b - c)$, in which a and b are the lengths of the legs and c is the length of the hypotenuse. A special case is considered in exercises 22 through 25.

The fact that the sums of the lengths of the opposite sides of a quadrilateral circumscribed about a circle are equal is revealed in exercises 26 and 27.

Korean Proof.

1. That $\angle PTO$ and $\angle PT'O$ are (equal because they are) right angles.
2. All radii of a circle are equal.
- •3. HL.
4. Corresponding parts of congruent triangles are equal.
5. The tangent segments to a circle from an external point are equal.

Theorem 67.

6. Two points determine a line.
- •7. Inscribed angles that intercept the same arc are equal.
8. AA.
- •9. Corresponding sides of similar triangles are proportional.
10. Multiplication.

Lunar Eclipse.

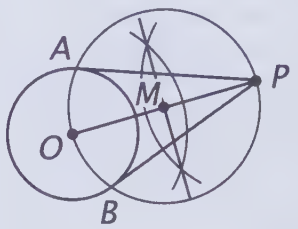
11. In a plane, two lines perpendicular to a third line are parallel to each other.
- •12. The tangent segments to a circle from an external point are equal.
13. SSS (or SAS or HL).
14. $\angle APS = \angle CPS$ because corresponding parts of congruent triangles are equal.
15. AA.

Intersecting Chords.

16. It gets larger.
17. It stays the same.

Tangent Construction.

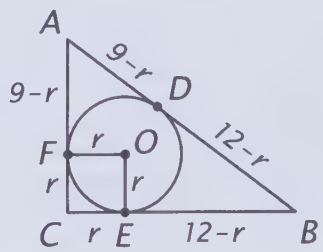
18.



- •19. Right angles. An angle inscribed in a semicircle is a right angle.
20. PA ⊥ OA and PB ⊥ OB.
21. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

Tangent Triangle.

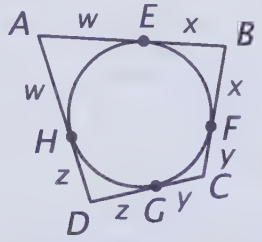
22.



23. ΔABC is a right triangle because $9^2 + 12^2 = 15^2$. (If the sum of the squares of two sides of a triangle is equal to the square of the third side, it is a right triangle.)
- •24. CFOE is a square.
25. $(9 - r) + (12 - r) = 15$, $21 - 2r = 15$, $2r = 6$, $r = 3$.

Tangent Quadrilateral.

26.



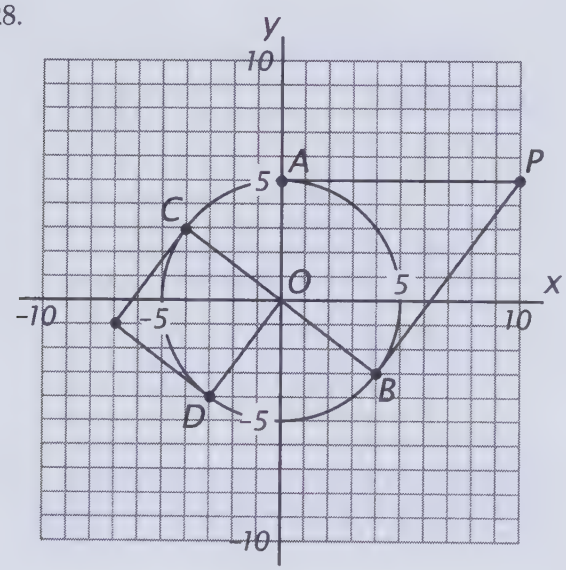
27. The sums of the lengths of the opposite sides of ABCD are equal.
AB + DC = $(w + x) + (y + z) = w + x + y + z$
and
BC + AD = $(x + z) + (w + y) = w + x + y + z$.

Set II (pages 519–521)

In his *Mathematical Magic Show* (Knopf, 1977), Martin Gardner wrote about the "ridiculous question" of exercises 35 through 37: "This is one of those curious problems that can be solved in a different way on the assumption that they have answers. Since P can be anywhere on the circle from X to Y, we move P to a limit (either Y or X). In both cases one side of triangle ABC shrinks to zero as side AB expands to 10, producing a degenerate straight-line 'triangle' with sides of 10, 10, and 0 and a perimeter of 20."

Grid Exercise 1.

28.



29. $d = \sqrt{(x-0)^2 + (y-0)^2} = 5$; so $x^2 + y^2 = 25$.

30. Point A (0, 5): $0^2 + 5^2 = 0 + 25 = 25$;
point B (4, –3): $4^2 + (-3)^2 = 16 + 9 = 25$.

•31. PA is tangent to circle O because it is perpendicular to radius OA at its outer endpoint.

32. The slope of PB is $\frac{4}{3}$ and the slope of OB is $-\frac{3}{4}$. Because the product of these slopes is –1, PB ⊥ OB; so PB is tangent to the circle.

•33. PA = 10; PB = $\sqrt{(10-4)^2 + (5-(-3))^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.

34. (–7, –1).

Ridiculous Question.

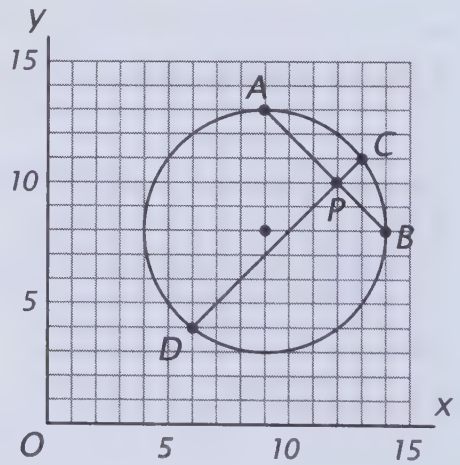
•35. 20 units.

36. The perimeter of ΔABC = AB + BC + AC = (AP + PB) + BC + AC. Because tangent segments to a circle from an external point are equal, YA = AP and XB = PB. Substituting, the perimeter of ΔABC =
YA + XB + BC + AC = YC + XC = 10 + 10 = 20.

37. Because, although it might at first seem impossible to figure out, the problem is actually very easy. (The solution is obvious if you think of folding AY onto AP and BX onto BP.)

Grid Exercise 2.

38.



39. $d = \sqrt{(x-9)^2 + (y-8)^2} = 5$;
so $(x - 9)^2 + (y - 8)^2 = 25$.

40. Point A (9, 13): $(9-9)^2 + (13-8)^2 = 0 + 25 = 25$;
point B (14, 8): $(14-9)^2 + (8-8)^2 = 25 + 0 = 25$;
point C (13, 11): $(13-9)^2 + (11-8)^2 = 16 + 9 = 25$;
point D (6, 4): $(6-9)^2 + (4-8)^2 = 9 + 16 = 25$.

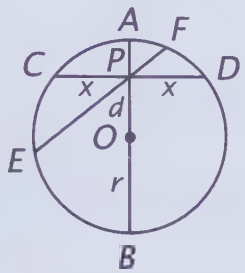
•41. (12, 10).

42. PA = $\sqrt{18}$ or $3\sqrt{2}$; PB = $\sqrt{8}$ or $2\sqrt{2}$;
PC = $\sqrt{2}$; PD = $\sqrt{72}$ or $6\sqrt{2}$.

43. PA · PB = 12; PC · PD = 12.
(PA · PB = $3\sqrt{2} \cdot 2\sqrt{2} = 6 \cdot 2 = 12$;
PC · PD = $\sqrt{2} \cdot 6\sqrt{2} = 6 \cdot 2 = 12$.)

Chord Division.

44.



•45. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

•46. BP.

47. AP.

48. AP · PB = CP · PD; so $(r-d)(r+d) = x \cdot x$,
$r^2 - d^2 = x^2$, $x^2 = r^2 - d^2$.

49. EP · PF = CP · PD = $x^2 = r^2 - d^2$.

50. r represents the radius of the circle and d represents the distance of point P from its center.

51. Draw chord EF through the center of the circle and choose point P to be the center of the circle.

52. 25. ($r^2 - d^2 = 5^2 - 0^2 = 25$.)

Earth's Shadow.

53. Because $\Delta SAP \sim \Delta EBP$, $\frac{EP}{SP} = \frac{EB}{SA}$.

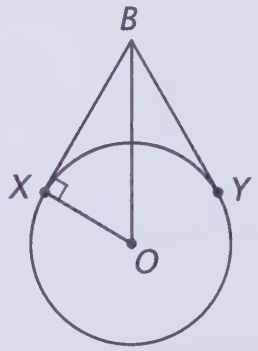
$$\frac{EP}{93{,}000{,}000 + EP} = \frac{3{,}960}{432{,}000},$$

$432{,}000EP = 368{,}280{,}000{,}000 + 3{,}960EP$,
$428{,}040EP = 368{,}280{,}000{,}000$,
$EP \approx 860{,}000$ miles.

Set III (page 521)

The story "The Brick Moon" by Edward Everett Hale originally appeared in the *Atlantic Monthly* in 1869–1870. Its plot is summarized in considerable detail in *Space Travel—A History*, by Wernher von Braun and Frederick I. Ordway III (Harper & Row, 1985). According to this summary, "its 37 inhabitants signaled the Earth in Morse code by jumping up and down on the outside of the satellite. People on Earth threw them books and other objects, some of which missed the Moon and went into orbit around it." The geometry of this problem is the same as that for existing artificial satellites, except that BO is different.

The Brick Moon.



ΔBXO is a right triangle; so $BX^2 + XO^2 = BO^2$.
Because XO = 4,000 and BO = 4,000 + 4,000 = 8,000,
$BX^2 = 8{,}000^2 - 4{,}000^2$, $BX^2 = 48{,}000{,}000$,
$BX \approx 6{,}928$ miles.

Chapter 12, Review

Set I (pages 523–524)

Friedrich von Martens, a German engraver-photographer living in Paris, built the camera with which he took the panoramic photograph of the city in 1844. As the overhead view suggests, the picture was taken on a curved plate.

Panoramic Camera.

1. Radii.

•2. A central angle.

3. A minor arc.

4. 150°.

Chords and Radius.

5. *Example figure:*



6. A diameter.

•7. 60°.

8. 180°.

Finding Lengths and Angles.

•9. 35°. ($\frac{1}{2}$70.)

•10. 16.5. ($8x = 11 \cdot 12$.)

11. 100°. [$\frac{1}{2}$(75 + 125).]

12. 8. (If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.)

13. 60°. (Inscribed angles that intercept the same arc are equal.)

14. 50°. (90 – 40.)

•15. 30°. [$\frac{1}{2}$(105 – 45).]

16. 4. (10 – 6.)

Two Equilateral Triangles.

17. 4.

•18. 60°.

19. 6.

20. 60°.

21. BC < DE (or BC = $\frac{2}{3}$DE).

22. $m\overset{\frown}{BC} = m\overset{\frown}{DE}$.

Not Quite Right.

23. ∠B and ∠D are right angles. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

24. ∠A and ∠C are supplementary. Because the sum of the angles of a quadrilateral is 360° and ∠B = ∠D = 90°, ∠A + ∠C = 180°.

•25. AB = AD. The tangent segments to a circle from an external point are equal.

26. BC = CD. All radii of a circle are equal.

Three Ratios.

•27. They are similar to ΔABC.

28. $\frac{1}{1}$, or 1.

•29. $\frac{1}{2}$.

30. $\frac{1}{2}$.

31. $\frac{1}{4}$.

Set II (pages 524–526)

The dome of Santa Maria del Fiore was designed by Filippo Brunelleschi and built in 1420–1446. The arch used is called "a quinto acuto," referring to the fact that a radius 4 units long springs from a base 5 units long. The arch is so strong that the dome was built without a scaffold. Its construction is considered to be one of the most impressive architectural accomplishments of the Renaissance.

The most famous ring of standing stones built in prehistoric England is Stonehenge. Although the measures of the arcs in the egg-shaped ring considered in exercises 45 through 50 are determined by the angles of the 3-4-5 right triangles, the radii are not determined by their sides. As a student who makes some drawings of this ring can discover, the longer the radii are in comparison with the sides of the triangles, the rounder the ring.

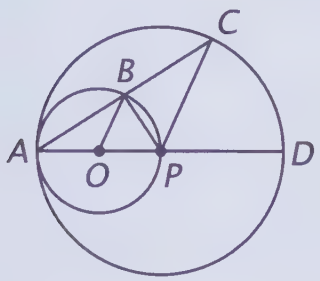
Exercises 55 through 62 demonstrate an appearance of the golden ratio, $\frac{1+\sqrt{5}}{2}$, in an unexpected place. (As the figure suggests, circles A, B, and C are tangent to one another as well as being tangent to a diameter of circle O. Circles A and C are also tangent to circle O.) Students who have done the Set III exercise of Chapter 11, Lesson 1, may recall the interesting result repeated here: if r is the golden ratio, then $\frac{1}{r} = r - 1$.

A surprising aspect of exercise 63, the "pond problem," is that the dimensions of the garden cannot be determined from the given information, because they are *not* determined by it! This

situation was previously encountered in Martin Gardner's "oil-well puzzle," included in exercises 47 through 50 of Chapter 11, Lesson 2. Both problems are connected to the fact that, given a point inside a rectangle, the sum of the squares of its distances from two opposite corners is equal to the sum of the squares of the distances from the other two corners.

Tangent Circles.

32.



33. $\angle ABP = 90°$. $\angle ABP$ is a right angle because it is inscribed in a semicircle.

•34. $PB \perp AC$.

35. $AB = BC$ because, if a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

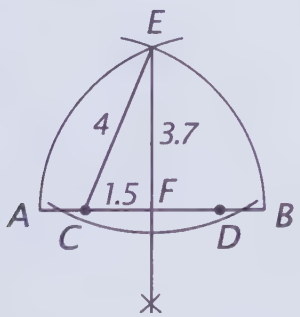
•36. Isosceles. (In the figure, they are also obtuse.)

37. $\Delta AOB \sim \Delta APC$ by AA. ($\angle A = \angle A$ and, because $OA = OB$ and $PA = PC$, $\angle A = \angle ABO$ and $\angle A = \angle C$; so $\angle ABO = \angle C$.)

38. $OB \parallel PC$ and $OB = \frac{1}{2}PC$ because OB is a midsegment of ΔAPC.

Arch Problem.

39.



•40. 4 cm. ($CE = CB = 5 - 1$.)

41. 1.5 cm. ($CF = \frac{1}{2}CD = \frac{1}{2}3$.)

•42. Approximately 3.7 cm. ($EF = \sqrt{4^2 - 1.5^2} = \sqrt{13.75} \approx 3.7$.)

43. Approximately 0.74. ($\frac{EF}{AB} \approx \frac{3.7}{5} \approx 0.74$.)

44. Approximately 68°. (In right ΔECF, $\cos \angle ECF = \frac{CF}{CE} = \frac{1.5}{4} = 0.375$; so $\angle ECF \approx 68°$. $m\widehat{AE} = m\widehat{BE} = \angle ECF \approx 68°$.)

Prehistoric Ring.

45. 180°. ($\widehat{EFG}$ is a semicircle.)

•46. 53°. (In right ΔADC, from the fact that $\sin \angle ADC = \frac{4}{5}$ or $\cos \angle ADC = \frac{3}{5}$ or $\tan \angle ADC = \frac{4}{3}$, we know that $m\widehat{GH} = \angle ADC \approx 53°$.)

47. 74°. [$m\widehat{HI} = \angle HCI = \angle BCD = 2\angle ACD \approx 2(90° - 53°) = 2(37°) = 74°$.]

48. 53°. (From the symmetry of the figure, $m\widehat{IE} = m\widehat{GH}$.)

49. They seem to be equal. ($m\widehat{EFG} = 180°$, and $m\widehat{GH} + m\widehat{HI} + m\widehat{IE} \approx 53° + 74° + 53° = 180°$.)

50. In ΔBCD, $\angle BDC + \angle BCD + \angle DBC = 180°$. Because $\angle HCI = \angle BCD$, $m\widehat{GH} = \angle BDC$, $m\widehat{HI} = \angle HCI$, and $m\widehat{IE} = \angle DBC$, it follows by substitution that $m\widehat{GH} + m\widehat{HI} + m\widehat{IE} = 180°$.

Circular Saw.

•51. C is the midpoint of AB.

•52. Approximately 9.7 in. (In right ΔAOC, $AO = 5$ and $OC = 5 - 3.75 = 1.25$; so $AC = \sqrt{5^2 - 1.25^2} \approx 4.84$. $AB = 2AC \approx 9.68$.)

•53. Approximately 75.5°. (In right ΔAOC, $\cos \angle AOD = \frac{OC}{AO} = \frac{1.25}{5} = 0.25$; so $\angle AOD \approx 75.5°$.)

54. Approximately 151°. ($m\widehat{ADB} = \angle AOB = 2\angle AOD \approx 151°$.)

Golden Ratio.

55. 2.

•56. $\sqrt{5}.\ (\sqrt{2^2+1^2}.)$

57. $\sqrt{5}+1.$

•58. $\dfrac{\sqrt{5}+1}{2}.$

59. 1.618034.

60. $\dfrac{2}{\sqrt{5}+1}.$

61. 0.618034.

62. It appears to be exactly 1 less than the ratio for exercise 59.

Pond Problem.

63. Applying the Pythagorean Theorem to some of the right triangles formed, we get
$(60 + r)^2 = w^2 + y^2$, $(52 + r)^2 = w^2 + z^2$,
$(28 + r)^2 = x^2 + z^2$, $(40 + r)^2 = x^2 + y^2$.
It follows that
$(60 + r)^2 + (28 + r)^2 = (52 + r)^2 + (40 + r)^2$
because both sums are equal to
$w^2 + x^2 + y^2 + z^2$.
So $3{,}600 + 120r + r^2 + 784 + 56r + r^2 =$
$2{,}704 + 104r + r^2 + 1{,}600 + 80r + r^2$,
$4{,}384 + 176r = 4{,}304 + 184r$, $80 = 8r$, $r = 10$.
The radius of the pond is 10 yards.

Algebra Review (page 528)

•1. $x = y$
$2x = y + 8;$
$2x = x + 8,$
$x = 8,$
$y = 8.$
$(8, 8).$

•2. $y = 2x$
$4x - y = 14;$
$4x - 2x = 14,$
$2x = 14,$
$x = 7,$
$y = 2(7) = 14.$
$(7, 14).$

•3. $y = x - 2$
$3(x + 1) = 4y;$
$3(x + 1) = 4(x - 2),$
$3x + 3 = 4x - 8,$
$11 = x,$
$y = 11 - 2 = 9.$
$(11, 9).$

•4. $y = x - 7$
$y = 5x - 19;$
$5x - 19 = x - 7,$
$4x = 12,$
$x = 3,$
$y = 3 - 7 = -4.$
$(3, -4).$

•5. $x = y + 1$
$2x = 3y + 3;$
$2(y + 1) = 3y + 3,$
$2y + 2 = 3y + 3,$
$-1 = y,$
$x = -1 + 1 = 0.$
$(0, -1).$

•6. $y = 3x - 1$
$x + 2y = 33;$
$x + 2(3x - 1) = 33,$
$x + 6x - 2 = 33,$
$7x = 35,$
$x = 5,$
$y = 3(5) - 1 = 15 - 1 = 14.$
$(5, 14).$

•7. $x + 11 = 5y$
$x = 2(3y - 8);$
$2(3y - 8) + 11 = 5y,$
$6y - 16 + 11 = 5y,$
$y = 5,$
$x = 2(3(5) - 8) = 2(15 - 8) = 2(7) = 14.$
$(14, 5).$

•8. $x + y = 14$
$y = 3x;$
$x + 3x = 14,$
$4x = 14,$
$x = 3.5,$
$y = 3(3.5) = 10.5.$
$(3.5, 10.5).$

•9. $x = 5y$
$2x - 7y = 27;$
$2(5y) - 7y = 27,$
$10y - 7y = 27,$
$3y = 27,$
$y = 9,$
$x = 5(9) = 45.$
$(45, 9).$

•10. $y = x + 3$
$x + 4y = 2;$
$x + 4(x + 3) = 2,$
$x + 4x + 12 = 2,$
$5x = -10,$
$x = -2,$
$y = -2 + 3 = 1.$
$(-2, 1).$

•11. $2x + y = -40$
$8y - 3 = x;$
$2(8y - 3) + y = -40,$
$16y - 6 + y = -40,$
$17y = -34,$
$y = -2,$
$x = 8(-2) - 3 = -16 - 3 = -19.$
$(-19, -2).$

•12. $9x = y + 5$
$4y - 3x = -20;$
$9x - 5 = y,$
$4(9x - 5) - 3x = -20,$
$36x - 20 - 3x = -20,$
$33x = 0,$
$x = 0,$
$9(0) - 5 = y,$
$-5 = y.$
$(0, -5).$

•13. $x + y = 35$
$x - y = 67;$
$2x = 102,$
$x = 51,$
$51 + y = 35,$
$y = -16.$
$(51, -16).$

•14. $5x - 7y = 92$
$5x + y = 4;$
$-8y = 88,$
$y = -11,$
$5x + (-11) = 4,$
$5x = 15,$
$x = 3.$
$(3, -11).$

•15. $3x - 5y = 51$
$x + 5y = 23;$
$4x = 74,$
$x = 18.5,$
$18.5 + 5y = 23,$
$5y = 4.5,$
$y = 0.9.$
$(18.5, 0.9).$

•16. $8x - 7y = 62$
$4x - 7y = 66;$
$4x = -4,$
$x = -1,$
$8(-1) - 7y = 62,$
$-8 - 7y = 62,$
$-7y = 70,$
$y = -10,$
$(-1, -10).$

•17. $x + y = 7$
$3x + 2y = 25;$
$3x + 3y = 21,$
$y = -4,$
$x + (-4) = 7,$
$x = 11.$
$(11, -4).$

•18. $4x + 3y = 31$
$2x - 9y = 5;$
$12x + 9y = 93,$
$14x = 98,$
$x = 7,$
$4(7) + 3y = 31,$
$28 + 3y = 31,$
$3y = 3,$
$y = 1.$
$(7, 1).$

•19. $2x + 5y = 29$
$4x - y = 25;$
$4x + 10y = 58,$
$11y = 33,$
$y = 3,$
$2x + 5(3) = 29,$
$2x + 15 = 29,$
$2x = 14,$
$x = 7.$
$(7, 3).$

•20. $5x + 4y = 53$
$x - 2y = 5;$
$2x - 4y = 10,$
$7x = 63,$
$x = 9,$
$9 - 2y = 5,$
$-2y = -4,$
$y = 2.$
$(9, 2).$

•21. $8x - 3y = 32$
$7x + 9y = 28;$
$24x - 9y = 96,$
$31x = 124,$
$x = 4,$
$8(4) - 3y = 32,$
$32 - 3y = 32,$
$-3y = 0,$
$y = 0.$
$(4, 0).$

•22. $6x + 6y = 24$
$10x - y = -15;$
$x + y = 4,$
$11x = -11,$
$x = -1,$
$10(-1) - y = -15,$
$-10 - y = -15,$
$-y = -5,$
$y = 5.$
$(-1, 5).$

•23. $5x - 7y = 54$
$2x - 3y = 22;$
$10x - 14y = 108,$
$10x - 15y = 110,$
$y = -2,$
$2x - 3(-2) = 22,$
$2x + 6 = 22,$
$2x = 16,$
$x = 8.$
$(8, -2).$

•24. $7x - 5y = 40$
$3x - 2y = 16;$
$14x - 10y = 80,$
$15x - 10y = 80,$
$x = 0,$
$7(0) - 5y = 40,$
$-5y = 40,$
$y = -8.$
$(0, -8).$

Chapter 13, Lesson 1

Set I (pages 532–534)

Exercises 15 through 21 illustrate a remarkable fact. Although it is easy to see why the lines through the pairs of intersections of three circles are concurrent if the circles have equal radii, the lines are concurrent even if the circles have different radii. If two circles intersect in two points, the line determined by those points is called the *radical axis* of the two circles. More on the theorem that the radical axes of three circles with noncollinear centers, taken in pairs, are concurrent can be found in two books by Howard Eves: *Fundamentals of Modern Elementary Geometry* (Jones and Bartlett, 1992) and *College Geometry* (Jones and Bartlett, 1995).

Exercises 22 through 25 lead to another surprise. If four lines intersect to form four triangles, the circumcircles of the triangles always intersect in a common point, named the Miquel point after an obscure nineteenth-century mathematician, Auguste Miquel. In *The Penguin Dictionary of Curious and Interesting Geometry* (Penguin, 1991), David Wells describes other remarkable properties of this point. The centers of the four circumcircles also lie on a circle that passes through it. It is also the focus of the unique parabola tangent to the four lines.

The regions into which a map is divided as in exercise 31 are called Dirichlet domains. According to Jay Kappraff, "a Dirichlet domain of a point from a set of points is defined to be the points of space nearer to that point than to any of the other points of the set." A detailed discussion of the geometry of Dirichlet domains and their connection to soap bubbles, spider webs, and phyllotaxis can be found in Kappraff's *Connections—The Geometric Bridge Between Art and Science* (McGraw-Hill, 1991).

Hot Tub.

1. Chords.
2. Inscribed angles.
3. Cyclic.

•4. Its circumcircle.

•5. ΔABC is inscribed in circle O.

•6. They are equidistant from it.

Circumcircles.

•7. Minor arcs.

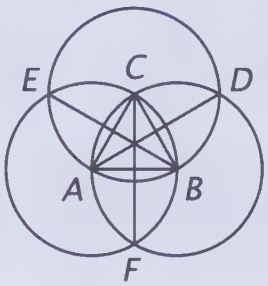
8. A semicircle.
9. A major arc.
10. An acute triangle.
11. A right triangle.
12. An obtuse triangle.

•13. A diameter (or a chord).

14. The midpoint of its hypotenuse.

RGB Color.

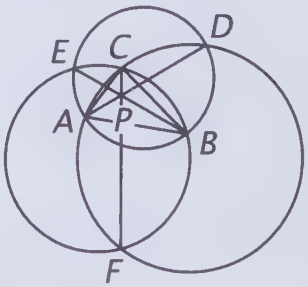
15.



16. They are concurrent.

•17. The lines are the perpendicular bisectors of the sides of ΔABC, and the perpendicular bisectors of the sides of a triangle are concurrent.

18. *Example figure:*



•19. No.

20. Yes.

21. By the Intersecting Chords Theorem. In the circle at the top, $AP \cdot PD = BP \cdot PE$ and, in the circle on the left, $BP \cdot PE = CP \cdot PF$; so $AP \cdot PD = BP \cdot PE = CP \cdot PF$.

Four Lines and Four Circles.

22. ΔABF, ΔACD, ΔBCE, ΔDEF.

23. They are their circumcircles.

•24. Three.

25. They intersect in a common point.

Equilateral Triangle.

26. Because AE, BF, and CD are altitudes, they are perpendicular to the sides of ΔABC. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

27. 30°-60° right triangles.

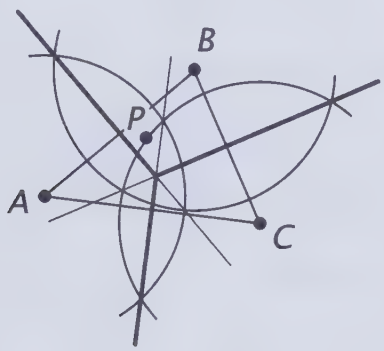
•28. OA = 2OD because, in a 30°-60° right triangle, the hypotenuse is twice the shorter leg.

29. The radius is $\frac{2}{3}$ the length of one of the altitudes.

Nearest School.

30. The school at B.

31.



•32. The circumcenter of the triangle with A, B, and C as its vertices.

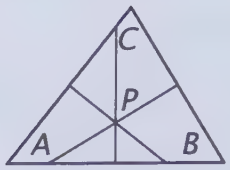
Set II (pages 534–535)

The triangle-folding exercises 33 through 40 are based on the fact that, if one point is reflected onto another, the line of reflection is the perpendicular bisector of the line segment connecting the two points.

Exercises 46 through 48 provide another example of a problem whose solution is based on Pythagorean triples and the Pythagorean Theorem yet was created more than a thousand years before Pythagoras was born. The tablet containing it was found in Susa, about 200 miles east of ancient Babylon.

Folding Experiment.

33.

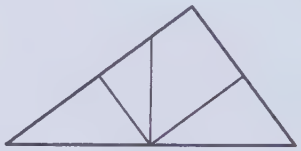


34. The folds are on the perpendicular bisectors of the sides of the triangle and they are concurrent.

35. It is equidistant from them.

•36. It is approximately 6.4 cm.

37.



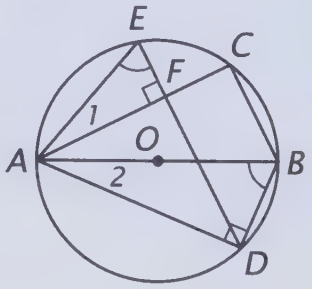
38. A right triangle because $9^2 + 12^2 = 15^2$; if the sum of the squares of two sides of a triangle is equal to the square of the third side, it is a right triangle.

•39. They are concurrent on the hypotenuse of the triangle.

•40. 7.5 cm.

Inscribed Triangles.

41.



•42. Both ∠E and ∠ABD intercept $\overset{\frown}{AD}$. Inscribed angles that intercept the same arc are equal.

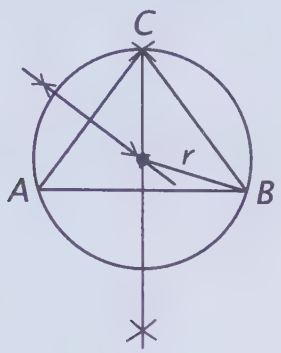
43. ∠ADB is a right angle because it is inscribed in a semicircle.

44. All right angles are equal.

45. If two angles of one triangle are equal to two angles of another triangle, the third pair of angles are equal.

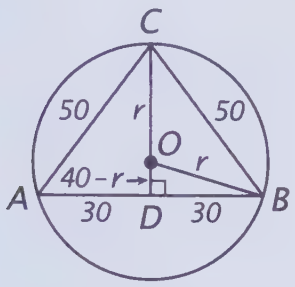
Babylonian Problem.

46.



•47. Approximately 31 mm.

•48. 31.25 mm.



[Because ΔACD is a right triangle with AC = 50 and AD = 30, CD = 40. In right ΔBDO, $(40 - r)^2 + 30^2 = r^2$; so $1{,}600 - 80r + r^2 + 900 = r^2$, $2{,}500 = 80r$, $r = 31.25$.]

Diameters and Sines.

49. ΔBCD is a right triangle because ∠BCD is inscribed in a semicircle.

50. Inscribed angles that intercept the same arc are equal.

•51. $\sin D = \frac{a}{2r}$.

52. $\frac{a}{\sin A} = 2r$. (Because $\sin A = \sin D$, $\sin A = \frac{a}{2r}$; so $2r \sin A = a$ and $\frac{a}{\sin A} = 2r$.)

53. It reveals that it is equal to the diameter of the triangle's circumcircle.

Set III (page 535)

The Slipping Ladder.

1.



2. Along an arc of a circle.

3. From the fact that the midpoint of the hypotenuse of a right triangle is its circumcenter, it follows that O is equidistant from A, B, and C; that is, OA = OB = OC. But OA and OB do not change in length as the ladder slides down the wall, and so OC does not change either. Because for every position of the ladder Ollie's feet remain the same distance from C, their path is along an arc of a circle.

Chapter 13, Lesson 2

Set I (pages 538–539)

Theorem 69 of this lesson appears in the *Elements* as Proposition 22 of Book III: "The opposite angles of quadrilaterals in circles are equal to two right angles." Strangely, although the converse is true and very useful, Euclid does not state or prove it.

A proof that the circumcircles of the equilateral triangles constructed on the sides of a triangle intersect in a common point (assumed in exercises 24 through 28) can be found on pages 82 and 83 of *Geometry Revisited,* by H. S. M. Coxeter and S. L. Greitzer (Random House, 1967). The point in which the circles intersect is called the Fermat point of the triangle. According to David Wells in *The Penguin Dictionary of Curious and Interesting Geometry* (Penguin, 1991), Fermat challenged Torricelli (of barometer fame) to find the point the sum of whose distances from the vertices of a triangle is a minimum. (In other words, to find the general solution to the Spotter's Puzzle.) That point turns out to be the Fermat point. According to Wells: "If all the angles of the triangle are less than 120° the desired point . . . is such that the lines joining it to the vertices meet at 120°. If the angle at one vertex is at least 120°, then the Fermat point coincides with that vertex." Wells also remarks that "the Fermat point can be found by experiment. Let three equal weights hang on strings passing through holes at the

vertices of the triangle, the strings being knotted at one point. The knot will move to the Fermat point."

Quilt Quadrilaterals.

•1. By seeing if a pair of opposite angles are supplementary.

2. The diamond.

Cyclic and Noncyclic.

3. The lines appear to be their perpendicular bisectors.

•4. They must be concurrent.

5. It is equidistant from them.

Euclid's Proof.

6. Two points determine a line.

7. The sum of the angles of a triangle is 180°.

•8. Inscribed angles that intercept the same arc are equal.

9. Betweenness of Rays Theorem.

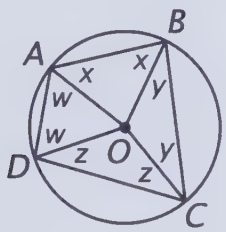
•10. Substitution.

11. Addition.

12. Substitution.

A Different Proof.

13.



•14. They are isosceles because all radii of a circle are equal.

15. 360°. The sum of the angles of a quadrilateral is 360°.

16. 180°.

17. The sum of each pair of opposite angles of ABCD is equal to $w + x + y + z$.

Isosceles Trapezoid.

•18. AB || DC because they are the bases of a trapezoid.

19. ∠A and ∠D are supplementary. Parallel lines form supplementary interior angles on the same side of a transversal.

20. ∠A = ∠B. The base angles of an isosceles trapezoid are equal.

•21. ∠B and ∠D are supplementary.

22. ABCD is cyclic. A quadrilateral is cyclic if a pair of its opposite angles are supplementary.

23. They prove that isosceles trapezoids are cyclic.

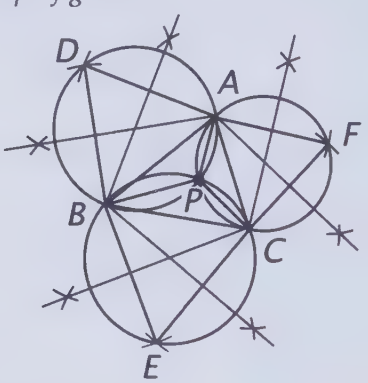
Equilateral Triangles on the Sides.

•24. They intersect in a common point, P.

25. They are cyclic.

26. They are each equal to 120°. They are supplementary to the angles opposite them in the quadrilaterals. The angles opposite them are 60° because they are also angles of the equilateral triangles.

27. *Example figure:*



28. *(Student answer.)* (Yes.)

Set II (pages 539–541)

Brahmagupta's formula for the area of a cyclic quadrilateral, discovered in the seventh century, is an extension of the formula for the area of a triangle in terms of its sides discovered by Heron of Alexandria six centuries earlier. (Heron's Theorem was introduced in the Set III exercises of Chapter 9, Lesson 3.) H. S. M. Coxeter and S. L. Greitzer include a derivation of Brahmagupta's

formula on page 58 of *Geometry Revisited.* Although they refer to it as "one of the simplest methods for obtaining Brahmagupta's formula," the method is based on several trigonometric identities including the Law of Cosines and is not easy.

J. L. Heilbron in *Geometry Civilized—History, Culture and Technique* (Clarendon Press, 1998), refers to Claudius Ptolemy as the astronomer "who was to astronomy what Euclid was to geometry" and refers to the proof of his theorem as "one of the most elegant in all plane geometry." From a special case of Ptolemy's Theorem comes the trigonometric identity

$$\sin(x-y) = \sin x \cos y - \cos x \sin y.$$

Cyclic or Not.

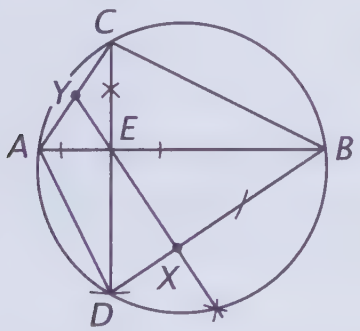
29. ABCD is cyclic, because its vertices are equidistant from point E. A circle is the set of all points in a plane equidistant from a given point; so a circle can be drawn with its center at E that contains all of the vertices.

•30. ABCD is not cyclic, because its opposite angles are not supplementary. (The sum of a right angle and an obtuse angle is more than 180°, and the sum of a right angle and an acute angle is less than 180°.)

31. ABCD is cyclic. Because ∠BCE is an exterior angle of ABCD, it forms a linear pair with ∠DCB; so ∠BCE and ∠DCB are supplementary. Because ∠BCE = ∠A, it follows that ∠A and ∠DCB are also supplementary; so ABCD is cyclic.

Brahmagupta's Theorem.

32. *Example figure:*



•33. Vertical angles are equal.

•34. Supplements of the same angle are equal. (∠7 and ∠5 are supplements of ∠6[∠8] and ∠6 and ∠8 are supplements of ∠7[∠5].)

35. Inscribed angles that intercept the same arc are equal.

36. Substitution.

37. Point Y is the midpoint of AC. Because ∠2 = ∠1, AY = YE and, because ∠3 = ∠4, YC = YE; so AY = YC.

38. If a cyclic quadrilateral has perpendicular diagonals, then any line through their point of intersection that is perpendicular to a side of the quadrilateral *bisects the opposite side.*

Area Formula.

39. The perimeter of a rectangle with sides a and b is $2a + 2b$; so, for a rectangle, $s = a + b$. $A = \sqrt{(a+b-a)(a+b-b)(a+b-a)(a+b-b)} = \sqrt{baba} = \sqrt{a^2b^2} = ab.$

•40. The perimeter of this quadrilateral is 176, so for it, $s = 88$. $A = \sqrt{(88-25)(88-39)(88-52)(88-60)} = \sqrt{63 \cdot 49 \cdot 36 \cdot 28} = 1{,}764.$

41. No. It is not cyclic and its area is evidently much smaller.

Ptolemy's Theorem.

•42. The Protractor Postulate.

43. Inscribed angles that intercept the same arc are equal.

44. AA.

45. Corresponding sides of similar triangles are proportional.

•46. Multiplication.

47. Addition.

48. Betweenness of Rays Theorem and substitution.

49. Inscribed angles that intercept the same arc are equal.

50. AA.

51. Corresponding sides of similar triangles are proportional.

52. Multiplication.

53. Addition.

•54. Substitution.

55. Multiplication.

Two Applications.

56. Because a rectangle is a cyclic quadrilateral and its opposite sides are equal, Ptolemy's Theorem becomes $a \cdot a + b \cdot b = c \cdot c$, or $a^2 + b^2 = c^2$, the Pythagorean Theorem!

57. Applying Ptolemy's Theorem to cyclic quadrilateral APBC, $PA \cdot BC + PB \cdot AC = PC \cdot AB$. Because ΔABC is equilateral, $AB = BC = AC$. By substitution, $PA \cdot AB + PB \cdot AB = PC \cdot AB$; dividing by AB gives $PA + PB = PC$.

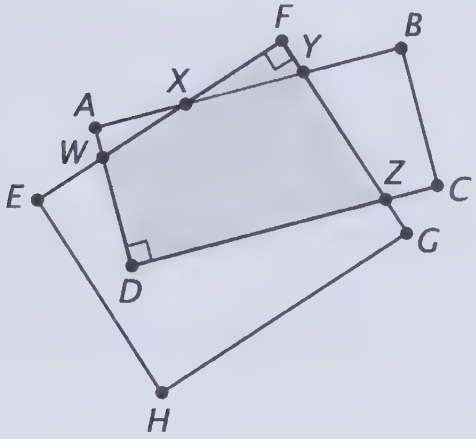
Set III (page 541)

The Set III puzzle is adapted from one by Stephen Barr, who has created many clever puzzles. Barr's puzzle was originally included by Martin Gardner in one of his "Mathematical Games" columns for *Scientific American* and is reprinted on page 110 of *Mathematical Carnival* (Knopf, 1975). Readers of Gardner's column discovered an extra set of points through which a circle could be drawn that neither Barr nor Gardner had noticed!

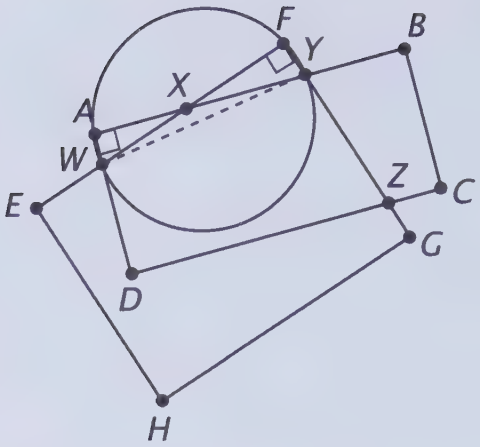
Overlapping Cards Puzzle.

1. These points are the vertices of the two rectangles. Because the opposite angles of a rectangle are supplementary, all rectangles are cyclic.

2.



One set of points consists of W, F, Z, and D. Drawing the hidden parts of the edges WX and YZ produces quadrilateral WFZD. Because ∠F and ∠D are right angles, they are supplementary; so WFZD is cyclic.



Another set of points consists of W, Y, F, and A. If WY is drawn and a circle is drawn with WY as its diameter, then it can be proved that points A and F lie on this circle owing to the fact that ∠A and ∠F are right angles.

Chapter 13, Lesson 3

Set I (pages 544–545)

Exercises 27 through 30 are related to a result known as Poncelet's porism. Jean Victor Poncelet, a nineteenth-century French mathematician, is credited for establishing the development of projective geometry as an independent subject. As Howard Eves explains in *An Introduction to the History of Mathematics* (Saunders, 1990), a porism is "a proposition stating a condition that renders a certain problem solvable, and then the problem has infinitely many solutions. For example, if r and R are the radii of two circles and d is the distance between their centers, the problem of inscribing a triangle in the circle of radius R, which will be circumscribed about the circle of radius r, is solvable if and only if $R^2 - d^2 = 2Rr$, and then there are infinitely many triangles of the desired sort." The general version of Poncelet's porism states that, if given two conics, an n-gon can be inscribed in one conic and circumscribed about the other, there are an infinite number of such n-gons.

Postage Stamp.

•1. The angle bisectors of a triangle are concurrent.

2. The incenter.

Circumcircle and Incircle.

•3. Two.

•4. They are perpendicular bisectors of the sides.

•5. The vertices of the triangle.

6. Three.

•7. Two angle bisectors.

8. A perpendicular from the triangle's incenter to one of its sides.

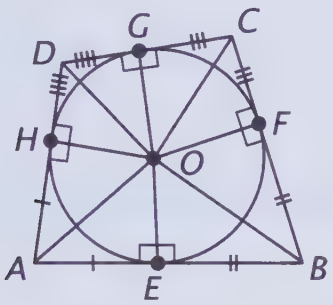
9. The sides of the triangle.

Circumscribed Quadrilateral.

•10. Its incircle.

•11. Its incenter.

12.



•13. They are perpendicular to the sides. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

14. They bisect the angles. They form pairs of triangles that are congruent by HL.

•15. They must be concurrent.

16. They must be equal.

•17. Only if the rectangle is a square.

18. Only if the parallelogram is a rhombus.

19. Only if the sum of the lengths of the bases is equal to the sum of the lengths of the legs.

Equilateral Triangle.

20. That circles have the same center.

•21. They are the perpendicular bisectors of the sides.

22. They are the bisectors of the angles.

23. Isosceles (obtuse) triangles and 30°-60° right triangles.

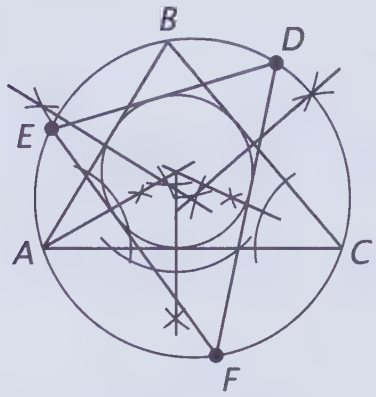
24. 2.

•25. 2π and 4π units.

26. π and 4π square units.

Construction Problem.

27. *Example figure:*



•28. EF seems to be tangent to the incircle.

29. *(Student answer.)* (Probably not.)

30. Yes.

Set II (pages 545–547)

In *Dutton's Navigation and Piloting* (Naval Institute Press, 1985), Elbert S. Maloney remarks that the LOP (line of position) is probably the most important concept in navigation. When three LOPs are taken, they are usually not concurrent, but intersect to form a triangle. The navigator ordinarily takes the point that is equidistant from the sides of this triangle (its incenter) as the "fix" or position of the ship. If there is a "constant error" in the LOPs, the method of "LOP bisectors" outlined in exercises 31 through 39 is used. As exercises 31 through 33 suggest and exercises 34 through 39 prove, the bisectors of an angle and the two remote exterior angles of a triangle are concurrent. In the method of LOP bisectors, the fix of the ship is taken at the point of concurrency.

In exercises 40 through 58, two special cases of a remarkable theorem are considered and, in exercises 59 through 62, the general case is developed. The theorem states that, for *any* polygon and its inscribed circle, the ratio of the

areas is equal to the ratio of the perimeters. A very nice treatment of this theorem is included in *Geometrical Investigations—Illustrating the Art of Discovery in the Mathematical Field,* by John Pottage (Addison-Wesley, 1983). Pottage presents his material in the form of a dialogue between Galileo's characters Salviata, Sagredo, and Simplicio. His book, like the work of George Polya, is a wonderful example of teaching mathematics by the Socratic method.

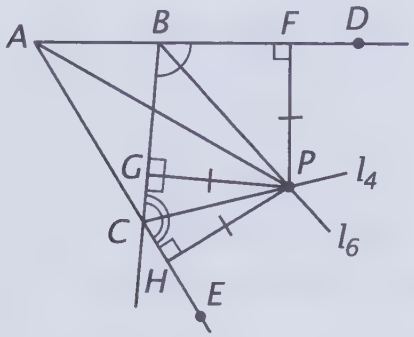
Ship Location.

31. l_5.

•32. Exterior angles.

33. They appear to be concurrent.

34.



35. $\Delta PFB \cong \Delta PGB$ (AAS) and $\Delta PGC \cong \Delta PHC$ (AAS).

36. They are equal because PF = PG and PG = PH.

37. $\Delta FAP \cong \Delta HAP$ (HL). (These triangles are right triangles whose hypotenuse is AP and whose legs PF and PH are equal.)

•38. It proves that AP bisects $\angle DAE$ (because $\angle DAP = \angle PAE$).

39. They prove that the three bisector lines are concurrent.

Incircle Problem 1.

•40. FDEO is a square (because it is equilateral and equiangular).

41. $AB = 2r$.

•42. $2\pi r$ units.

43. $8r$ units.

44. πr^2 square units.

45. $4r^2$ square units.

•46. $\frac{\pi}{4}$. $(\frac{2\pi r}{8r} = \frac{\pi}{4}.)$

47. $\frac{\pi}{4}$. $(\frac{\pi r^2}{4r^2} = \frac{\pi}{4}.)$

Incircle Problem 2.

48. ECDO is a square (because it is equilateral and equiangular).

•49. $AE = 3 - r$.

50. $DB = 4 - r$.

•51. $AB = 7 - 2r$. $(3 - r + 4 - r.)$

52. $7 - 2r = 5$, $2r = 2$, $r = 1$.

53. 2π units.

54. 12 units.

•55. π square units.

56. 6 square units. $(\frac{1}{2}3 \cdot 4.)$

57. $\frac{\pi}{6}$. $(\frac{2\pi}{12} = \frac{\pi}{6}.)$

58. $\frac{\pi}{6}$.

Incircle Problem 3.

•59. The area of a triangle is half the product of its base and altitude.

60. $\alpha ABCD =$
$\frac{1}{2}AB \cdot r + \frac{1}{2}BC \cdot r + \frac{1}{2}CD \cdot r + \frac{1}{2}DA \cdot r =$
$\frac{1}{2}r(AB + BC + CD + DA) = \frac{1}{2}rp.$

•61. $\frac{2\pi r}{p}$.

62. $\frac{2\pi r}{p}$. $(\frac{\pi r^2}{\frac{1}{2}rp} = \frac{2\pi r}{p}.)$

Trisector Challenge.

63. $\angle 1 = \angle 2$. Because BE and CE bisect two angles of ΔBDC, E is its incenter. Because the angle bisectors of a triangle are concurrent, DE must bisect $\angle BDC$.

Set III (page 547)

Every triangle has not only an incircle, but also three excircles. Each excircle is tangent to one side of the triangle and the lines of the other two sides. The center of each excircle is the point of concurrence of the bisectors of an angle and the two remote exterior angles of the triangle. The centers of the excircles are the vertices of a larger triangle whose altitudes lie on the lines that bisect the angles of the original triangle.

Excircles.

1. They appear to be its altitudes.
2. AY bisects the exterior angles of ΔABC at A; so $\angle 1 = \angle 2$. Because O is the incenter of ΔABC, AO bisects $\angle CAB$ and so $\angle 3 = \angle 4$. $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°$; so $\angle 2 + \angle 2 + \angle 3 + \angle 3 = 180°$, $2(\angle 2 + \angle 3) = 180°$, $\angle 2 + \angle 3 = 90°$, and so $AO \perp YA$. A line segment from a vertex of a triangle that is perpendicular to the line of the opposite side is an altitude of the triangle.

Chapter 13, Lesson 4

Set I (pages 550–551)

Martin Gardner points out a rather surprising fact (*Mathematical Circus*, Knopf, 1979): that the altitudes of a triangle are concurrent does not appear in Euclid's *Elements*. Gardner observes that, "although Archimedes implies it, Proclus, a fifth-century philosopher and geometer, seems to have been the first to state it explicitly." The fact that the medians of a triangle are concurrent does not appear in the *Elements* either, but Archimedes knew that the point in which they are concurrent is the triangle's center of gravity. More specifically, it is the center of gravity of a triangular plate of uniform thickness and density; it is usually *not* the center of gravity of a wire in the shape of the triangle.

"Ortho" Words.

1. Orthodontist.
2. Orthodox.
3. Orthopedic.

•4. The fact that the altitudes of a triangle form right angles with the lines of its sides.

Theorem 71.

•5. Two points determine a line.

•6. The Ruler Postulate.

7. Two points determine a line.

•8. A midsegment of a triangle is parallel to the third side.

9. A quadrilateral is a parallelogram if its opposite sides are parallel.

10. The diagonals of a parallelogram bisect each other.

•11. A line segment that connects a vertex of a triangle to the midpoint of the opposite side is a median.

12. Lines that contain the same point are concurrent.

Theorem 72.

13. An altitude of a triangle is perpendicular to the line of the opposite side.

•14. Through a point not on a line, there is exactly one line parallel to the line.

15. A quadrilateral is a parallelogram if its opposite sides are parallel.

16. The opposite sides of a parallelogram are equal.

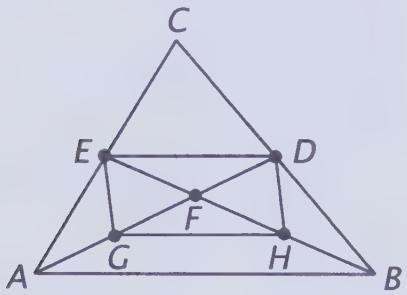
17. Substitution.

•18. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

19. The perpendicular bisectors of the sides of a triangle are concurrent.

Median Construction.

20. *(The construction lines and arcs have been omitted from the figure below.)*



21. ED || AB and $ED = \frac{1}{2}AB$. GH || AB and $GH = \frac{1}{2}AB$. A midsegment of a triangle is parallel to the third side and half as long.

22. EDHG is a parallelogram. A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.

•23. GD and EH bisect each other. The diagonals of a parallelogram bisect each other.

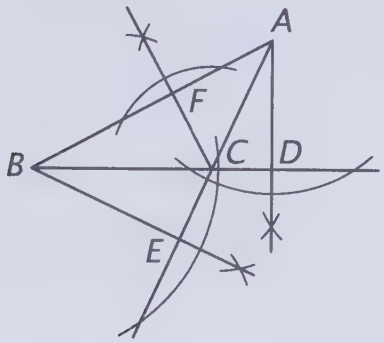
24. AG = GF and BH = HF because G and H are the midpoints of AF and FB, respectively. GF = FD and HF = FE because GD and EH bisect each other.

25. $\frac{AF}{FD} = \frac{2}{1} = 2$ and $\frac{BF}{FE} = \frac{2}{1} = 2$.

26. Points of trisection.

Altitude Construction.

27.



•28. ΔABC is obtuse.

•29. No. The altitude *segments* do not intersect.

30. Yes. We have proved that the lines that contain the altitudes of a triangle are concurrent.

31. Its orthocenter and its circumcenter.

Set II (pages 551–553)

Exercises 32 through 37 reveal Carnot's observation concerning the special relation between four points that are the three vertices of a triangle and its orthocenter: each point is the orthocenter of the triangle whose vertices are the other three. According to Howard Eves in *An Introduction to the History of Mathematics* (Saunders, 1990), Lazare Carnot was a general in the French Army and supported the French Revolution. In 1796, however, he opposed Napoleon's becoming emperor and had to flee to Switzerland. While living in exile, he wrote two important books on geometry. In one of them, he introduced several notations still used today, including ΔABC to represent the triangle having the vertices A, B, and C and $\overset{\frown}{AB}$ to represent arc AB. One of Carnot's sons became a celebrated physicist for whom the Carnot cycle of thermodynamics is named, and one of his grandsons became President of the French Republic.

Other Triangles, Other Orthocenters.

•32. GE, AF, and CD.

•33. At point B.

34. GF, AE, and BD.

35. At point C.

36. GD, BF, and CE.

37. At point A.

Medians as Bisectors.

38. The green line segments appear to be parallel to BC.

39. It appears to bisect them.

40. They are similar by AA. (Each pair of triangles has a common angle, and parallel lines form equal corresponding angles.)

•41. Corresponding sides of similar triangles are proportional.

•42. Substitution.

43. BM = MC.

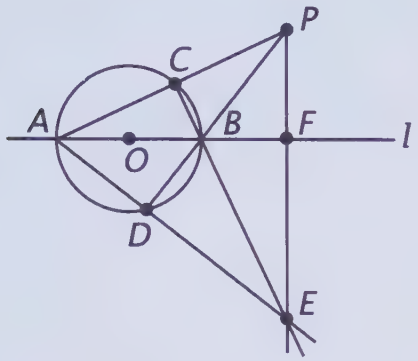
44. Substitution.

45. Multiplication.

46. They suggest that the triangle would balance on the edge of the ruler.

Doing Without a Compass.

47. *Example figure:*



48. ∠ACB and ∠ADB are right angles. An angle inscribed in a semicircle is a right angle.

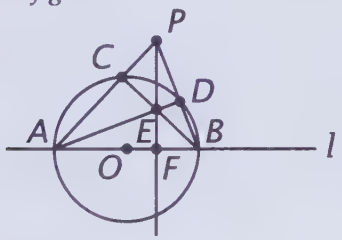
•49. PD and EC are altitudes of ΔAPE.

50. Its orthocenter.

51. AF is the third altitude of ΔAPE. The lines that contain the altitudes of a triangle are concurrent.

52. PE ⊥ *l*.

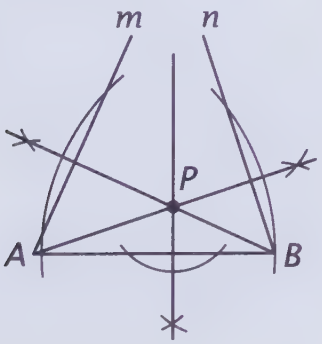
53. *Example figure:*



54. All of them.

Engineering Challenge.

55.

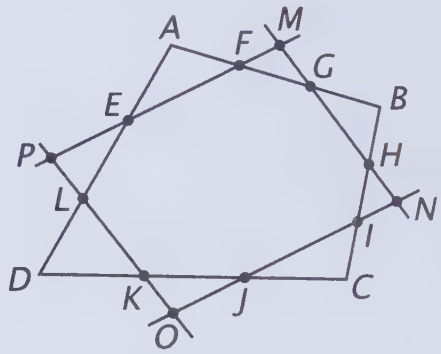


56. Constructing perpendiculars from A to line *n* and from B to line *m* produces two altitudes of the triangle formed by AB, *m*, and *n*. The point in which these altitudes intersect is the orthocenter of the triangle. A line drawn through this point perpendicular to AB therefore contains the third altitude of the triangle and passes through the point in which lines *m* and *n* intersect.

Set III (page 553)

The centroid of a uniform sheet in the shape of a quadrilateral was discovered by a German mathematician, Ferdinand Wittenbauer (1857–1922). The parallelogram whose sides lie in the lines through the points that trisect the sides of a quadrilateral is known as its "Wittenbauer parallelogram."

Balancing Point.



1. It seems to be a parallelogram.

2. The point in which the diagonals of the parallelogram intersect.

Chapter 13, Lesson 5

Set I (pages 557–558)

Although Ceva's Theorem is usually stated in terms of ratios, it appears in Nathan Altshiller Court's *College Geometry* (Barnes & Noble, 1952), in the following form: "The lines joining the vertices of a triangle to a given point determine on the sides of the triangle six segments such that the product of three nonconsecutive segments is equal to the product of the remaining three segments." This version of the theorem makes it immediately obvious that Obtuse Ollie's variations of it in exercises 25 and 26 are correct.

Which is Which?

•1. CF.

2. GF.

3. CD.

4. CE.

•5. No. GF is not a cevian, because neither of its endpoints is a vertex of the triangle.

Using Ceva's Theorem.

•6. 3.2. ($\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{x}=1$, $\frac{48}{15x}=1$, $15x=48$, $x=3.2$.)

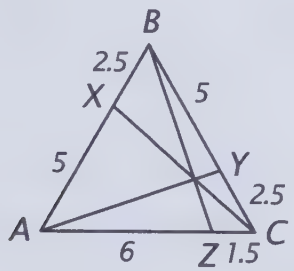
7. 2.4. ($\frac{3}{x}\cdot\frac{4}{x}\cdot\frac{x}{5}=1$, $\frac{12}{5x}=1$, $5x=12$, $x=2.4$.)

•8. Not concurrent.
$\frac{7}{13}\cdot\frac{11}{13}\cdot\frac{11}{5}=\frac{847}{845}\approx 1.002\neq 1$.

9. Not concurrent.
$\frac{14}{5}\cdot\frac{7}{17}\cdot\frac{13}{15}=\frac{1{,}274}{1{,}275}\approx 0.999\neq 1$.

Equilateral Triangle.

10.



•11. 2.

12. 2.

13. Because the cevians are concurrent,
$2\cdot 2\cdot\frac{CZ}{ZA}=1$; so $\frac{CZ}{ZA}=\frac{1}{4}$.

14. Because $\frac{CZ}{ZA}=\frac{1}{4}$, $ZA=4CZ$. Also,
$CZ+ZA=7.5$; so $CZ+4CZ=7.5$,
$5CZ=7.5$, $CZ=1.5$, and $AZ=6$.

Ratio Relations.

•15. $\frac{AP}{PY}\cdot\frac{BP}{PZ}\cdot\frac{CP}{PX}=2\cdot 2\cdot 2=8$.

16. $\frac{PX}{CX}+\frac{PY}{AY}+\frac{PZ}{BZ}=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$.

17. $\frac{CX}{PX}+\frac{AY}{PY}+\frac{BZ}{PZ}=3+3+3=9$.

•18. $\frac{AP}{AY}+\frac{BP}{BZ}+\frac{CP}{CX}=\frac{2}{3}+\frac{2}{3}+\frac{2}{3}=2$.

19. $\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=1$ (by Ceva's Theorem.)

20. $\frac{AB}{AX}+\frac{BC}{BY}+\frac{CA}{CZ}=2+2+2=6$.

Right Triangle.

•21. $\frac{BY}{YC}=\frac{6}{2}=3$.

22. $\frac{CZ}{ZA}=\frac{2}{4}=\frac{1}{2}$.

23. $\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=1$, $\frac{AX}{XB}\cdot 3\cdot\frac{1}{2}=1$,
$\frac{AX}{XB}=\frac{2}{3}$.

•24. Because ΔABC is a right triangle with legs 6 and 8, its hypotenuse is 10. Because $\frac{AX}{XB}=\frac{2}{3}$ and $AX+XB=10$, $AX=4$ and $XB=6$.

Ollie's Equations.

25. Alice is wrong. Ollie's equation is correct. One way to show this is to start with Alice's equation, $\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=1$. Clearing it of fractions gives $ace=bdf$. Dividing both sides by ace gives $1=\frac{bdf}{ace}=\frac{b}{c}\cdot\frac{d}{e}\cdot\frac{f}{a}$.

26. Ollie's second equation also is correct. One way to show this is to start with the expression $\frac{a}{f}\cdot\frac{e}{d}\cdot\frac{c}{b}$, getting $\frac{aec}{fdb}=\frac{ace}{bdf}=\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}$.

27. They suggest that you can begin with any one of the six segments and go in either direction around the triangle.

Set II (pages 559–560)

In his *A Course of Geometry for Colleges and Universities* (Cambridge University Press, 1970),

Dan Pedoe wrote: "The theorems of Ceva and Menelaus naturally go together, since the one gives the condition for lines through vertices of a triangle to be concurrent, and the other gives the condition for points on the sides of a triangle to be collinear." Menelaus is known for his book *Sphaerica*, in which the concept of a spherical triangle appears for the first time. The spherical version of the theorem of Menelaus considered in exercises 28 through 33 is, in fact, the basis for the development of much of spherical trigonometry.

Exercises 52 through 59 demonstrate, for the case of an acute triangle, how Ceva's Theorem can be used to prove that the altitudes of a triangle are concurrent.

The Theorem of Menelaus.

•28. $\Delta AXZ \sim \Delta CPZ$ and $\Delta BXY \sim \Delta CPY$.

29. Corresponding sides of similar triangles are proportional.

30. Multiplication.

31. Multiplication and division.

•32. Division (and substitution).

33. The equation in Ceva's Theorem.

Concurrent or Not?

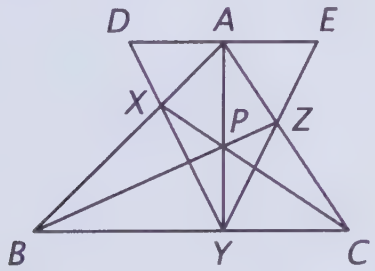
•34. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

35. $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = \frac{ZA}{CZ} \cdot \frac{BY}{BY} \cdot \frac{CZ}{ZA} = 1$. (By substituting $\frac{ZA}{CZ}$ for $\frac{AX}{XB}$ and BY for YC.)

36. It proves that AY, BZ, and CX are concurrent.

What Kind of Triangle?

37.



38. Isosceles.

39. AA. (The parallel lines form equal alternate interior angles in the triangles. Also, the vertical angles are equal.)

40. Corresponding sides of similar triangles are proportional.

•41. $\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{AZ} = 1$ (by Ceva's Theorem).

42. $\frac{AD}{BY} \cdot \frac{BY}{CY} \cdot \frac{CY}{AE} = 1$ by substitution. ($\frac{AX}{BX} = \frac{AD}{BY}$ and $\frac{CZ}{AZ} = \frac{CY}{AE}$.)

43. $\frac{AD}{AE} = 1$ because $\frac{AD}{BY} \cdot \frac{BY}{CY} \cdot \frac{CY}{AE} = 1$.

44. Because $\frac{AD}{AE} = 1$, AD = AE by multiplication.

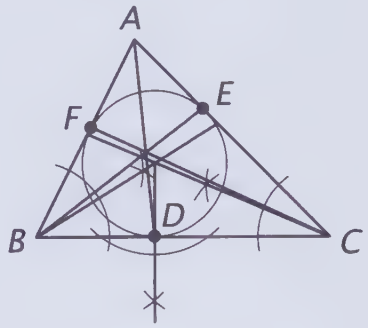
•45. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

46. $\Delta DYA \cong \Delta EYA$ by SAS. (AD = AE, $\angle DAY = \angle EAY$, and AY = AY.)

47. It proves that ΔDYE is isosceles. DY = EY because corresponding sides of congruent triangles are equal.

The Gergonne Point.

48.



49. AD, BE, and CF appear to be concurrent.

50. According to Ceva's Theorem, AD, BE, and CF are concurrent if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$.
The tangent segments to a circle from an external point are equal; so AF = AE, BF = BD, and CD = CE. Substituting in the product $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA}$ gives $\frac{AE}{BD} \cdot \frac{BD}{DC} \cdot \frac{CD}{EA} = 1$; so the three cevians are concurrent.

51. The point of concurrence of AD, BE, and CF (or, the point in which the cevians connecting the vertices of a triangle to the points of tangency of the incircle to the opposite sides are concurrent).

Another Look at Altitudes.

•52. $\Delta AXC \sim \Delta AZB$.

53. $\Delta BYA \sim \Delta BXC$.

54. $\Delta CZB \sim \Delta CYA$.

55. $\frac{AX}{AZ} = \frac{XC}{ZB}$.

56. $\frac{BY}{BX} = \frac{YA}{XC}$.

57. $\frac{CZ}{CY} = \frac{ZB}{YA}$.

58. $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = \frac{AX \cdot BY \cdot CZ}{XB \cdot YC \cdot ZA} = \frac{AX \cdot BY \cdot CZ}{AZ \cdot BX \cdot CY} = \frac{AX}{AZ} \cdot \frac{BY}{BX} \cdot \frac{CZ}{CY} = \frac{XC}{ZB} \cdot \frac{YA}{XC} \cdot \frac{ZB}{YA} = 1$; so AY, BZ, and CX are concurrent.

59. The lines containing the altitudes of a triangle are concurrent.

Set III (page 560)

If we go either clockwise or counterclockwise around a triangle and three cevians divide each side in the ratio $\frac{1}{2}$, then the smaller triangle formed by them has an area that is one-seventh the area of the original triangle. A generalization of this result is that, if the three cevians divide each side in the ratio $\frac{1}{x}$, then the smaller triangle has an area that is $\frac{(x-1)^2}{x^2+x+1}$ times the area of the original triangle. In fact, in the chapter on affine geometry in his *Introduction to Geometry* (Wiley, 1969), H. S. M. Coxeter considers an even broader generalization. If the three cevians divide the sides in the ratios $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, then the smaller triangle has an area that is $\frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(xz+z+1)}$ times the area of the original triangle! The "seven congruent triangles" approach to the version considered in the text comes from Hugo Steinhaus's *Mathematical Snapshots* (Oxford University Press, 1969).

Area Puzzle.

1. $\alpha\Delta ABD = 7x$.
($\alpha\Delta ABD = \alpha\Delta AFG + \alpha FGHB + \alpha\Delta BDH = x + 5x + x = 7x$.)

2. $\alpha\Delta ADC = 14x$. (Because ΔABD and ΔADC have equal altitudes and the base of ΔADC is twice as long as the base of ΔABD, $\alpha\Delta ADC = 2\alpha\Delta ABD = 14x$.)

3. $\alpha\Delta GHI = 3x$.
($\alpha\Delta GHI = \alpha\Delta ADC - \alpha AGIE - \alpha DHIC - \alpha\Delta EIC = 14x - 5x - 5x - x = 3x$.)

4. Because $\alpha\Delta GHI = 3x$ and $\alpha\Delta ABC = \alpha\Delta ABD + \alpha\Delta ADC = 7x + 14x = 21x$, $\alpha\Delta GHI = \frac{1}{7}\alpha\Delta ABC$.

5. The pairs of regions with the same number appear to be congruent and hence have the same area. From this, it appears that ΔABC is equal in area to the sum of the areas of the seven colored triangles. If these triangles are congruent, then $\alpha\Delta GHI$ (the area of the yellow triangle) is one-seventh of the area of ΔABC.

Chapter 13, Lesson 6

Set I (pages 562–563)

Exercises 6 through 34 lead to the discovery of the Euler line. Euler, the most prolific writer on mathematics in history, noted the remarkable relation of the circumcenter, orthocenter, and centroid of every nonequilateral triangle in 1765. Not only are the three points always collinear, but the distance from the centroid to the orthocenter is always twice the distance from the centroid to the circumcenter. It is interesting to note that Euler's proof of this relation was analytic. It was not until 1803 that Lazare Carnot published the first proof by Euclidean methods. The exercises are developed around a triangle chosen on a coordinate grid so that the coordinates of every point in the figure (except W) are integers. This was done for two reasons: (1) to help the student draw the figure accurately enough that the Euler line theorem can be discovered and (2) to review basic ideas such as distance and slope. Unfortunately, this approach obscures the fact that the results obtained are not limited to this particular triangle but are generally true.

Tilted Square.

•1. AB and DC.

•2. $-\frac{3}{2}$.

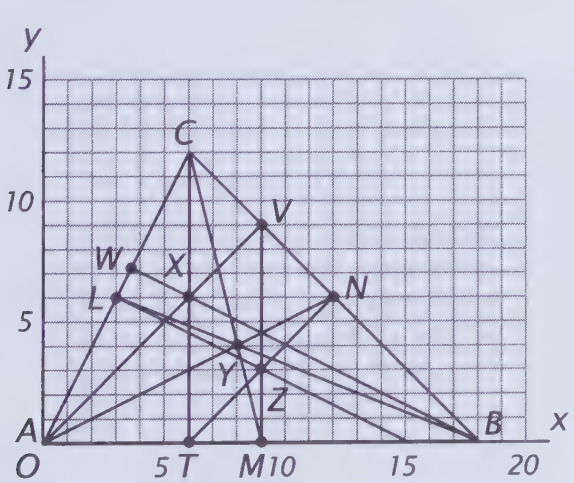
3. Because the sides are perpendicular.

•4. $\sqrt{13}$ units. ($\sqrt{3^2+2^2}$.)

5. B(10, 6), C(8, 9), D(5, 7).

Euler's Discovery.
The Centroid.

6.



•7. M(9, 0), L(3, 6).

•8. (8, 4).

9. (12, 6).

10. The coordinates of N are the averages of (or midway between) the corresponding coordinates of B and C.

•11. $CN = \sqrt{(6-12)^2+(12-6)^2} = \sqrt{36+36} = \sqrt{72}$ or $6\sqrt{2}$;
$NB = \sqrt{(12-18)^2+(6-0)^2} = \sqrt{36+36} = \sqrt{72}$ or $6\sqrt{2}$.

12. Its centroid.

13. Medians.

The Circumcenter.

14. (9, 9).

15. (6, 0).

•16. mNT = 1 and mCB = –1.

17. Yes. NT ⊥ CB; so their slopes are the opposites of the reciprocals of each other (or the product of their slopes is –1).

•18. (9, 3).

19. Its circumcenter.

20. mAC = 2 and mZL = $-\frac{1}{2}$.

21. Their slopes indicate that AC ⊥ ZL.

22. The perpendicular bisectors of its sides.

The Orthocenter.

23. CT ⊥ AB.

•24. At V.

25. mAV = 1.

26. Yes, because mCB = –1 and AV ⊥ CB. (Also, AV || NT because AV ⊥ CB and NT ⊥ CB; so mAV = mNT.)

•27. (6, 6).

28. Its orthocenter.

29. Its altitudes.

Euler's Conclusions.

30. They appear to be collinear.

31. $m\text{XY} = -1$ and $m\text{YZ} = -1$ (so, X, Y, and Z *are* collinear).

32. XY = 2YZ because $\text{XY} = \sqrt{8} = 2\sqrt{2}$ and $\text{YZ} = \sqrt{2}$.

33. Z.

34. $\text{ZA} = \text{ZB} = \text{ZC} = \sqrt{90} = 3\sqrt{10}$.

Set II (pages 564–565)

Students who did the Set III exercise of Chapter 6, Lesson 1 (page 218), may recall that the figure for exercises 35 through 42 is remarkable in that it shows an arrangement of eight points in the plane such that the perpendicular bisector of the line segment connecting any two of the points passes through exactly two other points of the figure. For most pairs of points in the figure, such as A and B or G and C, this is obvious from the symmetries of its parts.

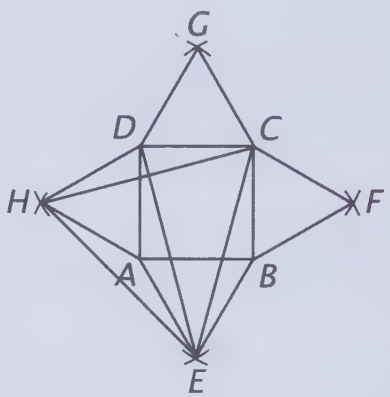
Exercises 43 through 51 explore another property possessed by all triangles. If squares are constructed on the sides of a triangle as in the familiar figure for the Pythagorean Theorem, the line segments connecting the centers of the squares to the opposite vertices of the triangle are always concurrent. Furthermore, these line segments have a special relation to the triangle whose vertices are the centers of the squares. Each one not only is perpendicular to one of its sides but also has the same length. Again, the use of the coordinate grid makes it easier to draw the figure accurately (especially the squares and their centers) but obscures the fact that the results are true for all triangles.

Exercises 52 through 60 concern a similar result for parallelograms. If squares are constructed outward on the sides of any parallelogram, the two line segments connecting the centers of the opposite squares are both equal and perpendicular to each other. This is connected to the fact that the quadrilateral whose vertices are the centers of the squares also is a square.

A partial proof of Napoleon's Theorem is considered in exercises 61 through 63. On the assumption that the three line segments connecting the vertices of the original triangle to the opposite vertices of the equilateral triangles are equal, are concurrent, and form equal angles with each other, it is fairly easy to prove that the triangle determined by the centers of the three equilateral triangles also is equilateral. In *Geometry Civilized* (Clarendon Press, 1998), J. L. Heilbron presents a complete trigonometric proof of the theorem based on the Law of Cosines. In *Hidden Connections, Double Meanings* (Cambridge University Press, 1988), David Wells wrote concerning Napoleon's Theorem: "To discover pattern and symmetry where there appears to be neither is always delightful and intriguing. It also suggests that we are not seeing all there is to be seen in the original diagram. There must certainly be another way of looking at it which will be more symmetrical from the start, and therefore make the conclusion more natural." On pages 49–52 of his book, Wells presents another way, using tessellations, to understand this remarkable theorem.

Triangles on Four Sides.

35.



•36. 48 units. (8 × 6.)

37. 150°. ($\angle\text{CBE} = 90° + 60° = 150°$.)

•38. 150°. ($\angle\text{HAE} = 360° - 60° - 90° - 60° = 150°$.)

39. They appear to be perpendicular. (Also, DE appears to be the perpendicular bisector of HC.)

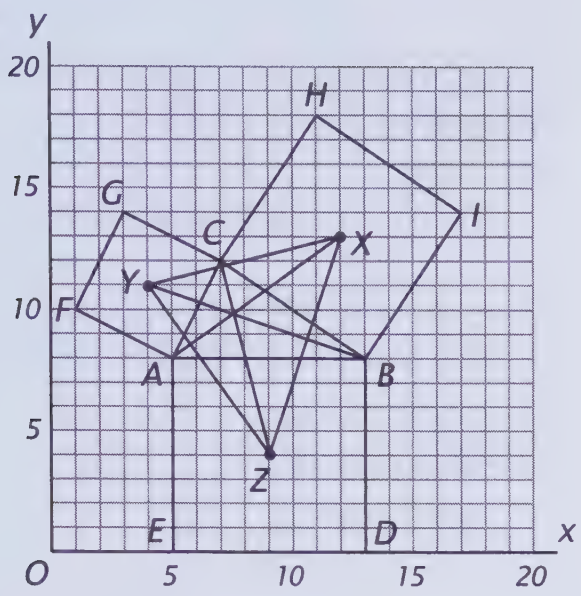
40. They are congruent by SAS. (They are also isosceles.)

41. It proves that ΔHCE is equilateral.

42. They prove that HC and DE are perpendicular because, in a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Squares on Three Sides.

43.



44. $m\text{AB} = 0$, $m\text{AC} = 2$, $m\text{CB} = -\frac{2}{3}$.

•45. No. AC and CB are not perpendicular, because the product of their slopes is not –1.

•46. G(3, 14), I(17, 14).

47. X(12, 13), Y(4, 11), Z(9, 4).

48. They appear to be concurrent.

49. They appear to be perpendicular to the sides of ΔXYZ.

50. It is true because $(m\text{AX})(m\text{YZ}) = (\frac{5}{7})(-\frac{7}{5}) = -1$, $(m\text{BY})(m\text{XC}) = (-\frac{1}{3})(3) = -1$, and $(m\text{CZ})(m\text{YX}) = (-4)(\frac{1}{4}) = -1$.

51. They are equal to the sides of ΔXYZ to which they are perpendicular.

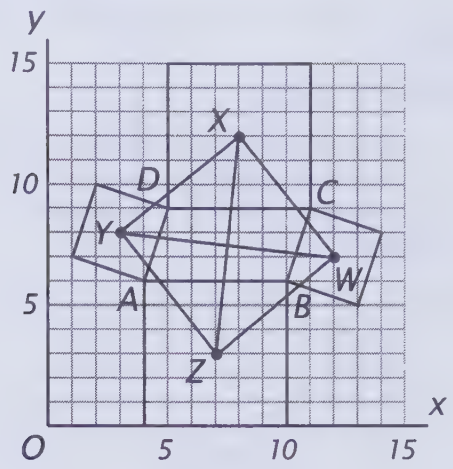
$\text{AX} = \text{YZ} = \sqrt{7^2+5^2} = \sqrt{74}$;

$\text{BY} = \text{XZ} = \sqrt{9^2+3^2} = \sqrt{90}$;

$\text{CZ} = \text{YX} = \sqrt{2^2+8^2} = \sqrt{68}$.

Squares on Four Sides.

52.



•53. $\text{AB} = \text{DC} = 6$; $\text{AD} = \text{BC} = \sqrt{10}$.

54. It is a parallelogram.

55. Both pairs of opposite sides are equal (or, two opposite sides are equal and parallel).

56. W(12, 7), X(8, 12), Y(3, 8), Z(7, 3).

•57. $m\text{XZ} = 9$, $m\text{WY} = -\frac{1}{9}$.

58. That XZ and WY are perpendicular.

59. $\text{XZ} = \text{WY} = \sqrt{1^2+9^2} = \sqrt{82}$.

60. WXYZ is a square.

Napoleon Triangles.

61. They are cyclic because their opposite angles are supplementary. (For example, in APCE, $\angle\text{E} = 60°$ and $\angle\text{APC} = 120°$.)

62. They are perpendicular to the triangle's sides because, in a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment. (For example, XA = XP and YA = YP because all radii of a circle are equal; so XY is the perpendicular bisector of AP.)

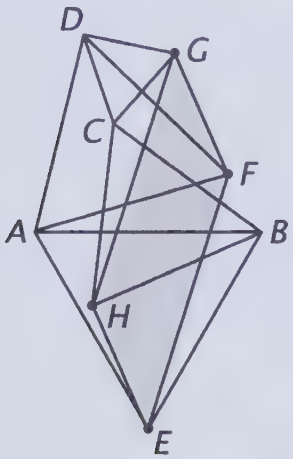
63. ΔXYZ is equilateral because it is equiangular. It is equiangular because each of its angles is 60°. (For example, $\angle\text{X} = 60°$ because, in quadrilateral PGXI, $\angle\text{XGP} = \angle\text{XIP} = 90°$ and $\angle\text{GPI} = 120°$.)

Set III (page 565)

Here, a surprising variation of Napoleon's Theorem is considered.

Alternating Triangles.

1. If equilateral triangles are drawn alternately outward and inward on the sides of a quadrilateral, their vertices determine a parallelogram.
2. Yes. The theorem still seems to apply.
Example figure:



Chapter 13, Review

Set I (pages 566–568)

Exercises 1 through 4 illustrate the fact that the feet of the three perpendiculars to the sides of a triangle from a point on its circumcircle are collinear. The line that they determine is called the Simson line, named after Robert Simson, a professor of mathematics at the University of Glasgow. Simson produced an important edition of Euclid's *Elements* in 1756. Proofs of the existence of the Simson line based on cyclic quadrilaterals can be found in J. L. Heilbron's *Geometry Civilized* and Nathan Altshiller Court's *College Geometry*. The Simson line leads to so many other ideas that it is the subject of an entire chapter of Court's book.

The configuration in exercises 28 through 31 has additional interesting properties. The perpendicular bisectors of the sides of the rectangle whose vertices are the incenters of the four overlapping triangles also bisect the four arcs of the circle corresponding to the sides of ABCD. Furthermore, each vertex of the rectangle is collinear with a midpoint of one of these arcs and a vertex of ABCD. A proof of the theorem that the incenters of the four triangles determined by the vertices of a cyclic quadrilateral determine a rectangle can be found on page 133 of Nathan Altshiller Court's *College Geometry*.

Cyclic Triangle.

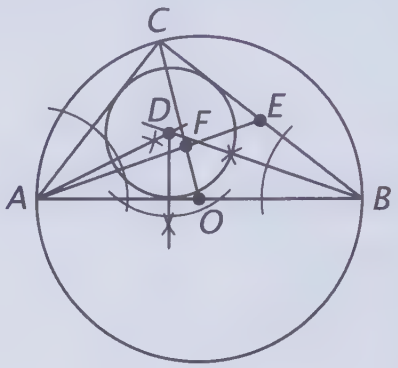
1. That there exists a circle that contains all of its vertices.

•2. ΔABC is inscribed in the circle.

3. They appear to be perpendicular to them.
4. They are collinear.

Circumcircle and Incircle.

5. *Example figure:*



•6. ΔABC is a right triangle because $\angle ACB$ is inscribed in a semicircle.

7. AE (or CO).
8. AC (or BC).
9. A and D (or B and D or C and D).
10. D.
11. O.

•12. F.

13. C.

Three Trapezoids.

•14. IJKL.

15. EFGH.
16. A quadrilateral is cyclic iff a pair of its opposite angles are supplementary. (For ABCD, we don't know whether a pair of opposite angles is supplementary or not.)

Double Identity.

17. Its circumcenter.

•18. Midsegments.

19. The sides of ΔDEF are parallel to the sides of ΔABC and half as long.

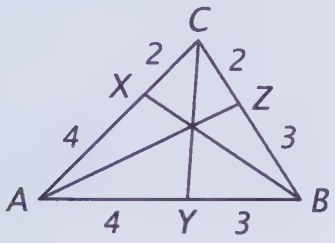
20. Its altitudes.

21. Its orthocenter.

22. A, B, and C.

Ceva's Theorem.

23.



•24. AX = 4, YB = 3, and ZC = 2.

25. $\frac{CX}{XA} = \frac{1}{2}$, $\frac{AY}{YB} = \frac{4}{3}$, and $\frac{BZ}{ZC} = \frac{3}{2}$.

26. $\frac{CX}{XA} \cdot \frac{AY}{YB} \cdot \frac{BZ}{ZC} = 1$.

27. It indicates that AZ, BX, and CY are concurrent.

Five Circles.

•28. Cyclic.

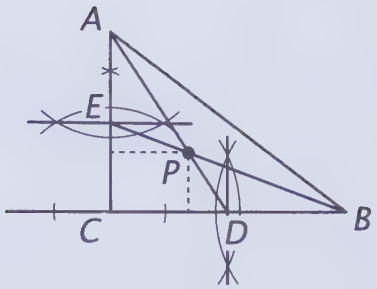
29. It is circumscribed about it.

30. They are the incircles of the (overlapping) triangles whose sides are the sides and diagonals of the quadrilateral.

31. They seem to be the vertices of a rectangle. (This is surprising because ABCD, the quadrilateral from which it comes, has no special property other than being cyclic.)

Centroid.

32.



33. 1 in.

•34. $\frac{3}{4}$ in.

35. They are one-third of the lengths.

Set II (pages 568–570)

The nineteenth-century Swiss mathematician Jacob Steiner has been called by Howard Eves "one of the greatest synthetic geometers the world has ever known." The "Steiner problem" named after him concerns the problem of connecting a set of coplanar points with a set of line segments having the least possible total length. (The Spotter's problem is the special case of this problem for a set of three points situated at the vertices of an equilateral triangle.) In *The Parsimonious Universe* (Copernicus, 1996), Stefan Hildebrandt and Anthony Tromba describe solving the Steiner problem by means of soap films: "Suppose we make a frame consisting of two parallel glass or clear plastic plates; these are connected by *n* parallel pins of the same size that meet both plates perpendicularly. If this framework is immersed in a soap solution and withdrawn, a system of planar soap films is formed. These films are attached to the pins, and they have two kinds of liquid edges, both of which are straight lines. One type adheres to the glass (or plastic), which it meets at 90° angles. . . . The other type is where three films meet, forming three angles of 120°. . . . Since the soap-film system minimizes area, the subsystem of edges on one plate must minimize length among all connections between the *n* given points." The exercises demonstrate that, for four suitably located points, equilateral triangles drawn on opposite sides of the quadrilateral determined by them can be used to construct a network of lines that meet at equal (120°) angles. Repeating the method with the other pair of opposite sides produces a different network with a possibly different total length. One of the two, however,

is the absolute minimum, the one obtained by the soap-film method.

Exercises 47 through 49 imply that two angle bisectors of a triangle meet at an angle determined entirely by the measure of the third angle of the triangle. In terms of the figure and with the use of a similar but more general argument, it can be shown that $\angle D = \frac{1}{2}\angle A + 90°$.

The two quadrilaterals in exercises 51 through 54 have another interesting relation. If perpendiculars are drawn from the point of intersection of the diagonals of a cyclic quadrilateral to its sides, their feet are the vertices of the inscribed quadrilateral of minimum perimeter. EFGH is therefore the quadrilateral of minimum perimeter that can be inscribed in ABCD.

Exercises 60 through 63 explore a more general case of the relations introduced in exercises 52 through 60 of Lesson 6. Martin Gardner included it in one of his "Mathematical Games" columns for *Scientific American*, reprinted in *Mathematical Circus* (Knopf, 1979). The column was on the subject of "simplicity" in science and mathematics, and Gardner wrote: "Mathematicians usually search for theorems in a manner not much different from the way physicists search for laws. They make empirical tests. In pencil doodling with convex quadrilaterals—a way of experimenting with physical models—a geometer may find that when he draws squares outwardly on a quadrilateral's sides and joins the centers of opposite squares, the two lines are equal and intersect at 90 degrees. He tries it with quadrilaterals of different shapes, always getting the same results. Now he sniffs a theorem. Like a physicist, he picks the simplest hypothesis. He does not, for example, test first the conjecture that the two lines have a ratio of one to 1.00007 and intersect at angles of 89 and 91 degrees, even though this conjecture may equally well fit his crude measurements. He tests first the simpler guess that the lines are always perpendicular and equal. His 'test,' unlike the physicist's, is a search for a deductive proof that will establish the hypothesis with certainty." Gardner goes on to remark that this result is known as Von Aubel's theorem and mentions several remarkable generalizations of it.

Exercises 64 through 67 establish the nice result that the area of a triangle is equal to the product of the lengths of its three sides divided by four times the radius of its circumcircle. (It is easier to begin with this conclusion and show that it is equivalent to the familiar formula for the area of a triangle than it is to do it the other way around.)

Soap-Film Geometry.

•36. $\angle AED + \angle AGD = 180°$. $\angle AED$ and $\angle AGD$ are supplementary because they are opposite angles of a cyclic quadrilateral.

37. $\angle AGD = 120°$ because $\angle AED = 60°$.

•38. $\angle EGD = \angle EAD$ because they are inscribed angles that intercept the same arc.

•39. $\angle EGD = 60°$ because $\angle EGD = \angle EAD$.

40. $\angle DGH = 120°$ because $\angle DGH$ forms a linear pair with $\angle EGD$.

41. $\angle AGH = 120°$ because $\angle AGH = 360° - \angle AGD - \angle DGH$.

•42. $\angle BHC = 120°$ because it is supplementary to $\angle BFC$.

43. $\angle FHC = 60°$ because $\angle FHC = \angle FBC$.

44. $\angle GHC = 120°$ because $\angle GHC$ forms a linear pair with $\angle FHC$.

45. $\angle BHG = 120°$ because $\angle BHG = 360° - \angle GHC - \angle BHC$.

46. It suggests that they form equal angles.

Irrelevant Information.

47. 125°. (In ΔABC, $\angle A = 70°$ and $\angle ABC = 60°$; so $\angle ACB = 50°$. Because D is the incenter of ΔABC, BD bisects $\angle ABC$ and so $\angle DBC = 30°$, and CD bisects $\angle ACB$ and so $\angle DCB = 25°$. In ΔDBC, $\angle DBC = 30°$ and $\angle DCB = 25°$; so $\angle D = 125°$.)

•48. 125°. (In ΔABC, $\angle A = 70°$ and $\angle ABC = 40°$; so $\angle ACB = 70°$. Reasoning as in exercise 47, we find that $\angle DBC = 20°$ and $\angle DCB = 35°$; so $\angle D = 125°$.)

49. 125°. (In ΔABC, $\angle ACB = 180° - 70° - \angle ABC = 110° - \angle ABC$. $\angle DBC = \frac{1}{2}\angle ABC$ and $\angle DCB = \frac{1}{2}(110° - \angle ABC) = 55° - \frac{1}{2}\angle ABC$; so $\angle D = 180° - \frac{1}{2}\angle ABC - (55° - \frac{1}{2}\angle ABC) = 125°$.)

Medians Theorem.

50. Because the medians bisect the sides of the triangle, AX = XB, BY = YC, and CZ = ZA; so $\frac{AX}{XB} = \frac{BY}{YC} = \frac{CZ}{ZA} = 1$. Therefore, $\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1 \cdot 1 \cdot 1 = 1$; so the medians are concurrent.

Irregular Billiard Table.

51. Cyclic.

52. They appear to be perpendicular to its sides.

•53. They appear to bisect its angles.

54. It would travel around the sides of quadrilateral EFGH (because its angles of incidence and reflection at the points in which it hits the sides of ABCD are equal).

Perimeter Problem.

•55. DB and FB.

56. AE and AF.

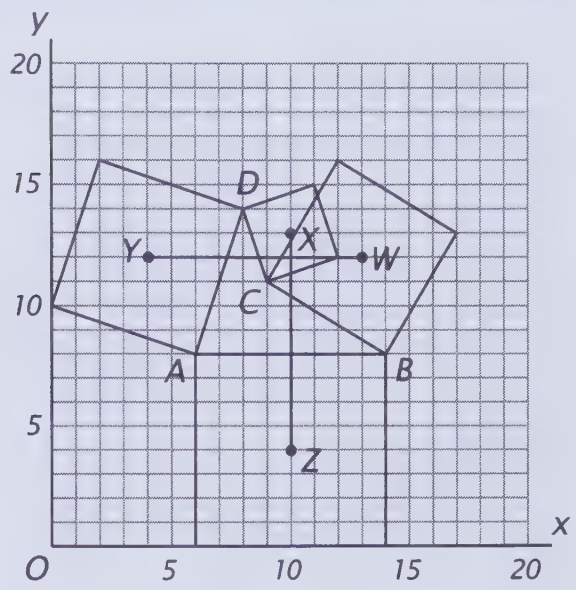
57. AB.

58. AB.

59. The perimeter of ΔABC is $AB + BC + CA = 2R + a + b$. $AB = (a - r) + (b - r) = 2R$; so $a + b = 2R + 2r$. Therefore, the perimeter of ΔABC is $2R + (2R + 2r) = 2r + 4R$.

Squares on the Sides.

60.



•61. ABCD is concave.

62. W(13, 12), X(10, 13), Y(4, 12), Z(10, 4).

63. XZ ⊥ WY and XZ = WY. (XZ = WY = 9.)

Area Problem.

64. ΔABD ~ ΔAEC.

65. ∠B = ∠E because they are inscribed angles that intercept the same arc. ∠ADB is a right angle because AD ⊥ BC, and ∠ACE is a right angle because it is inscribed in a semicircle; so ∠ADB = ∠ACE.

66. Because ΔABD ~ ΔAEC, $\frac{AB}{AE} = \frac{AD}{AC}$ (corresponding sides of similar triangles are proportional). So $AB \cdot AC = AD \cdot AE$ by multiplication.

67. $\frac{AB \cdot AC \cdot BC}{4r} = \frac{AD \cdot AE \cdot BC}{4r}$ by substitution. $AE = 2r$; so $\frac{AD \cdot 2r \cdot BC}{4r} = \frac{AD \cdot BC}{2}$. $\frac{AD \cdot BC}{2}$ is the area of ΔABC.

Chapter 14, Lesson 1

Set I (pages 574–576)

The two lithographs of Max Bill's *Fifteen Variations on a Single Theme* (1935–1938) shown in the text are Variations 14 and 8. The first picture in the series, the "Theme," consists of a continuous black spiral of 22 equal segments which form two sides of an equilateral triangle, three sides of a square, four sides of a regular pentagon, and so on, to seven sides of a regular octagon. In each of the 15 variations, the artist plays with various geometric possibilities inherent in this figure. Many of Bill's works were influenced by the figures of plane and solid geometry as well as topology.

In *Mathematical Models,* by H. Martyn Cundy and A. P. Rollett (Oxford University Press, 1961), the authors say concerning polygonal knots: "If a strip of paper is knotted once and carefully pressed flat the folds will form a regular pentagon. All polygons with an odd number of sides [starting with the pentagon] may be produced in this way. . . . The even-sided polygons require two strips of equal width; a reef-knot leads to the hexagon. . . ." Pictures of the first four polygonal knots can also be found in *The Penguin Dictionary of Curious and Interesting Geometry,* by David Wells (Penguin Books, 1991).

Geometry in Art.

1. Equilateral triangle, square, regular pentagon, regular hexagon, regular heptagon, regular octagon.
2. Each successive polygon shares a side with the preceding polygon.

•3. The radii of the polygons.

4. The circles that can be circumscribed about the polygons.

Geometry in Nature.

•5. Yes. If a triangle is equilateral, it is also equiangular.

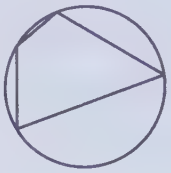
6. Yes.
7. No. Every triangle is cyclic, but not every triangle is regular.

•8. No.

9. Yes. (If it is equiangular, it is a rectangle; so its opposite angles are supplementary.)

•10. Yes. (*All* regular polygons are cyclic.)

11. No.
Example figure:



Cell Pattern.

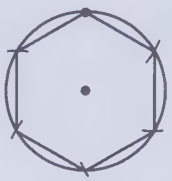
12. Regular hexagons.

•13. 60°. ($\frac{360°}{6}$.)

•14. Equilateral.

15. The sides are equal to the radius.

16. *Example figure:*



17. The perimeter of the hexagon is three times the diameter of the circle.

Regular Dodecagon.

•18. Three squares.

19. Four equilateral triangles.

•20. They are factors of it.

•21. Equilateral triangles and regular pentagons.

22. Squares and regular octagons.

23. No regular polygons.

Polygonal Knot.

24. A regular heptagon.
25. No.
26. Yes.

•27. Regular polygons having an even number of sides.

28. Regular polygons having at least six sides.

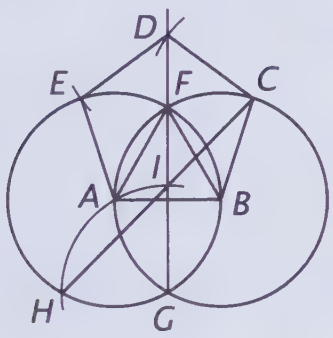
Set II (pages 576–578)

The approximate construction of the regular pentagon explored in exercises 29 through 32 is discussed in detail by J. L. Heilbron on pages 226 and 227 of *Geometry Civilized* (Clarendon Press, 1998). For a pentagon to be regular, each of its angles must have a measure of 108°. Unfortunately, in this particular construction, $\angle A = \angle B \approx 108.37°$, $\angle C = \angle E \approx 107.04°$, and $\angle D \approx 109.17°$. The method, popular with medieval masons, was also included by Renaissance artist Albrecht Dürer in his book titled *Course in the Art of Measurement with Compasses and Rulers* (1525).

Hidden in the figure for exercises 51 through 61 is the golden ratio. The appearance of the 36°-72°-72° isosceles triangles in the figure is the reason. Bisecting $\angle ABF$ produces a triangle similar to ΔABF. From this, it can be shown that the ratio of a leg of ΔABF to its base is the golden ratio. It follows immediately from the figure that the diagonals of a regular pentagon cut each other in this same ratio.

Mason's Pentagon.

29.



30. Yes. All five sides have been constructed equal to AB, the radius of the circles.

•31. If it isn't regular, it must not be equiangular.

32. Yes. ΔABF is regular because it is equilateral and therefore equiangular.

Folding an Octagon.

•33. Isosceles right triangles.

34. SAS.

35. Corresponding sides of congruent triangles are equal.

36. Isosceles (and obtuse).

37. ASA. (All of their acute angles equal 22.5°.)

38. They are equal.

•39. 22.5°. ($\angle AEI = \frac{1}{2}45° = 22.5°$.)

40. 135°. [$\angle EIH = 180° - 2(22.5°) = 135°$.]

•41. 135°. [$\angle IEJ = 90° + 2(22.5°) = 135°$.]

42. They are equal.

43. They prove that it is regular.

Hexagonal Fastener.

44. Radii.

•45. Apothems.

46. PA = PB (all radii of a circle are equal) and $\angle APC = \angle BPD$ (vertical angles are equal); so $\Delta APC \cong \Delta BPD$ by HA.

•47. 30°-60° right triangles.

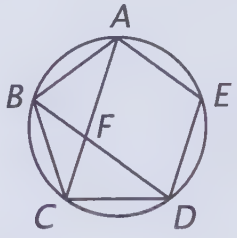
48. $AC = \frac{0.25 \text{ in}}{2} = 0.125$ in and $PC = 0.125\sqrt{3}$ in ≈ 0.2165 in.

49. $AB = 2PA = 0.50$ in and $CD = 2PC = 0.25\sqrt{3}$ in ≈ 0.433 in.

50. CD, because the wrench is used to grip the sides of the head, not its corners.

Inscribed Pentagon.

51.



•52. They are equal because the sides of the pentagon are chords of the circle, and equal arcs in a circle have equal chords.

53. 72°. ($\frac{360°}{5}$.)

•54. 72°. [$\angle ABD = \frac{1}{2}m\overset{\frown}{AD} = \frac{1}{2}(2 \cdot 72°) = 72°$.]

55. 72°. [$\angle AFB = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD}) = \frac{1}{2}(72° + 72°) = 72°$.]

56. 36°. ($\angle BAC = \frac{1}{2}m\widehat{BC} = \frac{1}{2}72° = 36°$.)

57. 36°.

58. Five.

59. They are all isosceles.

60. AEDF is a rhombus: AF = AB and FD = CD; so all four sides of AEDF are equal to the sides of the regular pentagon. Because AEDF is a rhombus, it follows that it is also a parallelogram.

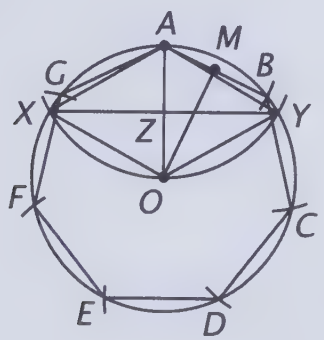
61. AEDB and AEDC are isosceles trapezoids.

Set III (page 578)

A chapter of Robert Dixon's book titled *Mathographics* (Dover, 1987) deals with "Euclidean approximations" to some of the impossible constructions. As Dixon explains concerning the regular heptagon: "To divide a circle into exactly seven equal parts would require us to be able to construct the angle of $\frac{360°}{7} \approx 51.428571°$, which we cannot do." He then presents and analyzes three comparatively simple constructions that are amazingly close approximations of the regular heptagon based on constructing angles of approximately 51.317813°, 51.340192°, and 51.470701°.

A Close Construction.

1.



2. Equilateral.

3. AO is the perpendicular bisector of XY (because A and O are equidistant from X and Y.)

4. A 30°-60° right triangle.

5. $OZ = \frac{1}{2}r$. ($OZ = \frac{1}{2}OY$.)

6. $ZY = \frac{1}{2}r\sqrt{3}$. ($ZY = \sqrt{3}OZ$.)

7. $AB = \frac{1}{2}r\sqrt{3}$. (AB = ZY by construction.)

8. $AM = \frac{1}{4}r\sqrt{3}$. ($AM = \frac{1}{2}AB$.)

9. 25.658906 . . .°. ($\sin \angle AOM = \frac{AM}{AO} = \frac{\frac{1}{4}r\sqrt{3}}{r} = \frac{\sqrt{3}}{4} = 0.433012702\ldots$)

10. 51.317812 . . .°. ($\angle AOB = 2\angle AOM$.)

11. 359.22468 . . .°.

12. That ABCDEFG is not regular. If ABCDEFG were regular, $7\angle AOB$ would equal 360°.

Chapter 14, Lesson 2

Set I (pages 581–582)

The figure showing the design of Palma Nuova suggests that the city was surrounded by nine bastions (the spade-shaped regions) for its defense, as well as possibly by a moat.

Students taking chemistry may wonder how each hydrogen atom in the hydrogen fluoride rings can have two bonds. In his book titled *The Architecture of Molecules* (W. H. Freeman and Company, 1964), Linus Pauling explains: "Many properties of substances can be easily explained by the assumption that the hydrogen atom, which is normally univalent, can sometimes assume ligancy two, and form a bridge between two atoms. This bridge is called the hydrogen bond. . . . Hydrogen fluoride gas has been found to contain not only molecules HF, but also polymers, especially $(HF)_5$ and $(HF)_6$ . . . . Each hydrogen atom is strongly bonded to one fluorine atom (bond length 1.00 Å) and less strongly to another (bond length 1.50 Å)." "Å" is the symbol for Angstrom, a unit of length equal to 10^{-8} cm (10^{-10} m).

The Susan B. Anthony dollar coin never became popular, owing in part to the fact that it was too easily confused with a quarter in both look and feel.

Perimeter Equations.

1. The radius.

•2. The number of sides.

3. 3.

•4. 3.0902. (10 sin 18°.)

5. 3.1411. (100 sin 1.8°.)

6. 3.1416. (1,000 sin 0.18°.)

7. 3.1416. (10,000 sin 0.018°.)

8. It increases.

9. It becomes more and more circular.

Angles and Radii.

•10. Radii.

11. $\Delta ABP \cong \Delta CBP$ by SSS.

12. $\angle 1 = \angle 2$ because corresponding parts of congruent triangles are equal.

13. No.

•14. PB bisects $\angle ABC$.

Italian City.

15. Nine.

•16. 9 sin 20°.

•17. 3.078.

18. 6,772 ft. [2(3.078)(1,100).]

Hydrogen Fluoride.

19. A regular pentagon and a regular hexagon.

20. They seem to be equal.

•21. The radius of the hexagon appears to be longer.

•22. 108°. $\left(\frac{3 \cdot 180°}{5}.\right)$

23. 120°. $\left(\frac{4 \cdot 180°}{6}, \text{ or } 2 \cdot 60°.\right)$

24. 12.5 angstroms. (5×2.5.)

25. 15 angstroms. (6×2.5.)

•26. $5 \sin 36° \approx 2.94$.

27. $6 \sin 30° = 3$.

•28. Approximately 2.13 angstroms. [$12.5 = 2(2.94)r$, $r \approx 2.13$.]

29. 2.5 angstroms. [$15 = 2(3)r$, $r = 2.5$.]

Dollar Coin.

30. 75 mm. [$c = 2\pi r = 2\pi(12) = 24\pi \approx 75$.]

•31. 74 mm. $\left[p = 2\left(11 \sin \frac{180}{11}\right)12 \approx 74.\right]$

32. 7 mm. $\left(\frac{74}{11} \approx 7.\right)$

Set II (pages 583–584)

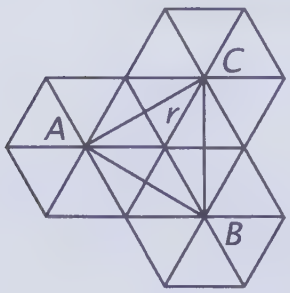
Most people who use them probably have no idea that the name "cell phone" refers to the cells into which the areas in which they are used are divided or that the cells have the same shape that bees use in building honeycombs: the regular hexagon.

Although regular pentagons cannot be used to fill a plane as hexagons can be, the figure for exercises 41 through 48 shows that 10 of them can fit perfectly around a regular decagon. In his book titled *Geometry in Architecture* (Wiley, 1984), William Blackwell observes that this is a nice arrangement for surrounding a large 10-sided pavilion with 10 smaller galleries.

Exercises 49 through 58 lead to Proposition 10 of Book XIII of the *Elements*: "If an equilateral pentagon is inscribed in a circle, the side of the pentagon is equal in square to that of the hexagon and that of the decagon inscribed in the same circle." A good discussion of how this proposition is proved is included by Benno Artmann in *Euclid—The Creation of Mathematics* (Springer, 1999). Artmann remarks concerning this theorem: "In other words, the side of the pentagon is the hypotenuse of a right triangle that has the sides of the hexagon and of the decagon as its legs—a really unexpected and beautiful insight!" After explaining that Euclid may have discovered this remarkable theorem in his investigation of the icosahedron, Artmann also observes: "It is common in mathematics that certain parts of complicated proofs become of independent interest, and afterwards one wonders how anybody could have thought of them."

Phone Cells.

33. $p = 2Nr = 2(6 \sin 30°)r = 2(3)r = 6r$.

34. 

35. Equilateral triangles. The sides of each cell are equal to its radius; so the perimeter of each cell is $6r$.

36. Equilateral.

•37. $r\sqrt{3}$. [r is the hypotenuse of each 30°-60° right triangle included by ΔABC; so the length of the shorter leg is $\frac{r}{2}$ and the length of the longer leg is $\frac{r}{2}\sqrt{3}$.
$AB = 2(\frac{r}{2}\sqrt{3}) = r\sqrt{3}$.]

38. $3r\sqrt{3}$.

•39. $p = 2Nr = 2(3 \sin 60°)r \approx 5.196r$.

40. Yes. $3r\sqrt{3} \approx 5.196r$.

Ten Pentagons.

•41. $\frac{10}{5} = 2$.

42. 36°. ($\angle X = \frac{1}{2} \cdot \frac{360°}{5} = 36°$.)

43. 18°. ($\angle XYZ = \frac{1}{2} \cdot \frac{360°}{10} = 18°$.)

•44. By the Law of Sines.

•45. $\frac{R}{r} = \frac{\sin 36°}{\sin 18°} \approx 1.9$.
(Because $\frac{\sin \angle X}{R} = \frac{\sin \angle XYZ}{r}$,
$\frac{R}{r} = \frac{\sin \angle X}{\sin \angle XYZ} = \frac{\sin 36°}{\sin 18°}$.)

46. X and Y are equidistant from V and Z. (Two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.)

47. The tangent of an acute angle of a right triangle is the ratio of the length of the opposite leg to the length of the adjacent leg.

•48. $\frac{A}{a} = \frac{\tan 36°}{\tan 18°} \approx 2.2$. (Because $\tan \angle X = \frac{WZ}{a}$ and $\tan \angle XYZ = \frac{WZ}{A}$,
$WZ = a \tan \angle X = A \tan \angle XYZ$;
so $\frac{\tan \angle X}{\tan \angle XYZ} = \frac{A}{a}$.)

Euclid's Discovery.

•49. 18°. ($\frac{1}{2} \cdot \frac{360°}{10} = 18°$.)

50. 30°. ($\frac{1}{2} \cdot \frac{360°}{6} = 30°$.)

51. 36°. ($\frac{1}{2} \cdot \frac{360°}{5} = 36°$.)

•52. $AX = 3.0901699\ldots$. ($\sin 18° = \frac{AX}{10}$; so $AX = 10 \sin 18°$.)

53. $BY = 5$. ($\sin 30° = \frac{BY}{10}$; so $BY = 10 \sin 30°$.)

54. $CZ = 5.8778525\ldots$. ($\sin 36° = \frac{CZ}{10}$; so $CZ = 10 \sin 36°$.)

55. $a^2 = (2\,AX)^2 = 38.19660\ldots$.

56. $b^2 = (2\,BY)^2 = 100$.

57. $c^2 = (2\,CZ)^2 = 138.19660\ldots$.

58. Euclid discovered that $a^2 + b^2 = c^2$. If a regular decagon, hexagon, and pentagon are inscribed in circles of a given radius (or the same circle), the sum of the squares of a side of the decagon and a side of the hexagon is equal to the square of a side of the pentagon.

Set III (page 584)

This exercise suggests that the ratio of the length of a diagonal to the length of a side of a regular pentagon is the golden ratio. We are more accustomed to seeing this ratio in its algebraic form, $\frac{1+\sqrt{5}}{2}$, than in its trigonometric form, $2 \sin 54°$.

Golden-Heather.

1. $\angle AOB = \frac{360°}{5} = 72°$; so $\angle AOX = 36°$.
   $\sin \angle AOX = \frac{AX}{OA}$; so $\sin 36° = \frac{AX}{1}$.
   $AB = 2AX = 2 \sin 36°$.
2. $\angle COE = 2\angle AOB = 144°$; so $\angle COY = 72°$.
   $\sin \angle COY = \frac{CY}{OC}$; so $\sin 72° = \frac{CY}{1}$.
   $CE = 2CY = 2 \sin 72°$.
3. $\frac{2 \sin 72°}{2 \sin 36°} = 1.6180339\ldots$ and
   $2 \sin 54° = 1.6180339\ldots$.
4. The golden ratio.

Chapter 14, Lesson 3

Set I (pages 587–588)

The expression $M = n \sin \frac{180}{n} \cos \frac{180}{n}$ has meaning only when n is an integer larger than 2, because n represents the number of sides of a polygon. Although a calculator readily finds sines and cosines for any angle, our right triangle definitions are valid only for acute angles. Nevertheless, with the use of a calculator when $n = 1$ or 2 and there is no proper polygon, M turns out to equal 0, suggesting that no area is enclosed.

Exercises 20 through 28 provide another preview of the fact that our work with regular polygons will be useful in measuring the circle.

Area Equations.

1. The radius.
2. The number of sides.
•3. 0. (1 sin 180° cos 180°.)
4. 0. (2 sin 90° cos 90°.)
•5. 1.30. (3 sin 60° cos 60°.)
6. 3.14. (180 sin 1° cos 1°.)
7. 3.

Inscribed Polygon.

8. The area of the circle.
•9. The perimeter of the nonagon.
10. The length of a side of the nonagon.
11. The circumference of the circle.

The Area of a Square.

12. A line segment that connects the center of the square to a vertex, or the distance from its center to one of its vertices.
13. s^2.
•14. $4a^2$. [$s = 2a$; so $s^2 = (2a)^2 = 4a^2$.]
15. An isosceles right triangle.
•16. $r = a\sqrt{2}$.
17. $r^2 = 2a^2$.
•18. $2r^2$. ($\alpha ABCD = 4a^2$ and $r^2 = 2a^2$; so $\alpha ABCD = 2r^2$.)
19. $(4 \sin 45° \cos 45°)r^2 = 2r^2$.

Close But Not Quite.

20. Its circumference.
21. 628 units. [$c = 2\pi(100) \approx 628$.]
•22. 628 units. [$p = 2(60 \sin 3°)100 \approx 628$.]
23. The circle.
24. 3.14. [$\frac{628}{2(100)}$.]
25. 3.14.
26. The circle.
27. 31,416 square units. ($a = \pi 100^2 \approx 31{,}416$.)
•28. 31,359 square units.
[$A = 60(\sin 3° \cos 3°)100^2 \approx 31{,}359$.]

Set II (pages 588–590)

József Kürschák, the man associated with the geometric proof that the area of a regular dodecagon having radius r is $3r^2$, was a professor of mathematics at the Polytechnic University in Budapest for many years. He is also remembered as the author of the *Hungarian Problem Books I and II*, still available in the New Mathematics Library of the Mathematical Association of America.

Honeycomb Geometry.

29. Equilateral.

30. An apothem.

•31. 30°-60° right triangles.

•32. $\frac{r}{2}$.

33. $\frac{r}{2}\sqrt{3}$.

•34. $\frac{\sqrt{3}}{4}r^2$.

35. $\frac{3\sqrt{3}}{2}r^2$.

$[\alpha ABCDEF = 6\alpha\Delta AOB = 6(\frac{\sqrt{3}}{4}r^2) = \frac{3\sqrt{3}}{2}r^2.]$

36. $(6 \sin 30° \cos 30°)r^2$.

37. In ΔBOP, $\sin \angle BOP = \frac{PB}{OB} = \frac{\frac{r}{2}}{r} = \frac{1}{2}$.

38. In ΔBOP, $\cos \angle BOP = \frac{OP}{OB} = \frac{\frac{r}{2}\sqrt{3}}{r} = \frac{\sqrt{3}}{2}$.

39. $(6 \sin 30° \cos 30°)r^2 = (6(\frac{1}{2})(\frac{\sqrt{3}}{2}))r^2 = \frac{3\sqrt{3}}{2}r^2$.

40. Yes. It agrees with the answer to exercise 35.

Kürschák Triangles.

41. ΔFHI and ΔBHI. Because they are equilateral and share a side, they are congruent by SSS. (Or, because they share a side and have 60° angles, they are congruent by ASA or SAS.)

•42. They are isosceles.

•43. That they are 15°. $(\frac{90° - 60°}{2}.)$

44. That they are 150°. [180° – 2(15°).]

45. $\angle AHI = \angle HIC = 150°$. [2(60°) + 2(15°).]

46. r. (DH = DI = AB = BC = CD = DA = r.)

•47. $AH = HI = IC = (2 \sin 15°)r \approx 0.518r$. [AH, HI, and IC are sides of a regular dodecagon with radius r. Its perimeter is $2(12 \sin 15°)r$; so the length of each side is $(2 \sin 15°)r$.]

48. $DE = DF = DG = (2 \sin 15°)r \approx 0.518r$. (The yellow triangles are isosceles.)

•49. $\frac{1}{4}r^2$. (BAHIC contains one-fourth of the examples of each type of triangle that make up the square.)

50. $\frac{3}{4}r^2$. $(r^2 - \frac{1}{4}r^2.)$

51. $\frac{1}{4}r^2$. $(\alpha\Delta DAH = \frac{1}{3}\alpha DAHIC.)$

Rats!

52. *(Student answer.)* (The square has the larger radius.)

53. *(Student answer.)* (The square has the larger perimeter.)

•54. 7.1 cm. $[100 = 4(\sin 45° \cos 45°)r^2, 100 = 2r^2, r^2 = 50, r = \sqrt{50} \approx 7.1.]$

55. 6.5 cm. $[100 = 5(\sin 36° \cos 36°)r^2, 100 \approx 2.38r^2, r^2 \approx 42.0, r \approx 6.5.]$

56. 40 cm. (Because $s^2 = 100$, $s = 10$, and $p = 4s = 40$.)

•57. 38 cm. $[p \approx 2(5 \sin 36°)(6.5) \approx 38.]$

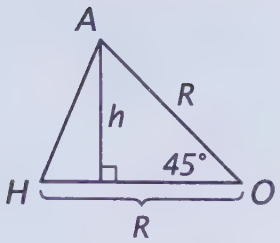
58. *(Student answer.)* (Both the radius and the perimeter of the pentagon are smaller than those of the square.)

Hebrew Exercise.

59. The problem is to prove that the area of a regular octagon having a radius of R is $2\sqrt{2}R^2$.

60. $A = (8 \sin 22.5° \cos 22.5°)R^2 \approx 2.828R^2$, and $2\sqrt{2}R^2 \approx 2.828R^2$.

61.



In ΔAOH, $h = \frac{R}{\sqrt{2}} = \frac{\sqrt{2}R}{2}$; so

$\alpha\Delta AOH = \frac{1}{2}HO \cdot h = \frac{1}{2}R(\frac{\sqrt{2}R}{2}) = \frac{\sqrt{2}}{4}R^2$.

The area of the octagon is

$8\alpha\Delta AOH = 8(\frac{\sqrt{2}}{4}R^2) = 2\sqrt{2}R^2$.

Set III (page 590)

More on the work of Marjorie Rice, including several beautiful Escher-like tessellations reproduced in full color, can be found in the chapter titled "In Praise of Amateurs" by Doris Schattschneider in *The Mathematical Gardner*, edited by David A. Klarner (Prindle, Weber & Schmidt, 1981) and on the Website members.aol.com/tessellations. The central tessellation with its sixfold rotation symmetry that is the subject of the exercises was discovered by Michael Hirschhorn, a teacher in New South Wales.

Congruent Pentagons.

1. They are equilateral. (The colors along the sides of the pentagons can be used to establish this.)

2. The boundary of the figure is a regular 18-gon because it is both equilateral and equiangular. (From the measures of the angles of the pentagonal pieces, we can determine that each of the angles of the 18-gon has a measure of 160°.)

3. Approximately 2.88 units. Because the perimeter of the 18-gon is 18 and

   $p = 2Nr = 2n \sin\frac{180}{n}r$, $18 = 2(18 \sin 10°)r$.

   So $r = \frac{1}{2 \sin 10°} \approx 2.88$.)

4. Approximately 25.53 square units.
   [$A \approx (18 \sin 10° \cos 10°)(2.88)^2 \approx 25.53$.]

5. Approximately 1.42 square units.
   ($\frac{25.53}{18} \approx 1.42$.)

Chapter 14, Lesson 4

Set I (pages 593–595)

Hats are the only instance of a clothing size in which circumference is measured to determine diameter. A tape measure is wrapped around the widest part of the head (about an inch above the eyebrows) to get the circumference of the inner band of the appropriate hat. Dividing this number by π gives the hat size. This method works quite well because the circle is a reasonable approximation of the cross section of a person's head.

The time that it takes the moon to travel once around Earth, about 27.3 days, is called the "sidereal" month. The word "sidereal" means "determined by the stars," and that is how the period of revolution was found. A "lunar" or "synodic" month is the time between two successive new moons, about 29.5 days, and a "solar" month is one-twelfth of a solar year (about 30.4 days).

In *The Innovators* (Wiley, 1996), David Billington wrote: "It has been argued that the steamboat was the first great American contribution to modern technology." Billington, a professor of civil engineering at Princeton, includes a wealth of information on Robert Fulton's calculations, including his patent formulas and the mathematics of his drag, power, and paddlewheel calculations.

The Babylonian-tablet exercises provide another example of Pythagorean triples appearing many centuries before Pythagoras was born. (If π is taken as 3.14, the convenient relation of the sides of the triangle, surely intended by the problem's creator, is lost.) It seems strange, however, that the writer of a problem of this level would think that π was equal to 3.

Francois Viète has been called the greatest French mathematician of the sixteenth century. In *A History of π* (Golem Press, 1971), Petr Beckmann reports that Viète was the first to discover a way to represent π as an expression of an infinite sequence of mathematical operations:

$$\pi = \frac{2}{\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \cdots}$$

Unfortunately, the expression is too cumbersome and time consuming to be of practical use in calculating π, and so Viète was also one of the last mathematicians to resort to using polygons to do so. The number of sides used, 393,216, results from 16 successive doublings of Archimedes' original hexagon.

Circumference.

1. Because $d = 2r$ and $c = 2\pi r = \pi(2r)$, $c = \pi d$ by substitution.

•2. π.

3. 2π.

Hat Sizes.

4. $\frac{22}{7} \approx 3.143$ and $\frac{22.75}{7.25} \approx 3.138$. (Both ratios are approximately the value of π.)

•5. The circumference of the hat.

6. The diameter of the hat.

The Moon's Orbit.

•7. Approximately 1,500,780 mi. [$c = 2\pi(238{,}857) \approx 1{,}500{,}783$.]

8. Approximately 656 hours. ($\frac{1{,}500{,}783}{2{,}287} \approx 656$.)

9. Approximately 27.3 days. ($\frac{656}{24} \approx 27.3$.)

Semicircles.

•10. πa.

11. πb.

12. $\pi(a + b)$.

13. Because $\pi(a + b) = \pi a + \pi b$, the length of semicircle C is equal to the sum of the lengths of semicircles A and B.

14. No.

Steamboat Geometry.

15. 42,240 ft/hour. ($8 \times 5{,}280$.)

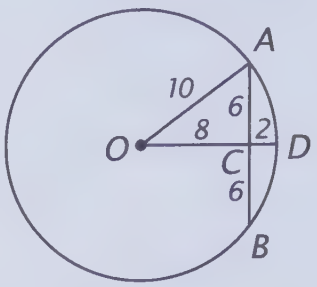
•16. 704 ft/minute. ($\frac{42{,}240}{60}$.)

•17. 14π ft.

18. 16. ($\frac{704 \text{ ft/minute}}{14\pi \text{ ft/revolution}} \approx$ 16 revolutions/minute.)

Babylonian Problem.

19.



•20. 10 units. [$c = 2\pi r$, $60 = 2(3)r$, $r = 10$.]

•21. 8 units. (OC = OD − CD = 10 − 2 = 8.)

22. 6 units. ($AC = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$.)

23. With the assumption that $\pi = 3$, the sides of ΔOAC form a Pythagorean triple: 6-8-10. (Or, the sides of ΔOAC are all integers.)

•24. $AC = \frac{1}{2}AB$. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

25. 12 units.

Viète's Calculation.

•26. 3.1415927. ($N = 393{,}216 \sin \frac{180}{393{,}216} \approx 3.1415927$.)

27. No. The value must be less than π because the perimeter of a regular polygon is always less than the circumference of a circle having the same radius.

Perimeters and Diameters.

•28. $2\sqrt{2}$. ($p = 4s$ and $d = s\sqrt{2}$; so $\frac{p}{d} = \frac{4s}{s\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.)

•29. 2.83.

30. 3. ($p = 6s$ and $d = 2s$; so $\frac{p}{d} = \frac{6s}{2s} = 3$.)

31. π.

32. 3.14.

33. Close to 3.14 (or close to, but less than, π.)

34. $p = 2Nr$ and $d = 2r$; so $\frac{p}{d} = \frac{2Nr}{2r} = N$.

$N = 100 \sin\frac{180}{n} \approx 3.1410759$.

Set II (pages 595–597)

Owing to Earth's rotation, the speed at which someone living on the equator is traveling, 1,040 miles per hour, is much faster than the official world land-speed record! Our speed due to Earth's motion around the sun, 67,000 miles per hour, is more than three times as great as that of the Space Shuttle.

Adam Kochansky published his construction as "An Approximate Geometrical Construction for Pi," and so he knew that it was not exact. The exercises exploit the calculator in finding the various distances. It isn't difficult to show (without using a calculator) that the exact length of BE is $\sqrt{\frac{40-6\sqrt{3}}{3}}$. In *Geometry Civilized* (Clarendon Press, 1998), J. L. Heilbron, noting that Kochansky liked to calculate, quotes him as saying: "The periphery thus found differs from the closest approximation to the true Archimedean value [of pi] by less than the ratio of one to ten times the current year 1685. . . ." Doing the calculation,

$$\sqrt{\frac{40-6\sqrt{3}}{3}} + \frac{1}{16{,}850} = 3.14159268\ldots,$$

gives the value of π, 3.14159265 . . . correctly to seven decimal places.

Going in Circles.

•35. Approximately 24,880 mi. [Student rounding may differ. $c = 2\pi(3{,}960) \approx 24{,}881$.]

•36. Approximately 1,040 mi.
($\frac{24{,}881}{24} \approx 1{,}037$.)

•37. 1,040 mi/hour.

38. Approximately 584,000,000 mi.
[$c = 2\pi(93{,}000{,}000) \approx 584{,}336{,}234$.]

39. Approximately 1,600,000 mi.
($\frac{584{,}336{,}234}{365.25} \approx 1{,}599{,}825$.)

40. Approximately 67,000 mi.
($\frac{1{,}599{,}825}{24} \approx 66{,}659$.)

41. Approximately 67,000 mi/hour.

•42. Approximately 19 mi/second.
($\frac{66{,}659}{3{,}600} \approx 18.5$.)

Two Squares.

43. HL. ΔABO and ΔDCO are right triangles with AB = DC because ABCD is a square; OB = OC because all radii of a circle are equal.

44. $OD = \frac{1}{2}x$. ($OD = \frac{1}{2}AD = \frac{1}{2}x$.)

•45. $r^2 = \frac{5}{4}x^2$. [In right ΔDCO,
$r^2 = x^2 + (\frac{1}{2}x)^2 = \frac{5}{4}x^2$.]

•46. $r^2 = \frac{1}{2}y^2$. (In right ΔGPH, $y^2 = r^2 + r^2 = 2r^2$; so $r^2 = \frac{1}{2}y^2$.)

47. $x^2 = \frac{2}{5}y^2$. (Because $r^2 = \frac{5}{4}x^2$ and $r^2 = \frac{1}{2}y^2$, $\frac{5}{4}x^2 = \frac{1}{2}y^2$; so $x^2 = \frac{2}{5}y^2$.)

48. The area of ABCD is two-fifths of the area of EFGH.

SAT Problem.

•49. $2a + 2b + 2c$.

50. $2\pi a + 2\pi b + 2\pi c$.

51. $\frac{1}{\pi}$. [$\frac{2a+2b+2c}{\pi(2a+2b+2c)} = \frac{1}{\pi}$.]

Videotape.

•52. 825 ft. ($\frac{1.375 \cdot 60 \cdot 60 \cdot 2}{12} = 825$.)

•53. $\pi \approx 3.14$ in. [$c = 2\pi(0.5) = \pi$.]

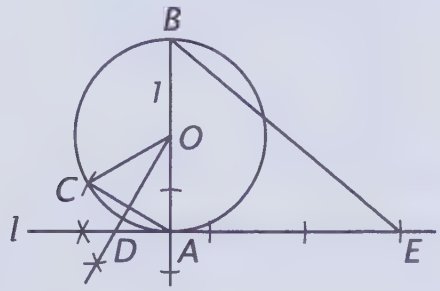
54. $3\pi \approx 9.42$ in. [$c = 2\pi(1.5) = 3\pi$.]

55. $2\pi \approx 6.28$ in. ($\frac{\pi + 3\pi}{2} = 2\pi$.)

56. About 1,576 times. ($\frac{825 \cdot 12}{6.28} \approx 1{,}576.$)

Kochansky's Construction.

57.



58. Equilateral.

59. A 30°-60° right triangle.

60. DA ≈ 0.5773503.
($DA = \frac{OA}{\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.5773503.$)

61. AE ≈ 2.4226497.
($AE = DE - DA \approx 3 - 0.5773503 = 2.4226497.$)

62. BE ≈ 3.1415333. ($BE^2 = AB^2 + AE^2$ so $BE = \sqrt{2^2 + AE^2} \approx 3.1415333.$)

63. It is a good approximation of π, correct to five digits (or four decimal places).

Set III (page 597)

Tire Change.

The ratio of the circumference of the larger tire to the circumference of the smaller tire is $\frac{33\pi}{28.9\pi} \approx 1.14$; so, for each revolution of the tires, the truck goes 1.14 times as far as before. 70 miles × 1.14 ≈ 79.8 miles. The truck would be going about 80 miles per hour.

Chapter 14, Lesson 5

Set I (pages 600–602)

The area calculated in exercise 3 for a typical hurricane is more meaningful when compared with the areas of states such as Florida (59,000 mi²) and Louisiana (48,000 mi²), both of which were hit hard by Hurricane Andrew in 1992.

Another unit of circular area measure is the "circular mil," defined as the area of a circle having a diameter of one "mil" (0.001 in). Circular units of area are obviously of little value in working with anything other than circles. Circular mils are used chiefly in the measurement of wire.

Exercises 13 through 16 provide an example of how a good foundation in geometry is important in calculus. Related rates and maximum/minimum problems are particularly obvious examples in that nearly every problem requires the use of plane or solid geometry.

Exercise 17 is actually the first theorem of Archimedes' *Measurement of a Circle*: "The area of a circle is equal to that of a right triangle in which one leg is equal to the radius and the other to the circumference of the circle."

Although the result of exercises 18 through 21, that in central pivot irrigation the number of subdivisions of a square region makes no difference in the area watered, may seem at first surprising, the reason is obvious when considered from the viewpoint of a dilation. No matter how small or large the cells might be, the ratio of the areas of each circle and its corresponding square, $\frac{\pi r^2}{4r^2}$, remains constant.

The use of the circle on a square grid to estimate π was first investigated by Gauss. According to David Hilbert and Stefan Cohn-Vossen in *Geometry and the Imagination* (Chelsea, 1952), Gauss "tried to determine the number $f(r)$ of lattice points in the interior and on the boundary of a circle of radius r, where the center of the circle is a lattice point and r is an integer. Gauss found the value of this number empirically for many values of n. . . . His interest was prompted by the fact that an investigation of this function yields a method for approximating the value of π." The details and an explanation of the procedure can be found on pages 33–35 of Hilbert's book.

Hurricanes.

•1. 300 mi.

•2. 940 mi. [$c = \pi(300) \approx 942.$]

•3. 71,000 mi². [$A = \pi(150)^2 \approx 70{,}686.$]

•4. 15 mi. ($700 = \pi r^2$, $r = \sqrt{\frac{700}{\pi}} \approx 15.$)

•5. 90 mi. [$c = 2\pi(15) \approx 94.$]

Square and Circular Inches.

6. The area of a square with sides of length 1 inch.

7. The area of a circle with a diameter of 1 inch.

8. A square inch.

•9. x^2. ($\frac{x^2}{1^2} = x^2$.)

10. 100 square inches.

11. x^2. [$\frac{\pi(\frac{x}{2})^2}{\pi(\frac{1}{2})^2} = x^2$.]

•12. 100 circular inches.

Ripples.

13. 60 cm.

•14. $3{,}600\pi$ cm^2. [$A = \pi(60)^2 = 3{,}600\pi$.]

•15. $60t$ cm.

16. $3{,}600\pi t^2$ cm^2. [$A = \pi(60t)^2 = 3{,}600\pi t^2$.]

Equal Areas.

17. $2\pi r$ (or, the circumference of the circle).
If the areas are equal, $\pi r^2 = \frac{1}{2}ABr$; so $AB = 2\pi r$.

Central-Pivot Irrigation.

18. *(Student answer.)*

•19. 36π square units. [The diameter of each circle is 6; so the radius is 3. The total area of the four circles is $4(\pi 3^2) = 36\pi$.]

20. 36π square units. [The diameter of each circle is 4; so the radius is 2. The total area of the nine circles is $9(\pi 2^2) = 36\pi$.]

21. They are equal.

Pi Square.

22. $\sqrt{\pi}$. ($\pi = s^2$; so $s = \sqrt{\pi}$.)

•23. π^2. [$A = \pi r^2 = \pi(\sqrt{\pi})^2 = \pi^2$.]

24. π. ($\frac{\pi^2}{\pi} = \pi$.)

Circle on a Grid.

•25. 6 units.

26. 3.13888 . . . . ($\frac{113}{6^2} = 3.13888\ldots$.)

•27. 3.1425. ($\frac{1{,}257}{20^2} = 3.1425$.)

28. About 3.1411. ($\frac{282{,}697}{300^2} \approx 3.14107\ldots$.)

29. They suggest that the limit is π.

30. About 31,416. ($\frac{p}{100^2} \approx \pi$, $p \approx 10{,}000\pi \approx 31{,}416$. The number of points is actually 31,417.)

Set II (pages 602–604)

Exercises 39 through 46 look at an early precursor of integration. Sawaguchi Kazuyuki published his method for finding the area of a circle in 1670, at about the time that Newton and Leibniz were discovering the rules of the calculus. As Petr Beckmann observes in *A History of π* (Golem Press, 1971), in theory the method can be used to find π to any desired degree of accuracy by choosing a sufficiently large number of strips. Beckmann also points out that, in practice, the series converges very slowly in addition to requiring the inconvenient extraction of square roots.

Area Problems.

•31. $3\pi x^2$. [The radius of circle A is $2x$; so its area is $\pi(2x)^2 = 4\pi x^2$. The radius of circle B is x; so its area is πx^2. The shaded area is $4\pi x^2 - \pi x^2 = 3\pi x^2$.]

32. $5\pi x^2$. [The radius of the larger circle is $3x$; so its area is $\pi(3x)^2 = 9\pi x^2$. The radius of the smaller circle is $2x$; so its area is $\pi(2x)^2 = 4\pi x^2$. The shaded area is $9\pi x^2 - 4\pi x^2 = 5\pi x^2$.]

•33. $(4 - \pi)x^2$. [The side of square ABCD is $2x$; so its area is $(2x)^2 = 4x^2$. The radius of circle O is x; so its area is πx^2. The shaded area is $4x^2 - \pi x^2 = (4 - \pi)x^2$.]

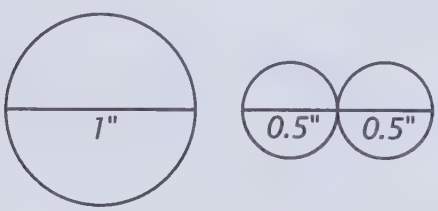
34. $(\pi - \frac{\sqrt{3}}{2})x^2$. [Because ΔABC is a 30°-60° right triangle, $AB = 2x$ and $AC = x\sqrt{3}$. Because $AB = 2x$, $OB = x$; so the area of the circle is πx^2. Because the legs of ΔABC are x and $x\sqrt{3}$, its area is $\frac{1}{2}x(x\sqrt{3}) = \frac{\sqrt{3}}{2}x^2$. The shaded area is $\pi x^2 - \frac{\sqrt{3}}{2}x^2 = (\pi - \frac{\sqrt{3}}{2})x^2$.]

•35. $(5\pi - 8)x^2$. [The legs of each small right triangle are $2x$ and x, and so $OB = \sqrt{(2x)^2 + x^2} = \sqrt{5x^2} = x\sqrt{5}$. The radius of circle O is $x\sqrt{5}$; so its area is $\pi(x\sqrt{5})^2 = 5\pi x^2$. The area of the rectangle is $(4x)(2x) = 8x^2$. The shaded area is $5\pi x^2 - 8x^2 = (5\pi - 8)x^2$.]

•36. $(3\sqrt{3} - \pi)x^2$. [The small triangle is a 30°-60° right triangle with shorter leg x, and so its longer leg is $x\sqrt{3}$ and $AB = 2x\sqrt{3}$. The area of equilateral ΔABC is $\frac{\sqrt{3}}{4}AB^2 = \frac{\sqrt{3}}{4}(2x\sqrt{3})^2 = 3\sqrt{3}x^2$. The radius of the circle is x and so its area is πx^2. The shaded area is $3\sqrt{3}x^2 - \pi x^2 = (3\sqrt{3} - \pi)x^2$.]

Cable Disaster.

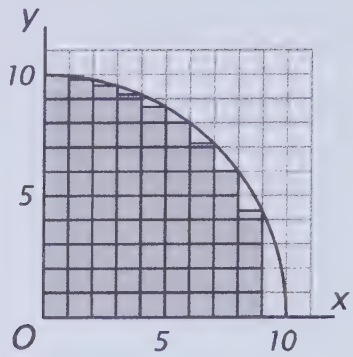
37. *Scale drawing:*



38. The strength of the cable depends, not on its diameter, but on the area of its cross section. The area of a cross section of a 1-inch cable is $\pi(\frac{1}{2})^2 = \frac{1}{4}\pi^2$ in^2. The total area of the cross sections of two $\frac{1}{2}$-inch cables is $2\pi(\frac{1}{4})^2 = \frac{1}{8}\pi^2$ in^2. The 1-inch cable should have been replaced by four $\frac{1}{2}$-inch cables.

Slicing a Circle.

39.



40. The height of the fifth strip is a leg of a right triangle with hypotenuse 10 and base 5. $h^2 + 5^2 = 10^2$; so $h = \sqrt{10^2 - 5^2} = \sqrt{100 - 25} = \sqrt{75}$. The width of the strip is 1; so its area is $1 \cdot \sqrt{75} = \sqrt{75}$.

41. $\sqrt{99}$, $\sqrt{96}$, $\sqrt{91}$, $\sqrt{84}$, $\sqrt{64} = 8$, $\sqrt{51}$, $\sqrt{36} = 6$, and $\sqrt{19}$. ($\sqrt{10^2 - 1^2} = \sqrt{99}$, $\sqrt{10^2 - 2^2} = \sqrt{96}$, $\sqrt{10^2 - 3^2} = \sqrt{91}$, etc.)

•42. 72.6.

•43. 290. [4(72.6) ≈ 290.]

44. It would be less than the actual area of the circle because the rectangular strips do not fill the quarter circle.

45. Narrower strips should fill more of the circle.

46. Keep dividing the circle into narrower and narrower strips.

Tangent Circles.

•47. $a + b$. [Its diameter is $2a + 2b = 2(a + b)$; so its radius is $a + b$.]

•48. $\pi a^2 + \pi ab$ or $\pi a(a + b)$. [The blue area is half the area of the largest circle plus half the area of the left circle minus half the area of the right circle. $\frac{1}{2}\pi(a + b)^2 + \frac{1}{2}\pi a^2 - \frac{1}{2}\pi b^2 = \frac{1}{2}\pi(a^2 + 2ab + b^2) + \frac{1}{2}\pi a^2 - \frac{1}{2}\pi b^2 = \pi a^2 + \pi ab = \pi a(a + b)$.]

49. $\pi ab + \pi b^2$ or $\pi b(a + b)$. [The yellow area is found in a similar way.
$\frac{1}{2}\pi(a + b)^2 + \frac{1}{2}\pi b^2 - \frac{1}{2}\pi a^2 =$
$\frac{1}{2}\pi(a^2 + 2ab + b^2) + \frac{1}{2}\pi b^2 - \frac{1}{2}\pi a^2 =$
$\pi ab + \pi b^2 = \pi b(a + b)$.]

50. $\frac{a}{b}$. [$\frac{\pi a(a+b)}{\pi b(a+b)} = \frac{a}{b}$.]

51. Yes. If $a = b$, the two areas should be equal.
We have shown that $\frac{\text{blue area}}{\text{yellow area}} = \frac{a}{b}$; so,
if $a = b$, $\frac{a}{b} = 1$, $\frac{\text{blue area}}{\text{yellow area}} = 1$, and so
blue area = yellow area.

52. 1. (Each border consists of three semicircles, one from each of the three circles.)

Set III (page 604)

In *The Age of Faith* (Simon & Schuster, 1950), Will Durant called Ramon Lull "one of the strangest figures of the many-sided thirteenth century." Lull wrote 250 books in Catalan, Latin, and Arabic on such subjects as love poetry, theology, warfare, education, philosophy, and science. According to Durant, "Amid all these interests he was fascinated by an idea that has captured brilliant minds in our own time—that all the formulas and processes of logic could be reduced to mathematical or symbolical form." More on Lull can be found in chapter 3 of *Science—Good, Bad, and Bogus*, by Martin Gardner (Prometheus Books, 1981).

Lull's Claim.

1. $2\pi r = 4s$.
2. $\pi r^2 = s^2$.
3. Solving the first equation for s, $s = \frac{\pi r}{2}$.
Substituting for s in the second equation,
$\pi r^2 = (\frac{\pi r}{2})^2$, $\pi r^2 = \frac{\pi^2 r^2}{4}$, $4\pi r^2 = \pi^2 r^2$, $\pi = 4$!

Chapter 14, Lesson 6

Set I (pages 606–608)

In *Animal Navigation* (Scientific American Library, 1989), Talbot H. Waterman remarks that dolphins, like bats, "use echolocation to find food, avoid obstacles, and maybe even to navigate over longer distances. Repeated sound bursts . . . emanate from the head in a 20 to 30° beam. These directional high intensity clicks are reflected back by objects not completely absorbing them. Such echoes, which give a detailed sound picture of the environment at ranges exceeding 300 meters, are markedly better than the best underwater visibility."

Franz Reuleaux was a French engineer and mathematician who wrote an important book in 1876 on mechanisms in machinery. One of Martin Gardner's "Mathematical Games" columns in *Scientific American* deals with the subject of curves of constant width, of which, other than the circle, the Reuleaux triangle is the simplest example. This column is included in Gardner's book titled *The Unexpected Hanging and Other Mathematical Diversions* (Simon & Schuster, 1969). The curve was known to earlier mathematicians, but Reuleaux was the first to demonstrate its constant-width properties, including the fact that it can be rotated inside a square without any space to spare. Exercises 18 through 22 prove that the perimeter of the Reuleaux triangle is the same as the circumference of a circle having the same width. It is also true that, of all curves of constant width having a given width, the Reuleaux triangle is the curve with the smallest area.

In *Human Information Processing—An Introduction to Psychology* (Harcourt Brace Jovanovich, 1977), Peter H. Lindsay and Donald A. Norman write concerning auditory space perception: "The cues used to localize a sound source are the exact time and intensity at which the tones arrive at the two ears. Sounds arrive first at the ear closer to the source and with greater intensity. The head tends to cast an acoustic shadow between the source and the ear on the far side. With some simple calculations, it is possible to determine the approximate maximum possible time delay between signals arriving at the two ears." These calculations are done in exercises 23 through 25. Lindsay and Norman observe that the human nervous system is able to distinguish the time at which a sound reaches the ear within an accuracy of 0.00003

second! They also explain that sound localization varies with frequency: low frequency sounds bend easily around a person's head, whereas high frequency sounds, whose wavelength is short compared with the size of the head, do not.

Although the degree seems to have been used to measure angles since angles were first measured, the radian was invented in 1871 by James Thomson, the brother of the physicist Lord Kelvin (William Thomson). Unlike the degree, based on the arbitrary division of a circle into 360 parts, the radian is a natural unit of angle measure. In *Trigonometric Delights* (Princeton University Press, 1998), Eli Maor remarks that the "reason for using radians is that it simplifies many formulas. For example, a circular arc of angular width θ (where θ is in radians) subtends an arc length given by $s = r\theta$; but if θ is in degrees, the corresponding formula is $s = \frac{\theta}{360} 2\pi r$. Similarly, the area of a circular sector of angular width is $A = \frac{1}{2}\theta r^2$ for θ in radians and $A = \frac{\theta}{360}\pi r^2$ for θ in degrees. The use of radians rids these formulas of the 'unwanted' factor $\frac{\pi}{180}$." Maor adds: "Even more important, the fact that a small angle and its sine are nearly equal numerically—the smaller the angle, the better the approximation—holds true only if the angle is measured in radians. . . . It is this fact, expressed as $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, that makes the radian measure so important in calculus."

Orange Slices.

•1. A sector.

•2. 45°. ($\frac{360°}{8}$.)

•3. $\frac{1}{4}\pi r$. ($\frac{2\pi r}{8}$.)

•4. $\frac{1}{8}\pi r^2$.

Sonar Beams.

•5. The length of $\overset{\frown}{AB}$ in meters.

6. The area of sector DAB in square meters.

7. 105 m. (From exercise 5.)

•8. 15,700 m^2. (From exercise 6.)

Latitude.

•9. About 6,220 mi. [$\frac{1}{4}2\pi(3{,}960) \approx 6{,}220$.]

10. About 69 mi. ($\frac{6{,}220}{90} \approx 69$.)

11. About 1,380 mi. ($20 \cdot 69 = 1{,}380$.)

Driveway Design.

•12. 2,009 ft^2. ($41 \cdot 49 = 2{,}009$.)

•13. 887 ft^2. [$2(\frac{1}{4}\pi(18)^2) + 8 \cdot 18 + 13 \cdot 18 \approx 887$.]

14. 1,122 ft^2. ($2{,}009 - 887 = 1{,}122$.)

Two Sectors.

15. $\frac{1}{2}\pi r^2$.

•16. πr^2. [$\frac{1}{4}\pi(2r)^2 = \pi r^2$.]

17. The yellow and blue areas are equal.

Reuleaux Triangle.

•18. 60°. ($m\overset{\frown}{AB} = \angle C = 60°$.)

•19. $\frac{1}{3}\pi x$. ($\frac{60}{360}2\pi x = \frac{1}{3}\pi x$.)

20. $3x$.

21. πx. ($3 \cdot \frac{1}{3}\pi x = \pi x$.)

•22. x. ($c = \pi d = \pi x$; so $d = x$.)

Sound Delay.

•23. About 9 in. [About $\frac{1}{2}(7) + \frac{1}{4}(7\pi) \approx 9$.]

24. 13,200 in. ($1{,}100 \cdot 12 = 13{,}200$.)

•25. About 0.0007 second.

($\frac{9 \text{ in}}{13{,}200 \text{ in / second}} \approx 0.0007$ second.)

The Radian.

26. $2\pi r$.

•27. 360°.

•28. $\frac{1}{2\pi}$. ($\frac{r}{2\pi r} = \frac{1}{2\pi}$.)

29. 57.3°. ($\frac{1}{2\pi} \cdot 360° \approx 57.3°$.)

•30. 57.3°. ($\angle AOB = m\overset{\frown}{AB}$.)

Set II (pages 608–611)

In a section titled "Lunatics" in *Geometry Civilized—History, Culture, and Technique* (Clarendon Press, 1998), J. L. Heilbron wrote: "Despite repeated failures, adventuresome geometers over the ages have tried their hands at squaring the circle. One of the earliest and most promising attempts was the work of Hippocrates of Chios, not the great physician but a man distinguished enough in his own line to be named in the old texts as the first to compile an *Elements of Geometry*. Hippocrates' attempt at circle-squaring involved 'lunes,' moon-shaped areas bounded by arcs of circles." Exercises 47 through 56 consider one version of Hippocrates' lunes: four of them are equal in area to the very square on which they are constructed.

Archimedes' "salt cellar" problem is theorem 14 from his *Liber Assumptorum*, or *Book of Lemmas*, a collection of theorems that has survived only in an Arabic translation.

Windshield Wipers.

•31. 46 in. [$\frac{120}{360} 2\pi(22) \approx 46$.]

•32. 469 in^2.
[$\frac{120}{360}\pi(22)^2 - \frac{120}{360}\pi(6)^2 \approx 469$.]

33. No, because the areas wiped usually overlap.

Land Area.

•34. $\angle AOC \approx 53°$. ($\sin \angle AOC = \frac{8}{10}$, $\angle AOC \approx 53.13°$.)

35. $\angle AOB \approx 106°$. ($2 \cdot 53° = 106°$.)

36. $m\overset{\frown}{AB} \approx 106°$.

•37. About 93 square units. [$\frac{106}{360}\pi(10)^2 \approx 93$.]

38. 48 square units. (Because OD = 10 – 4 = 6, $\alpha\Delta ABO = \frac{1}{2} AB \cdot OD = \frac{1}{2} 16 \cdot 6 = 48$.)

•39. About 45 square units. (93 – 48 = 45.)

40. Yes.

Running Track.

•41. 63.66 m. ($100 = \frac{1}{2}\pi d$, $d = \frac{200}{\pi} \approx 63.66$.)

42. 66.06 m. [63.66 + 2(1.2) = 66.06.]

•43. 103.77 m. [$\frac{1}{2}\pi(66.06) \approx 103.77$.]

44. 25. ($\frac{10{,}000}{400} = 25$.)

•45. About 188 m. (103.77 – 100 = 3.77; 25 laps is 50 semicircles; $50 \cdot 3.77 \approx 188$.)

46. About 24 m!
[$\frac{1}{2}2\pi(R + 0.15) - \frac{1}{2}2\pi R = 0.15\pi \approx 0.47$;
$50 \cdot 0.47 \approx 24$.]

Four Crescents.

•47. $2\sqrt{2}$. (ΔAOB is an isosceles right triangle with legs equal to 2.)

48. $\sqrt{2}$.

•49. 4π. [$\pi(2)^2 = 4\pi$.]

•50. 8. [$(2\sqrt{2})^2 = 8$.]

•51. $4\pi - 8$.

52. $\pi - 2$. ($\frac{4\pi - 8}{4} = \pi - 2$.)

•53. π. [$\frac{1}{2}\pi(\sqrt{2})^2 = \pi$.]

54. 2. [$\pi - (\pi - 2) = 2$.]

55. 8.

56. They prove that the four yellow crescents are together equal in area to the green square.

Drop-Leaf Table.

57. 20 in.

•58. 1,256.64 in^2. [$\pi(20)^2 = 400\pi \approx 1{,}256.64$.]

59. 60°. (Because CO = 2OH, ΔCOH is a 30°-60° right triangle; so ∠COH = 60°.)

60. 120°.

61. 418.88 in². $[\frac{120}{360}\pi(20)^2 \approx 418.88.]$

•62. 34.64 in.
$[CD = 2CH = 2(10\sqrt{3}) = 20\sqrt{3} \approx 34.64.]$

63. 173.2 in².
$[\alpha\Delta COD \approx \frac{1}{2}(34.64)(10) = 173.2.]$

•64. 246 in². (418.88 – 173.2 = 245.68.)

65. 765 in². [1,256.64 – 2(245.68) = 765.28.]

Salt Cellar.

•66. $2a + b$.

•67. $2a + 2b$.
$[CD = (2a + b) + b = 2a + 2b.]$

68. $a + b$.

69. $\pi(a + b)^2$.

70. The area of the "salt cellar" is equal to
$\frac{1}{2}\pi(2a + b)^2 - 2(\frac{1}{2}\pi a^2) + \frac{1}{2}\pi b^2 =$
$\frac{1}{2}\pi(4a^2 + 4ab + b^2) - \pi a^2 + \frac{1}{2}\pi b^2 =$
$2\pi a^2 + 2\pi ab + \frac{1}{2}\pi b^2 - \pi a^2 + \frac{1}{2}\pi b^2 =$
$\pi a^2 + 2\pi ab + \pi b^2 = \pi(a^2 + 2ab + b^2) = \pi(a + b)^2.$

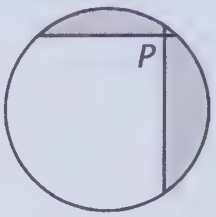
Set III (page 611)

This exercise is based on the "pizza problem" discussed in detail by Joseph D. E. Konhauser, Dan Yelleman, and Stan Wagon in *Which Way Did the Bicycle Go?* (Mathematical Association of America, 1996): "Suppose four cuts are made through a point P in a disk so that the eight angles at P are all equal to 45°. If the resulting eight pizza slices are colored alternately black and white, must the white area equal the black area?" That the answer is "yes" was first discovered and proved by L. J. Upton in 1968. As Konhauser, Yelleman, and Wagon explain: "It turns out that the equal-area property holds if and only if the number of chords is even and greater than or equal to 4 (equivalently, the number of pizza slices is 8, 12, 16, . . .). They include a proof by dissection for the 4-chords case and a proof using calculus and polar coordinates of the general case. There is a nice photograph of Stan Wagon on the last page of the book with a granite sculpture illustrating the dissection solution to the problem.

To prove, then, that Acute Alice's division of the pizza is a fair one is clearly not something of which a beginning geometry student is capable. To find a fairly obvious counterexample to Obtuse Ollie's claim, however, is fairly easy to do, as the figure below illustrates.

Pizza Puzzle.

Example figure for two cuts:



This figure shows that Ollie's method may easily result in one pair of opposite pieces having a greater area than the other pair. Strange as it may seem, regardless of where point P is chosen, Alice's method results in two sets of alternating pieces that are equal in area!

Chapter 14, Review

Set I (pages 612–614)

Of cholesterol, P. W. Atkins wrote in *Molecules* (Scientific American Library, 1987): "Although this molecule has an elaborate, rigid, hydrocarbon framework, its business end (more formally, its *functional group*) is primarily the –OH group. In other words, cholesterol is chemically an elaborate alcohol (hence the *–ol* in its name.)" In a line formula for an organic molecule, the line segments represent the bonds between the carbon atoms, which are understood to be located at their endpoints unless otherwise marked. The hydrogen atoms are usually ignored. The cholesterol molecule contains 27 carbon atoms, 1 oxygen atom, and 46 hydrogen atoms.

In Book IV of the *Elements*, Euclid discusses inscribed and circumscribed polygons—the triangle (Propositions 4 and 5), the square (Propositions 6 through 9), the regular pentagon

(Propositions 11 through 14), the regular hexagon (Proposition 15)—concluding with the regular 15-gon (Proposition 16). In his commentary on the *Elements* (Cambridge University Press, 1926), Sir Thomas Heath wrote: "Proclus refers to this proposition in illustration of his statement that Euclid gave proofs of a number of propositions with an eye to their use in astronomy. 'With regard to the last proposition in the fourth Book in which he inscribes the side of the fifteen-angled figure in a circle, for what object does anyone assert that he propounds it except for the reference of this problem to astronomy? For, when we have inscribed the fifteen-angled figure in the circle through the poles, we have the distance from the poles both of the equator and the zodiac, since they are distant from one another by the side of the fifteen-angled figure.' This agrees with what we know from other sources, namely that up to the time of Eratosthenes (c.284–204 B.C.) 24° was generally accepted as the correct measurement of the obliquity of the ecliptic."

Although the penny farthing bicycle reached its peak of popularity in about 1885, it is still popular in some parts of Europe. At the time of this writing, Hammacher Schlemmer was offering a version of the bicycle handmade in the Czech Republic for $4,999.95!

Concerning the Gothic arch, J. L. Heilbron wrote in *Geometry Civilized—History, Culture, and Technique* (Clarendon Press, 1998): "The construction of the equilateral triangle, which figures so prominently in Euclid's proofs, leads to a figure of great beauty and importance. . . . The curvilinear shape ACB is that of the basic Gothic arch. You can see it everywhere in churches built during the later middle ages. . . . At the threshold of the study of geometry, in connection with the very first proposition of Euclid, you have encountered one of the most important and versatile elements of architecture."

Molecule.

1. Regular hexagons and a regular pentagon.

•2. It must be convex, equilateral, and equiangular.

3. It becomes more circular.

A Regular 15-gon.

•4. $m\overset{\frown}{AB} = 120°$. $(\frac{360°}{3} = 120°.)$

5. $m\overset{\frown}{AD} = 72°$. $(\frac{360°}{5}.)$

•6. $m\overset{\frown}{DB} = 48°$.
$(m\overset{\frown}{DB} = m\overset{\frown}{AB} - m\overset{\frown}{AD} = 120° - 72° = 48°.)$

7. $m\overset{\frown}{BE} = 24°$.
$(m\overset{\frown}{BE} = m\overset{\frown}{DE} - m\overset{\frown}{DB} = 72° - 48° = 24°.)$

8. B and E, C and F.
$(m\overset{\frown}{BE} = m\overset{\frown}{CF} = 24° = \frac{1}{15}360°$. Because $m\overset{\frown}{EF} = 72° = 3(24°)$, BE and CF are sides of the *same* regular 15-gon.)

•9. 3.12. $(N = 15 \sin\frac{180}{15} = 15 \sin 12° \approx 3.12.)$

•10. 3.05. $(M = 15 \sin 12° \cos 12° \approx 3.05.)$

11. π.

•12. $6.24r$. $[p = 2Nr \approx 2(3.12)r = 6.24r.]$

13. $3.05r^2$. $(A = Mr^2 \approx 3.05r^2.)$

14. $0.42r$. $(s \approx \frac{6.24r}{15} \approx 0.42r.)$

Penny Farthing Bicycle.

•15. About 420 turns. $[\frac{5{,}280}{2\pi(2)} \approx 420.]$

16. About 3.77 ft. $[c = 2\pi(0.6) = 1.2\pi \approx 3.77.]$

17. About 1,401 turns. $(\frac{5{,}280}{3.77} \approx 1{,}401.)$

SAT Questions.

•18. $2\pi^2$. $[c = 2\pi(\pi) = 2\pi^2.]$

19. $\frac{1}{\pi}$. $(1 = \pi d, d = \frac{1}{\pi}.)$

•20. 4π. $[4\pi = 2\pi r, r = 2, A = \pi(2)^2 = 4\pi.]$

Semicircles on the Sides.

•21. $\frac{\pi}{8}a^2$, $\frac{\pi}{8}b^2$, and $\frac{\pi}{8}c^2$. $[A = \frac{1}{2}\pi(\frac{a}{2})^2 = \frac{\pi}{8}a^2$, etc.]

22. Yes. Because ΔABC is a right triangle, $a^2 + b^2 = c^2$. Multiplying each side of this equation by $\frac{\pi}{8}$ gives $\frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 = \frac{\pi}{8}c^2$.

Area Comparisons.

23. $\frac{1}{2}\pi r^2$.

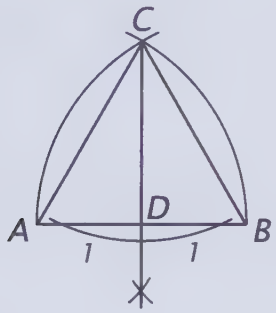
•24. $\frac{1}{2}\pi r^2$. $[\frac{1}{8}\pi(2r)^2 = \frac{1}{2}\pi r^2.]$

25. They are equal. Because $\alpha 1 + \alpha 2 = \alpha 2 + \alpha 3$, $\alpha 1 = \alpha 3$ by subtraction.

26. The yellow and red areas are equal. (This follows from the result of exercise 25 and the symmetry of the figure.)

Gothic Arch.

27.



•28. $\sqrt{3} \approx 1.732$. $[\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(2)^2.]$

•29. $\frac{2}{3}\pi \approx 2.094$. $[\frac{60}{360}\pi(2)^2 = \frac{2}{3}\pi.]$

30. $\frac{2}{3}\pi - \sqrt{3} \approx 0.362$.

•31. $\frac{4}{3}\pi - \sqrt{3} \approx 2.457$. $[\sqrt{3} + 2(\frac{2}{3}\pi - \sqrt{3}) = \sqrt{3} + \frac{4}{3}\pi - 2\sqrt{3} = \frac{4}{3}\pi - \sqrt{3}.]$

Set II (pages 614–616)

The answer to exercise 43, the width of the United States, is, of course, *very* approximate. The air distance between San Francisco and New York City, both with latitudes close to 40°, is 2,572 miles; the coast of Maine is roughly 400 miles east of New York City.

Exercises 50 through 54 are based on the surprising fact that, given two concentric circles in which a chord of the larger circle is tangent to the smaller circle, the area of the ring between the circles is determined solely by the length of the chord, regardless of the sizes of the circles! As the smaller circle shrinks to a point, the chord becomes the diameter of the larger circle and the expression for the area of the ring, $\pi R^2 - \pi r^2$, becomes πR^2, the area of the larger circle.

In exercises 55 through 62, the students discover a surprising property of the heart-shaped curve: *any* line through point O divides its border into two equal parts. We would expect this result of a line through the center of a figure that has point symmetry, but not of a figure that does not.

Cup Problem.

32. *(Student answer.)* (From the symmetry of the figure, $\overset{\frown}{AG}$ and $\overset{\frown}{BH}$ evidently have equal lengths, as do $\overset{\frown}{GD}$ and $\overset{\frown}{HC}$. The two lower arcs appear to be longer than the two upper ones.)

•33. 45°. $(m\overset{\frown}{AG} = \angle ABG = 45°.)$

•34. 0.79. $[\frac{45}{360}2\pi(1) = \frac{\pi}{4} \approx 0.79.]$

•35. $\sqrt{2} - 1$. (Because ABFE is a square, $BE = \sqrt{2}$. $BG = BA = 1$; so $EG = BE - BG = \sqrt{2} - 1$.)

36. 135°. $(m\overset{\frown}{GD} = \angle GED = 45° + 90° = 135°.)$

•37. 0.98. $[\frac{135}{360}2\pi(\sqrt{2} - 1) = \frac{3}{4}\pi(\sqrt{2} - 1) \approx 0.98.]$

38. *(Student answer.)*

Time Zones.

•39. About 1,037 mi. $[\frac{2\pi(3{,}960)}{24} \approx 1{,}037.]$

•40. About 3,034 mi. [In ΔABO, $\cos 40°$ (or $\sin 50°$) $= \frac{AB}{3{,}960}$; so $AB = 3{,}960 \cos 40° \approx 3{,}034.$]

41. About 19,060 mi. $[2\pi(3{,}034) \approx 19{,}060.]$

•42. About 794 mi. $(\frac{19{,}060}{24} \approx 794.)$

43. Approximately 3,000 mi. $(4 \cdot 794 = 3{,}176.)$

From Dodecagon to Square.

44. It is equilateral. The vertices of the dodecagon divide the circle into 12 equal arcs; so each arc has a measure of $\frac{360°}{12} = 30°$. Two of the angles of the triangle are inscribed angles of the circle and each intercepts an arc equal to four of these arcs; so each of these angles has a measure of $\frac{1}{2}(4 \cdot 30°) = 60°$.

•45. $\sqrt{2}$ (or about 1.414). ($A = Mr^2$; so $6 = 12 \sin\frac{180}{12}\cos\frac{180}{12}r^2$, $6 = 12 \sin 15° \cos 15° r^2$, $6 = 3r^2$, $r^2 = 2$, $r = \sqrt{2}$.)

46. $\sqrt{3}$ (or about 1.732). ($A = Mr^2$; so $6 = 4 \sin\frac{180}{4}\cos\frac{180}{4}r^2 = 4 \sin 45° \cos 45° r^2$, $6 = 2r^2$, $r^2 = 3$, $r = \sqrt{3}$.)

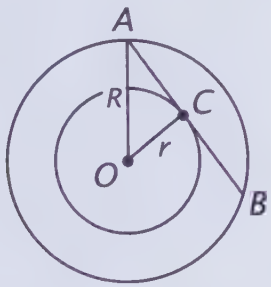
•47. 8.78. [$p = 2Nr = 2(12 \sin 15°)\sqrt{2} \approx 8.78$.]

48. 9.80. [$p = 2Nr = 2(4 \sin 45°)\sqrt{3} \approx 9.80$.]

49. It increases.

Chord and Ring.

50.



51. OC ⊥ AB and OC bisects AB. (If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact and, if a line through the center of a circle is perpendicular to a chord, it also bisects it.)

•52. $\pi R^2 - \pi r^2$ or $\pi(R^2 - r^2)$.

53. π. (Because AB = 2, AC = 1; so $r^2 + 1^2 = R^2$ and $R^2 - r^2 = 1$.)

54. The radii of the two circles are not needed to find the area of the ring. The area depends only on the length of the chord. (In general, if AB = 2x, then AC = x, $r^2 + x^2 = R^2$, $R^2 - r^2 = x^2$; so the area of the ring is πx^2.)

Half a Heart.

•55. 4π. $\{2[\frac{1}{2}2\pi(1)] + \frac{1}{2}2\pi(2) = 2\pi + 2\pi = 4\pi.\}$

56. $\angle AOC = (180 - x)°$.

57. $\angle OCE = (180 - x)°$. (In $\triangle CEO$, EO = EC; so $\angle OCE = \angle AOC$.)

•58. $\angle AEC = (360 - 2x)°$. ($\angle AEC$ is an exterior angle of $\triangle CEO$; so $\angle AEC = \angle AOC + \angle OCE$.)

•59. $\frac{180-x}{90}\pi$. $[\frac{360-2x}{360}2\pi(1) = \frac{180-x}{90}\pi.]$

60. $\frac{x}{90}\pi$. $[\frac{x}{360}2\pi(2) = \frac{x}{90}\pi.]$

61. 2π. $(\frac{180-x}{90}\pi + \frac{x}{90}\pi = \frac{180}{90}\pi = 2\pi.)$

62. Line CD bisects the border of the heart because the length of $\overset{\frown}{CAD}$, 2π, is half the length of the border of the heart, 4π.

Biting Region.

•63. About 83 ft. $[\frac{1}{4}2\pi(15-10) + \frac{3}{4}2\pi(15) + \frac{1}{4}2\pi(15-12) = \frac{106}{4}\pi \approx 83.3.]$

64. About 557 ft^2. $[\frac{1}{4}\pi(15-10)^2 + \frac{3}{4}\pi(15)^2 + \frac{1}{4}\pi(15-12)^2 = \frac{709}{4}\pi \approx 556.8.]$

Chapter 15, Lesson 1

Set I (pages 620–622)

The figures of exercises 1 through 6 are of special interest in the field of computational vision, a subfield of artificial intelligence. Martin A. Fischler and Oscar Firschein observe in their book titled *Intelligence—The Eye, the Brain, and the Computer* (Addison-Wesley, 1987): "We know that the shading and texture present in a single image can produce a vivid impression of three-dimensional structure. What is the computational basis of this effect? A crucial source of information about three-dimensional structure is provided by the spatial distribution of surface markings in an image. Since projection distorts texture geometry in a manner that depends systematically on surface shape and orientation, isolating and measuring this projective distortion in the image allows recovery of the three-dimensional structure of the textured surface."

In *Visual Explanations* (Graphics Press, 1997), Edward R. Tufte points out that, "For centuries, the profound, central issue in depicting information has been how to represent three or more dimensions of data on the two-dimensional display surfaces of walls, stone, canvas, paper, and recently, computer screens." About the figure of exercises 7 through 10, he remarks: "As early as 1642, pole-people were active in scaling recession for landscapes." Tufte also includes a nice "before and after" illustration for the Brighton pavilion (1808) that uses people holding poles to give a sense of scaling to a picture in which the perspective is ambiguous.

Ted Adelson's "corrugated Mondrian," the figure of exercises 13 through 20 is discussed in detail by Donald D. Hoffman in *Visual Intelligence* (Norton, 1998). According to Hoffman, the two grays in the numbered regions "measure precisely the same luminance on a photometer." He continues: "You also see this figure in three dimensions. It's possible, suggests psychologist Alan Gilchrist, that you group each surface with those that lie in the same 3D plane with it. Thus each vertical column would be a group. If so, then rectangle 2 would be the darkest in its group, and 1 the lightest in its group, explaining why A looks lighter than B." This hypothesis is supported by a flattened version of the figure in which the apparent difference of the two grays is not as pronounced.

Visual Planes.

•1. The second.

2. The right side.

3. Rectangles.

4. A horizontal plane.

5. The bottom part.

6. Circles.

Pole People.

•7. It appears to be oblique to the ground.

•8. A vertical plane.

9. That they are the same length.

•10. That they lie in a horizontal plane.

Posts and Walls.

•11. No. AB is perpendicular to the floor if it is perpendicular to *every* line in the floor that passes through point B.

12. Yes. The floor is perpendicular to DEFG because the floor contains a line, EH, that is perpendicular to DEFG.

3-D Shading.

•13. Parallelograms.

14. Rectangles.

•15. EFGH.

16. CDIJ.

17. BCJK and DEHI.

18. It can be viewed as being either behind or in front of the rest of the figure.

19. Rectangle 2.

20. *(Student answer.)* (The grays in the two rectangles are actually identical.)

Polarized Light.

•21. They intersect in line *l* (or they both contain it).

22. The planes appear to be perpendicular.

•23. One plane must contain a line that is perpendicular to the other plane.

Kilroy in Space.

24. CD || FE. (CD and FE lie in the same plane, CDEF, and can't intersect, because they lie in parallel planes.)

•25. CF is perpendicular to plane EFGH.

26. Planes ABCD and EFGH are parallel.

•27. If a plane intersects two parallel planes, it intersects them in *parallel lines.*

28. If a line is perpendicular to one of two parallel planes, it *is perpendicular to the other.*

29. Two planes perpendicular to the same line are *parallel to each other.*

30. *(Student answer.)* (It depends on whether the figure is interpreted as seen from above or below.)

Set II (pages 622–624)

Desargues's Theorem, a basic theorem of projective geometry, is considered in exercises 31 through 37. Gérard Desargues was a French architect, engineer and army officer who wrote an important book on the conic sections as well as a book on how to teach children to sing! Howard Eves calls him "the most original contributor to synthetic geometry in the 17th century." His theorem says that, "If two triangles, in the same plane or not, are so situated that lines joining pairs of corresponding vertices are concurrent, then the points of intersection of pairs of corresponding sides are collinear, and conversely." As the exercises show, this theorem is proved quite easily by treating the figure for the theorem as if it were a three-dimensional "pyramid" of lines.

Triangles in Perspective.

31. They appear to be collinear.

•32. Three noncollinear points determine a plane.

•33. If two points lie in a plane, the line that contains them lies in the plane.

•34. They lie in it.

35. They lie in it.

36. A line.

37. That they are collinear.

Impossible Slice.

38. If two planes intersect, their intersection is a line.

39. That three noncollinear points determine a plane. (If points A, B, and C lie on a circle, then they are noncollinear, but they don't determine a plane, because they lie in two different planes.)

Uneven Bars.

•40. No. Two horizontal lines can be skew. They can also intersect.

41. Yes. Part of the definition of parallel lines is that they lie in the same plane.

•42. About 91 cm. (In right ΔABE, $AB^2 = 43^2 + 80^2$; so $AB = \sqrt{8{,}249} \approx 91$.)

•43. About 62°. (In right ΔABE, $\tan \angle BAE = \frac{BE}{AE} = \frac{80}{43}$; so $\angle BAE \approx 62°$.)

Pyramid.

44. If a line is perpendicular to a plane, it is perpendicular to every line in the plane that passes through the point of intersection.

•45. SAS.

46. Corresponding parts of congruent triangles are equal.

•47. A quadrilateral is a parallelogram if its diagonals bisect each other.

48. The opposite sides of a parallelogram are equal.

49. SSS.

50. SSS (or SAS).

Lopsided Box.

51. ABEH ⊥ BCDE and ABCF ⊥ BCDE (because AB ⊥ BCDE), and CDGF ⊥ ABCF (because CD ⊥ ABCF).

52. $\angle ABE$ and $\angle ABC$ (because AB ⊥ BCDE), and $\angle DCF$, $\angle DCA$, and $\angle DCB$ (because DC ⊥ ABCF).

53. Yes; AD > AC. Because $\angle ACD$ is a right angle in ΔACD, $\angle ACD > \angle ADC$; so AD > AC.

Cube and Hypercube.

54. Six.

•55. Nine.

56. Eight.

57. 13.

Set III (page 625)

In a talk given in Amsterdam in 1963, Escher said: "If you want to focus the attention on something nonexistent, then you have to try to fool first yourself and then your audience, by presenting your story in such a way that the element of impossibility is veiled, so that a superficial listener doesn't even notice it. There has to be a certain enigma in it, which does not immediately catch the eye." (*The Magic of M. C. Escher*, Abrams, 2000).

The impossibility is in the way in which the columns of the first floor are connected to the second floor. As a result, someone climbing the ladder starts inside the building but ends up outside on reaching the top. The top floor is also at an angle with the second floor, which is impossible. The seated figure holds a distorted box that mirrors the impossible structure of the building.

Chapter 15, Lesson 2

Set I (pages 630–631)

The beetle on the box is another example of the Necker cube illusion, named after the Swiss scientist, Louis Necker, who first wrote about it in 1832. Martin Gardner included it in a *Scientific American* "Mathematical Games" column on optical illusions, republished in his *Mathematical Circus* (Knopf, 1979). The idea of adding a beetle to the figure was that of the mathematical physicist Roger Penrose and his father, L. S. Penrose. Of the Penrose version, Gardner wrote: "The beetle appears to be on the outside. Stare at the back corner of the box and imagine it to be the corner nearest you. The box will suddenly flip-flop, transporting the beetle to the floor *inside*."

It seems strange to think of the edge of a card (or a sheet of paper) as a rectangle having an area, but, as exercise 10 makes obvious, these objects have a thickness that can be easily calculated.

Concerning exercises 14 through 17, Martin Gardner wrote in his *Wheels, Life and Other Mathematical Amusements* (W. H. Freeman and Company, 1983): "Many geometric problems are solved by finding integral solutions for Diophantine equations. . . . Among the many geometrical Diophantine problems that are still unsolved, one of the most difficult and notorious is known as the problem of the 'integral brick' or 'rational cuboid.' The 'brick' is a rectangular parallelepiped. There are seven unknowns: The brick's three edges, its three face diagonals, and the space diagonal. . . . Can a brick exist for which all seven variables have integer values?. . . The problem has not been shown to be impossible, nor has it been solved. . . . The smallest brick with integral edges and face diagonals (only the space diagonal is nonintegral) has edges of 44, 117 and 240. This was known by Leonhard Euler to be the minimum solution. If all values are integral except a face diagonal, the smallest brick has edges of 104, 153 and 672 [the brick in exercises 14 through 17], a result also known to Euler."

Five different cross sections of a cube are considered in exercises 20 through 28. A good project for students who find this group of exercises interesting would be to try to discover (and possibly build models of) other possible cross sections. In *Curiosities of the Cube* (Crowell, 1977), Ernest R. Ranucci and Wilma E. Rollins observe: "The intersection of a plane with a cube can be an isosceles trapezoid. . . . Other cross sections include: a point, a line segment, scalene triangle, isosceles triangle, equilateral triangle, trapezoid, rectangle, square, parallelogram (are these always rectangles?), rhombus, pentagon, and hexagon."

Inside or Outside?

•1. Six.

•2. Eight.

•3. 12.

•4. F.

5. CD, EF, and GH.

6. AD, AE, BC, and BF.

•7. CG, DH, EH, and FG.

8. *(Student answer.)* (The box can be seen in two ways; so both answers are possible.)

Deck of Cards.

9. Changing the rounded corners of the cards to square ones.

•10. 0.025 cm. ($\frac{1.3}{52}$.)

11. 52.2 cm². (5.8 × 9.0.)

•12. 0.225 cm². ($\frac{1.3 \times 9.0}{52}$.)

13. 0.145 cm². ($\frac{1.3 \times 5.8}{52}$.)

Integer Lengths.

•14. 185 units.
($\sqrt{104^2+153^2} = \sqrt{34{,}225} = 185$.)

•15. 697 units.
($\sqrt{104^2+153^2+672^2} = \sqrt{485{,}809} = 697$.)

16. 680 units.
($\sqrt{104^2+672^2} = \sqrt{462{,}400} = 680$.)

17. About 689 units.
($\sqrt{153^2+672^2} = \sqrt{474{,}993} \approx 689$.)

SAT Problem.

•18. $8x + 10 = 106$.

19. 12. ($8x = 96$, $x = 12$.)

Slicing a Cube.

•20. A regular hexagon. (This slice is the perpendicular bisector of a diagonal of the cube. With the use of this fact and symmetry, it can be proved that the hexagon is regular.)

•21. ΔHEM and ΔHEN are congruent right triangles (SAS).

22. Isosceles.

23. ΔHDA and ΔFBA are congruent isosceles right triangles.

24. Equilateral.

•25. ∠EAC is a right angle. (EA is perpendicular to the plane of ABCD.)

26. A rectangle. No. (AE < EG.)

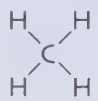
27. OP || EG and OP = $\frac{1}{2}$EG. (OP is a midsegment of ΔHEG.)

•28. An isosceles trapezoid.

Set II (pages 631–633)

The surprising result of cutting a hole in a cube discovered in exercises 38 through 42 is from a puzzle created by nineteenth-century English puzzlist Henry Ernest Dudeney. Included in his *536 Puzzles & Curious Problems* (Scribners, 1967), Dudeney called it "A Cube Paradox" and posed it in the following way: "I had two solid cubes of lead, one very slightly larger than the other. Through one of them I cut a hole (without destroying the continuity of its four sides) so that the other cube could be passed right through it. On weighing them afterwards it was found that the larger cube was still the heavier of the two! How was this possible?" Dudeney's answer: "It is a curious fact that a cube can be passed through another cube of smaller dimensions. Suppose a cube to be raised so that its diagonal is perpendicular to the plane on which it rests. Then the resulting projection will be a regular hexagon. [Dudeney shows a square hole cut through this hexagon to allow the passage of a cube of the same dimensions.] But it will be seen that there is room for cutting a hole that would pass a cube of even larger dimensions. Therefore, the one through which I cut a hole was not, as the reader may have hastily supposed, the larger one, but the smaller! Consequently, the larger cube would obviously remain the heavier. This could not happen if the smaller were passed through the larger."

About the structure of the methane molecule, Linus Pauling wrote in *The Architecture of Molecules* (W. H. Freeman and Company, 1964), "In 1858 the Scottish chemist A. S. Couper invented valence-bond formulas for chemical compounds and wrote the formula



for methane. Then in 1874 the 22-year-old Dutch chemist Jacobus Hendricus van't Hoff pointed out that some properties of substances could be simply explained by the assumption that the four bonds formed by a carbon atom are directed toward the corners of a tetrahedron, with the carbon atom at its center. . . . The structure theory of chemistry has been based on the tetrahedral

carbon atom ever since. Spectroscopic studies of methane have shown that the . . . six H–C–H bond angles have the value 109.5° that is characteristic of the regular tetrahedron." Pauling also comments on the fact that a tetrahedron can be inscribed in a cube helps in the evaluation of some of its dimensional properties. (This is explored in the "Divided Cube" exercises of the Chapter 15 review.)

Before Rubik's cube, there was the Soma cube. Its inventor, Piet Hein, was a Danish poet and scientist with many interests. He thought of the idea of the Soma cube while attending a lecture by Werner Heisenberg on quantum mechanics! About the "impossible" wall of exercises 52 through 54, Martin Gardner wrote in *Knotted Doughnuts and Other Mathematical Entertainments* (W. H. Freeman and Company, 1986): "It *is* possible . . . to build a wall that from the front looks exactly like the one in the illustration. If the wall is viewed from behind, however, the hidden corner [at the bottom center] is missing, and an extra cube protrudes at some other spot." With a set of Soma cubes in the classroom, some students might enjoy trying to build the figures of exercises 49 through 51 as well as the "impossible" wall.

An Inside Job.

29. Alice lined the brick up with two edges of the top of the table and moved it over one length. She then measured the distance between the corner of the table top and the upper left corner of the brick.

•30. 6 cm. ($21 = \sqrt{18^2 + 9^2 + x^2}$, $441 = 405 + x^2$, $x^2 = 36$, $x = 6$.)

31. Ollie's head!

Diagonals.

•32. 45°. (ΔBAC is an isosceles right triangle.)

•33. 60°. (If CE were drawn, ΔCAE would be an equilateral triangle.)

34. 90°. (CD is perpendicular to the plane of ΔABC.)

•35. About 35°. (Letting e be the edge of the cube, $\tan \angle CAD = \frac{CD}{AC} = \frac{e}{e\sqrt{2}} = \frac{1}{\sqrt{2}}$, $\angle CAD \approx 35°$.)

36. About 55°. (Because CD || AF, $\angle DAF = \angle ADC$. In right ΔACD, $\angle CAD \approx 35°$; so $\angle ADC \approx 55°$.)

37. No. $\angle CAD = \angle DAE \approx 35°$; so $\angle CAD + \angle DAE \approx 70°$, but $\angle CAE = 60°$. (Ray AD is not between rays AC and AE, because the three rays are not coplanar.)

Hole Through a Cube.

38. 16 cm^2. (4^2.)

•39. $3\sqrt{2}$ cm. (AD is the hypotenuse of isosceles right ΔADE.)

40. 18 cm^2. [$(3\sqrt{2})^2$.]

41. 4.24 cm. ($3\sqrt{2}$.)

42. Yes. We have just proved this possible with the results of exercises 38 through 41.

Bond Angles.

43. $AB = e\sqrt{2}$. (AB is the diagonal of a square whose side is e.)

•44. $AO = \frac{e}{2}\sqrt{3}$. (AO is half the diagonal of the cube. The diagonal of the cube is $e\sqrt{3}$.)

45. $AP = \frac{e}{2}\sqrt{2}$. ($AP = \frac{1}{2}AB$.)

•46. 54.74°. ($\sin \angle AOP = \frac{AP}{AO} = \frac{\frac{e}{2}\sqrt{2}}{\frac{e}{2}\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$, $\angle AOP \approx 54.74°$.)

47. 109.5°. ($\angle AOB = 2\angle AOP \approx 109.5°$.)

Soma Puzzles.

48. 27. (The first piece contains 3 cubes and the other six pieces each contain 4 cubes.)

•49. One. The structure appears to contain $27 + 1 = 28$ cubes.

50. Three. The structure appears to contain $3 \times 10 = 30$ cubes.

51. Three. The structure appears to contain $1 + 4 + 9 + 16 = 30$ cubes.

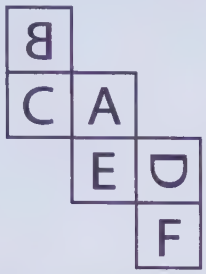
52. Ten.

53. Nine. (Five pieces can each provide one corner cube and the other two pieces can each provide two corner cubes.)

54. The pieces can provide only nine corner cubes, but the wall contains ten.

Set III (page 633)

Flattened Cube.



Chapter 15, Lesson 3

Set I (pages 635–637)

Chemists have synthesized several hydrocarbon compounds whose names are based on their polyhedral structures. In addition to *prismane* (a hexane in the shape of a triangular prism), there are *cubane*, *housane* (a pentagonal prism), and *dodecahedrane*. More spectacularly there are the *buckyballs* and their relatives—large cages formed by carbon atoms. These compounds have been studied intensively in recent years.

3-D Lettering.

•1. L, I, and E.

2. L is a hexagon, I is a quadrilateral (rectangle), and E is a dodecagon (12-gon).

•3. Prisms.

Prismane.

4. Because it is shaped like a prism.

•5. Two.

•6. Three.

7. Six.

8. Nine.

9. 60° and 90°.

Polyhedral Nets.

•10. The bases.

11. The lateral faces.

•12. n.

•13. $n + 2$.

14. Its lateral area.

15. A cube (or, a rectangular solid).

Inside a Prism.

16. The lateral faces.

17. 837 ft^2. (27×31.)

•18. 1,624 ft^2. [$2(27 \times 14) + 2(31 \times 14)$.]

A Hexagonal Surprise.

•19. A parallelogram.

•20. BAB'C.

21. CBC'D.

22. DCD'E.

•23. EDE'F.

24. FAF'E.

25. A parallelogram.

•26. They appear to be parallel and equal.

27. They appear to be congruent (and, if the green figure is viewed as three-dimensional, to lie in parallel planes).

28. A triangular prism.

Set II (pages 637–638)

In *Crystals and Crystal Growing* (Anchor Books, 1960), Alan Holden and Phylis Singer report that "there are six styles of crystalline architecture, each employing a different sort of building block; and every crystal belongs to one of these six 'crystal systems.'" All six systems are prisms. Three are rectangular solids (*cubic, tetragonal, and orthorhombic*), named according to whether the edges have 1, 2, or 3 different lengths. The other three include two oblique prisms with parallelogram faces (*monoclinic* and *triclinic*) and a right hexagonal prism (*hexagonal*).

Concerning the Euler characteristic, Howard Eves remarks in *An Introduction to the History of Mathematics* (Saunders, 1990), "One of the most interesting theorems relating to any convex (or more generally any *simply connected*) polyhedron, is that $v - e + f = 2$. This may have been known to Archimedes (ca. 225 B.C.), and was very nearly stated by Descartes about 1635. Since Euler later independently announced it in 1752, the result is often referred to as the Euler-Descartes formula." The stamp pictured in the text was issued by the German Democratic Republic in 1983 on the

200th anniversary of Euler's death. The letters used in the formula $e - k + f = 2$ are the initials of the German words for vertices (*ecken*), edges (*kanten*) and planes (*flächen*). We will return to the Euler characteristic in later lessons of this chapter.

Long House.

•29. Pentagons.

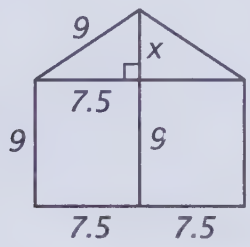
30. Rectangles.

31. Five.

32. 1,215 ft^2. (15×81.)

•33. 1,458 ft^2. [$2(9 \times 81)$.]

34. *Example figure:*



Approximately 14 ft. (Other roundings are acceptable.) ($x^2 + 7.5^2 = 9^2$, $x^2 = 24.75$, $x \approx 5$, $9 + x \approx 14$.)

Crystal Forms.

•35. They are rectangles.

36. The lateral edges are oblique to the planes of the bases.

37. They are parallelograms because their opposite sides are equal.

Feeding Trough.

•38. Yes. The bases of *every* prism are congruent.

39. Yes. AB = GH. AB = CD (given) and CD = GH (the opposite sides of a parallelogram are equal); so AB = GH (substitution).

40. Yes. AE || CG and AE = CG. AE and CG are lateral edges of a prism; so they are parallel. AE = CG because they are both equal to BF (or DH); the opposite sides of a parallelogram are equal.

41. ABFE and DCGH are rectangles because they are lateral faces of a right prism. (They are also congruent because all of their corresponding parts are equal.)

Magic Box.

42. A rectangle.

43. SAS. (AE = DH, $\angle$AEX = $\angle$DHY, and EX = HY.)

•44. AX > AE.

45. The area of AXYD is greater than the area of AEHD. (AD = AD, but AX > AE; AD · AX > AD · AE.)

•46. A right triangular prism.

47. ΔAXE and ΔDYH.

48. A right trapezoidal prism.

49. ABCD and XFGY; ABFX and DCGY.

Euler's Discovery.

•50. $2n$.

51. $n + 2$.

52. $3n$.

53. 2. [$V + F - E = 2n + (n + 2) - 3n = 2$.]

Set III (page 639)

This exercise is based on the work of the eighteenth-century French crystallographer René Just Haüy described in Elizabeth A. Wood's *Crystals and Light* (Van Nostrand, 1964). Wood writes: "It is in the nature of naturalists to describe in detail the odd things that they find, and the early naturalists wrote full descriptions of the crystals they found, recording carefully the angles between their faces. One of them [Haüy] noticed the special angular relationships between the faces of a crystal. . . . It was this small-whole-number relationship between the tangents of the interfacial angles of crystals that convinced the early naturalists that crystals were made up of building blocks." In *Crystals and Crystal Growing* (Anchor Books, 1960), Alan Holden and Phylis Singer elaborate on this idea: "Crystals . . . can always be imagined to be made of little building blocks. . . . The building block is somewhat fictitious, of course. Nobody supposes that the ultimate fine structure of crystals looks like a

collection of brick-shaped particles. The assembly of identical building blocks is merely a way of portraying the crystal's repetitive orderliness. Each building block represents a grouping of atoms or molecules which is regularly repeated throughout the crystal."

Topaz Angles.

1. $\tan \angle 1 = \frac{a}{b}$ and $\tan \angle 2 = \frac{2a}{b}$.
2. $\tan \angle 2 = 2 \tan \angle 1$.
3. 0.4557.
   ($\tan \angle 1 = \tan 24.5° \approx 0.4557$.)
4. 0.9114.
   [$\tan \angle 2 = 2 \tan \angle 1 \approx 2(0.4557) = 0.9114$.]
5. 42.3°.
6. Yes.

Chapter 15, Lesson 4

Set I (pages 642–644)

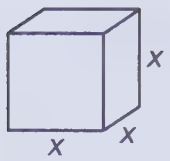
Tables of squares and cubes of numbers have been discovered on Babylonian tablets thought to date to 2100 B.C. The Pythagoreans may have been the first to associate these numbers with geometric figures.

According to Genesis 6:14–16, the ark had three stories and was divided into various rooms. The average height of each story would have been 15 feet.

Exercise 26 reveals that Cavalieri's Principle had been discovered long before the man for whom it was named was born. A good source of information on Cavalieri and his work is the article "Slicing It Thin" by Howard Eves included in *The Mathematical Gardner*, edited by David A. Klarner (Prindle, Weber & Schmidt, 1981).

The four candy bars weigh about the same: 3 Musketeers, 2.13 oz; Snickers, 2.07 oz; Butterfinger, 2.1 oz; Milky Way, 2.05 oz.

Squares and Cubes.

1. Its area.
- •2. "x cubed."
3. 

- •4. Its volume.
- •5. 27.
6. 1,728.
- •7. 36.
- •8. 1,296. (36^2.)
9. 46,656. (36^3.)

Largest Suitcase.

10. 2,772 in^3. ($22 \times 14 \times 9$.)
11. 1,728. (12^3.)
- •12. About 1.6 ft^3. ($\frac{2,772}{1,728}$.)

Noah's Ark.

13. 450,000 cubic cubits. ($300 \times 50 \times 30$.)
- •14. 1,518,750 ft^3. ($450,000 \times 1.5^3$.)
15. Approximately 500 boxcars.
    ($\frac{1,518,750}{3,000} = 506.25$.)

Weathering Rock.

- •16. 1 m^3. (1^3.)
- •17. 6 m^2. (6×1^2.)
18. 8.
19. 1 m^3.
- •20. 12 m^2. [$8(6 \times 0.5^2)$.]
21. 64.
22. 1 m^3.
23. 24 m^2. [$64(6 \times 0.25^2)$.]
24. It stays the same.
25. It increases (doubling at each stage).

Keng-Chih's Principle.

26. Cavalieri's Principle.

•27. A right hexagonal prism and an oblique hexagonal prism.

28. That the prisms have equal volumes.

Hole in the Ground.

29. *(Student answer.)* (Alice probably got 1,860,867 ft^3 because $123^3 = 1{,}860{,}867$.)

30. Because Ollie saw Alice using her calculator. If she had realized that a hole in the ground doesn't have any dirt in it, she wouldn't have been doing that.

Prism Volumes.

•31. 108 cubic units.
$[V = Bh = (\frac{1}{2}(4 \times 6)9 = 12 \times 9 = 108.]$

32. 440 cubic units.
$[V = Bh = (8 \times 5)11 = 40 \times 11 = 440.]$

•33. 288 cubic units.
$\{V = Bh = [\frac{1}{2}(4)(5 + 11)]9 = 32 \times 9 = 288.\}$

Candy Bars.

•34. 79 cm^3.
$(V = lwh = 3.0 \times 2.1 \times 12.5 = 78.75 \approx 79.)$

35. 66 cm^3.
$(V = lwh = 3.3 \times 2.0 \times 10.0 = 66.)$

•36. 55 cm^3. $\{V = Bh = [\frac{1}{2}(1.3)(2.3 + 3.5)]14.5 = 54.665 \approx 55.\}$

37. The 3 Musketeers bar. No; weight is not the same thing as volume.

Set II (pages 644–646)

Exercises 41 through 46 develop the formula for the cube of a binomial. A good class exercise would be to confirm the result algebraically.

Exercises 57 through 60 on the Toyota are based on material in A. K. Dewdney's *200% of Nothing* (Wiley, 1993). Dr. Dewdney wrote the "Mathematical Recreations" column in *Scientific American* for 8 years. As the exercises reveal, the commercial seems not only to confuse length with volume but also to have gotten the figures wrong.

Ream of Paper.

•38. 0.004 in. $(\frac{2}{500}.)$

39. 0.374 in^3. $[V = Bh = (8.5 \times 11)(0.004) = 0.374.]$

•40. 187.156 in^2.
$[A = 2(8.5 \times 11) + 2(8.5 \times 0.004) + 2(11 \times 0.004) = 187.156.]$

Binomial Cube.

•41. $(a + b)^3$.

42. Eight.

•43. Two of them.

•44. a^2b.

45. ab^2.

46. $a^3 + 3a^2b + 3ab^2 + b^3$.

Chinese Problem.

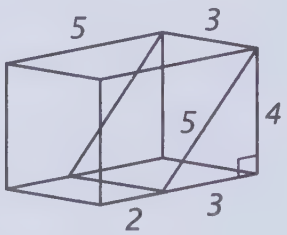
47. A right triangular prism.

48. $\frac{1}{2}abc$. $[V = Bh = \frac{1}{2}(ab)c.]$

•49. $ab + ac + bc + c\sqrt{a^2 + b^2}$.
$[A = 2(\frac{1}{2}ab) + ac + bc + c\sqrt{a^2 + b^2}.]$

SAT Problem.

50.



•51. 3 cm. $(5x = 15, x = 3.)$

52. 18 cm^3. $[V = Bh = \frac{1}{2}(3 \times 4)3 = 18.]$

53. 42 cm^3. $(5 \times 4 \times 3 = 60; 60 - 18 = 42.)$

Filling a Pool.

•54. 236.5 ft^2. $[A = \frac{1}{2}(3 + 5)16.5 + \frac{1}{2}(5 + 9)11.5 + \frac{1}{2}(9 + 7.5)9 + \frac{1}{2}(7.5 + 3)3 = 66 + 80.5 + 74.25 + 15.75 = 236.5.]$

•55. 4,730 ft^3. ($V = Bh = 236.5 \times 20 = 4{,}730$.)

56. A little under 40 hours.
($\frac{4{,}730}{2} = 2{,}365$ minutes ≈ 39.4 hours.)

Advertising Claim.

•57. 9 ft^3. ($0.75 \times 4 \times 3$.)

58. 13.5 ft^3. ($6 \times 0.75 \times 3$.)

59. 18 ft^3. ($6 \times 4 \times 0.75$.)

60. The extra room on the inside surely refers to volume, not length, so the wording should say "over two *cubic* feet more room." It is also strange that the commercial uses the number "two" when the volume should be increased by at least 9 cubic feet.

Measuring a Toad.

61. Its volume.

62. Push the toad under the water in a full measuring cup that can hold it comfortably. See how much water the toad displaces (the water will flow out).

Set III (page 646)

Benjamin Franklin's experiment in which he poured a spoonful of oil on the surface of a pond took place in 1765. He became curious about the effect of oil on water as a result of observing that the wakes of ships moving through greasy water were remarkably smooth. Although Franklin did not do the actual calculations, his experiment has been called a landmark in science in that it made possible the first rational measurement of the size of a molecule. A recreation of Franklin's experiment is pictured and described on pages 199–202 of Philip and Phylis Morrison's *The Ring of Truth* (Random House, 1987).

Molecule Experiment.

1. 0.0000002 cm. [$V = Bh$; so $h = \frac{V}{B}$.
$V = 4$ cm^3 and $B = \frac{1}{2}(4{,}000\ \text{m}^2) = 2{,}000\ \text{m}^2 =$
$2{,}000 \times 100^2\ \text{cm}^2 = 20{,}000{,}000\ \text{cm}^2$.
$h = \frac{4}{20{,}000{,}000} = \frac{2}{10{,}000{,}000} =$
0.0000002 cm.]

2. 5×10^{20} (or 500 quintillion). [If the edge of the cube is 0.0000002 cm $= 2 \times 10^{-7}$ cm, its volume is $(2 \times 10^{-7}\ \text{cm})^3 = 8 \times 10^{-21}\ \text{cm}^3$. If the cubes are packed together with no spaces between them, there would be as many as $\frac{4\ \text{cm}^3}{8 \times 10^{-21}\ \text{cm}^3} = 0.5 \times 10^{21} =$
5×10^{20} molecules.]

Chapter 15, Lesson 5

Set I (pages 649–650)

Ferrocene was first synthesized in 1951. At room temperature, it consists of yellow orange crystals that smell like camphor. It is used as a fuel additive, in rocket propellant, and in the production of antibiotics.

Leonardo da Vinci's sketch of a parachute appeared in a part of his *Codex Atlanticus* in which he explored mechanical flight. It was not until three centuries later that Louis Lenormand of France is credited for inventing the parachute. Its first trial took place in 1785, when a dog in a basket with a parachute was dropped from a balloon at a high altitude. Da Vinci's design was tested for the first time in 2000. According to BBC News, "a British man, Adrian Nicholas, dropped from a hot air balloon 3,000 meters (10,000 feet) above the ground, after ignoring expert advice that the canvas and wood contraption would not fly. . . . In the wider open spaces of Mpumalanga, South Africa, Mr. Nicholas safely floated down, saying the ride was smoother than with modern parachutes."

X-Ray Beam.

•1. A pyramid.

•2. Its apex.

3. Its base.

•4. A cross section.

•5. A lateral face.

Ferrocene.

6. Pentagonal (or regular pentagons).

7. At the apex of each pyramid.

8. The lateral edges.

9. Isosceles.

Pyramid Volumes.

•10. 128 cubic units.

$[V = \frac{1}{3}Bh = \frac{1}{3}(8^2)6 = 128.]$

11. 400 cubic units.

$[V = \frac{1}{3}Bh = \frac{1}{3}(10 \times 15)8 = 400.]$

•12. 70 cubic units.

$\{V = \frac{1}{3}Bh = \frac{1}{3}[\frac{1}{2}(5 \times 12)]7 = 70.\}$

A Regular Pyramid.

•13. They are equal. (They are radii of a regular polygon.)

14. They are right angles (and so are equal). (Because PO is the altitude, it is perpendicular to the plane of the pyramid's base; if a line is perpendicular to a plane, it is perpendicular to every line in the plane that passes through the point of intersection.)

15. They are right triangles. They are congruent (by SAS).

16. They are equal. (Corresponding parts of congruent triangles are equal.)

17. They are isosceles triangles. They are congruent (by SSS).

Leonardo's Claim.

18. A parachute (in the shape of a square pyramid).

•19. 576 yd^3. $[V = \frac{1}{3}Bh = \frac{1}{3}(12^2)12 = 576.]$

20. No. The man looks too big. He looks as tall as the pyramid above him, which has an altitude of 12 yd = 36 ft.

Triangle and Pyramid Compared.

21. $A = \frac{1}{2}bh.$

•22. $V = \frac{1}{3}Bh.$

•23. 55.125 square units.

$[A = \frac{1}{2}(10.5)(10.5) = 55.125.]$

24. 55. $(1 + 2 + 3 + \cdots + 10 = 55.)$

•25. 385.875 cubic units.

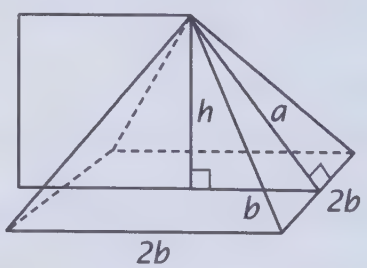
$[V = \frac{1}{3}(10.5)^2(10.5) = 385.875.]$

26. 385. $(1^2 + 2^2 + 3^2 + \cdots + 10^2 = 385.)$

Set II (pages 651–653)

Surprising as it might seem, blindness is less a disability in mathematics than it is in many other areas of study. The blind mathematician Nicholas Saunderson was mentioned earlier in the text (Chapter 4, Lesson 1) with regard to his invention of the pin-board. In his *Penguin Book of Curious and Interesting Mathematics* (Penguin, 1997), David Wells quotes French author Denis Diderot concerning Saunderson: "Wondrous things are told about him, and his progress in literature and in the mathematical sciences lends credence to all of them. He was the author of a very perfect book of its kind, the *Elements of Algebra*, in which the only clue to his blindness is the occasional eccentricity of his demonstrations, which would perhaps not have been thought up by a sighted person. To him belongs the division of the cube into six equal pyramids having their vertices at the center of the cube and the six faces as their bases; this is used for an elegant proof that a pyramid is one-third of a prism having the same base and height."

David Burton, in his *History of Mathematics—An Introduction* (Allyn & Bacon, 1985), observes that, if the dimensions of the Great Pyramid were chosen so that the square of its altitude is equal to the area of one of its lateral faces, then, given that a is the altitude of a face triangle and $2b$ is the length of a side of the base, $\frac{a}{b}$ is the golden ratio.



Expressing the claim algebraically, we have $h^2 = \frac{1}{2}(2b \cdot a) = ab$. Applying the Pythagorean Theorem to the figure above, we get $a^2 = h^2 + b^2$; so $h^2 = a^2 - b^2$. Substituting for h^2 gives

$a^2 - b^2 = ab$, from which we have $a^2 - ab - b^2 = 0$. Dividing by b^2 gives $(\frac{a}{b})^2 - (\frac{a}{b}) - 1 = 0$. Letting $x = \frac{a}{b}$, we have $x^2 - x - 1 = 0$. Using the quadratic formula to solve for x gives $x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{5}}{2}$. Because $x = \frac{a}{b} > 0$, we have $\frac{a}{b} = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$, the golden ratio.

The way in which the African salt funnel is used is described by Paulus Gerdes in *Geometry from Africa* (Mathematical Association of America, 1999): "The funnel is hung . . . and salt containing earth is put in it. A bowl is placed beneath the funnel, hot water is poured on the earth in the funnel, and saltwater is caught in the bowl. After evaporation, salt remains in the bowl." The method for making a model of the funnel from a square piece of paper is taken from Gerdes' book, which also includes a description of how African artisans weave the funnel.

Thomas Banchoff in his book titled *Beyond the Third Dimension* (Scientific American Library, 1990) calls the Egyptian formula for finding the volume of the frustrum of a square pyramid "a high point in the geometry of the ancient world." Two strange aspects of this formula are that: (1) it is suggested by Problem 14 of the Moscow Papyrus (ca. 1890 B.C.), a manuscript that makes no mention of how to find the volume of a complete pyramid, and (2) the method for finding the volume of a complete pyramid does not appear until the work of Democritus in the fifth century B.C., more than a thousand years later! It is interesting to note that a frustrum of a pyramid is pictured on the back of the U. S. dollar bill.

The Divided Cube.

27. x^3.

•28. $\frac{1}{6}x^3$. (The cube is divided into six congruent pyramids.)

29. x^2.

•30. $\frac{1}{2}x$.

31. $\frac{1}{2}x^3$. $[(x^2)(\frac{1}{2}x).]$

32. $V = \frac{1}{3}Bh$ (because $\frac{1}{6}x^3 = \frac{1}{3}(\frac{1}{2}x^3)$.

Pyramid Numbers.

•33. 231,000 ft^2. ($481^2 = 231{,}361$.)

•34. 612 ft. [$PM^2 = PO^2 + OM^2 = 481^2 + (\frac{756}{2})^2 = 374{,}245$; $PM = \sqrt{374{,}245} \approx 612$.]

•35. 231,000 ft^2. $[A = \frac{1}{2}bh \approx \frac{1}{2}(756)(612) \approx 231{,}336.]$

36. Yes. (The answers to exercises 33 and 35 are about the same.)

•37. 92,000,000 ft^3. $[V = \frac{1}{3}Bh = \frac{1}{3}(756)^2(481) = 91{,}636{,}272.]$

More on the Euler Characteristic.

•38. $n + 1$.

39. $n + 1$.

40. $2n$.

41. The Euler characteristic for pyramids is 2 because $V + F - E = (n + 1) + (n + 1) - 2n = 2$.

Salt Funnel.

42. *Student model of funnel:*



•43. It is a pyramid.

44. An equilateral triangle.

•45. Isosceles right triangles.

•46. $\frac{3}{4}s^2$.

•47. $\frac{1}{6}s^3$. $[V = \frac{1}{3}Bh = \frac{1}{3}(\frac{1}{2}s^2)s = \frac{1}{6}s^3.]$

The Frustum of a Pyramid.

•48. $\frac{1}{3}a^2(x+h)$.

49. $\frac{1}{3}b^2x$.

50. $\frac{1}{3}a^2(x+h) - \frac{1}{3}b^2x$, or $\frac{1}{3}(a^2x + a^2h - b^2x)$.

•51. OM $= \frac{1}{2}a$ and PN $= \frac{1}{2}b$.

•52. Corresponding sides of similar triangles are proportional.

•53. $\frac{x+h}{x} = \frac{\frac{1}{2}a}{\frac{1}{2}b} = \frac{a}{b}$.

54. $x = \frac{bh}{a-b}$. [$xa = xb + bh$, $xa - xb = bh$, $x(a-b) = bh$, $x = \frac{bh}{a-b}$.]

55. $\frac{1}{3}(a^2x + a^2h - b^2x) = \frac{1}{3}[(a^2 - b^2)x + a^2h] = \frac{1}{3}[(a^2 - b^2)\frac{bh}{a-b} + a^2h] = \frac{1}{3}[(a+b)bh + a^2h] = \frac{1}{3}(abh + b^2h + a^2h) = \frac{1}{3}(a^2 + ab + b^2)h$.

56. 504 cubic units. [$\frac{1}{3}(12^2 + 12 \times 3 + 3^2)8 = 504$.]

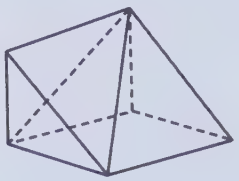
57. No. Trying the numbers for exercise 56 in the Babylonian expression gives $\frac{1}{2}(12^2 + 3^2)8 = 612 \neq 504$.

Set III (page 653)

When Daniel Lowen, then a 17-year-old honors student at Cocoa Beach High School, discovered the error in this PSAT problem, it made national news. The article reporting the mistake in the *New York Times* (March 17, 1981) carried the headline: "Youth Outwits Merit Exam, Raising 240,000 Scores." Lowen suspected that the question concerning the two pyramids might be a "trick question" and, when he got home, he made models of the pyramids to check his answer. It was the first time that an error in an SAT test had been discovered in its 33-year history.

How Many Faces?

1. Answer C (seven). Pyramid ABCD has four faces and pyramid EFGHI has five faces. When they are put together, the two faces that coincide are lost so that one would expect the resulting solid to have $4 + 5 - 2 = 7$ exposed faces.

2. 

Answer A (five). When the pyramids are put together in the way described, two pairs of faces lie in common planes, so only two faces are formed rather than four. (This can be proved with elementary geometry by showing that the dihedral angles of the solids are supplementary.)

Chapter 15, Lesson 6

Set I (pages 656–658)

Finding volumes of solids of revolution is an important application of integration in the calculus. The formulas for the shell and disk methods are based on right circular cylinders. Some of your students might be interested in seeing the disk equation $V = \int_a^b \pi[f(x)]^2 dx$ and comparing it with the geometric formula for the volume of a cylinder.

The Channel tunnel consists of two large tunnels each containing a single track railway line and a smaller service tunnel midway between them. The tunnels are 50 kilometers long and were bored at an average depth of 45 meters below the ocean floor. More than 5 million cubic meters of rock were removed in building them, a volume twice that of the Great Pyramid.

Primitive Perspective.

1. A cone, a cylinder, and a pyramid.
2. *Example figures:*



Italian into English.

3. Volume, area, base.
4. Cylinder.
5. Cone.

•6. Product.

7. A third.
8. Altitude.
9. The second.

Solids of Revolution.

•10. Its axis.

•11. The altitude.

•12. The lateral surface.

•13. The bases.

14. It is perpendicular to them.

•15. 45π in^3. [$V = Bh = (\pi 3^2)5 = 45\pi$.]

16. No. Its volume is 75π in^3. [$V = Bh = (\pi 5^2)3 = 75\pi$.]

17. A right triangle.

18. No. If it were revolved about the hypotenuse, the resulting solid would consist of two cones with a common base.

Hockey Puck.

•19. 7 in^2. [$A = \pi r^2 = \pi(1.5)^2 = 2.25\pi \approx 7$.]

20. 7 in^3. [$V = Bh = 2.25\pi(1) \approx 7$.]

Cylindrical Tunnel.

21. 64 m^2. [$A = \pi r^2 = \pi(4.5)^2 = 20.25\pi \approx 64$.]

•22. 4,800 m^3. [$V = Bh = 20.25\pi(75) \approx 4{,}800$.]

Sliced Solids.

23. Cones.
24. *Example figures:*



•25. They have equal volumes.

•26. Cavalieri's Principle.

Sausages in a Can.

•27. 35π cm^3. [$7Bh = 7(\pi 1^2)5 = 35\pi$.]

•28. 110 cm^3.

29. 54π cm^3. [$Bh = (\pi 3^2)6 = 54\pi$.]

30. 170 cm^3.

•31. 65%. ($\frac{35\pi}{54\pi} \approx \frac{110}{170} \approx 0.65$.)

Triangles or Cones.

32. The triangles have equal areas. (They are both composed of two 3-4-5 right triangles, or $\frac{1}{2}bh = \frac{1}{2}8 \cdot 3 = \frac{1}{2}6 \cdot 4 = 12$.)

33. The cone with the base of radius 4 has a larger volume. Its volume is $\frac{1}{3}Bh = \frac{1}{3}(\pi 4^2)3 = 16\pi$, whereas the volume of the cone with the base of radius 3 is $\frac{1}{3}Bh = \frac{1}{3}(\pi 3^2)4 = 12\pi$.

Set II (pages 658–661)

The pipes for exercises 34 and 35 are drawn to scale, making it possible to guess without making any calculations that the two pipes in arrangement B can carry the most water.

Pipe Puzzle.

•34. That they have the same length.

35. Arrangement B. The areas of the cross sections of the pipes are: A, $\pi 4^2 = 16\pi$; B, $2(\pi 3^2) = 18\pi$; C, $3(\pi 2^2) = 12\pi$.

Cake Pans.

36. The round pan has a volume of 95 in^3 and the square pan has a volume of 96 in^3. [$V_1 = Bh = (\pi 4.5^2)1.5 \approx 95$ and $V_2 = Bh = (8^2)1.5 = 96$.]

37. They would be *about* the same, although in theory the height of the round cake would be greater. The same amount of batter is used in each pan, but the area of the base of the round pan is slightly smaller.

Engine Displacement.

•38. The diameter of its base.

39. Its altitude.

40. $V = Bh = \pi r^2 h$. Letting the bore $= d = 2r$ and the stroke $= h$, we have $r = \frac{d}{2}$; so $V = \pi(\frac{d}{2})^2 h = \frac{\pi}{4} d^2 h$.

•41. 352 in^3. [$8V = 8(\frac{\pi}{4})(4.0)^2(3.5) \approx 352$.]

Stage Lighting.

•42. πh^2. ($A = \pi r^2 = \pi h^2$.)

43. $\frac{1}{3}\pi h^3$. [$V = \frac{1}{3}Bh = \frac{1}{3}(\pi h^2)h = \frac{1}{3}\pi h^3$.]

•44. It is multiplied by four. [$A = \pi r^2 = \pi(2h)^2 = 4\pi h^2$.]

45. It is multiplied by eight. {$V = \frac{1}{3}Bh = \frac{1}{3}[\pi(2h)^2]2h = \frac{8}{3}\pi h^3$.}

Paper Towels.

46. 176. [80.6 $ft^2 = (80.6)(12^2)\ in^2 = 11{,}606.4\ in^2$; $\frac{11{,}606.4}{(11)(6)} \approx 175.9$.]

47. 311 in^3. [$V = Bh = (\pi 3^2)11 = 99\pi \approx 311$.]

•48. 285 in^3. [$99\pi - \pi(\frac{1.75}{2})^2 11 \approx 285$.]

49. 1.6 in^3. ($\frac{285}{176} \approx 1.6$.)

Cylinder and Cone Areas.

50. A rectangle.

51. $2\pi r$.

•52. $2\pi rh$. ($A = bh = 2\pi rh$.)

53. Its total area. [$2\pi r(r + h) = 2\pi r^2 + 2\pi rh$, or the areas of the two bases plus the lateral area.]

•54. A sector of a circle.

•55. $2\pi r$. (The length of arc CD is the circumference of the base of the cone.)

56. $\frac{r}{l}$. ($\frac{\text{lateral area}}{\text{area of circle O}} = \frac{\text{length of arc CD}}{\text{circumference of circle O}} = \frac{2\pi r}{2\pi l} = \frac{r}{l}$.)

•57. πrl. ($\frac{\text{lateral area}}{\pi l^2} = \frac{r}{l}$; so lateral area $= \frac{r}{l}\pi l^2 = \pi rl$.)

58. $\pi r\sqrt{r^2 + h^2}$. ($l^2 = r^2 + h^2$; so $l = \sqrt{r^2 + h^2}$.)

59. Its total area. [$\pi r(r + l) = \pi r^2 + \pi rl$, or the area of its base plus its lateral area.]

Coin Wrappers.

60. Because the width of the flat wrapper is 34 mm, its circumference when it is formed into a cylinder is 2(34) = 68 mm. Because $c = \pi d$, $d = \frac{c}{\pi} = \frac{68}{\pi} \approx 21.6$. The diameter of the cylinder is about 21.6 mm, just large enough for nickels with a diameter of 21 mm.

61. About 29 mm. ($c = \pi d = 18\pi$, $w = \frac{c}{2} = 9\pi \approx 28.3$.)

62. Slightly more than $\frac{\pi x}{2}$ mm. ($w = \frac{c}{2} = \frac{\pi x}{2}$.)

Rolling Cones.

63. Its apex, V, stays in place. The points on the edge of the base of the cone sweep out a circle with center V and radius l.

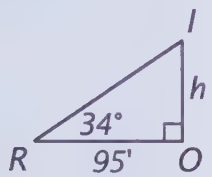
64. $\angle PVO = 30°$. If the cone returns to the same spot after two rotations, $2c = 2(2\pi r) = 2\pi l$; so $l = 2r$, and so $\angle PVO = 30°$.

Set III (page 661)

The picture of the cone of iron ore was taken by photographer Alex MacLean, the founder of Landslides, a Boston-based firm that specializes in lower-altitude aerial photography. The cone was located near the loading docks of Ashtabula, Ohio, ready to be taken by train to steel mills in southern Ohio and West Virginia. MacLean's photographs have been published in two beautiful books: *Look at the Land—Aerial Reflections of America*, with text by Bill McKibben (Rizzoli, 1993) and *Taking Measures Across the American Landscape*, with text by James Corner (Yale University Press, 1996).

Iron Ore.

1. About 190 ft. [From the width of the train cars, we know that 4 mm corresponds to 10 feet; so 1 mm corresponds to 2.5 feet. The diameter of the pile is 75 mm, or $75(2.5) = 187.5 \approx 190$ ft.]
2. About 64 ft.



($\tan 34° = \frac{h}{95}$, $h = 95 \tan 34° \approx 64$.)

3. About 28,000 ft^2. [$A = \pi r^2 \approx \pi(95)^2 \approx 28{,}000$.]
4. About 600,000 ft^3.
   [$V = \frac{1}{3}Bh \approx \frac{1}{3}\pi(95)^2(64) \approx 600{,}000$.]
5. About 137. ($\frac{600{,}000\ \text{ft}^3}{4{,}400\ \text{ft}^3} \approx 137$.)

Chapter 15, Lesson 7

Set I (pages 665–667)

The earth obviously has a shadow *cone*, rather than the shadow cylinder depicted in Anno's whimsical painting. We have evidence of this shadow cone every time we observe an eclipse of the moon.

For our sun to become a black hole, its mass would have to be compressed into a sphere with a radius of about two-thirds of a mile (1 kilometer). If the earth were to become a black hole, it would have to be compressed to about the size of a marble.

Although Euclid considers spheres in the *Elements*, he does not show how to find their surface areas or volumes. He does establish in the last theorem (Proposition 18) of Book XII, however, that the ratio of the volumes of two spheres is equal to the cube of the ratio of their diameters. It was Archimedes who developed the formulas for the volume and area of a sphere in his work *On the Sphere and Cylinder*, which contains 53 theorems on these solids.

In *Beyond the Third Dimension* (Scientific American Library, 1990), Thomas Banchoff describes *Flatland* as "simultaneously a social satire and an introduction to the idea of higher dimensions. . . . [Its author, Edwin] Abbott used dimensional analogies to great effect in raising questions about the way we see the world, especially when we come into contact with the truly transcendental. For over a century, mathematicians and others have speculated about the nature of higher dimensions, and in our day the concept of dimension has begun to play a larger and larger role in our conception of a whole range of activities."

Your students may be surprised to learn that a baseball floats on water. In his book titled *Sport Science* (Simon & Schuster, 1984), Peter J. Brancazio observes that, in a list of balls commonly used in sports, including the football, baseball, golf ball, handball, soccer ball, basketball, and volleyball, only the golf ball and handball sink in water!

The fact established in exercises 24 and 25 that the two boxes contain the same amount of chocolate is contrary to many people's intuition, but, if we think of each sphere as inscribed in a cube, the result is obvious. The ratio of the volumes of the sphere and cube is a constant and does not depend on how large they might be.

Earth Shadow.

•1. They are equal.

•2. It is four times as large.

Black Hole.

•3. About 12.6 km^2.
   [$A = 4\pi r^2 = 4\pi(1)^2 = 4\pi \approx 12.6$.]

•4. About 4.2 km^3.
   [$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi \approx 4.2$.]

5. 3 km. ($4\pi r^2 = \frac{4}{3}\pi r^3$, $12\pi r^2 = 4\pi r^3$, $3 = r$.)

6. 36π, or about 113 km^2, and 36π, or about 113 km^3. [$4\pi(3)^2 = 36\pi \approx 113$; $\frac{4}{3}\pi(3)^3 = 36\pi$.]

Archimedes' Discoveries.

•7. $\frac{1}{3}\pi x^3$. ($V = \frac{1}{3}\pi x^2 x = \frac{1}{3}\pi x^3$.)

8. $\frac{2}{3}\pi x^3$. ($V = \frac{1}{2} \cdot \frac{4}{3}\pi x^3 = \frac{2}{3}\pi x^3$.)

9. πx^3. ($V = \pi x^2 x = \pi x^3$.)

10. The volume of the hemisphere is twice that of the cone, and the volume of the cylinder is three times that of the cone. The sum of the volumes of the cone and hemisphere is equal to the volume of the cylinder.

Flatland.

•11. All radii of a sphere are equal.

•12. If a line is perpendicular to a plane, it is perpendicular to every line in the plane that passes through the point of intersection.

13. HL.

14. Corresponding sides of congruent triangles are equal.

15. It implies that the curve is a circle. The set of all points in a plane that are equidistant from a given point in the plane is a circle.

Sound Wave.

16. At its center.

17. Its area.

•18. It decreases.

Volume and Density.

•19. 212 cm^3. [$V = \frac{4}{3}\pi(3.7)^3 \approx 212$.]

20. 42 cm^3. [$V = \frac{4}{3}\pi(2.15)^3 \approx 42$.]

•21. 0.7 gram/cm^3. ($d = \frac{m}{v} = \frac{145\text{ g}}{212\text{ cm}^3} \approx 0.7\text{ g/cm}^3$.)

22. A baseball will float on water.

23. It is more than 42 grams. ($d = \frac{m}{v} > 1$, $\frac{m}{42} > 1$, $m > 42$.)

Chocolate Packing.

24. *(Student answer.)*

25. The volume of chocolate in each box is $\frac{8}{3}\pi d^3$; so they contain the same amount of chocolate.

First box: $16[\frac{4}{3}\pi(\frac{d}{2})^3] = \frac{64}{3}\pi\frac{d^3}{8} = \frac{8}{3}\pi d^3$.

Second box: $2(\frac{4}{3}\pi d^3) = \frac{8}{3}\pi d^3$.

26. *(Student answer.)*

27. The first box contains more wrapping paper. The amounts are related to (but not the same as) the surface areas of the chocolates. If the wrappings are similar to one another, they will be proportional to the squares of the radii (or diameters) of the spherical chocolates that they enclose. Therefore, the box that contains more wrapping paper can be found by comparing the total areas of the chocolates in each box.

First box: $16(4\pi(\frac{d}{2})^2) = 64\pi\frac{d^2}{4} = 16\pi d^2$.

Second box: $2(4\pi d^2) = 8\pi d^2$.

Set II (pages 667–669)

Martin Gardner observed in one of his "Mathematical Games" columns for *Scientific American* (reprinted in *Fractal Music, Hypercards and More . . .* (W. H. Freeman and Company, 1992) that Max Bill's *Half Sphere Around Two Axes* "is based on an old folk method of quickly slicing an apple into congruent halves" and that "it is not as easy as it looks." Some of your students might enjoy the challenge of trying to cut an apple in this way.

In his "Games" column in the October 1983 issue of *Omni* magazine Scot Morris features the "Hollow Earth," telling the story of Cyrus Teed and his strange cult and commenting on the geometry explored in exercises 32 through 37. Morris wrote: "What's most infuriating is that a little mathematical fiddling turns this crazy theory into a proposition that is virtually impossible to refute. The trick is done by inversion, a purely

geometric transformation that lets a mathematician turn shapes inside-out. When a sphere is inverted, every point outside is mapped to a corresponding point inside, and vice versa." Morris asked the great geometer H. S. M. Coxeter if there were "any way to prove that we *aren't* inside a hollow earth" and Coxeter couldn't think of any. Morris observes that "just as the geometry of space inverts, so do all the laws of physics. Toward the center of a hollow Earth, light slows down and everything shrinks—atoms, astronauts, spaceships, and measuring rods." More on this topic can be found at various sites on the internet; the entire text of the *Omni* article is at www.sumeria.net.

Of the two methods for finding the volume of water in exercises 44 and 45, the first method is exact. The second method, treating the spherical shell as if it were a right prism wrapped around the sphere, provides a good approximation only if h, the thickness of the shell, is very small in comparison with r, the radius of the inner sphere.

Exercise 46 illustrates the fact that an expression that measures area must produce *square* units, whereas an expression that measures volume must produce *cubic* units.

This is even more evident in the "volume" formulas given for exercises 50 through 54 for spheres in different dimensions. Because our experience is limited to three dimensions, we have special names for only *linear*, *square*, and *cubic* units. The formulas in the table are derived from general formulas given by Clifford Pickover in *Surfing Through Hyperspace* (Oxford University Press, 1999). As the powers of π in them suggest, there are two different general formulas, depending on whether the number of dimensions is odd or even. The volume of a k-dimensional sphere where k is even is

$$V = \frac{\pi^{\frac{k}{2}} r^k}{(\frac{k}{2})!}$$

whereas, if k is odd, the volume is

$$V = \frac{\pi^{\frac{k-1}{2}} [\frac{k+1}{2}]! 2^{k+1} r^k}{(k+1)!}.$$

(I am tempted to put an exclamation point at the end of the second formula, but that would only cause confusion!)

Half Sphere.

•28. $\frac{2}{3}\pi r^3$. $[\frac{1}{2}(\frac{4}{3}\pi r^3) = \frac{2}{3}\pi r^3.]$

29. $2\pi r^2$. $[\frac{1}{2}(4\pi r^2) = 2\pi r^2.]$

•30. $\frac{3}{2}\pi r^2$. $[3(\frac{1}{2}\pi r^2) = \frac{3}{2}\pi r^2.]$

31. $3.5\pi r^2$. $(2\pi r^2 + \frac{3}{2}\pi r^2 = 3.5\pi r^2.)$

Hollow Earth.

32. ΔOTP is a right triangle because, if a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

•33. Either leg of a right triangle is the geometric mean between the hypotenuse and its projection on the hypotenuse (or by similar triangles).

34. 65 mi. $[\frac{240{,}000}{3{,}960} = \frac{3{,}960}{OP'}$,
$240{,}000\ OP' = (3{,}960)^2$, $OP' \approx 65.]$

•35. 3,895 mi.
$(AP' = AO - OP' = 3{,}960 - 65 = 3{,}895.)$

36. 900 ft. $[\frac{93{,}000{,}000}{3{,}960} = \frac{3{,}960}{OP'}$,
$93{,}000{,}000\ OP' = (3{,}960)^2$,
$OP' \approx 0.17$ mi ≈ 900 ft.]

37. Near the center of the earth.

Melting Snowball.

•38. About 15 hours.
$[V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(6\text{ cm})^3 \approx 905\text{ cm}^3$;
905 minutes ≈ 15 hours.]

•39. About 13 hours.
$[V = \frac{4}{3}\pi(3\text{ cm})^3 \approx 113\text{ cm}^3$;
$905\text{ cm}^3 - 113\text{ cm}^3 = 792\text{ cm}^3$;
792 minutes ≈ 13 hours.]

40. About 452 cm². $[A = 4\pi(6)^2 \approx 452.]$

41. About 113 cm². $[A = 4\pi(3)^2 \approx 113.]$

42. No. It takes about 13 hours for the diameter of the snowball to decrease 6 inches and only 2 hours for it to decrease the remaining 6 inches. (The diameter decreases at an increasing rate.)

43. No. The surface area of the snowball decreases about $452 - 113 = 339$ cm^2 in 13 hours (an average of 26 cm^2 per hour) and 113 cm^2 in 2 hours (about 56 cm^2 per hour). (The surface area also decreases at an increasing rate.)

Earth Under Water.

•44. 325,000,000 mi^3.
$(V = \frac{4}{3}\pi(3{,}960 + 1.65)^3 - \frac{4}{3}\pi(3{,}960)^3 =$
$\frac{4}{3}\pi(3{,}961.65^3 - 3{,}960^3) \approx 325{,}000{,}000.)$

45. 325,000,000 mi^3.
$(V \approx 4\pi r^2 h = 4\pi(3{,}960)^2(1.65) \approx 325{,}000{,}000.)$

Doughnuts.

46. The first expression must give its volume because (a units)(b units)^2 = ab^2 cubic units; the second expression must give its area because (a units)(b units) = ab square units.

•47. The second expression.

48. It would be doubled.
$[2\pi^2(2a)b^2 = 4\pi^2ab^2 = 2(2\pi^2ab^2.]$

49. It would be multiplied by four.
$[2\pi^2a(2b)^2 = 8\pi^2ab^2 = 4(2\pi^2ab^2.]$

Spheres in Different Dimensions.

50. A circle.

51. Its area.

52. Its diameter.

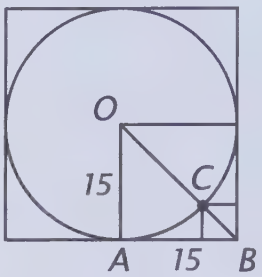
53.

Dimensions	*Volume*
1	2
2	3.14
3	4.19
4	4.93
5	5.26
6	5.17
7	4.72

54. As the number of dimensions increases, the volume of the sphere at first increases and then decreases.

Set III (page 669)

Frank Marshall, the producer of *Raiders of the Lost Ark*, said concerning the rolling-boulder sequence in the movie: "A huge boulder comes crashing down through a chute and chases after Indy. We had to make it look like it rolled free, but we also had to be able to control it. The boulder was on this contraption like an arm; it was free-spinning and the arm was hidden. The boulder tumbled down the chute, then it had to be taken back if we wanted to do another take. But we couldn't do it again very quickly because we had to put in the stalactites that got broken off as it rolled out."



1. You could lie down along the edge of the tunnel. In the figure above, ΔOAB is an isosceles right triangle; so $OB = 15\sqrt{2}$ and $CB = OB - OC = 15\sqrt{2} - 15 = 15(\sqrt{2} - 1) \approx$ 6.2 feet. The sides of the small square are about $\frac{6.2}{\sqrt{2}} \approx 4.4$ feet; so there is plenty of room.

2. *(Student answer.)* (It might not work if the corners of the passage were rounded.)

Chapter 15, Lesson 8

Set I (pages 672–674)

About the Statue of Liberty, Robert Osserman in his book titled *Poetry of the Universe* (Anchor Books, 1995) explains: "When the Statue of Liberty was built, the first step was the construction of a small-scale model. Building the statue to scale from the model in one step would have been a very risky operation. The sculptor decided to construct a number of intermediate models, scaling the statue up by a modest amount at each

step. One reason for caution is that each time the scale is doubled, the volume—and therefore the weight—goes up by a factor of eight. On the other hand, the strength of the supports, which depends on cross-sectional area, is multiplied only by a factor of four. That leads to the inevitable conclusion that whatever shape you design, if built on a large enough scale, will collapse under its own weight."

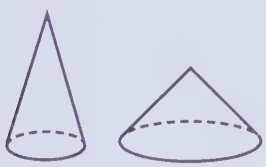
A similar lesson on this subject appears in J. E. Gordon's *The Science of Structures and Materials* (Scientific American Library, 1988) under the title "The Danger of Scaling Up": "Engineering design practice is based to a large extent upon experience and precedent. One successful project leads to another, and things generally get bigger and bigger. The tendency is to scale up a successful design, keeping the working stresses the same. Not very long ago the author had a head-on collision over the conference table with NASA about its wind-energy program. A number of safe and successful windmills are 175 feet in diameter. NASA proposed, in effect, to scale up this design to a diameter of 400 feet. I had to point out that, for reasons of modern fracture mechanics, extensive redesign would be needed if the bigger windmills were to operate safely."

The difficulty of comparing the volumes of solids from two-dimensional drawings of them is demonstrated by exercises 22 through 26 on olive sizes. The volume of a supercolossal olive is almost twice that of a jumbo olive, something that might be hard to guess from the side-view drawings of exercises 19 and 21.

The three pyramids at Giza were built in a period of about 150 years (ca. 2650 B.C.–2500 B.C.) and are named after the kings who built them.

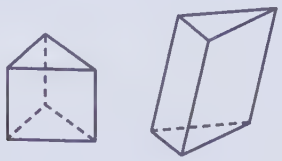
Similar or Not?

1. Two cones are not always similar.
Example figures:



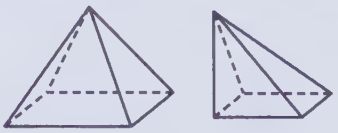
•2. Two cubes are always similar.

3. Two triangular prisms are not always similar.
Example figures:



4. Two spheres are always similar.

5. Two square pyramids are not always similar.
Example figures:



Scaling Up.

•6. It is multiplied by 9. (3^2.)

•7. It is multiplied by 27. (3^3.)

8. It is multiplied by 400. (20^2.)

9. It is multiplied by 8,000. (20^3.)

•10. It is multiplied by n^2.

•11. It is multiplied by n^3.

Giant Windmill.

12. 3.2. ($\frac{400}{125}$.)

•13. 10.24. (3.2^2.)

•14. 32.768. (3.2^3.)

15. Approximately 82,000 pounds (or 41 tons!) ($32.768 \times 2{,}500 = 81{,}920$.)

Gulliver in Lilliput.

16. By cubing 12. (They were basing the amount of food that Gulliver needed on his volume.)

•17. That Gulliver's body was similar to their bodies.

18. 144 times as much. (Basing the amount of cloth on surface areas gives $12^2 = 144$.)

Earth and Mars.

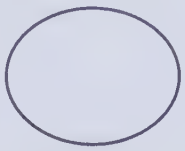
19. About 1.9 times. ($\frac{3{,}960}{2{,}100} = 1.8857\ldots$)

•20. About 3.6 times. [$(1.8857\ldots)^2 \approx 3.6$.]

21. About 6.7 times. [$(1.8857\ldots)^3 \approx 6.7$.]

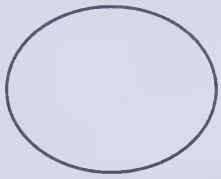
Olive Sizes.

22. *Exact size:*



•23. About 22 mm. ($\frac{x}{18} = \frac{28}{23}$, $23x = 504$, $x \approx 22$.)

24. *Exact size:*



•25. About 1.8 times. [$(\frac{28}{23})^3 \approx 1.8$.]

26. About 9 grams. (1.8×5.)

Three Pyramids.

27. Cheops, 0.62; Chephren, 0.67; Mycerinus, 0.61.

28. Cheops and Mycerinus, because the ratio of their height to their base edge is closest to being the same.

Set II (pages 674–676)

There are two stories associated with the problem of the duplication of the cube, one of the three famous problems of antiquity. In *Science Awakening* (Oxford University Press, 1961), B. L. van der Waerden quotes from a supposed letter from Eratosthenes to king Ptolemy:

"It is said that one of the ancient tragic poets brought Minos on the scene, who had a tomb built for Glaucus. When he heard that the tomb was a hundred feet long in every direction, he said:

'You have made the royal residence too small, it should be twice as great. Quickly double each side of the tomb, without spoiling the beautiful shape.'

He seems to have made a mistake. For when the sides are doubled, the area is enlarged fourfold and the volume eightfold. The geometers then started to investigate how to double a given body, without changing its shape; and this problem was called the duplication of the cube, since they started with a cube and tried to double it. After they had looked for a solution in vain for a long time, Hippocrates of Chios observed that, if only one could find two geometric means between two line segments, of which the larger one is double the smaller, then the cube would be duplicated. This transformed the difficulty into another one, not less great."

The difficulty is in obtaining the length of the edge of the larger cube solely by geometric construction. Without this restriction, the problem can be easily solved. Exercises 37 through 43 present a simple solution, but, unfortunately, the figure used cannot be constructed by straightedge and compass.

Exercises 47 and 48 comparing the shape of liquid in a conical cup to its container has an interesting biological connection. In *Cat's Paws and Catapults* (Norton, 1998), Steven Vogel wrote: "The need for growth without loss of function can impose severe geometrical limitations. Consider the possible shapes of mollusks, a widespread and diverse group that includes scallops, slugs, snails, and squid. Mollusk shell doesn't grow, so a shell cannot enlarge except by adding incremental additions to edges and inner surfaces. For most shapes [such as cylinders and rectangular solids], such incremental additions would quickly lead to awkward changes in proportions. . . . Hollow cones are much better than cylinders or most other shapes if what matters is growth in size without simultaneous change in shape. And mollusk shells are basically cones. . . ."

The iron-doughnut problem is a classic puzzle in physics. Martin Gardner observes that opticians take advantage of the fact that a ring remains similar to itself as it expands when they heat the frame of a pair of glasses in order to remove a lens.

Doubling a Cube.

29. $6x^2$.

•30. $24x^2$. [$6(2x)^2 = 24x^2$, or $2^2 \cdot 6x^2 = 24x^2$.]

31. x^3.

•32. $8x^3$. [$(2x)^3 = 8x^3$, or $2^3 \cdot x^3 = 8x^3$.]

33. 4. ($\frac{24x^2}{6x^2} = 4$, or 2^2.)

34. 8. ($\frac{8x^3}{x^3} = 8$, or 2^3.)

•35. The square.

36. The cube.

Plato's Method.

37. The altitude to the hypotenuse of a right triangle is the geometric mean between the segments into which it divides the hypotenuse.

•38. $x = \sqrt[3]{2}$; $y = \sqrt[3]{4}$. [$\frac{1}{x} = \frac{x}{y}$ and $\frac{x}{y} = \frac{y}{2}$, $y = x^2$ and $y^2 = 2x$, $(x^2)^2 = 2x$, $x^3 = 2$, $x = \sqrt[3]{2}$; $y = (\sqrt[3]{2})^2 = \sqrt[3]{4}$.]

•39. PA, 1; PB, 2; PC, 4; PD, 8.

•40. It is twice as large.

41. $x = \sqrt[3]{3}$; $y = \sqrt[3]{9}$. [$\frac{1}{x} = \frac{x}{y}$ and $\frac{x}{y} = \frac{y}{3}$, $y = x^2$ and $y^2 = 3x$, $(x^2)^2 = 3x$, $x^3 = 3$, $x = \sqrt[3]{3}$; $y = (\sqrt[3]{3})^2 = \sqrt[3]{9}$.]

42. PA, 1; PB, 3; PC, 9; PD, 27.

43. It is three times as large.

Rubber Bands.

44. No. Although the corresponding lengths and widths of the rubber bands are proportional, we don't know whether their corresponding thicknesses are proportional.

•45. Eight. ($\frac{2\frac{1}{2}}{1\frac{1}{4}} = \frac{\frac{1}{8}}{\frac{1}{16}} = 2$; $2^3 = 8$. If the bands were made from the same material, their weights would have the same ratio as their volumes.)

46. The fact that the ratio $\frac{5{,}780}{1{,}420} \approx 4.1$ indicates that the weight, and hence volume, of a #31 band is about four times that of a #10 band. The rubber bands are evidently not similar in shape and apparently have approximately the same thickness.

Two Cups.

47. Yes. $\Delta ABE \sim \Delta CDE$ by AA because $\angle A = \angle DCE$ and $\angle AEB = \angle CED$.

48. Yes. Because $\Delta ABE \sim \Delta CDE$, $\frac{AB}{CD} = \frac{AE}{CE}$; so the corresponding dimensions of the cones are proportional.

49. No. The rectangles are *not* similar because $\frac{AB}{CD} = 1$, whereas $\frac{BF}{DF} > 1$. Alternatively, the rectangles are not similar, because they do not have the same shape.

50. No.

Height and Weight.

•51. About 47 pounds. [$(\frac{4}{6})^3 \times 160 \approx 47$.]

•52. About 93 pounds. [$(\frac{5}{6})^3 \times 160 \approx 93$.]

53. About 254 pounds. [$(\frac{7}{6})^3 \times 160 \approx 254$.]

54. About 379 pounds. [$(\frac{8}{6})^3 \times 160 \approx 379$.]

Heated Ring.

55. *(Student answer.)*

56. The diameter of its hole gets larger. Because it is given that the shape stays the same, *all* dimensions must increase in order to remain proportional.

Teddy Bear Trouble.

57. No. She would need $2^2 = 4$ times as much fur to cover the surface, $2^3 = 8$ times as much kapok to fill the volume, 2 times as much length of ribbon and the same number of eyes as for the smaller bears.

58. The kapok.

59. Five. (She has enough stuffing to make 40 small bears; $\frac{40}{8} = 5$.)

60. The ribbon.

Set III (page 676)

According to the *Guinness Book of World Records*, a clam weighing 734 pounds was found off Ishigaki Island, Okinawa, Japan, in 1956.

Colossal Clam.

We will assume that the clams are similar and that the ratio of their weights is equal to the ratio of their volumes.

Let x = the weight of the larger clam in ounces:

$$\frac{x}{1} = \left(\frac{48}{2.4}\right)^3;$$

so $x = 20^3 = 8{,}000$. The larger clam would weigh 8,000 ounces, or 500 pounds.

Chapter 15, Lesson 9

Set I (pages 678–680)

The prefixes used in naming the five regular polyhedra are of Greek origin: *tetra*, *hexa*, *okta*, *dodeka*, and *eikosa*. Surprisingly, however, of the terms referring to "four" most commonly used in mathematics, only "tetrahedron" comes from Greek. Terms such as "quadrilateral," "quadrant," "quadratic," "quadrillion," and "quadruple" come from Latin. It is also interesting to note that, although the names of the five regular polyhedra refer to *all* polyhedra having those numbers of faces, they are used almost exclusively to mean the *regular* polyhedra. Terms such as "pentahedron" or "decahedron," for which no regular polyhedrons exist, are almost never used.

Exercises 6 through 11 consider two- and three-dimensional versions of the same idea. It seems appropriate that the sides of the "midpoint triangle" in two dimensions are $\frac{1}{2}$ the sides of the original triangle, whereas the edges of the "midpoint tetrahedron" in three dimensions are $\frac{1}{3}$ the edges of the original tetrahedron. A rather surprising difference between the two situations, however, is that, although the large triangle can be "filled" with four of the smaller triangles, the large tetrahedron cannot be "filled" with *any* number of the smaller tetrahedrons. In *Martin Gardner's Sixth Book of Mathematical Games from Scientific American* (W.H. Freeman and Company, 1971), Gardner explains: "Because equilateral triangles also tile a plane, one might suppose that regular and congruent tetrahedrons would also pack snugly to fill space. This seems so intuitively evident that even Aristotle, in his work *On the Heavens*, declared it to be the case. The fact is that among the Platonic Solids the cube alone has this property. If the tetrahedron also had it, it would long ago have rivaled the cube in popularity for packaging."

In *The Penguin Book of Curious and Interesting Mathematics* (Penguin, 1997), David Wells reports that, in a questionnaire of mathematicians regarding the "most beautiful theorems" of mathematics, the fact that there are five regular polyhedra ranked near the top of the list. Benno Artmann (*Euclid—The Creation of Mathematics*, Springer, 1999) points out that "Euclid's treatment of the regular polyhedra is especially important for the history of mathematics because it contains the first example of a major classification theorem. Such theorems start with a definition (or axiomatic description) and end with a list of objects satisfying the description. There is no official definition of a regular solid in the list of definitions at the beginning of Book XI. However, we find one in the very last theorem of the *Elements*, which asserts the completeness of the preceding list of polyhedra." Exercises 30 through 36 suggest the form of Euclid's argument.

Numbers and Names.

•1. Eight.

2. 12.

•3. Quadrilateral.

4. Cube.

5. An icosahedron is a regular polyhedron that has 20 faces. An icosagon is a polygon that has 20 sides.

Two Triangles.

•6. Yes. (The sides of the smaller triangle are midsegments of the larger triangle; so they are parallel to its sides. The parallel lines form equal corresponding angles; so the triangles are similar by AA. Alternatively, the sides of the smaller triangle are half as long as the sides of the larger triangle; so the triangles are similar by SSS.)

7. They are twice as long. (This also follows from the Midsegment Theorem.)

8. The perimeter of the larger triangle is twice the perimeter of the smaller one. (The ratio of the perimeters of two similar polygons is equal to the ratio of a pair of corresponding sides.)

•9. The area of the larger triangle is four times the area of the smaller one. (The ratio of the areas of two similar polygons is equal to the square of the ratio of a pair of corresponding sides.)

Two Tetrahedrons.

•10. The surface area of the larger tetrahedron is $3^2 = 9$ times the surface area of the smaller one.

11. The volume of the larger tetrahedron is $3^3 = 27$ times the volume of the smaller one.

da Vinci Polyhedron.

12. A dodecahedron.

13. Regular pentagons.

•14. 12.

15. Five.

•16. Three.

17. Two.

•18. 20.

19. 30.

Escher Box.

•20. An icosahedron.

21. Equilateral triangles.

22. 20.

23. Three.

24. Five.

25. Two.

•26. 12. $\left(\frac{20\times3}{5}.\right)$

27. 30. $\left(\frac{20\times3}{2}.\right)$

Euclid's Claim.

28. Regular polygons.

•29. That they are congruent.

•30. 180°. ($3 \times 60°$.)

31. 240°. ($4 \times 60°$.)

32. 300°. ($5 \times 60°$.)

33. Because, if six equilateral triangles surround a point, they must lie in a plane (because the sum of the six angles is 360°).

•34. 270°. ($3 \times 90°$.)

35. 324°. ($3 \times 108°$.)

36. The figure is flat if the sum is 360°; it is solid if the sum is less than 360°.

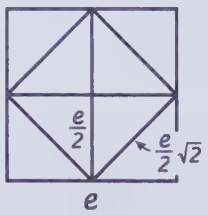
Set II (pages 680–682)

The golden ratio appears as the ratio of the lengths of the edges of the dodecahedron and its inscribed cube in exercise 45. The exercise reveals that this ratio comes from the relation of the length of a diagonal of a regular pentagon to one of its sides.

Exercises 47 through 60 reveal something rather surprising. As the number of sides of a regular polygon inscribed in a circle increases, the perimeter and area of the polygon increase. As the number of faces of a regular polyhedron inscribed in a sphere increases, its surface area and volume do not necessarily increase! Howard Eves wrote in the first volume of *In Mathematical Circles* (Prindle, Weber & Schmidt, 1969): "The uninitiated will almost always intuitively believe that of the regular dodecahedron (a solid having 12 faces) and a regular icosahedron (a solid having 20 faces) inscribed in the same sphere, the icosahedron has the greater volume. . . . The reverse is actually the case." Not only is the inscribed dodecahedron closer than the icosahedron to the sphere in volume, it is also closer to it in surface area. If these solids are *circumscribed* about a sphere rather than *inscribed* in one, however, it is the icosahedron whose surface area and volume are closer to those of the sphere. Another remarkable fact about the solids inscribed in the sphere is that the ratio of their volumes is equal to the ratio of their surfaces areas.

Octahedron and Cube.

37.



•38. $\frac{e}{2}\sqrt{2}$. (From the isosceles right triangle.)

39. $\frac{1}{2}e^2$. $[A = (\frac{e}{2}\sqrt{2})^2 = \frac{1}{2}e^2.]$

40. $\frac{e}{2}$.

•41. $\frac{1}{6}e^3$. $\{V = 2(\frac{1}{3}Ah) = 2[\frac{1}{3}(\frac{1}{2}e^2)(\frac{e}{2})] = \frac{1}{6}e^3.\}$

42. It is one-sixth of the volume of the cube.

Dodecahedron Construction.

43. By HL. (ΔABP and ΔCBP are right triangles, AB = BC because ABCDE is regular, and BP = BP.)

•44. $\angle ABP = 54°$ and $\angle BAP = 36°$.
($\angle ABP = \angle CBP$; so
$\angle ABP = \frac{1}{2}\angle ABC = \frac{1}{2}108° = 54°$;
$\angle BAP = 90° - 54° = 36°$.)

•45. 1.618. (In right ΔABP,
$\sin \angle ABP$ [or $\cos \angle BAP$] $= \frac{AP}{AB}$; $AC = 2AP$;
so $\frac{AC}{AB} = \frac{2AP}{AB} = 2 \sin 54° = 1.618$. This is the golden ratio.)

•46. About 1.8. [The volume of the cube is $1^3 = 1$ and the volume of the dodecahedron is $1 + 6(0.135) = 1.81$.]

Round About.

47. That there exists a circle that contains all of its vertices.

48. That there exists a sphere that contains all of a regular polyhedron's vertices.

49. The polygon with 20 sides.

50. *(Student answer.)* (The polyhedron with 20 faces seems like a reasonable guess.)

51. 12.6 square units.
$[A = 4\pi r^2 = 4\pi(1)^2 = 4\pi \approx 12.6.]$

•52. 0.714 unit.

•53. 10.5 square units.

54. 1.051 units.

55. 9.6 square units!

56. *(Student answer.)* (Again the icosahedron seems like a reasonable guess.)

57. 4.2 cubic units.
$[V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi \approx 4.2.]$

•58. 2.8 cubic units. $[\frac{1}{4}(0.714)^3(15 + 7\sqrt{5}) \approx 2.8.]$

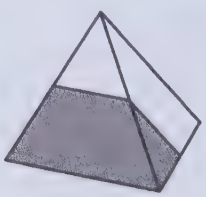
59. 2.5 cubic units. $[\frac{5}{12}(1.051)^3(3 + \sqrt{5}) \approx 2.5.]$

60. The dodecahedron!

Set III (page 682)

About this "Pyramid Puzzle," Jerry Slocum and Jack Botermans wrote in *Puzzles Old and New* (University of Washington Press, 1986): "This put-together puzzle consists of two identical pieces and at first sight it seems too easy to include in a serious puzzle book. However, appearances can be deceptive and this apparently simple puzzle has frustrated many a self-confident individual." The puzzle has been marketed in various versions, including one in which the two pieces have each been cut in half so that there are four pieces and another one with four pieces, each piece consisting of four connected spheres.





Tetrahedron Puzzle.

The solution is shown at the right above.

Chapter 15, Review

Set I (pages 684–686)

In *Visual Intelligence* (Norton, 1998), Donald Hoffman considers the ambiguity of seeing depth: "The image at the eye has two dimensions; therefore it has countless interpretations in three dimensions. . . . *Anytime* you see depth you construct it, not just when you view strange drawings . . . but also in everyday life. . . . Quite simply, your visual system is biased. It constructs only those 3D worlds that conform to its rules. Most others it simply ignores. These rules are powerful. They prune the possible depths you can see from infinity down to one or two. This is useful, for if you saw all options then you would see depth flip or distort with your every glance." Vasarely's sculpture of the "flipping cubes" is a dramatic example of the problems that we can have in viewing a two-dimensional picture as three-dimensional.

The two figures by Gaetano Kanizsa are almost identical yet are interpreted very differently. The second, seen as three-dimensional, has the same depth flip: Is the top of the box the quadrilateral at the upper left or the one at the upper right?

A *cord* is an unusual unit of volume for several reasons: it is only used to measure *wood*, it is associated with a specific *shape*, and that shape is *not a cube*. *The American Heritage Dictionary of the English Language* defines it as: "A unit of quantity for cut fuel wood, equal to 128 cubic feet in a stack measuring 4 by 4 by 8 feet."

The building considered in exercises 13 through 16 is the Baptistery of the Cathedral of Santa Maria del Fiore in Florence, Italy. It dates from the eleventh century, and its octagonal shape was evidently inspired by a building that had *previously* existed on the site.

The "inverse square law" considered for light intensity in exercises 17 through 19 appears in Newton's law of universal gravitation and in Coulomb's law for electric charges. The authors of *Project Physics* (Holt Rinehart Winston, 1981) remark that "these two great laws arise from completely different sets of observations and apply to completely different kinds of phenomena. Why they should match so exactly is to this day a fascinating puzzle."

Exercises 20 through 24 are based on one of the methods used by astronomers to estimate the number of stars in our galaxy, the Milky Way. It is assumed in this method that the density of stars throughout the galaxy is the same as that in our "neighborhood," 10 stars in a sphere with a radius of 10 light-years. The Milky Way itself is not spherical, of course, but shaped somewhat like a pancake with a bulge in the middle. Current estimates suggest that it may contain as many as 400 billion stars.

A simple method for determining the volume of a regular tetrahedron in terms of the length of its edge is explored in exercises 25 through 31. A more challenging approach would be to treat it as a triangular pyramid so that the length of one of its altitudes has to be considered.

Flipping Cubes.

•1. 4 and 8.

2. A line.

•3. They appear to be perpendicular.

4. 1, 7, and 11.

5. It depends on how the figure (an optical illusion) is seen. (For example, if 3 is seen as the top of a cube and 9 as the bottom of a cube, the planes do not appear to be parallel. If 3 is seen as the top of a cube and 9 is seen as the floor of a cubical space, the planes do appear to be parallel.)

6. That they do not intersect.

Seeing Things.

7. *(Student answer.)* (The most likely interpretation is that this figure is two-dimensional: an irregular hexagon with three concurrent diagonals.)

8. *(Student answer.)* (The most likely interpretation is that this figure is three-dimensional: a polyhedron with six quadrilateral faces.)

Cord of Wood.

9. 128. ($V = lwh = 8 \cdot 4 \cdot 4 = 128$.)

•10. 16 ft. ($128 = 4 \cdot 2 \cdot x$, $8x = 128$, $x = 16$.)

Box Pattern.

11. The hexagons.

12. A right hexagonal prism.

Octagonal Building.

•13. A prism and a pyramid.

14. 1,080,000 ft^3.
($V = Bh = 12{,}000 \times 90 = 1{,}080{,}000$.)

•15. 80,000 ft^3. ($V = \frac{1}{3}Bh = \frac{1}{3}12{,}000 \times 20 = 80{,}000$.)

16. 1,160,000 ft^3. (1,080,000 + 80,000.)

Screen Illumination.

17. They are twice as long.

•18. It is four times as large. ($2^2 = 4$.)

19. The same amount of light is spread over an area four times as large; so the light is one-fourth as bright.

Stars.

20. 500. ($\frac{5{,}000}{10}$.)

•21. 250,000. (500^2.)

22. 125,000,000. (500^3.)

•23. 1,250,000,000. ($10 \times 125{,}000{,}000$.)

24. 125,000,000,000. ($100 \times 1{,}250{,}000{,}000$.)

Divided Cube.

•25. $e\sqrt{2}$.

26. They are equilateral and congruent.

•27. A regular tetrahedron.

28. Five. (The regular tetrahedron and the four pyramids on its faces.)

•29. $\frac{1}{6}e^3$. [$V = \frac{1}{3}Bh = \frac{1}{3}(\frac{1}{2}e^2)e = \frac{1}{6}e^3$.]

•30. $\frac{1}{3}e^3$. [$e^3 - 4(\frac{1}{6}e^3) = \frac{1}{3}e^3$.]

31. One-third.

Set II (pages 686–688)

More from Archimedes' *On the Sphere and Cylinder* is included in exercises 32 through 42. About 150 years after Archimedes' death, the great Roman statesman Cicero found his tombstone. He wrote: "When I was quaestor [administrator] in Sicily I managed to track down Archimedes' grave. . . . I remembered having heard of some simple lines of verse which had been inscribed on his tomb, referring to a sphere and cylinder modelled in stone on top of the grave. . . . Finally, I noticed a little column, just visible above the scrub: it was surmounted by a sphere and cylinder. . . . The verses were still visible, though approximately the second half of each line had been worn away." Although the tomb no longer exists, Archimedes' discoveries of the remarkable connections between the area and the volume of a sphere and its circumscribed cylinder are part of his lasting memorial.

The "twisted ring" of exercises 43 through 45 is equivalent to a Möbius strip. In one of his "Mathematical Games" columns for *Scientific American*, Martin Gardner wrote: "The cross section of a Möbius band actually is a rectangle, very much longer than it is wide. The band itself may be regarded as a four-sided prism, twisted so that one end has rotated 180 degrees before the two ends are joined. Viewed in this way it is a solid ring with two distinct 'faces.'" More concerning Möbius strips as twisted prisms can be found in Gardner's article titled "Twisted Prismatic Rings" in *Fractal Music, Hypercards and More . . .* (W. H. Freeman and Company, 1992).

About Cavalieri's method for calculating the volume of a wine barrel, John Pottage comments in *Geometrical Investigations* (Addison-Wesley, 1983): "Cavalieri's barrel rule may well have been the source of an important sequence of mathematical development. . . . It was probably this formula for the capacity of a cask of wine which suggested to James Gregory . . . the more general formula for approximate integration, often written in the form

$$\int_0^{2h} y\,dx = \frac{1}{3}h(y_0 + 4y_1 + y_2)$$

which, in its turn, led to the outstanding generalizations due to Newton and Cotes, and subsequently to the minor generalization known as 'Simpson's rule.'"

About the Schlegel diagrams of polyhedra, Thomas Banchoff wrote in *Beyond the Third Dimension* (Scientific American Library, 1990): "The central projections of regular polyhedra form beautiful patterns which reveal the structure and symmetries of these objects. If a regular polyhedron has all of its vertices on a sphere in three-space, then we may use central projection from the north pole to obtain an image on the horizontal plane at the south pole. . . . The images of the edges of a polyhedron are said to form a *Schlegel diagram* of the polyhedron, named for Viktor Schlegel, the German mathematician who

invented this type of diagram in 1883." These diagrams can be manipulated into various perspectives in the *Poly* program for computers available from Pedagoguery Software Inc., www.peda.com. The diagrams are used in exercises 57 through 61 to verify that the Euler characteristic of the regular polyhedra is the same number that we proved for prisms and pyramids: 2. (See the commentary for exercises 50 through 53 of Chapter 15, Lesson 3.)

According to Archimedes.

32. A circle whose radius is equal to the radius of the sphere.

33. If the radius of a great circle and the sphere is r, the area of a great circle is πr^2 and the surface area of the sphere is $4\pi r^2$.

34. The volume of the cone is $\frac{1}{3}Bh = \frac{1}{3}\pi r^2 r = \frac{1}{3}\pi r^3$ and the volume of the sphere is $\frac{4}{3}\pi r^3$; so the theorem is true.

35. $\frac{4}{3}\pi x^3$.

•36. $2\pi x^3$. [$V = Bh = \pi x^2(2x) = 2\pi x^3$.]

•37. $\frac{2}{3}$. ($\frac{\frac{4}{3}\pi x^3}{2\pi x^3} = \frac{2}{3}$.)

38. $4\pi x^2$.

39. $4\pi x^2$. [$A = bh = 2\pi x(2x) = 4\pi x^2$.]

•40. $6\pi x^2$. ($4\pi x^2 + 2\pi x^2 = 6\pi x^2$.)

41. The two areas are equal.

42. $\frac{2}{3}$. ($\frac{4\pi x^2}{6\pi x^2} = \frac{2}{3}$.)

Twisted Ring.

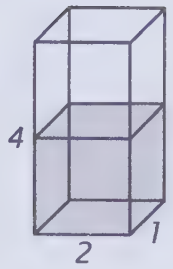
43. Two.

44. 10 cubic units. (10×1^3.)

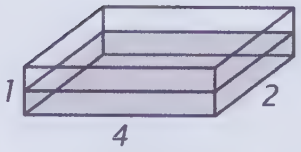
•45. 40 square units. (Each cube has four exposed faces; so each cube has an exposed surface area of $4 \times 1^2 = 4$ square units. $10 \times 4 = 40$.)

SAT Problem.

46.



47.



48. 2 feet and 0.5 foot, respectively. ($1 \times 2 \times d_1 = 4$, $d_1 = 2$; $2 \times 4 \times d_2 = 4$, $d_2 = 0.5$.)

Wine Barrel.

49. The area of its base.

•50. The area of the horizontal cross section halfway up the barrel.

•51. $\pi r^2 h$. [$\frac{1}{3}(2\pi r^2 + \pi r^2)h = \frac{1}{3}(3\pi r^2)h = \pi r^2 h$.]

52. A cylinder.

53. $\frac{1}{3}\pi r^2 h$. [$\frac{1}{3}(2\pi 0^2 + \pi r^2)h = \frac{1}{3}\pi r^2 h$.]

54. A cone. [If $R = 0$, the wine-barrel figure becomes an "hour glass" of two cones with a common apex, bases of radius r, and altitudes of $\frac{1}{2}h$. $2(\frac{1}{3}\pi r^2)(\frac{1}{2}h) = \frac{1}{3}\pi r^2 h$; so the two cones are together equal in volume to a single cone with the same base and altitude h.]

55. $\frac{4}{3}\pi R^3$. [$\frac{1}{3}(2\pi R^2 + \pi 0^2)2R = \frac{4}{3}\pi R^3$.]

56. A sphere.

Euler Characteristic.

57. Yes. (The three small triangles and the triangle containing them.)

•58. Four.

•59. Six.

60.

Polyhedron	*F*	*V*	*E*
Tetrahedron	4	4	6
Cube	6	8	12
Octahedron	8	6	12
Dodecahedron	12	20	30
Icosahedron	20	12	30

61. It is 2 for all of them.

Chapter 16, Lesson 1

Set I (pages 692–693)

The beach ball of exercises 16 through 24 is a "spherical dodecahedron," the result of projecting the edges of a regular dodecahedron on the surface of its circumscribing sphere. Because of the duality of the regular dodecahedron and regular icosahedron, the ball is also a "spherical icosahedron." The sphere is divided into 120 congruent right triangles, of which 6 are contained in each equilateral triangle of the icosahedron and 10 are contained in each regular pentagon of the dodecahedron. More on the mathematics of this topic can be found in chapter 1 of *Spherical Models,* by Magnus J. Wenninger (Cambridge University Press, 1979).

Our Spherical Earth.

•1. At the center of the sphere.

•2. A radius of the sphere.

•3. Two hemispheres.

4. A diameter.

Basic Differences.

5. Both geometries.

6. Euclidean geometry.

•7. Euclidean geometry.

8. Sphere geometry.

9. Euclidean geometry.

•10. Both geometries.

11. Euclidean geometry.

Shortest Routes.

12. Arc 3.

13. It is the largest.

14. The length of the minor arc of the great circle that contains them.

15. B~cause the shortest route between two points is along a great circle.

Beach Ball.

•16. 186°. (36° + 60° + 90°.)

17. No. In these triangles, their sum is more than 90°.

•18. $\angle A = 2(36°) = 72°$, $\angle B = \angle C = 60°$; $\angle A + \angle B + \angle C = 192°$.

19. $\angle A = 36°$, $\angle E = 90°$, $\angle F = 2(36°) = 72°$; $\angle A + \angle E + \angle F = 198°$.

20. $\angle C = 60°$, $\angle D = 36°$, $\angle F = 3(36°) = 108°$; $\angle C + \angle D + \angle F = 204°$.

21. $\angle A = \angle D = \angle F = 2(36°) = 72°$; $\angle A + \angle D + \angle F = 216°$.

•22. Yes.

23. The greater the area of the triangle, the greater the sum of its angles seems to be.

24. A very small triangle.

Set II (pages 693–695)

Exercises 25 through 28 describe some of the content of *Sphaerica*, the major work of Menelaus. The following summary of the first book of *Sphaerica* is from Morris Kline's *Mathematical Thought from Ancient to Modern Times* (Oxford University Press, 1972): "In the first book, on spherical geometry, we find the concept of a spherical triangle, that is, the figure formed by three arcs of great circles on a sphere, each arc being less than a semicircle. The object of the book is to prove theorems for spherical triangles analogous to what Euclid proved for plane triangles. Thus the sum of two sides of a spherical triangle is greater than the third side and the sum of the angles of a triangle is greater than two right angles. Equal sides subtend equal angles. Then Menelaus proves the theorem that has no analogue in plane triangles, namely, that if the angles of one spherical triangle equal respectively the angles of another, the triangles are congruent. He also has other congruence theorems and theorems about isosceles triangles."

The bowling-ball exercises were inspired by some remarks on geodesics by David Henderson in *Experiencing Geometry in Euclidean, Spherical and Hyperbolic Spaces* (Prentice Hall, 2001). In chapter 2, titled "Straightness on Spheres", Henderson describes some activities to help in visualizing great circles as "the only straight lines on the surface of a sphere." He suggests: (1) stretching a rubber band on a slippery sphere—it will stay in place only on a great circle, (2) rolling a ball on a straight chalk line—the chalk will form a great circle on the ball, (3) putting a stiff strip of paper "flat" on a sphere—it will lie properly

only along a great circle, (4) using a small toy car whose wheels are fixed so that it rolls along a straight line on a plane—the car will roll around a great circle on a sphere but not around other curves. Henderson's book is a valuable resource for anyone who teaches geometry.

Spherical Triangles.

•25. Arcs of great circles.

26. The sum of any two sides of a triangle is greater than the third side.

27. No. In Euclidean geometry, the triangles would be similar but not necessarily congruent.

28. The greater the area of the triangle, the greater the sum of its angles.

Seemingly Parallel.

29. They do not intersect.

•30. Both curves cannot correspond to lines. (Lines are great circles and great circles divide the sphere into two equal hemispheres. If one of these curves divides the sphere into two equal hemispheres, the other one clearly does not.)

Euclidean and Sphere Geometries.

•31. One.

•32. No.

33. That they are parallel.

34. No. The corresponding angles formed by lines AB and AC with line BC are equal because they are right angles, but the lines intersect in point A.

35. That they are parallel.

36. No.

37. It is greater than either remote interior angle (and equal to their sum).

38. No. In ΔABC, for instance, the exterior angle at B is equal to the remote interior angle at C.

39. It is 180°.

•40. No. The sum of the angles of ΔABC is greater than 180°.

Wet Paint.

41. It is finite in length, has a definite width and thickness, and so on.

42. It makes sense if the "line" on the ball is a great circle.

43. The radius (or diameter) of the sphere.

Euclidean and Right.

44. It has 90° angles rather than 60° angles. Also, the sum of its angles is 270°, not 180°.

•45. It has three right angles rather than one.

•46. It is four times as long. (The two perpendicular lines through point A divide the line through points B and C into four equal parts.)

47. 4π units. [$c = 2\pi(2) = 4\pi$.]

48. 3π units.

49. 16π square units. [$A = 4\pi r^2 = 4\pi(2)^2 = 16\pi$.]

50. 2π square units. (ΔABC is one of eight congruent triangles into which the lines containing its sides divide the sphere.)

Set III (page 695)

The following material is from the "Games" column of *Omni* magazine, reprinted in *The Next Book of Omni Games,* by Scot Morris (New American Library, 1988):

"Ross Eckler, editor of *Word Ways*, sent us an interesting question based on his geographical research. As you move from place to place during your life, including visits to foreign countries, visualize the corresponding paths of your antipodal points—the point traced exactly through the center of the earth to the opposite side of the globe.

What fraction of the time, would you say, is your antipodal trace on water, and what fraction of the time is it on land?

Most people confronted with this problem allow for the fact that about two thirds of the globe is covered with water. So they answer that their antipodal point would be on water anywhere from 65 to 90 percent of the time. The truth of matter is that for most Americans the answer is likely to be closer to 100 percent.

The reason for this is the extremely nonrandom distribution of land and water. For example, there

are only a few square miles of the continental United States that map onto land: two specks in Southeast Colorado that correspond to St. Paul and Amsterdam Islands, in the Indian Ocean, and a larger speck on the Montana–Canada border that corresponds to Kerguelen Island, also in the Indian Ocean. None of these points are near major highways, however, so most people have never been there. Western Europe and Israel—common tourist destinations—also map into water, with the exception of parts of Spain.

Interestingly, the Hawaiian islands are totally land-antipodal, mapping into Botswana; most of Alaska, however, maps into water, except for a region near Point Barrow that is opposite of Antarctica.

Have you taken a trip to South America? If so, your antipodal point might have dried off. Bogota, Quito, Lima, and much of Chile and Argentina are opposite China, Sumatra, and neighboring lands.

A visit to Australia does no good whatever. The entire continent neatly fits into the Atlantic Ocean, with Perth missing Bermuda by only a few miles.

The chances are very high, therefore, that at no time during your life could you have dug a hole through the center of the earth and struck land on the other side."

Polar Points.

1. The sphere of the earth has been mapped twice so that each pair of antipodal points coincide.
2. For most points of the earth that are on land, the corresponding antipodal points are in an ocean.

Chapter 16, Lesson 2

Set I (pages 698–699)

The western and eastern borders of Wyoming are each about 280 miles in length. The southern and northern borders are about 370 miles and 350 miles, respectively. A nice class exercise might be to use the latitudes of the southern and northern borders (41°N and 45°N) to calculate the ratio of the circumferences of the circles of latitude and show that it is in reasonable agreement with the ratio of these lengths. Students who know their geography may enjoy naming the states that border Wyoming. (Montana on the north, Utah and Colorado on the south, Idaho and Utah on the west, and South Dakota and Nebraska on the east.) Another question of interest would be to name other states whose borders are Saccheri quadrilaterals. (There is only one: Colorado.)

Theorem 85, that the summit angles of a Saccheri quadrilateral are equal, is the first theorem proved by Saccheri in his *Euclid Freed of Every Flaw*. The first English edition of Saccheri's book, titled *Girolamo Saccheri's Euclides Vindicatus* and edited and translated by George Bruce Halsted with the original Latin on the facing pages, was published by Open Court in 1920. It was republished with added notes in 1986 by the Chelsea Publishing Company and can be ordered from the Mathematical Association of America on the Internet (www.maa.org).

Wyoming.

- •1. Its legs.
- •2. The base angles.
- •3. The summit.
4. The summit angles.
- •5. WY and MO.
- •6. $\angle Y$ and $\angle O$.

Theorem 87.

- •7. Two points determine a line.
8. The legs of a Saccheri quadrilateral are equal.
- •9. All right angles are equal.
10. SAS.
11. Corresponding parts of congruent triangles are equal.
12. SSS.
13. Corresponding parts of congruent triangles are equal.

Theorem 89.

- •14. The Ruler Postulate.
15. Two points determine a line.
- •16. It is a birectangular quadrilateral whose legs are equal.
- •17. The summit angles of a Saccheri quadrilateral are equal.

18. The "whole greater than part" theorem.

19. Substitution.

•20. An exterior angle of a triangle is greater than either remote interior angle.

21. The transitive property.

Theorem 90.

22. The "three possibilities" property.

•23. A Saccheri quadrilateral.

24. The summit angles of a Saccheri quadrilateral are equal.

•25. The fact that $\angle D > \angle C$.

•26. If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

27. The fact that $\angle D > \angle C$.

28. $CB > DA$.

Set II (pages 699–700)

Of Nasir Eddin (1201–1274), Howard Eves wrote in *An Introduction to the History of Mathematics* (Saunders, 1990): "He wrote the first work on plane and spherical trigonometry considered independently of astronomy. Saccheri (1667–1733) started his work on non-Euclidean geometry through a knowledge of Nasir Eddin's writings on Euclid's parallel postulate. His was the only attempt to prove this postulate in the period from the ancient Greeks up to the Renaissance." An outline of Nasir Eddin's approach to the postulate is included on pages 208–210 of the first volume of *The Thirteen Books of Euclid's Elements,* with introduction and commentary by Sir Thomas L. Heath (Dover, 1956).

Theorem 86 is the second theorem proved by Saccheri in his *Euclid Freed of Every Flaw.*

A curious fact about the flag of Poland: if it is flown upside down, it becomes the flag of both Monaco and Indonesia!

Nasir Eddin.

•29. CD.

30. ABDC is a birectangular quadrilateral in which $\angle 2 > \angle 3$ ($\angle 2$ is obtuse and $\angle 3$ is acute); so $CD > AB$. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

Theorem 88.

•31. $DM = CM$ because $\Delta ADM \cong \Delta BCM$ (SAS).

32. The midpoint of a line segment divides it into two equal segments.

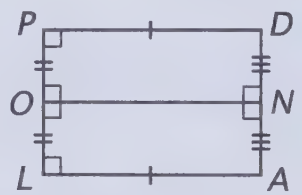
•33. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

34. $\angle D = \angle C$ because the summit angles of a Saccheri quadrilateral are equal; so $\Delta ADN \cong \Delta BCN$ (SAS); $AN = BN$ because corresponding parts of congruent triangles are equal.

35. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

Polish Flag.

36.



37. A Saccheri quadrilateral.

•38. DA.

39. The summit angles of a Saccheri quadrilateral are equal.

•40. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

41. Birectangular.

Alter Ego.

•42. In a plane, two lines perpendicular to a third line are parallel.

43. A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.

•44. The opposite angles of a parallelogram are equal.

45. Because a quadrilateral each of whose angles is a right angle is a rectangle.

46. It is equal to and parallel to the base.

Irregular Lots.

47. AF > CD. ABEF is a birectangular quadrilateral in which $\angle 1 > \angle F$, and so AF > BE; BCDE is a birectangular quadrilateral in which $\angle D > \angle 2$, and so BE > CD. The conclusion that AF > CD follows from the transitive property.

48. $\angle D > \angle F$. ACDF is a birectangular quadrilateral in which AF > CD. If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

Set III (page 701)

Accompanying the photograph by Alex S. MacLean of the jog in the road in Castleton, North Dakota, from *Taking Measures Across the American Landscape* (Yale University Press, 1996) is this caption by Hildegard Binder Johnson: "It did not take long for legislators to understand that a township could not be exactly six miles on each side if the north-south lines were to follow the lines of longitude, which converged, or narrowed, to the north. The grid was, therefore, corrected every four townships to maintain equal allocations of land. These slippages, measured every twenty-four miles (or every thirty minutes if one was driving), poignantly register the spherical and magnetic condition of our planet, doglegging to counter the diminishing distances as one moves north."

Offsets.

1. No. The sides of the townships lie along map directions; so the lines containing their legs all intersect at the North (and South) Pole.

2. They are shorter than the bases. It is given that AB = DF = 6 miles; DE < DF, and so DE < AB.

3. They become shorter and shorter. (If a township had its summit very close to the North Pole, it would look very much like an isosceles triangle.)

Chapter 16, Lesson 3

Set I (page 704)

Lobachevskian Theorem 1, that the summit of a Saccheri quadrilateral is longer than its base, is part of the third theorem proved by Saccheri in his *Euclid Freed of Every Flaw.* (Saccheri covers all three cases, stating that the summit is equal to, less than, or greater than the base, depending on whether the summit angles are right, obtuse, or acute.)

Lobachevskian Theorem 1.

•1. The summit angles of a Saccheri quadrilateral are acute in Lobachevskian geometry.

2. An acute angle is less than 90°.

•3. Substitution.

•4. A line segment has exactly one midpoint.

5. Two points determine a line.

•6. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

•7. A quadrilateral that has two sides perpendicular to a third side is birectangular.

8. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

•9. The Addition Theorem of Inequality.

10. Substitution.

Lobachevskian Theorem 2.

11. $\Delta ADM \cong \Delta CPM$ by SAS (AM = MC, DM = PM, and $\angle AMD = \angle CMP$). $\Delta BEN \cong \Delta CPN$ for the same reason.

12. Corresponding parts of congruent triangles are equal.

13. Substitution.

•14. A birectangular quadrilateral whose legs are equal is a Saccheri quadrilateral.

•15. The summit of a Saccheri quadrilateral is longer than its base in Lobachevskian geometry.

16. Substitution.

17. Substitution.

18. Multiplication (or division).

Set II (pages 705–706)

Although our study of non-Euclidean geometry is limited to two dimensions, exercises 37 through 46 illustrate some of its strange consequences in three dimensions. The simple pattern of equilateral triangles that can be folded to form a regular tetrahedron in Euclidean geometry results in an irregular tetrahedron in Lobachevskian geometry. The pattern of squares that can be folded to form an open box in the shape of a cube in Euclidean geometry does not exist in Lobachevskian geometry. The Set III exercise of Lesson 2 illustrated the difficulty of trying to subdivide the surface of a sphere (the Riemannian plane) into squares and the same is true of the Lobachevskian plane. The existence of squares and cubes depends on the Euclidean Parallel Postulate; so areas and volumes in non-Euclidean geometry are not measured in square or cubic units. The existence of a square in a geometry forces that geometry to be Euclidean because the square can tile the plane, and subdividing the resulting squares produces a Cartesian grid that forces a Pythagorean pattern on the entire plane.

Lobachevsky and his Geometry.

19. *(Student answer.)* (Below the portrait of Lobachevsky is his name. The last word is "geometry.")

•20. Through a point not on a line, there is more than one line parallel to the line.

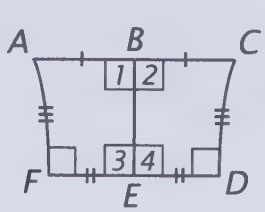
21. The summit angles of a Saccheri quadrilateral are acute.

22. The summit of a Saccheri quadrilateral is longer than its base.

23. A midsegment of a triangle is less than half as long as the third side.

Lobachevskian Quadrilateral.

24.



25. A Saccheri quadrilateral.

26. Acute. The summit angles of a Saccheri quadrilateral are acute in Lobachevskian geometry.

27. Right. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

•28. Birectangular.

29. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

30. In Lobachevskian geometry, the line segment that connects the midpoints of the summit and base of a Saccheri quadrilateral is shorter than either leg.

Riemannian Quadrilateral.

31. Obtuse.

32. *(Student answer.)* (Yes.)

33. $DC < AB$.

•34. In Riemannian geometry, the summit of a Saccheri quadrilateral is shorter than its base.

Riemannian Triangle.

35. In Riemannian geometry, a midsegment of a triangle is more than half as long as the third side.

36. No. There are no parallel lines in Riemannian geometry.

Triangular Pyramid.

•37. They are all half as long.

38. They are equilateral and congruent.

39. The edges of the base are less than half as long as the sides of ΔACE.

40. Its lateral faces are congruent isosceles triangles in which the base is shorter than the legs. Its base is an equilateral triangle.

Open Box.

41. It would have the shape of a cube.

•42. They are congruent squares.

43. Saccheri quadrilaterals.

44. They would be longer than the sides of CFIL. (In Lobachevskian geometry, the summit of a Saccheri quadrilateral is longer than its base.)

45. A square.

46. A square is also a Saccheri quadrilateral but, in Lobachevskian geometry, the summit of a Saccheri quadrilateral is longer than its base. (Also, in Lobachevskian geometry, the summit angles of a Saccheri quadrilateral are acute, not right.)

Set III (page 706)

In the section on non-Euclidean geometries in *Mathematics—The Science of Patterns* (Scientific American Library, 1994), Keith Devlin observes that, with adjustments such as those considered in Lesson 1, the surface of a sphere is a good model for representing Riemannian geometry. He then asks, what surface would be a good model for representing Lobachevskian geometry? Devlin writes: "The answer turns out to involve a pattern that is familiar to every parent. Watch a child walking along, pulling a toy attached to a string. If the child makes an abrupt left turn, the toy will trail behind, not making a sharp corner, but curving around until it is almost behind the child once again. This curve is called a tractrix." A pseudosphere is a surface produced by rotating a tractrix about its asymptote; in contrast with a sphere, which has a constant positive curvature, a pseudosphere has a constant negative curvature. Remarkably, a pseudosphere matched with the appropriate sphere has the same volume and surface area. For more on this subject, see *A Book of Curves*, by E. H. Lockwood (Cambridge University Press, 1963).

Directions for crocheting a pseudosphere are given by David Henderson in *Experiencing Geometry in Euclidean, Spherical, and Hyperbolic Spaces* (Prentice Hall, 2001).

As with geometry on the surface of a sphere, lines on a pseudosphere are represented by geodesics, the shortest paths on the surface between pairs of points. These paths are easily pictured on a sphere because they are along arcs of great circles. Unfortunately, for the pseudosphere, the only simple geodesics are the meridians, illustrated by lines AB and DC in the figure illustrating the Saccheri quadrilateral, and the circles orthogonal to them, of which AD and BC in the figure are arcs. H. S. M. Coxeter in *Introduction to Geometry* (Wiley, 1969) points out that, because of this shortcoming, the pseudosphere is "utterly useless as a means for drawing significant hyperbolic figures."

Pseudosphere.

From the figure, it appears that AD < BC; so BC is the summit of the Saccheri quadrilateral. It follows that AB = DC because they are the legs. Also, $\angle BAD$ and $\angle ADC$ are right angles and $\angle ABC$ and $\angle DCB$ are acute angles because they are, respectively, the base angles and the summit angles of the Saccheri quadrilateral. Lines AB and DC are parallel because, in a plane, two lines perpendicular to a third line are parallel (this theorem comes before the Parallel Postulate in Euclidean geometry).

Chapter 16, Lesson 4

Set I (pages 709–710)

Lobachevskian Theorem 3, that the sum of the angles of a triangle is less than 180°, is part of Proposition 15 proved by Saccheri in his *Euclid Freed of Every Flaw*. Saccheri states it thus: "By any triangle ABC, of which the three angles are equal to, or greater, or less than two right angles, is established respectively the hypothesis of right angle, or obtuse angle, or acute angle."

Saccheri's Proposition 16 includes the corollary about the sum of the angles of a quadrilateral being less than 360°.

English mathematician John Wallis (1616–1703) tried to prove Euclid's Fifth Postulate but assumed in doing so that, for every triangle, there exists a similar triangle of any arbitrary size. Saccheri was aware of Wallis's attempt and discusses it in his book. The proof of Lobachevskian Theorem 4 outlined in exercises 14 through 20 is, in fact, the one used by Saccheri in establishing that Wallis's assumption and Euclid's Fifth Postulate are logically equivalent.

Lobachevskian Theorem 3.

1. $\Delta ADM \cong \Delta CPM$ by SAS (AM = MC, DM = PM, and $\angle AMD = \angle CMP$). $\Delta BEN \cong \Delta CPN$ for the same reason.

•2. $\angle D$ and $\angle E$.

3. AD and BE.

•4. A Saccheri quadrilateral.

•5. The summit angles of a Saccheri quadrilateral are acute in Lobachevskian geometry.

6. $\angle DAB + \angle EBA < 180°$.

7. $\angle 1 = \angle 5$ and $\angle 4 = \angle 6$.

•8. Substitution.

9. Substitution.

Corollary.

•10. The sum of the angles of a triangle is less than 180° in Lobachevskian geometry.

11. Addition.

12. Betweenness of Rays Theorem.

13. Substitution.

Lobachevskian Theorem 4.

•14. Corresponding angles of similar triangles are equal.

15. SAS.

16. Substitution. ($\angle D = \angle A$ and $\angle 1 = \angle A$; $\angle F = \angle C$ and $\angle 2 = \angle C$.)

•17. The angles in a linear pair are supplementary.

18. Addition.

19. Substitution.

20. The sum of the angles of a quadrilateral is less than 360° in Lobachevskian geometry.

Set II (pages 710–712)

Exercises 31 and 32 lead to the theorem that, in Lobachevskian geometry, an exterior angle of a triangle is greater than the sum of the remote interior angles. Exercises 33 and 34 reveal that, in Lobachevskian geometry, an angle inscribed in a semicircle is acute. Both theorems are considered by Saccheri, the first as a corollary to his Proposition 15 and the second as his Proposition 18.

The following excerpt is from the preface to the second English edition of *Euclid Freed from Every Flaw* (Chelsea, 1986): "In this book Girolamo Saccheri set forth in 1733, for the first time ever, what amounts to the axiom systems of non-Euclidean geometry. It is in this book that, for the first time in history, theorem after theorem of [Lobachevskian] non-Euclidean geometry is stated and proved. . . . Why—one cannot help asking—did Saccheri not evaluate correctly what he had achieved, why did he not claim credit for the discovery of non-Euclidean geometry?. . . In his era, it would seem, the existence of a valid geometry alternative to Euclid's was, quite literally, unthinkable. Not impossible, not wrong, but *unthinkable*."

Exercises 35 through 50 show an example of an attempt to "double a triangle" in Lobachevskian geometry. For the right triangle considered, when the hypotenuse is doubled, one leg of the resulting triangle is less than doubled, whereas the other leg is more than doubled!

Lobachevskian Triangles.

•21. No. The sums of the angles of the two triangles may not be equal.

22. Their sum is less than 90° (or, they are not complementary).

23. Each angle is less than 60°.

24. The longer the sides, the smaller the angles.

Triangle on a Sphere.

25. $\angle B = \angle C$.

•26. $\angle A + \angle B + \angle C > 180°$.

•27. Riemannian geometry.

Triangle on a Pseudosphere.

28. $\angle B = \angle C$.

29. $\angle A + \angle B + \angle C < 180°$.

30. Lobachevskian geometry.

Exterior Angles.

•31. $\angle 1 = \angle A + \angle B$.

32. The conclusion about $\angle 1$ is not true in Lobachevskian geometry. $\angle 1 + \angle 2 = 180°$ (the angles in a linear pair are supplementary) and $\angle A + \angle B + \angle 2 < 180°$ (the sum of the angles of a triangle is less than 180° in Lobachevskian geometry); so $\angle A + \angle B + \angle 2 < \angle 1 + \angle 2$ (substitution). It follows that $\angle A + \angle B < \angle 1$ (subtraction); so, in Lobachevskian geometry, $\angle 1 > \angle A + \angle B$.

Angle Inscribed in a Semicircle.

33. It is a right angle.

34. The conclusion about $\angle ABC$ is not true in Lobachevskian geometry. $\angle 1 + \angle ABC + \angle 4 < 180°$ (the sum of the angles of a triangle is less than 180° in Lobachevskian geometry). $OA = OB = OC$ (all radii of a circle are equal); so $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (if two sides of a triangle are equal, the angles opposite them are equal). Also, $\angle ABC = \angle 2 + \angle 3$, and so $\angle 2 + (\angle 2 + \angle 3) + \angle 3 < 180°$ (substitution); so $2(\angle 2 + \angle 3) < 180°$ and $\angle 2 + \angle 3 < 90°$ (division). So $\angle ABC < 90°$. An angle inscribed in a semicircle in Lobachevskian geometry is acute.

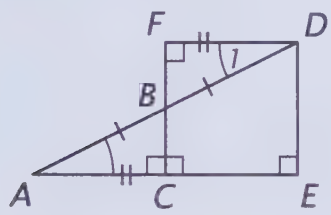
Magnification and Distortion.

•35. $\Delta ADE \sim \Delta ABC$ (AA).

36. $\angle D = \angle ABC$.

37. Each side of ΔADE is twice as long as the corresponding side of ΔABC.

38.



39. $\angle DBF \cong \Delta ABC$. $\angle 1 = \angle A$, $BD = AB$, and $\angle FBD = \angle CBA$; so they are congruent by ASA.

40. $BF = BC$.

•41. $\angle F$ is a right angle. ($\angle F = \angle ACB$.)

•42. CFDE is a birectangular quadrilateral (in two different ways).

•43. $\angle FDE$ is acute. (The sum of the angles of a quadrilateral is less than 360° in Lobachevskian geometry. The other three angles of CFDE are right angles.)

44. $DE > FC$. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

45. $FC = 2BC$. ($FB = BC$ because $\Delta DBF \cong \Delta ABC$.)

•46. $DE > 2BC$. (Substitution.)

47. $CE < FD$. (If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.)

48. $CE < AC$. ($FD = AC$; so $CE < AC$ by substitution.)

49. $AE < 2AC$. [$CE < AC$, and so $AC + CE < AC + AC$ (addition); so $AE < 2AC$.]

50. $AD = 2AB$, $DE > 2BC$ (exercise 46), and $AE < 2AC$ (exercise 49).

Set III (page 712)

Marvin Jay Greenberg, in a section of *Euclidean and Non-Euclidean Geometries—Development and History* (W. H. Freeman and Company, 1980) titled "What Is the Geometry of Physical Space?" wrote: "Certainly, engineering and architecture are evidence that Euclidean geometry is extremely useful for ordinary measurement of distances that are not too large. However, the representational accuracy of Euclidean geometry is less certain when we deal with larger distances. For example, let us interpret a 'line' physically as the path traveled by a light ray. We could then consider three widely separated light sources forming a physical triangle. We would want to measure the angles of this physical triangle in order to verify whether the sum is 180° or not (such an experiment would presumably settle the question of whether space is Euclidean or [non-Euclidean].

Gauss allegedly performed this experiment, using three mountain tops as the vertices of his triangle. The results were inconclusive. Why? Because any physical experiment involves experimental error. Our instruments are never completely accurate. Suppose the sum did turn out to be 180°. If the error in our measurement were at most 1/100 of a degree, we could conclude only that the sum was between 179.99° and 180.01°. We could never be sure that it actually was 180°.

Suppose, on the other hand, that measurement gave us a sum of 179°. Although we would conclude only that the sum was between 178.99° and 179.01°, we would be certain that the sum was less than 180°. In other words, the only conclusive result of such an experiment would be that space is [Lobachevskian]! . . .

To repeat the point: because of experimental error, a physical experiment can never prove conclusively that space is Euclidean—it can prove only that space is non-Euclidean."

Furthermore, the fact that the same quasar can be observed along *two different directions* indicates that light paths violate our assumptions about straight lines in Euclidean space.

Astronomical Triangle.

1. No. The difference could be due to errors in measurement.
2. It would be easier to prove that physical space is non-Euclidean. It is impossible to measure anything exactly; so it is impossible to know that the sum of the three angle measures is *exactly* 180°.

Chapter 16, Review

Set I (pages 714–715)

The beach ball of exercises 12 through 24 is another "spherical polyhedron," the result of projecting the edges of a regular polyhedron on the surface of its circumscribing sphere. Because of the duality of the cube and the regular octahedron, it is both a "spherical cube" and a "spherical octahedron." The sphere is divided into 48 congruent right triangles, of which 8 are contained in each square of the cube and 6 are contained in each equilateral triangle of the octahedron.

Sphere Geometry.

•1. A great circle.

•2. False.

3. False.

•4. True (because antipodal points are considered to be identical).

5. False.

6. False.

Saccheri Quadrilateral.

•7. AD.

8. AD > BC (or, AD is the longest side of ABCD).

9. ABCD cannot be a rhombus, because it is not equilateral.

10. Acute.

11. ABCD cannot be a rectangle, because ∠A and ∠D are not right angles. They are acute.

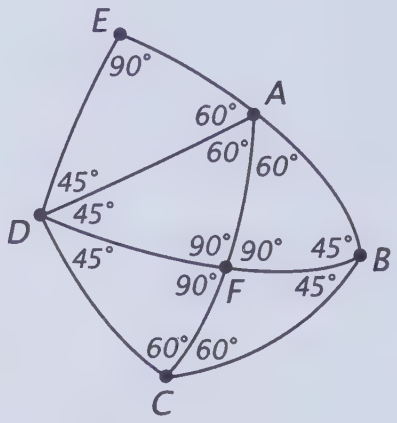
Beach Ball.

•12. 360°.

13. 360°.

14. 360°.

15.



•16. ∠ABC = 90°; ∠BCD = 120°; ∠CDA = 90°; ∠DAB = 120°.

•17. The quadrilateral is a parallelogram.

18. No. There are no parallelograms in sphere geometry, because there are no parallel lines.

•19. It is a rhombus.

20. They are perpendicular to each other and they bisect each other. (It is given that all of the triangles are congruent.)

21. Yes.

22. ∠E = 90°; ∠EAF = 120°; ∠AFD = 90°; ∠FDE = 90°.

•23. No.

24. Yes.

Set II (pages 715–717)

The ladder-rung exercises demonstrate that, in Lobachevskian geometry, two parallel lines are not everywhere equidistant. The earliest known attempt to prove Euclid's Fifth Postulate was made by Posidonius in the first century B.C.. Posidonius did it by changing Euclid's definition of parallel lines to say the two lines are parallel if

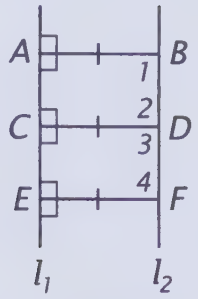
the distance between them is always the same. Concerning this idea, Saccheri wrote in *Euclid Freed of Every Flaw*: "Thence now I may proceed to explain, why in the Preface to the Reader I have said: *not without a great sin against rigid logic two equidistant straight lines have been assumed by some as given.* Where I should point out that none of those is carped at, whom I have mentioned even indirectly in this book of mine, because they are truly great geometers, and verily free from this sin. But I say: *great sin against rigid logic*: for what else is it to assume as given *two equidistant straight lines*: unless either to assume; that every line equidistant in the same plane from a certain supposed straight line is itself also a straight line; or at least to suppose, that some one thus equidistant may be a straight line, as if therefore it were allowable to make assumption, whether by hypothesis, or by postulate, of any such distance of one from another? But it is certain neither of these can be made traffic of as if *per se* known."

It is probably no surprise that the Pythagorean Theorem does not hold in the non-Euclidean geometries, because geometric squares do not even exist in them. In terms of the lengths of the legs a and b and hypotenuse c of a right triangle, in Lobachevskian geometry, $c^2 > a^2 + b^2$, and in Riemannian geometry, $c^2 < a^2 + b^2$.

Maurits Escher created four Circle Limit prints based on Poincaré's model of Lobachevskian geometry, one of which, the picture of angels and devils, appears in Chapter 12, Lesson 5. The geometer H. S. M. Coxeter had sent Escher a drawing of the Poincaré model in 1958 and Escher replied: "Since a long time I am interested in patterns with 'motives' getting smaller and smaller till they reach the limit of infinite smallness." Coxeter tells of this in *The Mathematical Gardner*, edited by David A. Klarner (Prindle, Weber & Schmidt, 1981), and describes the model: "the 'straight lines' of the [Lobachevskian] plane appear as arcs of circles orthogonal to a boundary circle drawn in the Euclidean plane. Angles are represented faithfully but distances are distorted, with the points of [the boundary circle] being infinitely far away. Although the [fish] get 'smaller gradually from the center towards the outside circle-limit', we enter the spirit of [Lobachevskian] geometry when we stretch our imagination in order to pretend that these [fish] are all . . . congruent."

Lobachevsky's Ladder.

25.



•26. Saccheri quadrilaterals.

•27. Acute.

28. $\angle 2 + \angle 3 < 180°$.

29. The angles in a linear pair are supplementary.

30. It proves that they cannot all be of equal lengths.

Pythagoras Meets Lobachevsky.

31. The Pythagorean Theorem.

•32. Substitution.

33. $(\frac{1}{2}CA)^2 + (\frac{1}{2}CB)^2 = MN^2$, and so

$\frac{1}{4}CA^2 + \frac{1}{4}CB^2 = MN^2$; so

$\frac{1}{4}(CA^2 + CB^2) = MN^2$. If $CA^2 + CB^2 = AB^2$,

$\frac{1}{4}AB^2 = MN^2$ by substitution; so

$\frac{1}{2}AB = MN$.

•34. A midsegment of a triangle is less than half as long as the third side.

35. The Pythagorean Theorem is not true in Lobachevskian geometry.

Lobachevsky Meets Escher.

36. Two circles are orthogonal iff they intersect in two points and their tangents at the points of intersection are perpendicular to each other.

•37. Two points determine a line.

38. Through a point not on a line, there is more than one parallel to the line.

39. The summit angles of a Saccheri quadrilateral are acute (or, the summit of a Saccheri quadrilateral is longer than its base).

•40. The sum of the angles of a triangle is less than 180°.

41. A midsegment of a triangle is less than half as long as the third side.

The Geometry of the Universe.

42. It is finite. The volume of the space of the universe would be comparable to the area of the surface of a sphere, which is finite.

43. The astronaut would eventually return to the point from which he or she started. A straight line in space would be comparable to a line on the surface of a sphere, which is a great circle.

Final Review

Set I (pages 718–722)

German Terms.

1. Radius.
2. A diameter is a chord of a circle that contains its center.
3. A secant is a line that intersects a circle in two points; a tangent is a line that intersects a circle in exactly one point.

Regular Polygons.

4. A regular hexagon and a regular dodecagon.
5. They are convex, equilateral, equiangular, and cyclic.
6. 60° and 120°.
7. (Isosceles) trapezoids.
8. 720°. [Each angle has a measure of $2(60°) = 120°$; $6(120°) = 720°$.]
9. 1,800°. [Each angle has a measure of $60° + 90° = 150°$; $12(150°) = 1{,}800°$.]

What Follows?

10. Its base and altitude.
11. Is perpendicular to the chord.
12. Lies in the plane.
13. The triangle is a right triangle.
14. Have equal areas.
15. Are equal.
16. Tangent to the circle.
17. A line.
18. $e\sqrt{3}$.
19. In the same ratio.
20. The hypotenuse and its projection on the hypotenuse.
21. The area of its base and its altitude.
22. The ratio of any pair of corresponding sides.
23. Is –1.
24. Half the positive difference of its intercepted arcs.

Pythagorean Squares.

25. $w + x + y$.

26. The two areas are equal. Because the squares are from the sides of a right triangle, $(v + w) + (y + z) = (w + x + y)$; so $v + z = x$ by subtraction.

The Value of Pi.

27. ΔABC is a right triangle. $\angle ACB$ is a right angle because it is inscribed in semicircle.

28. $\frac{AD}{CD} = \frac{CD}{DB}$. (The altitude to the hypotenuse of a right triangle is the geometric mean between the segments into which it divides the hypotenuse.)

29. $CD = \sqrt{10} \approx 3.16$. ($\frac{2}{CD} = \frac{CD}{5}$, $CD^2 = 10$, $CD = \sqrt{10}$.)

30. No. $\sqrt{10} > \pi$ because $\pi \approx 3.14$.

Euclid in Color.

31. The opposite angles of a cyclic quadrilateral are supplementary.

32. Inscribed angles that intercept the same arc are equal.

33. The sum of the angles of a triangle is 180°.

34. Two right angles, or 180°.

Largest Package.

35. 15,000 in^3. ($10 \times 20 \times 75$.)

36. No. The length of its longest side is 75 in and the distance around its thickest part is $2(10 + 20) = 60$ in; $75 + 60 = 135 > 130$ in.

37. 26 in. ($e + 4e = 130$, $5e = 130$, $e = 26$.)

38. 17,576 in^3. (26^3.)

Four Centers.

39. The orthocenter.

40. The incenter.

41. The centroid.

42. The circumcenter.

43. B (the incenter).

44. D (the circumcenter).

45. C (the centroid).

46. B (the incenter).

Formulas.

47. To find the circumference of a circle (using its diameter).

48. To find the length of a diagonal of a rectangular solid (using its length, width, and height).

49. To find the area of a trapezoid (using its altitude and two bases).

50. To find the altitude of an equilateral triangle (using its side).

Kindergarten Toy.

51. A, a cube; B, a cylinder; and C, a sphere.

52. $V = e^3$, $A = 6e^2$.

53. $V = \pi r^2 h$, $A = 2\pi rh + 2\pi r^2$ [or $A = 2\pi r(h + r)$].

54. $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$.

Steep Roof.

55. $m\text{OA} = \frac{21-0}{12-0} = \frac{21}{12} = \frac{7}{4} = 1.75$;

$m\text{AB} = \frac{21-0}{12-24} = \frac{21}{-12} = -\frac{7}{4} = -1.75$.

56. 60°. ($\tan \angle AOB = \frac{21}{12} = 1.75$; $\angle AOB \approx 60.3°$.)

57. 24 ft. ($OA = AB = \sqrt{12^2 + 21^2} = \sqrt{585} \approx 24.2$.)

58. No. (It is not equilateral, because OB is slightly longer than the other two sides or, equivalently, because $\angle AOB$ is not 60°.)

Greek Crosses.

59. $12x$.

60. $5x^2$.

61. $10x^2$.

62. $x\sqrt{10}$.

63. $4x\sqrt{10}$.

Baseball in Orbit.

64. About 138 mi. [$c = 2\pi r = 2\pi(22) = 44\pi \approx 138$.]

65. About 83 minutes. ($\frac{138 \text{ mi}}{100 \text{ mi / hour}}$ = 1.38 hours ≈ 82.8 minutes.)

Two Quadrilaterals.

66. $2w + 2x + 2y + 2z = 360°$ (the sum of the angles of a quadrilateral is 360°); so $2(w + x + y + z) = 360°$ and $w + x + y + z = 180°$ (division).

67. In ΔAFD, $\angle F + w + z = 180°$ and, in ΔBHC, $\angle H + x + y = 180°$ (the sum of the angles of a triangle is 180°); so $\angle F = 180° - w - z$ and $\angle H = 180° - x - y$ (subtraction).

68. $\angle F + \angle H = (180° - w - z) + (180° - x - y) = 360° - w - x - y - z = 360° - (w + x + y + z)$ (addition); so $\angle F + \angle H = 360° - 180° = 180°$ (substitution).

69. $\angle F$ and $\angle H$ are supplementary (if the sum of two angles is 180°, they are supplementary); so quadrilateral EFGH is cyclic (a quadrilateral is cyclic if a pair of its opposite angles are supplementary).

Long Shadows.

70. Approximately 5,500 ft.
($\tan 12.8° = \frac{1{,}250}{x}$, $x = \frac{1{,}250}{\tan 12.8°} \approx 5{,}500$.)

71. Approximately 26 ft. ($\tan 12.8° = \frac{6}{y}$, $y = \frac{6}{\tan 12.8°} \approx 26.4$; or $\frac{1{,}250}{5{,}500} = \frac{6}{y}$, etc.)

Tree Geometry.

72. k.

73. k^2.

74. k^3.

75. k^3.

76. Those of the larger tree would be $10^2 = 100$ times the area of those of the smaller tree.

77. Those of the larger tree would be $10^3 = 1{,}000$ times the weight of those of the smaller tree.

78. The larger tree.

SAT Problem.

79. $3\pi s$ units. [$2(\frac{3}{4}2\pi s$.]

80. $\frac{3}{2}\pi s^2 + s^2$ [or $(\frac{3}{2}\pi + 1)s^2$] square units.
[$2(\frac{3}{4}\pi s^2) + s^2 = \frac{3}{2}\pi s^2 + s^2$.]

Sight Line.

81. About 23°. ($\tan \angle 1 = \frac{90}{210}$, $\angle 1 \approx 23°$.)

82. About 18°. ($\tan \angle 2 = \frac{70}{210}$, $\angle 2 \approx 18°$.)

Set II (pages 723–727)

Dog Crates.

83. 19 in. ($V = lwh$, $w = \frac{V}{lh} = \frac{12{,}540}{30 \times 22} = 19$.)

84. No. They are not similar, because their corresponding dimensions are not proportional: $\frac{30}{36} \neq \frac{22}{26}$.
($\frac{30}{36} \approx 0.83$ and $\frac{22}{26} \approx 0.85$.)

85. 51,364 in^3.
[$V = (\frac{48}{30})^3(12{,}540) = 51{,}363.84$.]

Tangent Circles.

86. $\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1$ (or any equivalent equation) by Ceva's Theorem.

87. Yes. If the radius of the three circles is r, $\frac{r}{r} \cdot \frac{r}{r} \cdot \frac{r}{r} = 1$.

88. Yes. Letting the radii of the three circles be r_1, r_2, and r_3, we have $\frac{r_1}{r_2} \cdot \frac{r_2}{r_3} \cdot \frac{r_3}{r_1} = 1$.

Squares on the Legs.

89. The altitude to the hypotenuse of a right triangle forms two triangles similar to it and to each other (or AA).

90. The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides.

91. They are equal because $\alpha ABEF = AB^2$ and $\alpha BCHG = BC^2$.

92. Yes. DAFEB ~ DBGHC. (Their corresponding angles are equal and their corresponding sides are proportional.)

Formula Confusion.

93. *(Student answer.)* [Because $c = 2\pi r$, the formula $A = \frac{1}{2}rc$ is equivalent to $A = r(2\pi r) = \pi r^2$, which gives the area of the circle correctly.]

94. No. Because $A = 4\pi r^2$, the formula $V = \frac{1}{2}rA$ is equivalent to $V = \frac{1}{2}r(4\pi r^2) = 2\pi r^3$, but the volume of a sphere is $\frac{4}{3}\pi r^3$.

95. Yes. Because $A = 2\pi rh$, the formula $V = \frac{1}{2}rA$ is equivalent to $V = \frac{1}{2}r(2\pi rh) = \pi r^2 h$, which gives the volume of the cylinder correctly.

Regular 17-gon.

96. 3.12. $(17 \sin\frac{180}{17}.)$

97. 3.07. $(17 \sin\frac{180}{17} \cos\frac{180}{17}.)$

98. 62 cm. $[p = 2Nr \approx 2(3.12)(10) \approx 62.]$

99. 307 cm^2. $[A = Mr^2 \approx 3.07(100) = 307.]$

100. The corresponding measurements of the circle would be larger.

101. 63 cm and 314 cm^2.
$[c = 2\pi r = 2\pi(10) = 20\pi \approx 63.$
$A = \pi r^2 = \pi(10)^2 = 100\pi \approx 314.]$

Packing Circles.

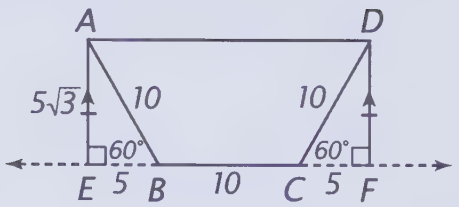
102. $\sqrt{3}r^2$. $[\frac{\sqrt{3}}{4}(2r)^2 = \sqrt{3}r^2.]$

103. $\frac{1}{2}\pi r^2$. $[3(\frac{1}{6}\pi r^2) = \frac{1}{2}\pi r^2.]$

104. About 91%. $(\frac{\frac{1}{2}\pi r^2}{\sqrt{3}r^2} = \frac{\pi}{2\sqrt{3}} \approx 0.91.)$

Rain Gutter.

105.



106. ΔAEB and ΔDFC are 30°-60° right triangles; $\Delta AEB \cong \Delta DFC$ (AAS).

107. In a plane, two lines perpendicular to a third line are parallel.

108. Corresponding parts of congruent triangles are equal.

109. ADFE is a parallelogram because AE || DF and AE = DF (a quadrilateral is a parallelogram if two opposite sides are both parallel and equal); so AD || EF (the opposite sides of a parallelogram are parallel).

110. $5\sqrt{3} \approx 8.66$ cm. (ΔABE is a 30°-60° right triangle with hypotenuse AB = 10; so EB = 5 and $AE = 5\sqrt{3}$.)

111. 130 cm^2. (ABCD is a trapezoid with BC = 10, AD = 20, and $AE = 5\sqrt{3}$; so $\alpha ABCD = \frac{1}{2}5\sqrt{3}(10 + 20) = 75\sqrt{3} \approx 130.$)

Incircle.

112. The angle bisectors of a triangle are concurrent.

113. The incenter.

114. The tangent segments to a circle from an external point are equal.

115. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

116. In a plane, two lines perpendicular to a third line are parallel to each other.

117. ODCE is a parallelogram because both pairs of its opposite sides are parallel. OD = EC and OE = DC because the opposite sides of a parallelogram are equal.

118. AB = 29. (ΔABC is a right triangle, and so $AB^2 = CA^2 + CB^2 = 20^2 + 21^2 = 400 + 441 = 841$; so $AB = \sqrt{841} = 29.$)

119. $20 - r$. [ODCE is also a square; so $CD = r$. $AD = AF$ (the tangent segments to a circle from an external point are equal) and $AD = AC - CD = 20 - r$.]

120. $21 - r$.

121. $AB = AF + FB$; so $29 = (20 - r) + (21 - r)$, $29 = 41 - 2r$, $2r = 12$, $r = 6$.

Angles and Sides.

122. By the Law of Cosines.

123. $\cos C = \frac{1}{2}$. ($c^2 = a^2 + b^2 - 2ab \cos C$, $c^2 = c^2 + c^2 - 2c^2 \cos C$, $2c^2 \cos C = c^2$, $2 \cos C = 1$, $\cos C = \frac{1}{2}$.)

124. $\angle C = 60°$.

125. Yes. If $a = b = c$, ΔABC is equilateral; so each of its angles has a measure of 60°.

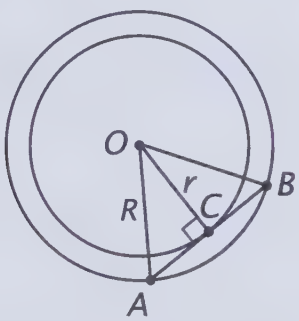
126. $\cos C = 0$. ($c^2 = a^2 + b^2 - 2ab \cos C$, $c^2 = c^2 - 2ab \cos C$, $2ab \cos C = 0$, $\cos C = 0$.)

127. $\angle C = 90°$.

128. Yes. If $c^2 = a^2 + b^2$, ΔABC is a right triangle with hypotenuse c, and so $\angle C = 90°$.

Circular Track.

129.



130. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

131. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

132. ΔACO is a right triangle; so $AO^2 = AC^2 + CO^2$, $R^2 = 100^2 + r^2$, and so $R^2 - r^2 = 10{,}000$.

133. 31,416 m^2. [$A_{track} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = 10{,}000\pi \approx 31{,}416$.]

da Vinci Problem.

134. $\frac{1}{4}\pi$. ($\frac{1}{4}\pi 1^2$.)

135. $\frac{1}{2}$. ($\frac{1}{2}1^2$.)

136. $\frac{1}{4}\pi - \frac{1}{2}$.

137. $2\pi - 4$. [$8(\frac{1}{4}\pi - \frac{1}{2})$.]

138. 2π. [In isosceles right ΔOPF, $PF = 1$; so $OF = \sqrt{2}$; $\pi(\sqrt{2})^2 = 2\pi$.]

139. 4. [$2\pi - (2\pi - 4)$.]

140. 1. ($\frac{1}{4}4$.)

A Problem from Ancient India.

141. 10 units. ($FB = HA + AB$. $FB = \sqrt{6^2 + (12 - x)^2}$, $HA = 6$, and $AB = x$. $\sqrt{36 + (144 - 24x + x^2)} = 6 + x$, $180 - 24x + x^2 = 36 + 12x + x^2$, $144 = 36x$, $x = 4$. $FB = 6 + 4 = 10$.)

Circumradius.

142. Every triangle is cyclic.

143. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

144. $\Delta BOH \cong \Delta COH$ (HL); so $\angle BOH = \angle COH$; so ray OH bisects $\angle BOC$; so $\angle BOH = \frac{1}{2}\angle BOC$.

145. A central angle is equal in measure to its intercepted arc.

146. An inscribed angle is equal in measure to half its intercepted arc.

147. $\angle BOH = \frac{1}{2}\angle BOC = \frac{1}{2}m\overset{\frown}{BC} = \angle A$.

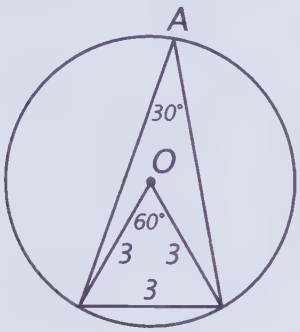
148. The sine of an acute angle in a right triangle is equal to the ratio of the opposite leg to the hypotenuse.

149. $\sin \angle BOH = \frac{\frac{a}{2}}{r}$ and $\angle BOH = \angle A$; so
$\sin A = \frac{\frac{a}{2}}{r}$, $r \sin A = \frac{a}{2}$, $a = 2r \sin A$,
$\frac{a}{\sin A} = 2r$.

150. By the Law of Sines.

151. 3 cm. ($\frac{a}{\sin A} = 2r$, $\frac{3}{\sin 30°} = 2r$, $6 = 2r$, $r = 3$.)

152. *Example figure:*



Conical Mountain.

153. 70 m. ($\Delta AFC \sim \Delta AGE$ by AA; so $\frac{AF}{AG} = \frac{FC}{GE}$,
$\frac{x}{x+100} = \frac{35}{85}$, $85x = 35x + 3{,}500$, and so
$50x = 3{,}500$, $x = 70$.)

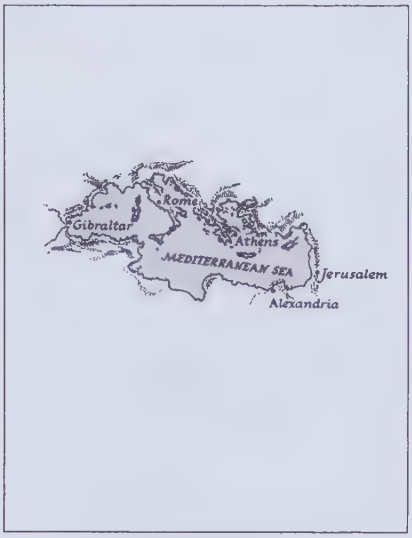
154. 1,286,220 m^3.
[$V = \frac{1}{3}Bh = \frac{1}{3}\pi(85)^2(170) \approx 1{,}286{,}220$.]

155. 89,797 m^3. [$V = \frac{1}{3}Bh = \frac{1}{3}\pi(35)^2(70) \approx 89{,}797$.]

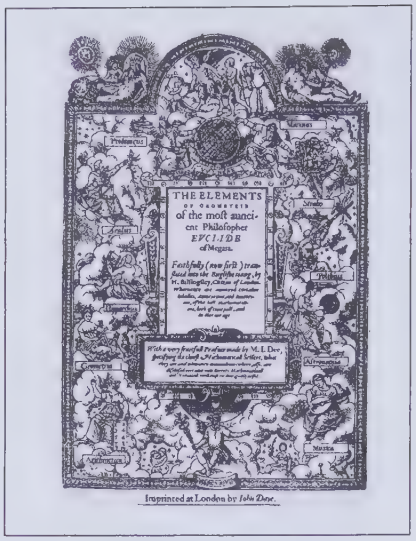
156. 89,797 m^3. [$V = \frac{1}{2} \cdot \frac{4}{3}\pi(35)^3 \approx 89{,}797$.]

157. 1,100,000 m^3.
(1,286,220 – 89,797 – 89,797 = 1,106,626.)

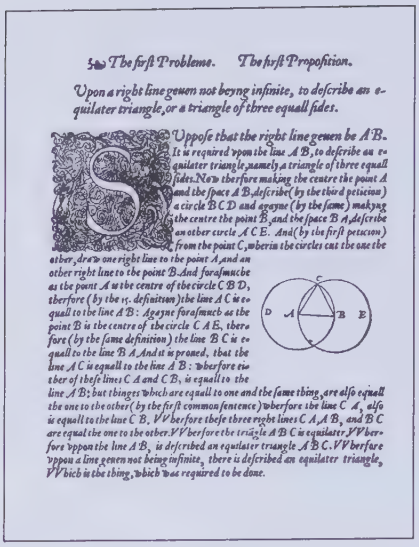
LESSON PLANS



TRANSPARENCY 0-1



TRANSPARENCY 0-2



TRANSPARENCY 0-3

Introduction

Euclid, the Surfer and the Spotter

Transparencies

0-1 Map around the Mediterranean
0-2 Title page of Euclid's *Elements*
0-3 Proposition 1 from the *Elements*
0-4 The surfer and the spotter
0-5 The spotter's puzzle
0-6 The surfer's puzzle

The introductory section includes a brief commentary on Euclid and the *Elements*, the construction of an equilateral triangle, and the puzzles of the surfer and the spotter. Although this material could be assigned as homework, I recommend doing and discussing it in class. You will probably need to provide your students with rulers and compasses.

The puzzle of the spotter is to find the location of the point inside or on an equilateral triangle for which the sum of the distances from that point to the vertices of the triangles is a minimum. After some experimentation, your students should conclude that this point is at the "center" of the triangle. The three paths from the center to the vertices would have the same length: their sum is about 20.8 km. The worst place on the island for the spotter to locate is at the point for which the sum of its distances to the vertices of the triangle is a maximum. There are actually *three* of these points—namely, the vertices themselves. The sum of the lengths of the paths from one vertex to the other two is 24.0 km.

The surfer's puzzle consists of trying to find the location of the point inside or on an equilateral triangle for which the sum of the distances from that point to the *sides* of the triangle is a minimum. Before they try any points, your students will probably assume that the answer to this puzzle is the same as before. It is a remarkable fact, however, that the sum of the three distances is the same for *every* point either inside or on the triangle! It is about 10.4 km. Hence, there is no best or worst place for the surfer to locate.

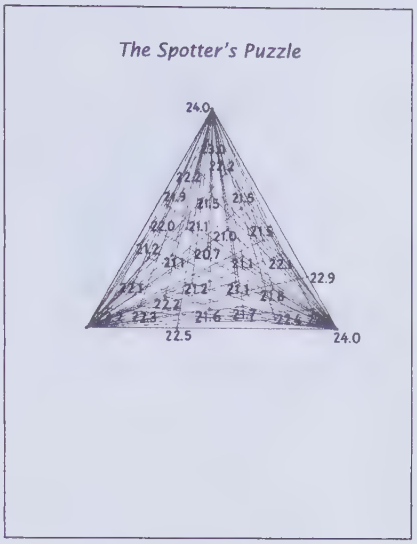
My students are generally content with their analysis of the spotter's puzzle. Although they haven't proved why the center is the best place and the corners are the worst, these answers seem reasonable.

The result of the surfer's puzzle, on the other hand, is surprising and hence disconcerting. *Why* should the sum of the three distances be independent of the position of the point? Certainly no amount of experimentation is capable of either proving or explaining this claim. This realization suggests that, to be confident of the conclusions that we may draw in geometry, we must have some basis for understanding them and for convincing others that they are correct. Deductive reasoning provides this basis, and it is the subject of the second chapter of the course.

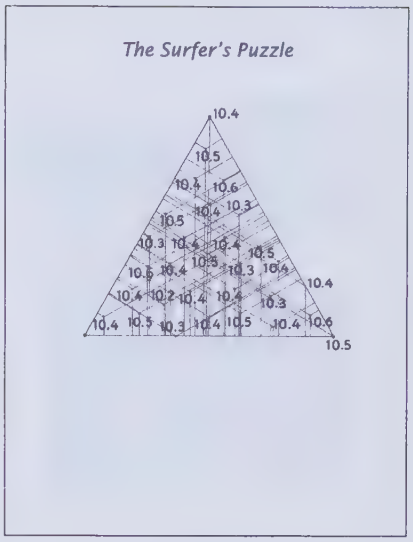
Transparencies 0-1 through 0-4 are for use in presenting the introductory lesson. Transparencies 0-5 and 0-6 and their overlays illustrate many dif-



TRANSPARENCY 0-4



TRANSPARENCY 0-5

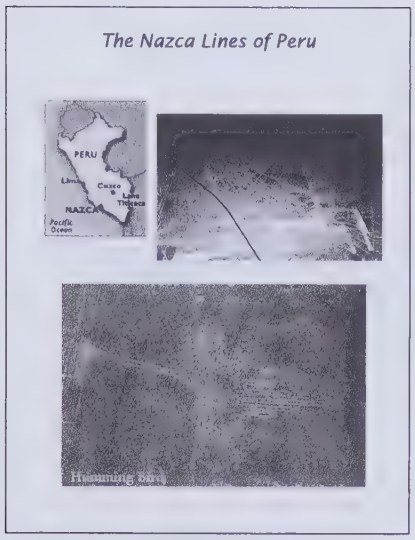


TRANSPARENCY 0-6

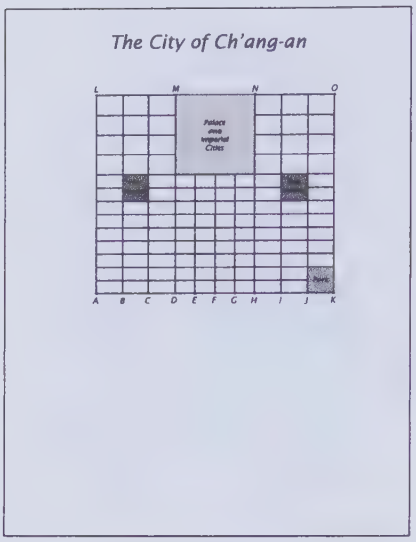
ferent points for each puzzle and the distance sums that correspond to the points.

The spotter's puzzle is a special case of a problem proposed by Pierre Fermat; its solution is based on the fact that the perpendicular segment from a point to a line is the shortest segment connecting them. The solution to the surfer's puzzle can be understood by means of the theory of area. Both puzzles will be reconsidered later in the course.*

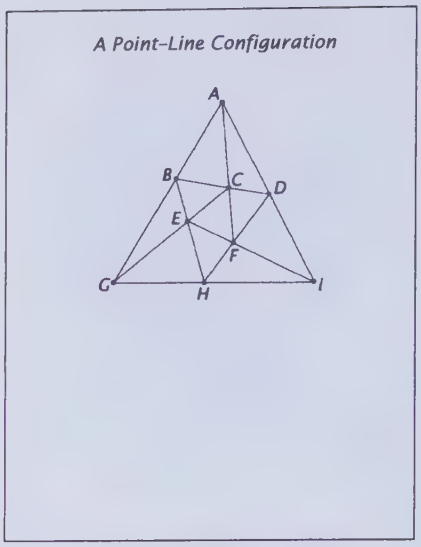
*An explanation of the surfer's puzzle is developed in Chapter 9, Lesson 3, exercises 29–34 of the text. The spotter's puzzle is discussed in the second review lesson for Chapter 13 of the *Teacher's Guide*.



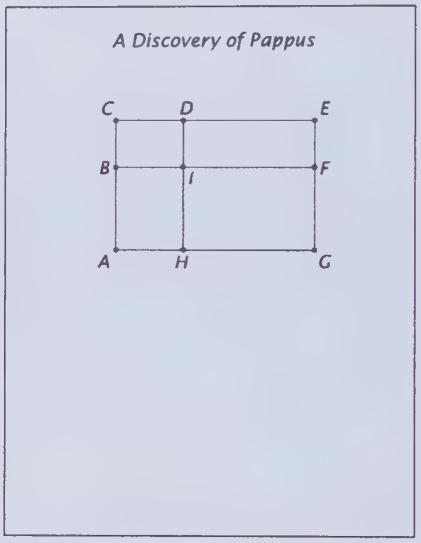
TRANSPARENCY 1-1-1



TRANSPARENCY 1-1-2



TRANSPARENCY 1-1-3



TRANSPARENCY 1-1-4

Chapter 1

An Introduction to Geometry

Lesson 1
Lines in Designing a City

Lesson Outline

Terms: Point, line, line segment, plane, collinear, noncollinear, coplanar, concurrent.

Transparencies

1-1-1 The Nazca lines of Peru
1-1-2 The city of Ch'ang-an (Set I)
1-1-3 A point–line configuration (Set II)
1-1-4 A discovery of Pappus (Set III)

Introduction

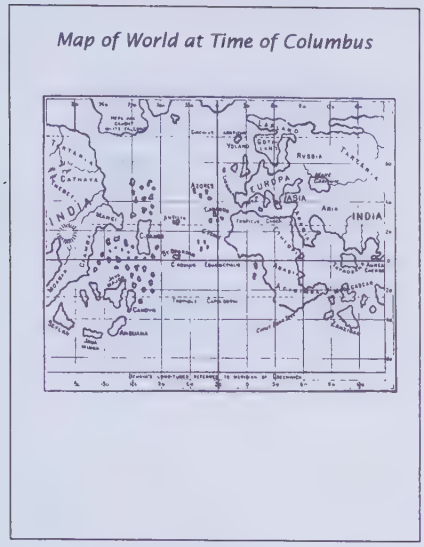
One of the world's unsolved mysteries concerns the desert of southern Peru. Stretching for miles across the land are gigantic drawings of birds and other animals, as well as lines—many lines—leading in every direction. The drawings and lines were put there by the Nazca civilization more than 1,500 years ago. For what purpose, no one knows.

Strange as it may seem, the people who drew these figures probably never saw them in their entirety, because they can only be seen from heights of at least 1,000 feet. They weren't noticed until the 1920s when airplanes began flying over the Peruvian desert.*

*Further information about the Nazca (also spelled Nasca) lines can be found in *Pathways to the Gods*, by Tony Morrison (Harper & Row, 1978), *The World's Last Mysteries* (A Reader's Digest Book, 1978), and various sites on the Internet, including www.crystalinks.com/nasca.html.

Some of the lines extend for many miles and are incredibly straight. The Nazcas are thought to have stretched strings from posts to aid in their construction. How many posts would be needed to determine the direction of one of these strings? (Two.) The positions of the posts are good models of geometric *points*, and the stretched string is a good model of part of a geometric *line*. The flatness of the Nazca plain on which the figures were created makes it a good model of a geometric *plane*.

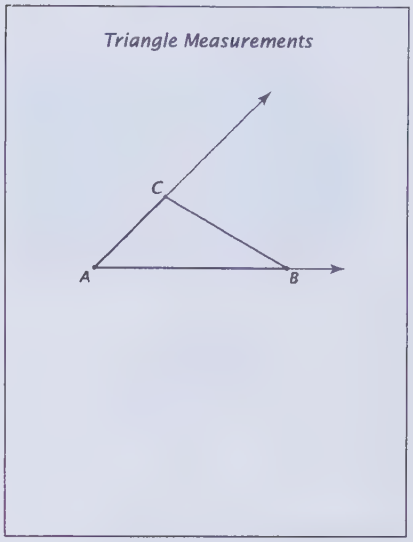
This introduction leads naturally into the rest of the ideas included in the lesson. Guide your students in listing and making sketches illustrating the basic terms before they begin the exercises.



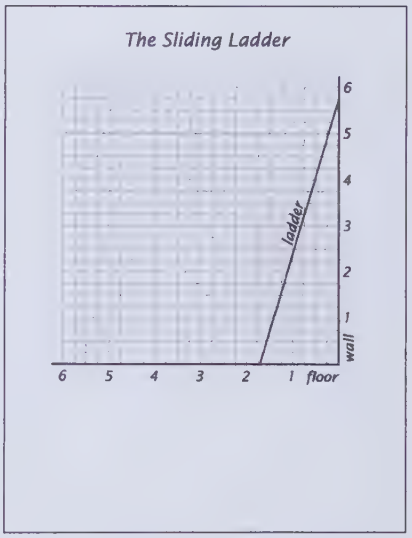
TRANSPARENCY 1-2-1



TRANSPARENCY 1-2-2



TRANSPARENCY 1-2-3



TRANSPARENCY 1-2-4

Lesson 2
Angles in Measuring the Earth

Lesson Outline

Terms: Ray, angle (sides and vertex), protractor, degree.

Transparencies

1-2-1 Map of world at time of Columbus
1-2-2 Map of Alexandria and Syene
1-2-3 Triangle measurements (Set I, Exs. 1–12)
1-2-4 The sliding ladder (Set II)

Introduction

Good opening questions: Do you know what the distance around the earth is? How long ago do you suppose it was first determined?

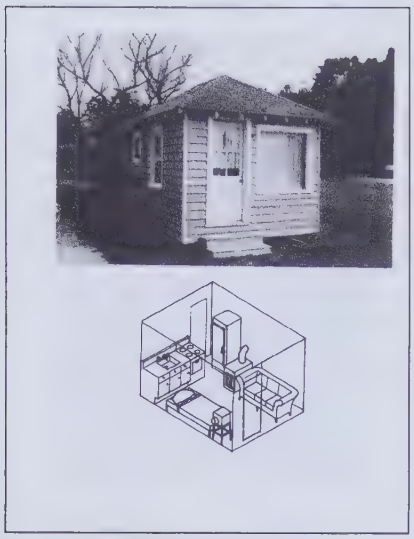
In the foreword to the book titled *The Log of Christopher Columbus,* edited by Robert H. Fuson (International Marine Publishing Company, 1987), Luis Marden wrote:

"No literate person in 1492 thought the world was flat; the Greeks had proved the sphericity of the earth long before Christ. In fact, in that same year of 1492 Martin Behaim in Germany produced a terrestrial globe, still extant. The question under debate was the size of the globe. In the third century B.C., the Greek astronomer Eratosthenes, using only his brain and a shadow-casting column, came astonishingly close to the true circumference, but 500 years later the geographer Ptolemy described an earth one fourth smaller than the true dimension."

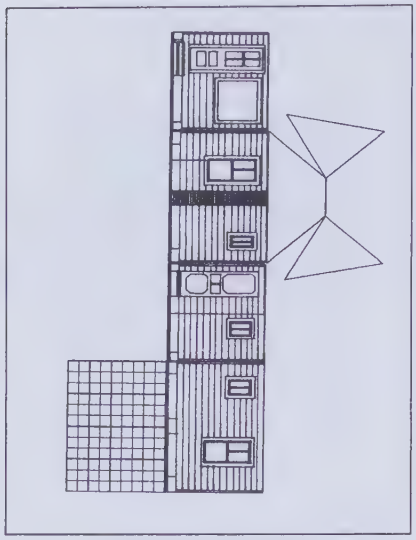
Columbus thought that the circumference of the earth was approximately 20,000 miles. If he had known the correct value, he might not have attempted the journey. As Fuson remarks, Columbus's belief that the earth was much smaller than it is "was one of the most fortunate mistakes in history."

The map on Transparency 1-2-1, based on Behaim's world globe of 1492, makes clear why when he sailed west from Spain and arrived at Cuba, Columbus thought it was Japan (Cipango). Not until considerably later did the Europeans realize that the earth was large enough to contain another continent and ocean.

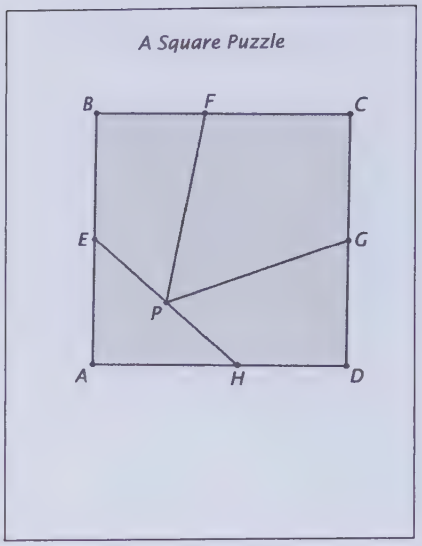
How could someone have accurately determined the size of the earth more than seventeen centuries before Columbus made his historic voyage? Explain the method that Eratosthenes used as described on pages 14 and 15 of the text by first showing Transparency 1-2-2 and then having your



TRANSPARENCY 1-3-1



TINY HOUSE PATTERN



TRANSPARENCY 1-3-2

students draw the following figure. Use the semicircular edge of a protractor to draw a circle to represent the earth. Mark a point, S, at the top of the circle to represent Syene and draw a ray upward from the center of the circle through this point to show the direction of the sun when it is directly overhead. Use the protractor to draw an angle with this ray as one of its sides and with a measure of 7.5° to find the position, A, of Alexandria on the circle. Draw a ray from point A parallel to the original ray to complete the figure and then guide your students in discovering what Eratosthenes did.

Lesson 3
Polygons and Polyhedra: Pyramid Architecture

Lesson Outline

Terms: Polygon (triangle, quadrilateral, pentagon, hexagon, octagon), polyhedron, perimeter, area, rectangle, square.

Transparencies and Worksheet

1-3-1 Tiny house
Tiny house pattern

1-3-2 A square puzzle (Set III) *(Cut out a set of the four pieces for use in illustrating how they can be rearranged to form the equilateral triangle.)*

Introduction

Before class, tape off a rectangle measuring 9 ft by 12 ft on the floor of your classroom to illustrate just how small the tiny house featured in this lesson actually is. The house is located near Whitestone, Virginia. Designed as a 1950s ranch house, it consists of just one room 12 ft long and 9 ft wide. Its interior is shown in the diagram below the photograph. (Its bathroom is a separate outhouse, not visible in the picture.)*

Hand out scissors and copies of the Tiny House Pattern so that your students can put together a simple model of the house. Before they do so, observe that the figures representing the walls, floor, and sections of the roof are examples of *polygons* [*triangles and quadrilaterals (rectangles, squares, and trapezoids)*]. The surfaces of the house, after it has been put together, provide an example of a *polyhedron* (one having nine faces). If the pattern is cut out as a single piece and just one piece of tape used to connect the unattached walls, it can be folded flat again after the lesson has been completed. The rectangle representing the floor of the house has been divided into small squares, each corresponding to 1 square foot. The model can thus be used to develop the rest of the lesson by having your students consider the perimeter and area of the floor.

Lesson 4
Constructions: Telling Time with Shadows

Lesson Outline

Terms: Straightedge, compass, circle (radius, center), construction, bisect.

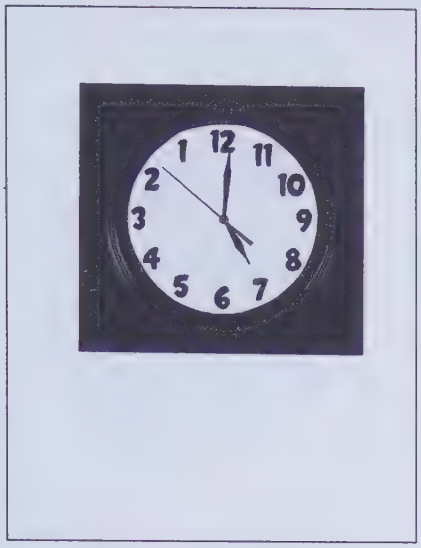
Constructions: To bisect a line segment, to bisect an angle.

Transparencies

1-4-1 Counterclockwise clock

1-4-2 Sundial

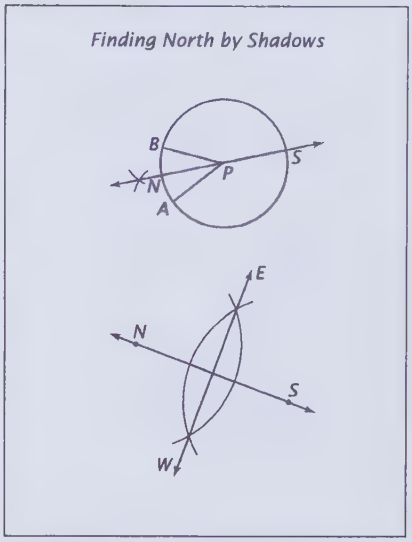
**Tiny Houses,* by Lester Walker (Overlook Press, 1987).



TRANSPARENCY 1-4-1



TRANSPARENCY 1-4-2



TRANSPARENCY 1-4-3

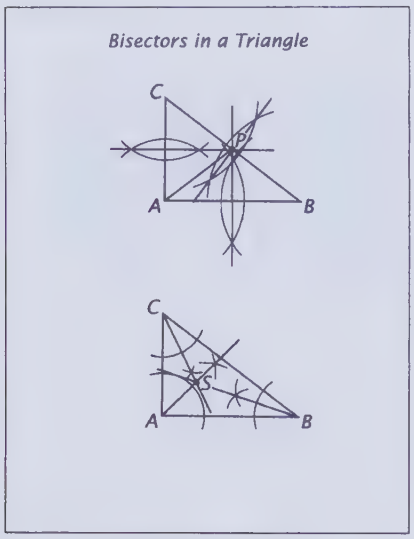
1-4-3 Finding north by shadows (Set I, Exs. 3–9)
1-4-4 Bisectors in a triangle (Set I, Exs. 10–16)

Introduction

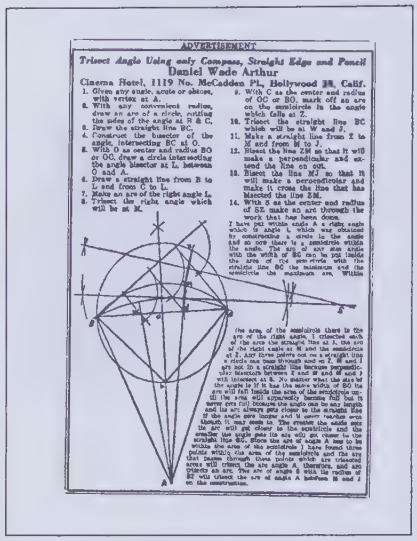
An amusing way to begin this lesson is to have a "backward clock" in view in your classroom as your students arrive. Such clocks are available for about $15 from many different sources on the Internet and can be found by typing "backward clock" on your search engine. The clock shown on Transparency 1-4-1 is an example of one of these clocks. Ask your students how a clock with its numbers in reverse order could keep correct time. (Its hands go counterclockwise.) The manufacturer of one of these clocks states: "Does the frantic pace of today's world make you want to turn back the hands of time and give yourself a few extra hours? We've solved this problem for you with our new Backward Clock. The hour, minute and second hands move backward to give you the extra time you need. The numbers on this clock are arranged counterclockwise to help you keep track of how much time you're saving."

Why do clocks ordinarily run "clockwise"? In his book titled *Why Do Clocks Run Clockwise? and Other Imponderables* (Harper & Row, 1987), David Feldman gives the following explanation: "Before the advent of clocks, we used sundials. In the northern hemisphere, the shadows rotated in the direction we now call 'clockwise.' The clock hands were built to mimic the natural movement of the sun. If clocks had been invented in the southern hemisphere . . . 'clockwise' would be the opposite direction." The sundial pictured on Transparency 1-4-2 illustrates this resemblance nicely. In their book on *Time* (Life Science Library, 1980), Samuel A. Goudsmit and Robert Claiborne observe: "As long as the principal timekeeper was the motion of the sun—for some 4,000 years from ancient Egypt to the 16th Century—the main timekeeping device was the sundial. . . . The sundial is remarkably reliable, so good that for centuries it was employed to correct mechanical clocks."

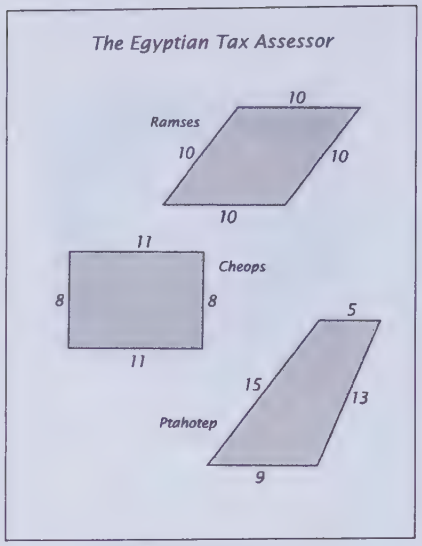
Have your students use a compass to draw a circle and a horizontal diameter. Show them how to construct a vertical diameter and label the endpoints of the diameters according to map directions with north at top. Point out that their drawing lies in the horizontal plane of the paper. Imagine a vertical post with its base at the center of the circle. In what direction would its shadow point when the sun is rising in the east? (West.) Where is the sun in the sky at noon? (In the Northern Hemisphere, it is toward the south.) In what direction does the shadow of the post point at noon? (North.) In what direction does the shadow point when the sun is setting in the west? (East.) The answers to these questions confirm the fact that, in the Northern Hemisphere, shadows turn clockwise during the day.



TRANSPARENCY 1-4-4



TRANSPARENCY 1-5-1



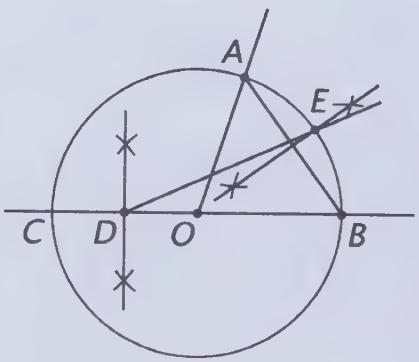
TRANSPARENCY 1-5-2

Lesson 5
We Can't Go On Like This

Lesson Outline
Theme: The need to know *why.*
Transparencies
1-5-1 Angle-trisection advertisement
1-5-2 The Egyptian tax assessor

Introduction
[This introduction is especially appropriate in that it both reviews the bisection constructions of Lesson 4 and leads naturally into the new lesson.] A very strange advertisement (Transparency 1-5-1) once appeared in the *Los Angeles Times*. It was paid for by a man named Daniel Wade Arthur whose address was a cheap hotel in Hollywood. The ad appeared in the Sunday paper and was rather large; so Mr. Arthur had to have paid a lot of money to run it. He didn't have any hope of making any money from it, because he wasn't trying to sell anything. The subject of the ad was a geometry problem! Mr. Arthur was explaining how to trisect an angle using only a compass, straightedge, and pencil. His method was very complicated and would take a lot of time and effort to carry out, which makes one wonder if anyone who saw the ad in the paper bothered to even read it.

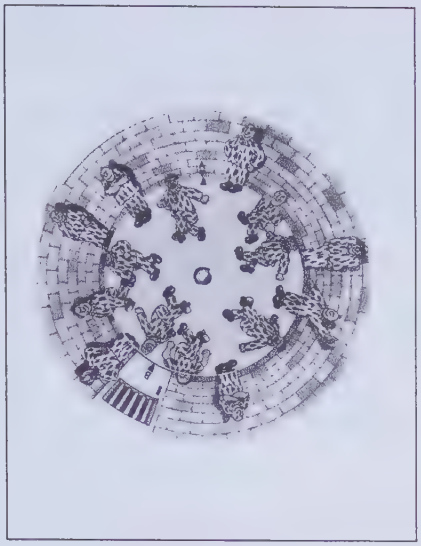
A French mathematician published a much simpler way to "trisect an angle" in 1934.* To try it out, have your students use their protractors to draw an angle of 72° and then carry out the construction as illustrated in the figure above and described in the following steps:



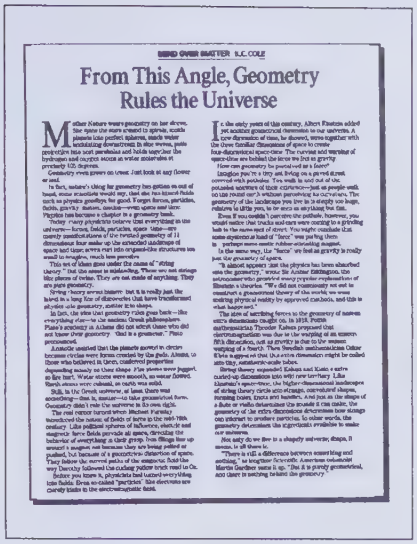
1. Draw a circle with its center at the vertex of the angle, O, and let A and B be the points in which it intersects the sides of the angle.
2. Extend line OB to intersect the circle at C.
3. Bisect CO and let D be its midpoint.
4. Draw AB, bisect it and let E be the point in which the bisecting line intersects the circle.
5. Draw line DE.

∠EDB *appears* to have a measure of 24°, and so ∠AOB *appears* to have been trisected. However, if ∠AOB is exactly 72°, then it can be proved that ∠EDB has a measure of slightly *more* than 24°. So the method doesn't quite work, but the mathematician who thought of it realized this. He knew that another mathematician had proved such a construction to be impossible in 1837, almost a hundred years earlier. Unfortunately for Mr. Arthur, the man who

*The mathematician, Maurice d'Ocagne (1862–1938), was sufficiently famous in France that a street in Paris is now named for him. For an approximation, his method is especially accurate for angles less than 20°. An analysis of it can be found on page 55 of *The Trisection Problem*, by Robert C. Yates (Franklin Press, 1942; reprinted by N.C.T.M., 1971).



TRANSPARENCY 1-R-1



MIND OVER MATTER N.C. COLE

From This Angle, Geometry Rules the Universe

Mother Nature wears geometry on her sleeve. She spins the stars around in spirals, molds planets into perfect spheres, sends water undulating downstream in sine waves, pulls projectiles into neat parabolas and holds together the hydrogen and oxygen atoms in water molecules at precisely 105 degrees.

Geometry even powers us now. Just look at any flower or seed.

In fact, nature's liking for geometry has gotten us out of bread, some scientists would say, that she has mixed fields such as physics goodbye for good. Forget forces, particles, fields, gravity, motion, cosmic—even space and time: Physics has become a chapter in a geometry book.

Today many physicists believe that everything in the universe—forces, fields, particles, space, time—are merely manifestations of the twisted geometry of 11 dimensions: four make up the stretched landscape of space and time; seven fold into origami-like structures too small to imagine, much less perceive.

This set of ideas goes under the name of "string theory," but the name is misleading. These are not strings like pieces of twine. They are not made of anything. They are pure geometry.

String theory seems bizarre—but it is really just the latest in a long line of discoveries that have troubled physicists since geometry, earliest in its shape.

In fact, the idea that geometry rules goes back—like everything else—to the ancient Greek philosophers. Plato's prophecy in Athens did not affect those who did not know their geometry. "God is a geometer," Plato pronounced.

Aristotle insisted that the planets moved in circles because circles were forms created by the gods. Atoms, to those who believed in them, conferred properties depending mainly on their shape. Fire atoms were jagged, so fire hurt. Water atoms were smooth, so water flowed. Earth atoms were cubical, so earth was solid.

Still, in the Greek universe, at least, there was something—that is, matter—to take geometrical form. Geometry didn't rule the universe in its own right.

The real rupture turned when Michael Faraday theorized the nature of fields of force in the mid-19th century. Like puffball spheres of influence, electric and magnetic force fields pervade all space, dictating the behavior of everything in their grasp. Iron filings line up around a magnet not because they are being pulled or pushed, but because of a geometric distortion of space. They follow the curved paths of the magnetic field the way I once followed Uri during yellow brick road in Oz. Before you knew it, physicists had bulldozed everything into fields. Even so-called "particles" like electrons are merely kinks in the electromagnetic field.

In the only event of this century, Albert Einstein added yet another geometrical dimension to our universe. A new dimension of time, he showed, were together with the three familiar dimensions of space to create four-dimensional space-time. The curved and wrinkles of space-time are behind the force we feel as gravity.

How can geometry be portrayed as a force?

Imagine you're a tiny ant living on a curved sheet covered with potholes. You walk in and out of the potholed sections of that curved sheet—just as people walk on the round earth without perceiving its curvature. The geometry of the landscape you live in is simply too large, relative to little you, to be seen as anything but flat.

Even if you couldn't perceive the potholes, however, you would notice that tracks and cars were starting to a grinding halt in the snow-laced of desert. You might conclude that some mysterious at least of "force" was pulling them in—perhaps some exotic rubber-attracting magnet.

In the same way, the "force" we feel as gravity is really just the geometry of space.

"It almost appears that the physics has been absorbed into the geometry," wrote Sir Arthur Eddington, the astronomer who popularized many popular explanations of Einstein's theories. "If a did not consciously set out to construct a geometrical theory of the world, we were making physical reality by apparent carelessness, and this is what happened."

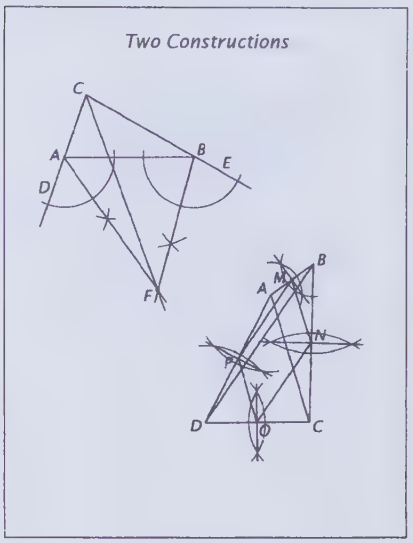
The idea of ascribing forces to the geometry of nature extra dimensions caught on. In 1919, Polish mathematician Theodor Kaluza proposed that electromagnetism was due to the warping of an extra fifth dimension, just as gravity is due to the warping of a fourth. Then Swedish mathematician Oskar Klein suggested that the extra dimension might be rolled into tiny, subatomic-scale tubes.

String theory expanded Kaluza and Klein's naive curled-up dimensions into with new territory. Like Alumna's space-time, the higher-dimensional landscape of string theory circle and oranges, convoluted shapes, twisting tubes, knots and bands. And just as the shape of a flute or violin determines the sounds it can make, the geometry of the extra dimensions determines how strongly one interact to another particles. In other words, the geometry determines the experiments available to make our universe.

Not only do we live in a shapely universe, shape, it seems, is all there is.

"There is still a difference between something and nothing," he laughter Scientific American columnist Martin Gardner notes it up. "But it is purely geometrical, and there is nothing behind the geometry."

TRANSPARENCY 1-R-2



TRANSPARENCY 1-R-3

spent his money on the angle trisection ad, this means that his method doesn't quite work either. If his method *had* worked, his money would have been well spent because his name would have become famous in the history of mathematics for succeeding in doing something that people have been trying to figure out ever since the time of Euclid.

The situation presented by the problem of trisecting an angle with just a straightedge and compass leads naturally to several questions. How do you *prove* that something is impossible? How is it possible to tell the difference between *exactly right* and *almost?* How do you prove something in the first place? To eventually be able to answer such questions is a major goal of the study of geometry.

Students who would like to know more about the problem of angle trisection and how people have tried to solve it may be interested in reading the chapter titled "How to Trisect an Angle" in *Mathematical Carnival,* by Martin Gardner (Knopf, 1975) and looking through *A Budget of Trisections,* by Underwood Dudley (Springer Verlag, 1987).

Review of Chapter 1

First day

Transparencies

1-R-1 Find the escapee
1-R-2 Geometry rules the universe
1-R-3 Two constructions (Set II)
1-R-4 Russian second-grade problem

Introduction

A light-hearted way to begin the review lesson is with a "Geometrical Vanish." In his book titled *Mathematics, Magic and Mystery* (Dover, 1956), Martin Gardner includes two chapters on puzzles of this sort. He describes these puzzles as geometrical paradoxes that consist of "rearranging parts of a figure. After the rearrangement is completed, a portion of the original figure (either part of its area or one of a series of pictures drawn on the figure) has apparently vanished without trace. When the pieces are returned to their original form, the missing area or picture mysteriously appears once more." American puzzlist Sam Loyd invented and patented an extremely clever puzzle of this sort in 1896. Called "Get Off the Earth," more than 10 million copies of it were sold in Loyd's lifetime.

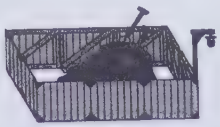
Australian physicist Dugald Keith was inspired by Loyd's puzzle to create a puzzle with the title "Find the Escapee." Pictured on Transparency 1-R-1 and its overlay, the puzzle shows 15 prisoners in a circular jail cell when the arrow on the overlay points to A; when the overlay is turned so that the arrow points to B, one of the prisoners has vanished.

Alternative Introduction

A recent article in the *Los Angeles Times* (Transparency 1-R-2) claims that "Geometry Rules the Universe." Although it would not be appropriate to read it to your class in its entirety, your students may find the following excerpts to be of interest (italics added for emphasis):

"Mother Nature wears geometry on her sleeve. She spins the stars around in *spirals,* molds planets into perfect *spheres,* sends water undulating downstream in *sine waves,* pulls projectiles into neat *parabolas* and holds together the hydrogen and oxy-

Russian Second-Grade Problem



A rectangular lot is enclosed by a fence of wooden panels. How many panels are needed if the lot is 6 meters long and 4 meters wide and the length of a panel is 2 meters? Write an expression based on the problem and solve it.

TRANSPARENCY 1-R-4

gen atoms in water molecules at precisely *105 degrees*.

Geometry even grows on trees: Just look at any flower or leaf.

In fact, nature's thing for geometry has gotten so out of hand, some scientists would say, that she has kissed fields such as physics goodbye for good. Forget forces, particles, fields, gravity, matter, motion—even space and time. Physics has become a chapter in a *geometry book*.

Today, many physicists believe that everything in the universe—forces, fields, particles, space, time—are merely manifestations of the twisted geometry of 11 dimensions: four make up the extended landscape of space and time; seven curl into origami-like structures too small to imagine, much less perceive.

This set of ideas goes under the name of 'string theory.' But the name is misleading. These are not strings like pieces of twine. They are not made of anything. They are *pure geometry*.

String theory seems bizarre, but it is really just the latest in a long line of discoveries that have transformed physics into *geometry*, matter into *shape*.

In fact, the idea that geometry rules goes back—like everything else—to the ancient Greek philosophers. Plato's academy in Athens did not admit those who did not know their geometry. 'God is a geometer,' Plato pronounced."

[The article goes on to describe Faraday's introduction of the idea of force fields, geometrical distortions of space, in the nineteenth century; Einstein's addition of another geometrical dimension, time, in the twentieth century to form four-dimensional space-time, used to explain gravity; and the use of string theory to explain why elementary particles exist.]

"In other words, . . . geometry determines the ingredients available to make our universe.

Not only do we lie in a shapely universe; shape, it seems, is all there is.

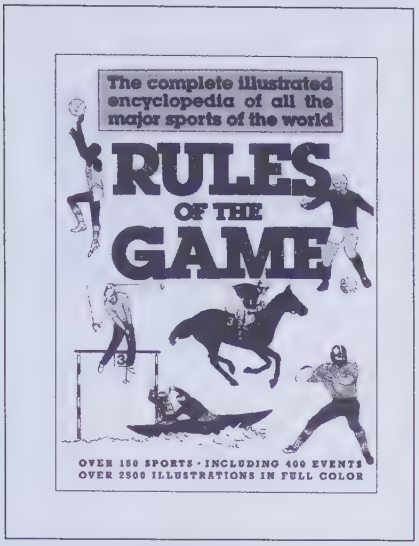
'There is still a difference between something and nothing,' as longtime Scientific American columnist Martin Gardner sums it up. 'But it is purely geometrical, and there is nothing behind the geometry.'"

Second day

Introduction

People immigrating to the United States are sometimes surprised at the difference between what was taught in their former countries and what their children encounter in their new country. Here is an example that may surprise you. This problem (Transparency 1-R-4) appeared in a book for second-graders used for many years throughout the former Soviet Union. What expression do you think the students were expected to write? What is the solution to the problem? ($\frac{6+4+6+4}{2}$ or $\frac{2(6+4)}{2}$; 10 panels.)

Suppose that the problem were stated in terms of variables rather than numbers: How many panels are needed if the lot is x m long and y m wide and the length of a panel is z m? ($\frac{2x+2y}{z}$.) In writing this expression, what are we assuming about z, the length of a panel, with respect to x and y, the dimensions of the lot? (That z divides evenly into x and y.) These more general questions will suggest to some of your students the need for some review of algebra. The first algebra review appears at the end of this chapter and might be done in class as time allows after the chapter review has been completed.

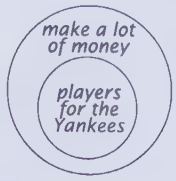


TRANSPARENCY 2-1-1

1. If an opponent touches the ball, all the passing side is eligible to receive the ball.
2. A game is forfeited if a team does not have at least two players on court.
3. If the kicker breaks the law, the defending side is award an indirect free kick.
4. A runner is out if he fails to beat a throw to the base to which he is running.
5. If the game is tied at 14/14, it continues until one team has a two-point lead.
6. A player is offside if he is within his opponents' half when the ball is played by a team mate.

TRANSPARENCY 2-1-2

All players for the Yankees make a lot of money.



a) If you play for the Yankees, then you make a lot of money.
b) If you make a lot of money, then you play for the Yankees.
c) If you don't play for the Yankees, then you don't make a lot of money.
d) If you don't make a lot of money, then you don't play for the Yankees.

TRANSPARENCY 2-1-3

If you have lived a long time, then you are an octagenarian.



a) If you are an octagenarian, then you have lived a long time.
b) If you have not lived a long time, then you are not an octagenarian.
c) If you are not an octagenarian, then you have not lived a long time.
d) You are an octagenarian if you have lived a long time.

TRANSPARENCY 2-1-4

Chapter 2

The Nature of Deductive Reasoning

Lesson 1
Conditional Statements

Lesson Outline

Terms: Conditional statement (hypothesis, conclusion), Euler diagram.

Transparencies

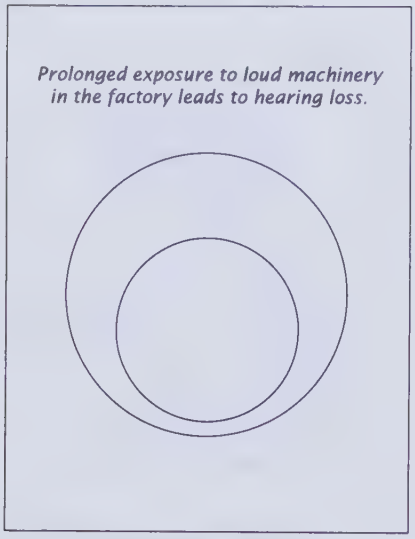
2-1-1 Rules of the Game book cover
2-1-2 Rules that are conditional statements
2-1-3 Yankees problems (Set II, Exs. 31–33)
2-1-4 Octagenarian problems (Set II, Exs. 34–36)

Introduction

Show Transparencies 2-1-1 and 2-1-2. Every game is played by rules. Here are some examples of rules for various sports that are listed in a book titled *Rules of the Game* (The Diagram Group, St. Martin's Press, 1990). Which sports can you identify?

1. If an opponent touches the ball, all the passing side is eligible to receive the ball. (Football.)
2. A game is forfeited if a team does not have at least two players on court. (Basketball.)
3. If the kicker breaks the law, the defending side is awarded an indirect free kick. (Soccer.)
4. A runner is out if he fails to beat a throw to the base to which he is running. (Baseball.)
5. If the game is tied at 14/14, it continues until one team has a two-point lead. (Volleyball.)
6. A player is offside if he is within his opponents' half when the ball is played by a team mate. (Hockey.)

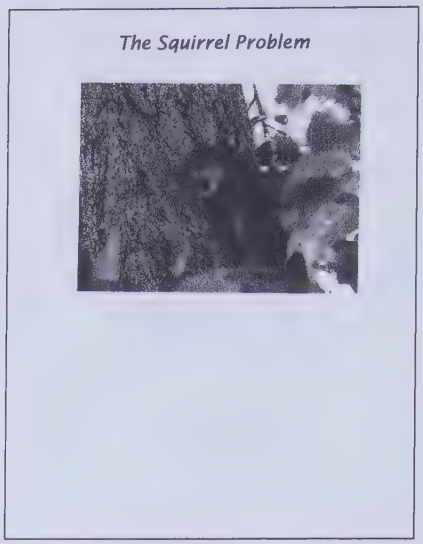
What do all the statements of these rules have in common? (Each statement tells what happens in the game if something else has happened. Each statement contains the word "if.") Explain that these statements are examples of *conditional statements* and develop the rest of the lesson as in the text. In introducing the Euler diagram, be sure to generalize so that the students understand the meaning of the symbols $a \rightarrow b$. Call attention to the fact that the exterior of each circle represents the opposite of what its interior represents. Some students are unable to interpret Euler diagrams correctly, because they do not realize that the two circles separate the



TRANSPARENCY 2-2-1



TRANSPARENCY 2-2-2



TRANSPARENCY 2-2-3

plane into three regions, two of which represent "not *a*" and two of which represent "not *b*."

Lesson 2
Definitions

Lesson Outline
Terms: Definition, converse, if and only if.
Transparencies
2-2-1 Hearing-loss argument
2-2-2 If you keep on biting your nails
2-2-3 The squirrel problem

Review
Rob Eastaway and Jeremy Wyndham relate the following legal case in their book titled *Why Do Buses Come in Threes? The Hidden Mathematics of Everyday Life* (Wiley, 1998):

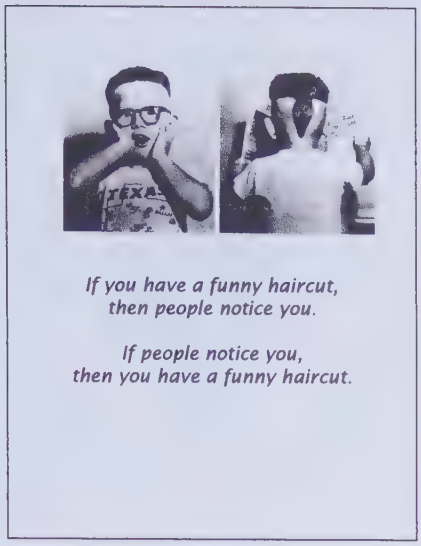
"A man was suing his company for having caused his deafness. It was agreed by both sides that 'prolonged exposure to loud machinery in the factory led to hearing loss.' . . . [T]he plaintiff's argument was that this *proved* that his damaged hearing was due to prolonged exposure to loud machinery."

Ask your students if they agree with this argument. Show Transparency 2-2-1. How would you change this sentence to "if-then" form? (If you are exposed to loud machinery in the factory for a long time, then this exposure leads to hearing loss.) What words belong in each circle of the Euler diagram? ("Loud machinery exposure" in the smaller circle and "hearing loss" in the larger circle.) If you have a hearing loss, does that mean that it was caused by exposure to loud factory machinery? This question leads naturally into a discussion of the *converse* of a conditional statement, part of the new lesson.

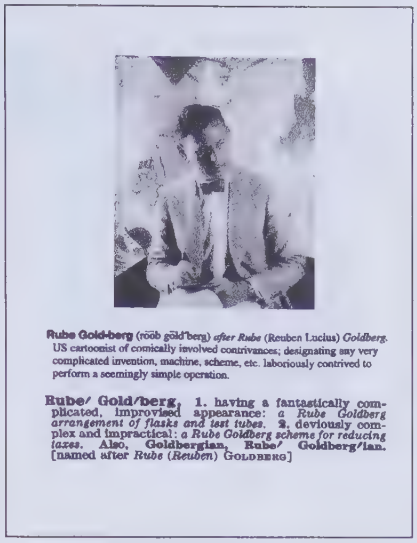
Alternative Review
As a quick review, show Transparency 2-2-2 and ask, Of what kind of statement is "See what will happen to you if you keep on biting your nails" an example? (A conditional statement.) What is its hypothesis? (You keep on biting your nails.) What is the rest of the statement called? (The conclusion.) If an Euler diagram were drawn to represent the statement, which words would go in the smaller circle? (You keep on biting your nails.)

Introduction
The American philosopher William James told of camping with some friends in the mountains and returning from a walk to find them engaged in a heated argument. Scot Morris relates the story in his *Next Book of Omni Games* (New American Library, 1988): "It seems that one of the men had come upon a squirrel clinging to a tree trunk [Transparency 2-2-3]. As the man circled the tree, the squirrel did the same, so that it always faced the man. After going once around the tree, they were both back to where they started. It was clear that the man and the squirrel had both gone around the tree. The question was, Did the man go around the squirrel?" Before giving your students James's answer, you might poll them about what they think. Morris summarizes it as follows: "If by around we mean 'First to the south of, then to the west of, then to north of, then to east of . . .' then the man does go around the squirrel. If around means 'First in front of, then to the left of, then in back of, then to the right of . . .' then, of course, he doesn't."

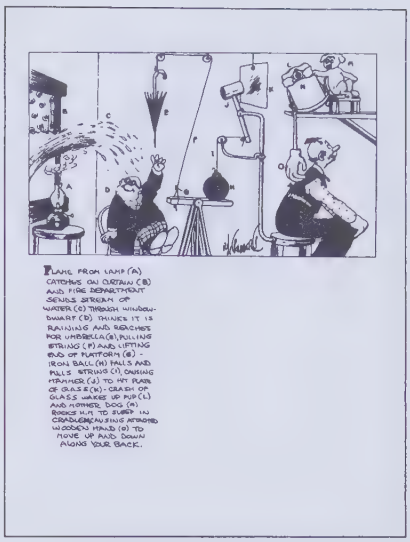
This story leads into the rest of the new lesson.



TRANSPARENCY 2-3-1



TRANSPARENCY 2-3-2



TRANSPARENCY 2-3-3

A convenient example to use that is different from the one in the text is the definition of "concurrent lines." You might have your students write it in all three forms as you introduce the ideas of converse and iff:

If lines are concurrent, then they contain the same point.

If lines contain the same point, then they are concurrent.

Lines are concurrent if and only if they contain the same point.

Background for the Teacher

In the chapter titled "The Role of Definitions" in his book titled *Euclid—The Creation of Mathematics* (Springer, 1999), Benno Artmann states: "After Blaise Pascal . . . it became firmly established that in mathematics definitions are what are called nominal definitions. The notion 'square' is nothing but an abbreviation of the phrase 'equilateral and equiangular quadrilateral,' and the same is true for any other mathematical concept. In principle one should always be able to replace the shorthand notion by the terms that were used in its definition. In fact, many routine proofs do just this. And more than that: Definitions are never right or wrong, but can only be more or less useful."

Lesson 3
Direct Proof

Lesson Outline

Terms: Syllogism, direct proof (premises, conclusion), theorem.

Ideas: Pattern of direct proof.

Transparencies

2-3-1 If you have a funny haircut
2-3-2 Rube Goldberg dictionary entry
2-3-3 Rube Goldberg cartoon
2-3-4 Aristotle and the syllogism
2-3-5 The yellow rows of taxis (Set I, Exs. 17 and 18)
2-3-6 Tick-tack-toe proofs (Set II, Exs. 29–32)

Review

Show just the photograph at the upper left of Transparency 2-3-1 and then the photograph at the upper right. These pictures are of three-year-old Robert Eugene Powell of Wichita Falls, Texas, whose father cut his hair in the shape of a coonskin cap when Davy Crockett was popular in the early days of television. Compare the statements below the photographs: (1) If you have a funny haircut, then people notice you. (2) If people notice you, then you have a funny haircut. Do these statements say the same thing? (Observe that, although the first statement may be true, the second certainly is not.) In what way is the second statement related to the first? (It is the converse of the first.) There is one type of statement in mathematics for which the converse is always true. What kind of statement is it? (A definition.) Why is the converse of a definition always true? (The word being defined and its definition have the same meaning.)

Introduction

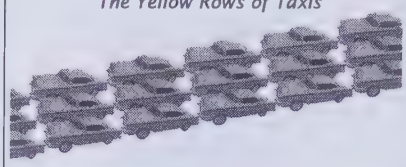
Show Transparency 2-3-2. Rube Goldberg was an American artist whose name now appears in many dictionaries owing to his drawings of crazy inventions. There are even Rube Goldberg contests the purpose of which is to build the most complicated

Aristotle and the Syllogism



TRANSPARENCY 2-3-4

The Yellow Rows of Taxis



If you go to Dallas,
you will take a plane.

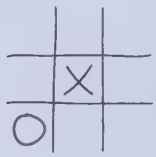
If you take a plane,
you will go to the airport.

If you go to the airport,
you will see all the cabs lined up.

If you see all the cabs lined up,
you will see the yellow rows of taxis.

TRANSPARENCY 2-3-5

Tick-tack-toe Proofs





TRANSPARENCY 2-3-6

machine possible to perform a simple everyday task. (Your students might enjoy *Rube Goldberg Inventions* by Maynard Frank Wolfe (Simon & Schuster, 2000) as well as the Web site www.rube-goldberg.com.) The cartoon on Transparency 2-3-3 is for use in illustrating the idea of a direct proof as in the lesson in the text. Explain that the cartoon depicts an outlandish device for scratching a person's back. If the curtain catches fire, the fire department will send water through the window; if water comes through the window, the man by the window will think it is raining—and so on—until, if the pup's cradle is rocked, the wooden hand will scratch the man's back. If all of the premises are true, we can conclude that, "if the curtain catches fire, then the wooden hand will scratch the man's back."

After representing the pattern of the argument in symbols, you can use Transparency 2-3-4 to illustrate the shortest such argument, the syllogism. The photograph is of a marble bust of Aristotle, the Greek philosopher and scientist who lived about a hundred years before Euclid. The work of Aristotle on the subject of logic is credited for contributing indirectly to Euclid's *Elements.* It was in Aristotle's writings that the syllogism was first discussed. The words "syllogism" and "logic" both derive from the Greek word "logos," meaning "word" or "reason."

Lesson 4
Indirect Proof

Lesson Outline
Terms: Indirect proof.
Ideas: Pattern of indirect proof.

Sea Lion Logic



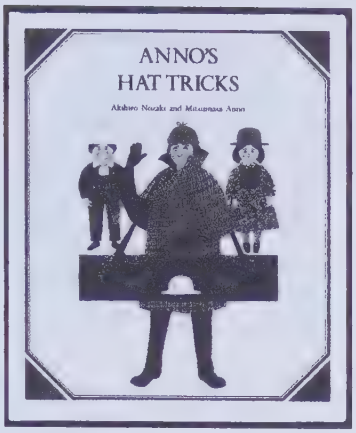
$a = b$
$b = c$
Therefore, $a = c$

$a \rightarrow b$
$b \rightarrow c$
Therefore, $a \rightarrow c$

TRANSPARENCY 2-4-1

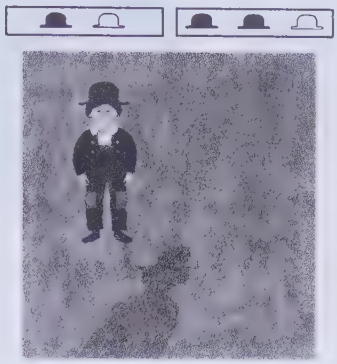
Transparencies
2-4-1 Sea lion solving logic problem
2-4-2 Anno's Hat Tricks book cover
2-4-3 Anno's hat tricks
2-4-4 The third hat trick explained
2-4-5 Trading desks proof (Set II, Ex. 15)
2-4-6 Poker proof (Set II, Ex. 17)
2-4-7 Balanced weights puzzles (Set II, Ex. 18)

Review
Scientific researchers working with sea lions have found that these animals can solve simple problems in logic. The photograph on Transparency 2-4-1 shows a seal comparing three different symbols. If the seal has learned that symbols *a* and *b* are equivalent and that symbols *b* and *c* are equivalent, it concludes that symbols *a* and *c* are equivalent.



TRANSPARENCY 2-4-2

Anno's Hat Tricks



TRANSPARENCY 2-4-3

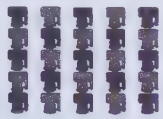
The Third Hat Trick Explained

Facts:
There are three hats: two red, one white.
You see a red hat on Tom.
When Tom looks at your hat,
he doesn't know what color his hat is.

Proof that your hat is red:
Suppose that
your hat is not red.
If your hat is not red,
then it is white.
If your hat is white,
then Tom would know
that his hat is red.
This contradicts the fact
that he doesn't know
what color his hat is.
Therefore,
what we supposed is false, and
your hat must be red.

TRANSPARENCY 2-4-4

Trading Desks Proof



Theorem
The pupils can't obey the teacher.

Proof
Suppose that

If they can obey the teacher, the
13 pupils at the black desks
will move to the brown desks.
This contradicts the fact

Therefore,
our assumption is false and

TRANSPARENCY 2-4-5

Poker Proof

Theorem
In this deal
of the cards,
no one holds
a royal flush.



Proof
Suppose that

If someone holds a royal flush,
then one of the "10" cards would
be in his or her hand.
This contradicts the fact

Therefore,

TRANSPARENCY 2-4-6

Balanced Weights Puzzles

Separate the weights into two sets
so that one set will exactly balance
the other.

Puzzle 1



Puzzle 2



TRANSPARENCY 2-4-7

Compare the two arguments below the photograph: $a = b$, $b = c$, therefore, $a = c$; and $a \rightarrow b$, $b \rightarrow c$, therefore, $a \rightarrow c$. The first of these arguments is of the type with which you became acquainted in algebra: in it, a, b, and c represent numbers, and the three statements are equations. The second argument is of the type with which you have become acquainted in geometry. What do a, b, and c represent in it? (Phrases.) What are the three individual statements called? (Conditional statements.) What is the pattern of the three statements called? (A syllogism.) What kind of proof does a syllogism demonstrate? (A direct proof.) The first two statements are the *premises* of the argument. What is the third statement called? (The conclusion.) What is a statement that is proved by reasoning deductively from already accepted statements called? (A theorem.)

Introduction

The book *Anno's Hat Tricks* with text by Akihiro Nozaki and pictures by Mitsumasa Anno (Transparency 2-4-2) includes a series of entertaining puzzles that provide a good introduction to indirect proof. The following list outlines how the puzzles develop:

1. A hatter has one red hat and one white hat. He puts a hat on Tom and a hat on you. Tom looks at your hat and concludes that his own hat is red. What color is your hat? (White.)
2. The hatter takes *two* red hats and one white hat. He puts a hat on Tom and a hat on you. You see that Tom's hat is red. Tom looks at your hat and concludes that his

A Puzzle from Scientific American

A boy and a girl are sitting on the front steps of their commune. "I'm a boy," said the one with black hair. "I'm a girl," said the one with red hair. If at least one of them is lying, who is which?

TRANSPARENCY 2-5-1

Pepper's Ghost Illusion



TRANSPARENCY 2-5-2

Intersecting Lines Theorem



Figure 1 *Figure 2*

Theorem
Two lines cannot intersect in more than one point.

Proof
Suppose that

If they intersect in more than one point, then two points do not determine a line.
This contradicts

Therefore,
our assumption is false and

TRANSPARENCY 2-5-3

own hat is red. What color is your hat? (White. Otherwise, Tom wouldn't have known that his hat was red.)

3. The hatter again takes two red hats and one white hat. He puts a hat on Tom and a hat on you. You see that Tom's hat is red. Tom looks at your hat but doesn't know what color his hat is. What color is your hat? (Red. Otherwise, Tom would know that his hat was red.)

Although these tricks are considered in the text, your class might enjoy seeing these puzzles enacted with two students: one playing Tom and facing the class; the other playing you (the reader of the book) and facing Tom. Alternatively, you might show Transparency 2-4-3 with the upper right box showing three hats covered. After the first puzzle, cover the upper left box, uncover the upper right box and proceed with the remaining puzzles. Transparency 2-4-4 contains the final argument put into the format of an indirect proof as in the text. The most effective way to use it is to begin with the transparency covered with a sheet of paper and then slide the paper downward to enable your students to anticipate the progression of the proof.

Lesson 5
A Deductive System

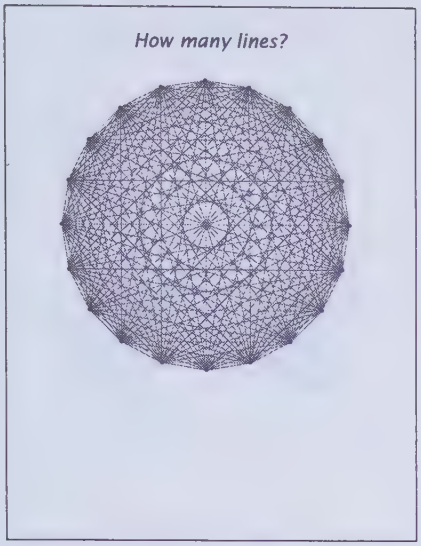
Lesson Outline
Terms: Undefined terms, postulate, theorem, deductive system.
Postulate 1: Two points determine a line.
Postulate 2: Three noncollinear points determine a plane.
Transparencies
2-5-1 A puzzle from *Scientific American*
2-5-2 Pepper's ghost illusion
2-5-3 Intersecting lines theorem (Set II, Ex. 41)
2-5-4 How many lines? (Set III)

Review
The puzzle on Transparency 2-5-1 once appeared in Martin Gardner's "Mathematical Games" column in *Scientific American* (July 1971) and provides a good review of indirect reasoning:

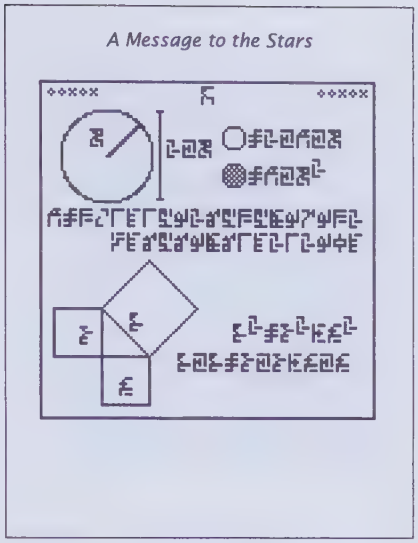
"A boy and a girl are sitting on the front steps of their commune. 'I'm a boy,' said the one with black hair. 'I'm a girl,' said the one with red hair. If at least one of them is lying, who is which?"

As Gardner explains, there are four possible true–false combinations for the two statements: TT, TF, FT, and FF. The first is eliminated because it contradicts the fact that at least one of the two is lying. The second and third are eliminated because they lead to the conclusions that both are boys or both are girls, each of which contradicts the fact that one is a boy and one is a girl. So we are left with the conclusion that both lied. The person with black hair is a girl and the person with red hair is a boy.

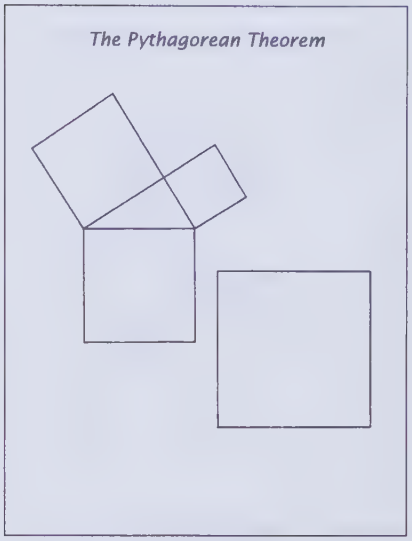
Introduction
A very popular entertainment in London in 1862 was Pepper's Ghost Illusion. Something similar to it is used in the Haunted House in Disneyland. The picture on Transparency 2-5-2 shows how the ghost illusion worked. A large sheet of glass was set up on a stage so that it was tilted toward the audience. An actor playing a ghost stood below the level of



TRANSPARENCY 2-5-4



TRANSPARENCY 2-6-1



TRANSPARENCY 2-6-2

the stage. Light projected onto the actor was reflected onto the glass, creating the illusion of a ghost interacting with the actors on the stage. The lighting in the theater was such that the audience saw the ghost but not the glass, and the resulting effect caused a sensation.*

This illusion was based on clever lighting and some three-dimensional geometry. To understand how it works, it is convenient to think of the positions of the real ghost and the illusion on the stage as *points*, the light producing the illusion as traveling along *lines*, and the plate of glass and the floor of the stage as parts of *planes*.

After this introduction, the rest of the lesson can be presented as in the text. Having your students make the following outline should help them understand the components of a deductive system.

A Deductive System
Undefined terms: to avoid circularity.
Definitions.
Postulates (assumed statements): to avoid circularity.
Theorems (proved statements): proved directly or indirectly by using rules of logic.

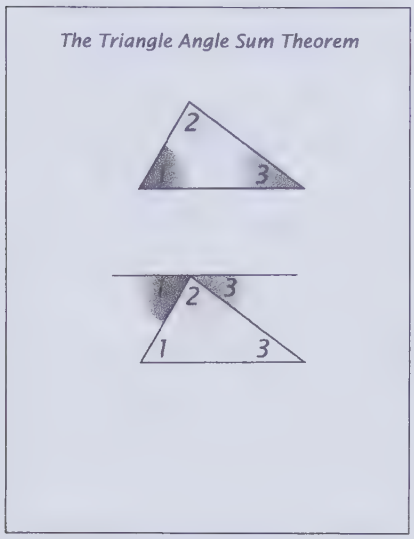
Lesson 6
Some Famous Theorems of Geometry

Lesson Outline
Terms: Hypotenuse, circumference, diameter, pi.
The Pythagorean Theorem: The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.
The Triangle Angle Sum Theorem: The sum of the angles of a triangle is 180°.
Two Circle Theorems: If the diameter of a circle is d, its circumference is πd. If the radius of a circle is r, its area is πr^2.
Transparencies
2-6-1 A message to the stars
2-6-2 The Pythagorean Theorem
2-6-3 The Triangle Angle Sum Theorem

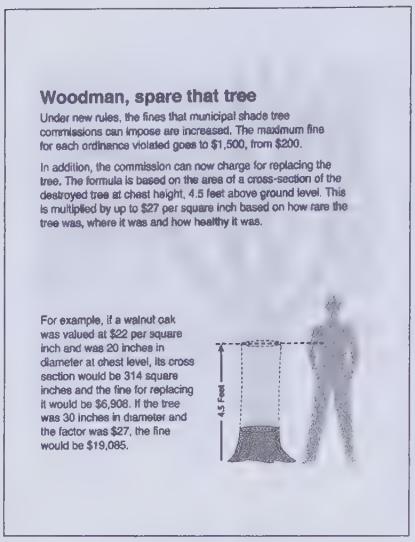
Introduction
In 1999, two Canadians, astrophysicist Yvan Dutil and cryptographer Stephane Dumas, sent a message into space in the hope of reaching extraterrestrial intelligence. Called the Evpatoria Message or Cosmic Call, it consisted of 23 pages of information about our mathematics, ourselves, Earth, and our solar system.† Page 5 is about geometry and is reproduced on Transparency 2-6-1. I suggest making copies of this page so that your students can mark it as you consider it in class. The figures and the symbols accompanying them represent three of the most useful and famous theorems of geometry. Can you guess what either figure is about? (The first figure shows a circle with a radius. The symbol by the radius represents its length, R, and the vertical segment to the right of the circle represents the length of its diameter. To the immediate right of this segment are three symbols representing 2 × R. The two rows of symbols to the right of these symbols

*_The Great Illusionists_, by Edwin A. Dawes (Chartwell Books, 1979).

†The entire message can be found on the Web site www3.sympatico.ca/stephane_dumas/CETI/#1999_MESSAGE.



TRANSPARENCY 2-6-3



TRANSPARENCY 2-R-1



TRANSPARENCY 2-R-2

represent $C = 2 \times \pi \times R$ and $A = \pi \times R^2$. The two long rows of symbols directly below list some of the digits of π. ($\pi = 3.14159 \ldots$)

The lower figure, showing squares drawn on the three sides of a right triangle, represents the Pythagorean Theorem. The three symbols on the sides, starting with the hypotenuse and reading clockwise, are c, b, and a. The two rows of symbols to the right of the figure represent $c^2 = a^2 + b^2$ and $c \times c = a \times a + b \times b$.

Although formal proofs of the theorems of Lesson 6 will not be considered until later in the course, you will want to present the area demonstration of the Pythagorean Theorem illustrated at the top of page 66 of the text. Transparency 2-6-2 can be used for this purpose. In advance of class, cut from thick sheets of colored plastic the three squares and four copies of the triangle shown in the top figure of the transparency. The triangles should be one color and the squares another. Place the three squares and one of the triangles on the transparency figure and point out that we want to show that it is reasonable to think that the sum of the areas of the two smaller squares is equal to the area of the largest one. Then arrange the triangle and large square in the frame at the bottom of the transparency as shown in the third figure at the top of page 66 of the text. Fill in the remaining spaces with the other three triangles. Explain that we can remove the large square and rearrange the four triangles so that there is exactly enough room for the two smaller squares. In this demonstration, you might leave the triangle at the lower left in place and slide the other three triangles to the appropriate positions so that the two smaller squares can be fitted into place.

Transparency 2-6-3 is for use in illustrating the Triangle Angle Sum Theorem.

Review of Chapter 2

Transparencies

2-R-1 Woodman, spare that tree
2-R-2 Backgammon sculpture
2-R-3 Some rules of backgammon
2-R-4 Putting knowledge to work
2-R-5 Dialogue with an expert system
2-R-6 Expert system's explanation
2-R-7 Panda proof (Set II, Exs. 40 and 41)
2-R-8 Clue proof (Set II, Exs. 42 and 43)
2-R-9 Circle in the square (Set II, Exs. 44–49)

First day

Introduction

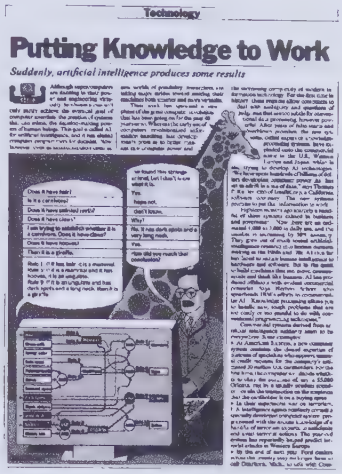
A newspaper article tells of rules establishing fines for cutting down city shade trees. According to the article, "The formula is based on the area of a cross-section of the destroyed tree at chest height, 4.5 feet above ground level. This is multiplied by up to $27 per square inch based on how rare the tree was, where it was and how healthy it was."

Checking the numbers in the example in the article provide a good exercise for your students in reviewing the formula for the area of a circle. The example is somewhat strange for two reasons. First, a tree with a 20-inch diameter has a radius of 10 inches. The cross-sectional area of 314 square inches clearly comes from letting $\pi = 3.14$. Students using the π key on a calculator will get an area of 314.159 . . . square inches and a fine of $6,911.50 rather than

Some Rules of Backgammon

1. A player may double when it is his turn to throw.
2. If a player throws a double, the numbers are used twice.
3. If a player cannot use either number thrown on the dice, the player's turn ends.
4. A point is a blot iff it is occupied by a single checker.
5. A player may not bear a checker off if he has checkers outside his home board.
6. If a checker has been borne off, it cannot be returned to the game.

TRANSPARENCY 2-R-3


Technology

Putting Knowledge to Work

Suddenly, artificial intelligence produces some results

TRANSPARENCY 2-R-4

Dialogue with an Expert System

I've found this strange animal,
but I don't know what it is.
Does it have hair?
Yes.
Is it a carnivore?
I hope not.
Does it have pointed teeth?
I don't know.
Does it have claws?
Why?
I am trying to establish whether it is a carnivore. Does it have claws?
No. It has dark spots and a very long neck.
Does it have hooves?
Yes.
Then it is a giraffe.
How did you reach that conclusion?

TRANSPARENCY 2-R-5

$6,908. Second, if we continue to let $\pi = 3.14$, then a tree with a 30-inch diameter has a cross-sectional area of $(3.14)(15)^2 = 706.5$ square inches; a fine factor of $27 gives $19,075.50, rather than $19,085. On the other hand, if the π key on the calculator is used, the fine becomes about $19,085, the number reported in the article!

Second day

Introduction

Show Transparency 2-R-2 and ask if anyone can guess what game the two people in the picture are playing. (It looks like backgammon.) Amazingly, the photograph is of a sculpture found by Chinese archeologists in a second-century tomb. Some of the rules of backgammon are listed on Transparency 2-R-3:

1. A player may double when it is his turn to throw.
2. If a player throws a double, the numbers are used twice.
3. If a player cannot use either number thrown on the dice, the player's turn ends.
4. A point is a blot iff it is occupied by a single checker.
5. A player may not bear a checker off if he has checkers outside his home board.
6. If a checker has been borne off, it cannot be returned to the game.

What kind of statements are these rules? (Conditional statements.) What is the hypothesis of statement 1? (When it is his turn to throw.) What is the conclusion of statement 3? (The player's turn ends.) Which statement is a definition? (Statement 4.) What does the symbol "iff" in this statement mean? (If and only if.) Is the converse of every definition true? (Yes.) Why? (Because the word being defined and its definition have the same meaning.) In general, if a conditional statement is true, is it correct to conclude that its converse is true? (No.) Most rules of games are assumed to be true without proof. What are such statements called in geometry? (Postulates.) What do we call statements that are proved? (Theorems.)

Alternate Introduction

An important branch of computer science is artificial intelligence or AI. Artificial intelligence has been defined as "the study of the computations that make it possible to perceive, reason, and act." Its goal is to understand and emulate the ability of humans to reason; included in every AI program are the rules of deductive reasoning. AI programs called expert systems are now used in a wide variety of applications ranging from business to engineering to medical care. An example of a simple expert system illustrated on Transparency 2-R-4 shows how a series of questions lead to the identification of an animal. The dialogue is reproduced on Transparency 2-R-5:

I've found this strange animal, but I don't know what it is.
Does it have hair?
Yes.
Is it a carnivore?
I hope not.
Does it have pointed teeth?
I don't know.

Expert System's Explanation

Rule 1: If it has hair, it is a mammal.

Rule 5: If it is a mammal and it has hooves, it is an ungulate.

Rule 9: If it is an ungulate and has dark spots and a long neck, then it is a giraffe.

TRANSPARENCY 2-R-6

Panda Proof

Theorem
If a panda went to a restaurant, then he eats shoots and leaves.

Proof
If a panda went to a restaurant, he would have a sandwich.

If he took out a gun, he would shoot the waiter.

If he leaves without paying, then the panda eats shoots and leaves.

TRANSPARENCY 2-R-7

Clue Proof

Mr. Boddy has been the victim of foul play. It happened after 4 P.M.

Theorem
Colonel Mustard did it.

Proof

If Colonel Mustard didn't do it, then Miss Scarlet did it.

If it was done in the dining room,

then it happened before noon.
This contradicts the fact

Therefore,
our assumption is false and

TRANSPARENCY 2-R-8

Does it have claws?
Why?
I am trying to establish whether it is a carnivore. Does it have claws?
No. It has dark spots and a very long neck.
Does it have hooves?
Yes.
Then it is a giraffe.
How did you reach that conclusion?

The expert system's answer is reproduced on Transparency 2-R-6:

Rule 1: If it has hair, it is a mammal.
Rule 5: If it is a mammal and it has hooves, it is an ungulate.
Rule 9: If it is an ungulate and has dark spots and a long neck, then it is a giraffe.

These rules can be used to provide a brief review of conditional statements. They also provide an example of a chain of statements similar to but somewhat more complicated than $a \rightarrow b$, $b \rightarrow c$, $c \rightarrow d$. The flow chart at the bottom of the figure on Transparency 2-R-4 is comparable to the structure of a complex proof in geometry.

Circle in the Square



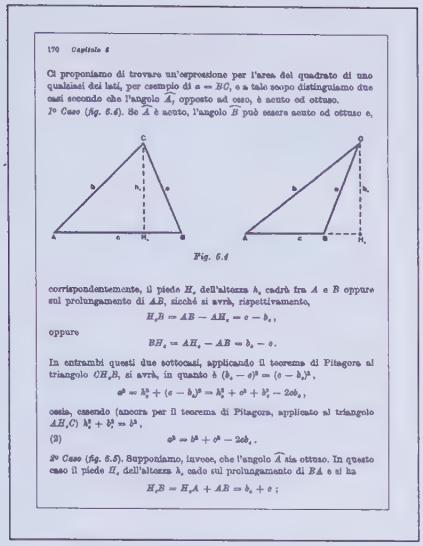
Circumference of the circle

Perimeter of the square

Area of the circle

Area of the square

TRANSPARENCY 2-R-9



170 Capitolo 4

Ci proponiamo di trovare un'espressione per l'area del quadrato di uno qualunque dei lati, per esempio di $a = BC$, e a tale scopo distinguiamo due casi secondo che l'angolo $\widehat{A}$, opposto ad esso, è acuto od ottuso.

1° Caso (fig. 5.4). Se $\widehat{A}$ è acuto, l'angolo $\widehat{B}$ può essere acuto od ottuso e,

Fig. 5.4

corrispondentemente, il piede H_a dell'altezza h_a cadrà fra A e B oppure sul prolungamento di AB, sicché si avrà, rispettivamente,

$$H_aB = AB - AH_a = c - b_a,$$

oppure

$$BH_a = AH_a - AB = b_a - c.$$

In entrambi questi due sottocasi, applicando il teorema di Pitagora al triangolo CH_aB, si avrà, in quanto è $(b_a - c)^2 = (c - b_a)^2$,

$$a^2 = h_a^2 + (c - b_a)^2 = h_a^2 + c^2 + b_a^2 - 2cb_a,$$

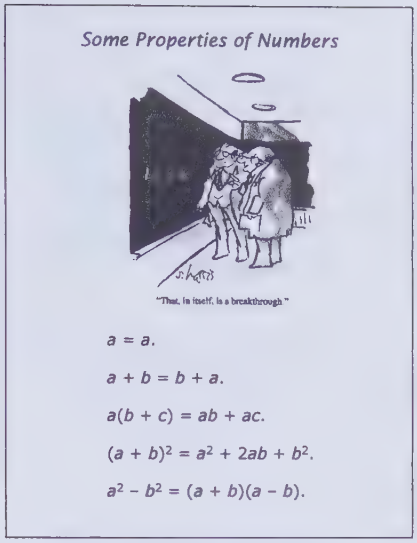
ossia, essendo (ancora per il teorema di Pitagora, applicato al triangolo AH_aC) $h_a^2 + b_a^2 = b^2$,

$$a^2 = b^2 + c^2 - 2cb_a. \tag{2}$$

2° Caso (fig. 5.5). Supponiamo, invece, che l'angolo $\widehat{A}$ sia ottuso. In questo caso il piede H_a dell'altezza h_a cade sul prolungamento di BA e si ha

$$H_aB = H_aA + AB = b_a + c;$$

TRANSPARENCY 3-1-1



Some Properties of Numbers

"That, in itself, is a breakthrough."

$a = a.$

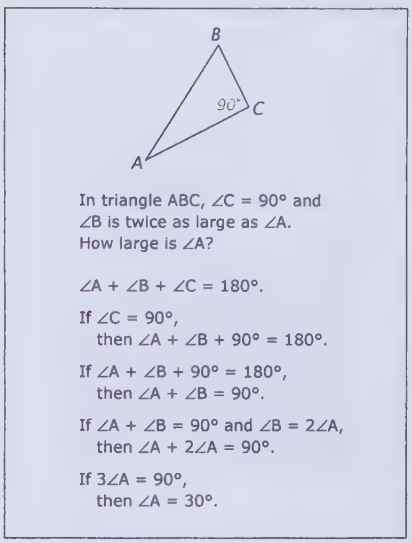
$a + b = b + a.$

$a(b + c) = ab + ac.$

$(a + b)^2 = a^2 + 2ab + b^2.$

$a^2 - b^2 = (a + b)(a - b).$

TRANSPARENCY 3-1-2



In triangle ABC, $\angle C = 90°$ and $\angle B$ is twice as large as $\angle A$. How large is $\angle A$?

$\angle A + \angle B + \angle C = 180°.$

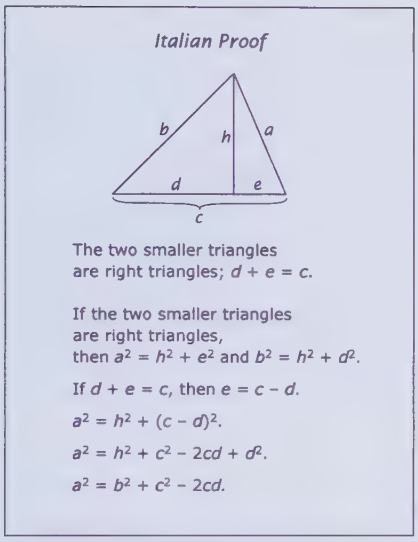
If $\angle C = 90°$,
then $\angle A + \angle B + 90° = 180°.$

If $\angle A + \angle B + 90° = 180°$,
then $\angle A + \angle B = 90°.$

If $\angle A + \angle B = 90°$ and $\angle B = 2\angle A$,
then $\angle A + 2\angle A = 90°.$

If $3\angle A = 90°$,
then $\angle A = 30°.$

TRANSPARENCY 3-1-3



Italian Proof

The two smaller triangles are right triangles; $d + e = c.$

If the two smaller triangles are right triangles, then $a^2 = h^2 + e^2$ and $b^2 = h^2 + d^2.$

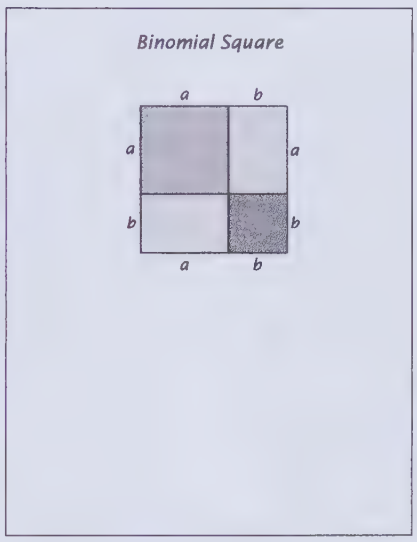
If $d + e = c$, then $e = c - d.$

$a^2 = h^2 + (c - d)^2.$

$a^2 = h^2 + c^2 - 2cd + d^2.$

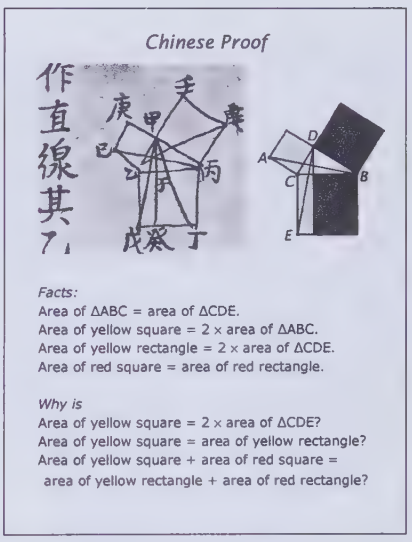
$a^2 = b^2 + c^2 - 2cd.$

TRANSPARENCY 3-1-4



Binomial Square

TRANSPARENCY 3-1-5



Chinese Proof

Facts:
Area of ΔABC = area of ΔCDE.
Area of yellow square = 2 × area of ΔABC.
Area of yellow rectangle = 2 × area of ΔCDE.
Area of red square = area of red rectangle.

Why is
Area of yellow square = 2 × area of ΔCDE?
Area of yellow square = area of yellow rectangle?
Area of yellow square + area of red square =
area of yellow rectangle + area of red rectangle?

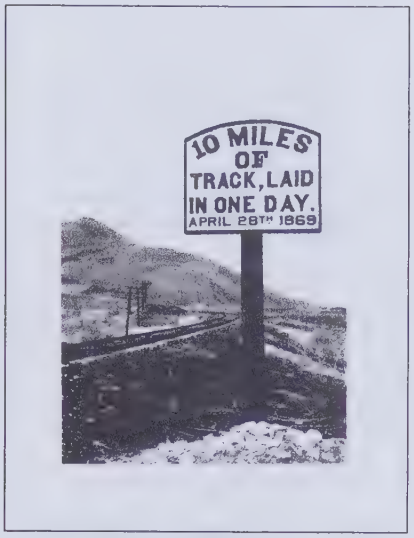
TRANSPARENCY 3-1-6

Chapter 3
Lines and Angles

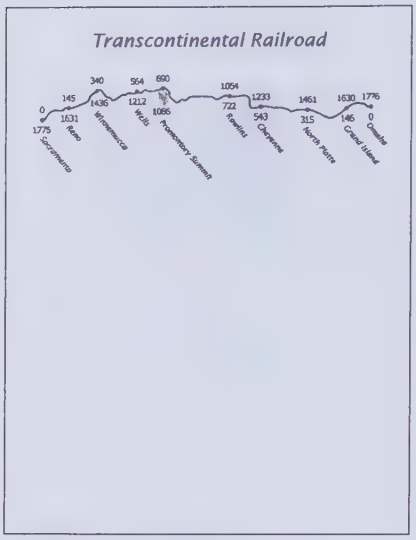
Lesson 1
Number Operations and Equality

Lesson Outline
Terms: Axiom, equal.
Properties of equality: Reflexive, substitution, addition, subtraction, multiplication, division.
Transparencies
3-1-1 Page from Italian geometry book
3-1-2 Some properties of numbers
3-1-3 Properties of equality applied to a geometric figure
3-1-4 Italian proof
3-1-5 Binomial square (Set II, Exs. 25–30)
3-1-6 Chinese proof (Set II, Exs. 31–35)

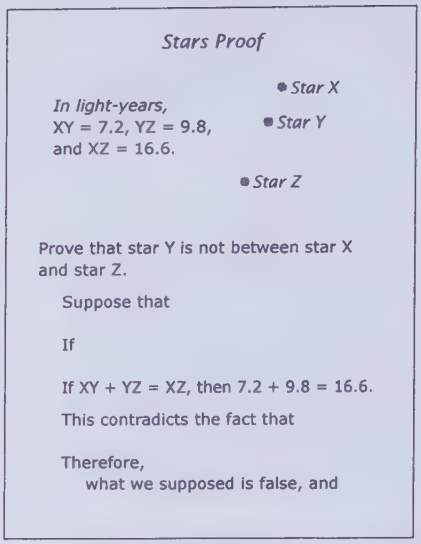
Introduction
The lessons in Chapter 2 ended with one on some famous theorems of geometry. Each of these theorems included information that can be stated in the form of an equation: $c^2 = a^2 + b^2$, $\angle 1 + \angle 2 + \angle 3 = 180°$, $c = \pi d$, $a = \pi r^2$. The page from the Italian geometry book reproduced on Transparency 3-1-1 illustrates the fact that equations are also used in writ-



TRANSPARENCY 3-2-1

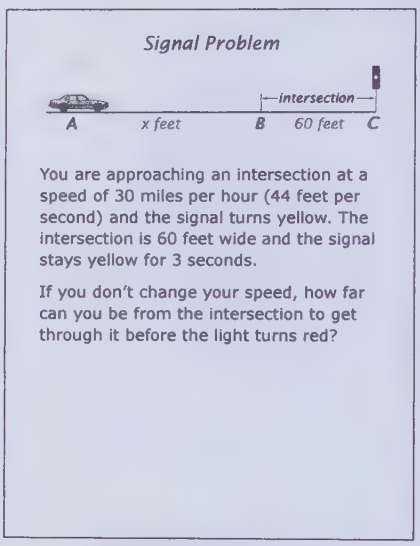


TRANSPARENCY 3-2-2



TRANSPARENCY 3-2-3

ing geometric proofs. (Observe also that there are two references to "il teorema di Pitagora" on this page.) In this lesson, we consider some of the properties of numbers and equations and add them to our logical system. Transparencies 3-1-2 and 3-1-3 are for possible use in developing the lesson as in the text. In using Transparency 3-1-3, keep the statements beginning with $\angle A + \angle B + \angle C = 180°$ covered until your students have had a chance to try to figure out the measure of $\angle A$. Then reveal the lines of the proof one at a time so that your students can anticipate the steps, as well as giving the reasons. If your students are up to it, you might have them take a further look at some of the proof on the page from the Italian geometry book. (Although it is a development of the Law of Cosines, I suggest that you not mention this fact to your students at this point.) Transparency 3-1-4 can be used for this purpose. Again, uncover the steps as your students consider how they are explained.



TRANSPARENCY 3-2-4

Lesson 2
The Ruler and Distance

Lesson Outline

Terms: Ruler, coordinate, distance, between (points).

Postulate 3. The Ruler Postulate: The points on a line can be numbered so that positive number differences measure distances.

Definition: Betweenness of points.

Theorem 1. The Betweenness of Points Theorem. If A-B-C, then AB + BC = AC.

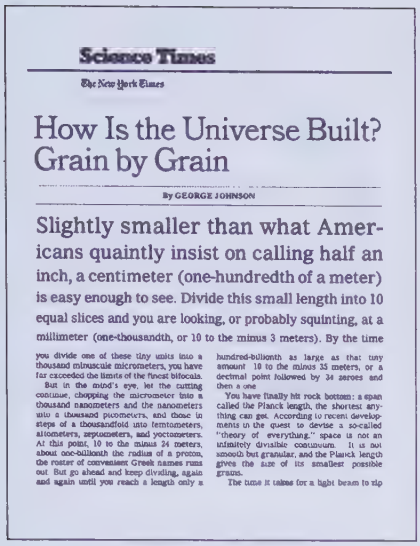
Transparencies

3-2-1 Transcontinental railroad sign
3-2-2 Transcontinental railroad map
3-2-3 Stars proof (Set II, Ex. 45)
3-2-4 Signal problem (Set II, Ex. 46)

Introduction

Show Transparency 3-2-1. Although the date on this sign is April 28, 1869, it was an event 2 weeks later that became famous in American history. Do you know what the event was? (The transcontinental railroad was completed on May 10.) Before the completion of this railroad, it took at least 4 months to travel across the United States. Afterward, the trip could be made in just 1 week.

Hand out duplicated copies of Transparency 3-2-2 to your students. Before 1865, railroad tracks had been built from the east coast as far west as Omaha,



Science Times

The New York Times

How Is the Universe Built? Grain by Grain

By GEORGE JOHNSON

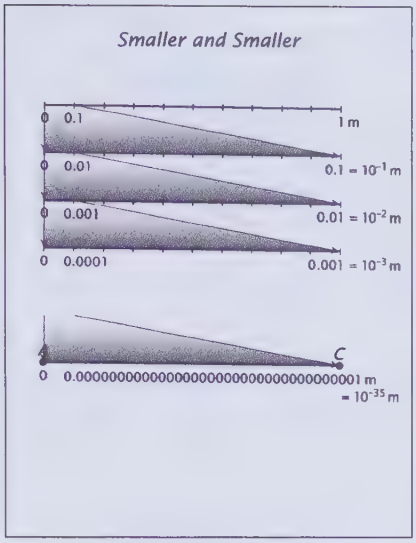
Slightly smaller than what Americans quaintly insist on calling half an inch, a centimeter (one-hundredth of a meter) is easy enough to see. Divide this small length into 10 equal slices and you are looking, or probably squinting, at a millimeter (one-thousandth, or 10 to the minus 3 meters). By the time you divide one of these tiny units into a thousand minuscule micrometers, you have far exceeded the limits of the finest bifocals.

But in the mind's eye, let the cutting continue, chopping the micrometer into a thousand nanometers and the nanometers into a thousand picometers, and those in steps of a thousandfold into femtometers, attometers, zeptometers, and yoctometers. At this point, 10 to the minus 24 meters, about one-billionth the radius of a proton, the roster of convenient Greek names runs out. But go ahead and keep dividing, again and again until you reach a length only a hundred-billionth as large as that tiny amount 10 to the minus 35 meters, or a decimal point followed by 34 zeroes and then a one.

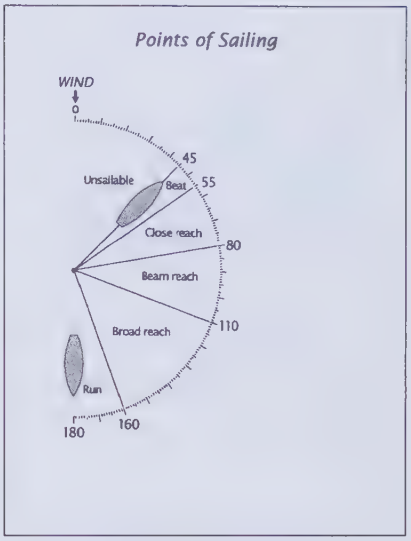
You have finally hit rock bottom: a span called the Planck length, the shortest anything can get. According to recent developments in the quest to devise a so-called "theory of everything," space is not an infinitely divisible continuum. It is not smooth but granular, and the Planck length gives the size of its smallest possible grains.

The time it takes for a light beam to zip

TRANSPARENCY 3-3-1



TRANSPARENCY 3-3-2



TRANSPARENCY 3-3-3

Nebraska. In that year, Union Pacific began laying tracks westward at the same time that Central Pacific was laying tracks eastward from Sacramento. It was Central Pacific that managed to put down 10 miles of tracks in just 1 day just before the two tracks met at Promontory Summit just north of the Great Salt Lake. Two sets of numbers appear on the map on Transparency 3-2-2. What do you suppose they represent? (The distances in miles of the indicated points along the railroad from Sacramento and from Omaha.) How many miles of track did Central Pacific lay from Sacramento to Promontory Summit? (690.) How many miles of track did Union Pacific lay from Omaha to Promontory Summit? (1,086.) How many miles of track connected Reno to Cheyenne? (1,088.) Explain that the two sets of numbers are two coordinate systems. The question about the distance from Reno to Cheyenne reveals that the distance between two points can be found as the *positive difference* between their coordinates. Further examples from the railroad map can be used to develop the rest of the lesson, including the Ruler Postulate and the ideas concerning betweenness of points. Some students may enjoy learning more about the construction of the transcontinental railroad from the Web site, www.pbs.org/wgbh/amex/tcrr/sfeature/sf_map.html, which also features an attractive interactive map.

Lesson 3
The Protractor and Angle Measure

Lesson Outline

Terms: Protractor, degree, rotation of rays, half-rotation of rays, coordinate, measure of angle, acute, right, obtuse, straight, between (rays).

Postulate 4. The Protractor Postulate: The rays in a half-rotation can be numbered from 0 to 180 so that number differences measure angles.

Definition: Betweenness of rays.

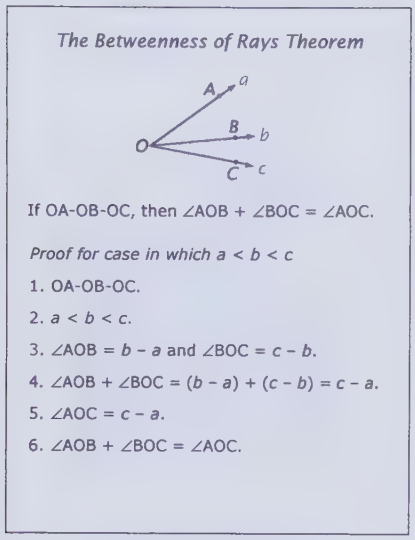
Theorem 2: The Betweenness of Rays Theorem. If OA-OB-OC, then $\angle AOB + \angle BOC = \angle AOC$.

Transparencies

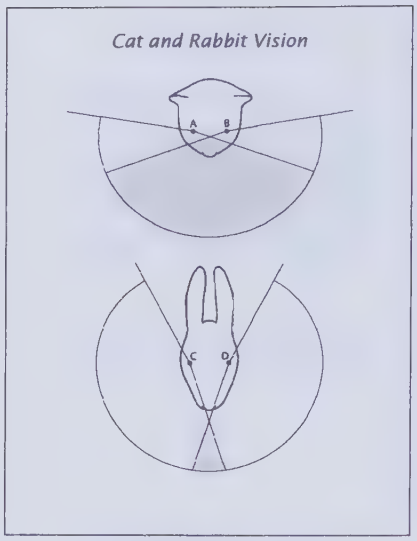
3-3-1 How is the universe built?
3-3-2 Smaller and smaller
3-3-3 Points of sailing
3-3-4 Betweenness of Rays Theorem (Set II, Ex. 54)

Review

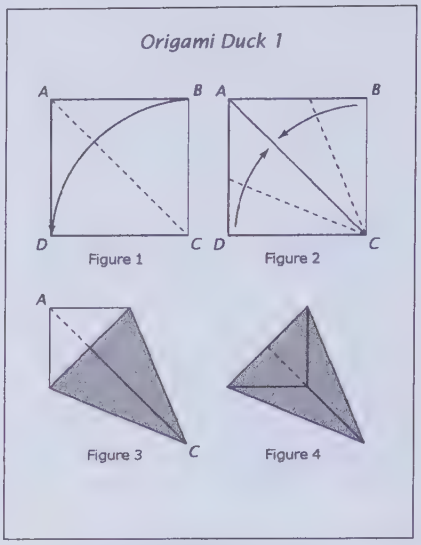
On Transparency 3-3-1 is the beginning of an article from the *New York Times* (December 7, 1999), about the structure of the universe. It concerns a property of physical space that seems almost impossible to believe because it is so contrary to our intuition. The Ruler Postulate is based on thinking of a ruler as an infinite number line. The points on the line are numbered so that to every point there corresponds exactly one real number called its *coordinate* and to every real number there corresponds exactly one point. Show Transparency 3-3-2. The figure at the top illustrates the division of a length of 1 meter into 10 equal parts called decimeters (1 dm = 0.1 m = 10^{-1} m. The second figure shows the division of 1 decimeter into 10 equal centimeters (1 cm = 0.01 m = 10^{-2} m). The third figure shows 1 centimeter divided into 10 equal millimeters, the smallest length that you see marked on a ruler (1 mm = 0.001 m = 10^{-3} m). If we imagine continuing this process, we even-



TRANSPARENCY 3-3-4



TRANSPARENCY 3-4-1



TRANSPARENCY 3-4-2

tually get to a length of 10^{-35} m, and here is the surprise. As the *Times* article puts it: "You have finally reached rock bottom: a span called the Planck length, the shortest anything can get. According to recent developments in the quest to devise a so-called 'theory of everything,' space is not an infinitely indivisible continuum. It is not smooth but granular, and the Planck length gives the size of its smallest possible grains."

It seems natural to be able to choose points between the points A and C that mark the endpoints of the Planck length. Just choose any number less than 10^{-35} and let it be the coordinate of the point. By our definition of betweenness of points, if the coordinate of the point, B, is n and $0 < n < 10^{-35}$, then A-B-C. The Ruler Postulate treats geometric "lines" as being "continuous," meaning that they can be subdivided indefinitely. Physical "lines," on the other hand, do not seem to be continuous in that they cannot be subdivided indefinitely! As J. L. Heilbron remarks in his *Geometry Civilized* (Clarendon Press, 1998): "None of these things—points, lines, or planes—exist in the word of immediate experience. And yet they exist. By the powerful act of your imagination, you have been able to conceive them clearly and distinctly." This is a good reminder that, although geometry provides a powerful model of the physical world, the two are not the same.

Introduction

Hand out duplicated copies of Transparency 3-3-3. The protractor diagram shows some of the "points" of sailing. The arrow at the top indicates the direction of the wind. Most sailboats cannot sail nearer than 45° toward the wind. A boat sailing at angle between 45° and 55° is said to be "beating" the wind. The protractor also shows the directions included in "close reaching," "beam reaching," "broad reaching," and "running" with the wind. The diagram can be used to develop the lesson as in the text. You will probably not have time to make more than a brief outline of the statements of the definitions and theorem.

Lesson 4
Bisection

Lesson Outline

Terms: Bisect, midpoint, angle bisector, congruent, corollary.

Corollary to the Ruler Postulate: A line segment has exactly one midpoint.

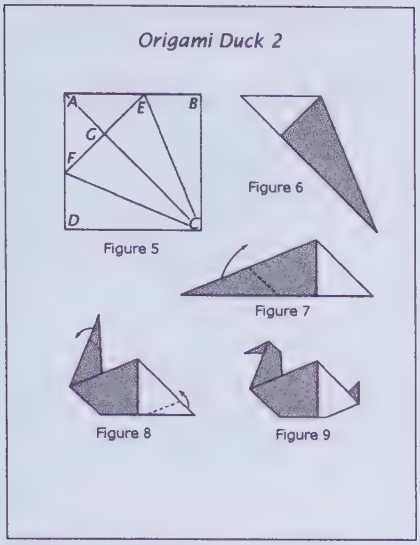
Corollary to the Protractor Postulate: An angle has exactly one ray that bisects it.

Transparencies

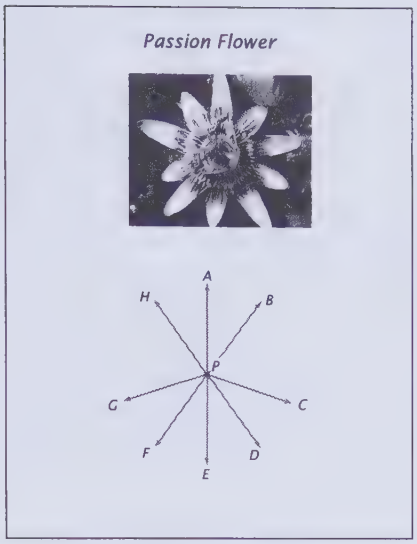
3-4-1 Cat and rabbit vision
3-4-2 Origami duck 1
3-4-3 Origami duck 2
3-4-4 Passion flower (Set II, Exs. 41–47)

Introduction

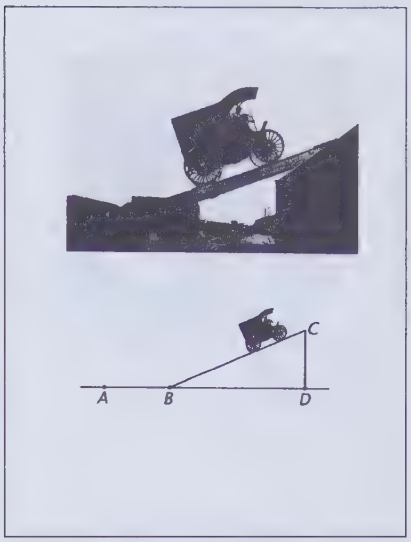
Hand out duplicated copies of Transparency 3-4-1. Cats and rabbits do not see the world in the same way. The eyes of a cat are positioned toward the front of its head. This positioning causes the visual fields of the eyes to overlap, and so the animal has good binocular vision (that is, it is able to see in three-dimensions). The eyes of a rabbit, on the other hand, are on either side of its head. As a result, the



TRANSPARENCY 3-4-3



TRANSPARENCY 3-4-4



TRANSPARENCY 3-5-1

visual fields of the eyes have relatively little overlap, and so the rabbit has a much more panoramic view of its surroundings. In general, predatory animals have better binocular vision, whereas nonpredatory animals have better panoramic vision. To which animal is your vision more comparable? (The cat.) How does the vision of a horse compare with your vision? (It is flatter but more panoramic.)

How do the angles of the visual fields of the individual eyes of a cat and rabbit compare? Have your students use their protractors to measure the angles. (They are the same, about 170°.) To find the centers of the visual fields of the eyes, we can bisect these angles. Use your protractor to do so. We could also have bisected the angles by using a straightedge and compass. Suppose you had neither these tools nor a protractor. Can you think of another way that you might accurately bisect the angles? (By folding the paper.) The rest of the lesson can now be developed as in the text.

Alternate introduction

Because the lesson on bisection is developed around paper folding, your students might enjoy being introduced to it by folding the origami duck pictured on page 101 of the text. I suggest using squares of paper measuring about 5 inches on a side. Illustrations for doing the folding are shown on Transparencies 3-4-2 and 3-4-3, and the directions, all from Peter Engel's *Folding the Universe* (Vintage Books, 1989), are given below:

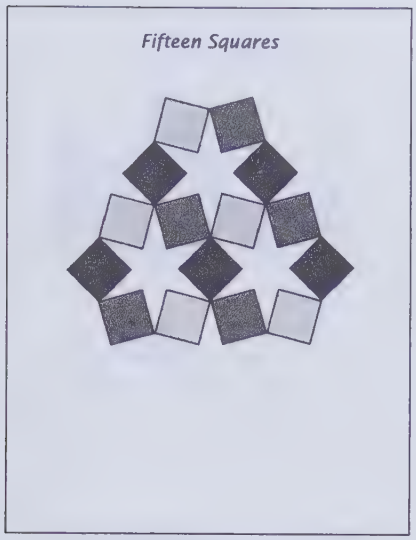
1. Fold corner B over onto corner D. Unfold the paper flat again (Figure 1).
2. Fold sides BC and DC onto the fold AC (Figures 2 and 3).
3. Fold corner A over onto the line where sides BC and DC meet as shown by Figure 4.
4. Unfold the paper flat again and label the points indicated on Figure 5.

Before proceeding any further, ask the following questions. Which angles were bisected by the first fold? (The angles at A and C.) Which angles were bisected next? ($\angle DCA$ and $\angle BCA$.) Can you find any more angles that have been bisected? ($\angle FCE$ and the straight angles at G. We know that $\angle FCE$ has been bisected because $\angle DCA = \angle BCA$ and so $\frac{1}{2}\angle DCA = \frac{1}{2}\angle BCA$; $\angle FCA = \frac{1}{2}\angle DCA$ and $\angle ECA = \frac{1}{2}\angle BCA$; so $\angle FCA = \angle ECA$ by substitution.) Can you find any line segments that have been bisected? (FE.) Introduce the word *congruent* in this discussion.

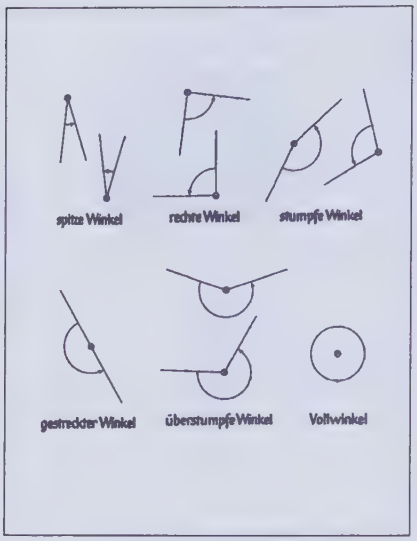
The remaining steps are a bit more complicated:

5. Fold the paper so that it looks like figure 3 again.
6. Refold along AC so that label A appears on the outside rather than the inside (Figure 6).
7. Turn the figure so that it is in the position shown in Figure 7.
8. Fold the lower left part inward as shown in Figure 8 to form the neck.
9. Fold the top of the neck inward to form the head and the lower right part inward to form the tail (Figure 9).

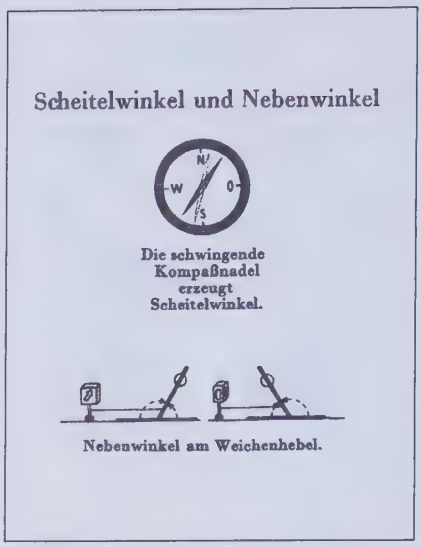
The rest of the lesson can now be presented as in the text.



TRANSPARENCY 3-5-2



TRANSPARENCY 3-6-1



TRANSPARENCY 3-6-2

Lesson 5
Complementary and Supplementary Angles

Lesson Outline

Terms: Complementary (complement), supplementary (supplement).

Theorem 3. Complements of the same angle are equal.

Theorem 4. Supplements of the same angle are equal.

Transparencies

3-5-1 *Popular Mechanics* stunt
3-5-2 Fifteen squares (Set III)

Introduction

In the early days of automobiles, motorists frequently had to get out of their cars and push them when they came to a slight grade. The photograph on Transparency 3-5-1 is from a 1902 issue of *Popular Mechanics* magazine and shows a car being driven up a ramp to the top of a house. The purpose of the photograph was to demonstrate the power and grade-climbing ability of the car, both of which were unusual for the time. Modern cars are more powerful but also a lot heavier. It is hard to imagine such a stunt being carried out today!

The drawing below the photograph represents the ramp, the side of the house, and the ground . Guide your students in making a copy of it by doing the following:

1. Draw a horizontal line to represent the ground. Mark a point B on it to represent the bottom of the ramp and a point D to represent the base of the house.
2. Use your protractor to draw a line to represent the ramp, given that it rises 23° with the ground.
3. Use the protractor to draw a line at D to represent the side of the house.
4. Label the point C in which these two lines intersect to represent the top of the ramp.
5. Choose and label a point A on the ground line as shown in the figure.

Ask your students to measure $\angle C$, the angle that the ramp makes with the side of the house. Can you think of a way to figure out the size of $\angle C$ without measuring it? (By the Triangle Angle Sum Theorem: the sum of the angles of triangle BCD is 180°; $\angle BCD = 90°$ and $\angle CBD = 23°$; so $\angle C = 180° - 90° - 23° = 67°$.) Can you tell the measure of $\angle ABC$ without measuring it? ($\angle ABD$ is a straight angle; so $\angle ABC = 180° - 23° = 157°$.) So $\angle CBD + \angle C = 90°$ and $\angle CBD + \angle ABC = 180°$.) These results lead naturally into a discussion of complementary and supplementary angles and the rest of the lesson.

Lesson 6
Linear Pairs and Vertical Angles

Lesson Outline

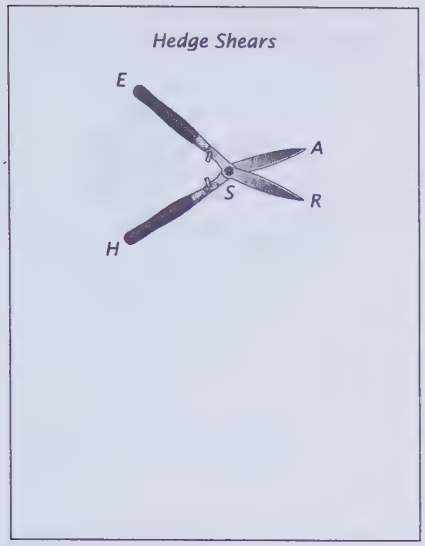
Terms: Opposite rays, linear pair, vertical angles.

Theorem 5. The angles in a linear pair are supplementary.

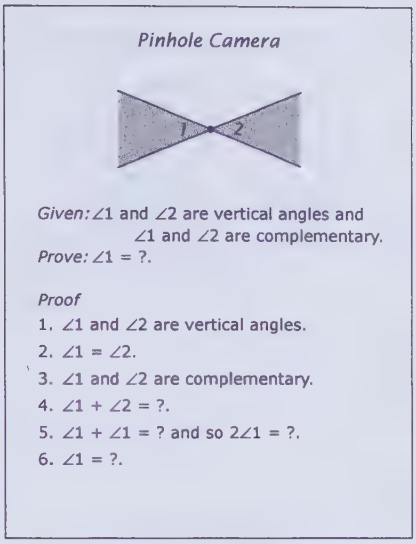
Theorem 6. Vertical angles are equal.

Transparencies

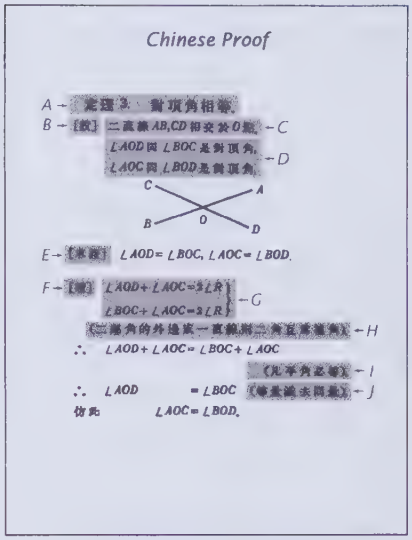
3-6-1 German angles
3-6-2 German angle relations
3-6-3 Hedge shears (Set II, Exs. 40–43)
3-6-4 Pinhole camera (Set II, Exs. 46 and 47)
3-6-5 Chinese proof (Set III)



TRANSPARENCY 3-6-3



TRANSPARENCY 3-6-4



TRANSPARENCY 3-6-5

Introduction

Transparencies 3-6-1 and 3-6-2 show some terms about angles from a German geometry book. The illustrations on the first transparency name angles according to their measures: *Winkel* means "angle"; *spitze* means "pointed" or "acute"; *rechte,* "right"; *stumpfe,* "blunt," "stupid," or "obtuse"; and *gestreckter,* "enlongated" or "straight." The next term names a type of angle that we will not encounter until later: *überstumpe* in German means "beyond stupid" or "beyond obtuse." Our term for it is "reflex." The final term refers to the angle through which something turns in making one full revolution: *Voll* appropriately means "full."

The terms on the second transparency refer to angle relations: *Scheitel* means "part," as in the hair, and *neben* means "beside." The caption of the first illustration states that the swinging compass needle produces "scheitelwinkel," angles that we call vertical. The second illustration shows two positions of a railroad switch, and the caption states that "nebenwinkel," angles that we called a linear pair, appear in a yield lever. With this introduction, the idea of opposite rays can be introduced, and the rest of the lesson can be developed as in the text.

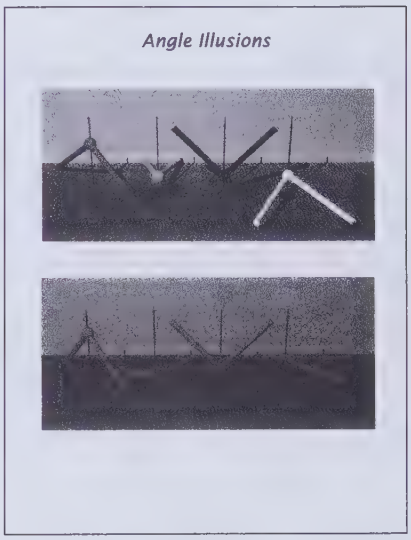
Lesson 7
Perpendicular and Parallel Lines

Lesson Outline

Terms: Perpendicular, parallel.

Theorem 7. Perpendicular lines form four right angles.

Corollary to the definition of a right angle. All right angles are equal.



TRANSPARENCY 3-7-1

Theorem 8. If the angles in a linear pair are equal, then their sides are perpendicular.

Transparencies

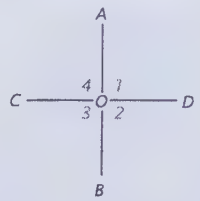
3-7-1 Angle illusions

3-7-2 A deceptive figure (Set II, Exs. 23–27)

Introduction

The photographs on Transparency 3-7-1 were created by scientists at Duke University and are from Al Seckel's beautiful book titled *The Great Book of Optical Illusions* (Firefly Books, 2002). The first photograph shows models of four angles mounted on sticks. Which angle do you think is the largest and which angle do you think is the smallest? (All four angles are right angles!) As Seckel reports: "According to Duke University neuroscientists Dale Purvis,

A Deceptive Figure



In this figure, ∠1 is a right angle but ∠2, ∠3, and ∠4 are not.

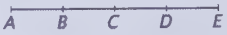
TRANSPARENCY 3-7-2

A Midpoint Problem from the SAT

Points X and Y are the endpoints of a line segment, and the length of the segment is less than 25. There are five other points on the line segment, R, S, T, U, and V, which are located at distances of 1, 3, 6, 10, and 13, respectively, from point X. Which of the points could be the midpoint of XY?

TRANSPARENCY 3-R-1

Another Midpoint Problem from the SAT



Note: Figure not drawn to scale.

On line segment AE above, if the lengths of AC, BD, and CE are equal, which of the following must be true?

I. C is the midpoint of AE.
II. B is the midpoint of AC.
III. BC and DE are equal in length.

TRANSPARENCY 3-R-2

ing to Duke University neuroscientists Dale Purvis, R. Beau Lotto, and Suragit Nundy, the orientation of the angle exerts a strong bias in the judgment of its extent, depending on the frequency of occurrence of particular angles at that orientation in our experience. Angles at the orientation of the red angle tend to be large and [those at the orientation of] the green angle tend to be small, so we overestimate the red as large and we underestimate the green."

How do the angles in the lower photograph appear to compare in size? (Although they may *appear* to be right angles and hence equal, they actually differ considerably in size.)*

These questions lead directly into the definition of perpendicular lines and the rest of the new lesson.

Review of Chapter 3

Transparencies

3-R-1 A midpoint problem from the SAT
3-R-2 Another midpoint problem from the SAT
3-R-3 Chair design (Set II, Exs. 37–40)
3-R-4 The geometry of a pencil point

First day

Introduction

Transparencies 3-R-1 and 3-R-2 contain two problems that have appeared on SAT tests in the past. Both are about midpoints. The first is a good exercise in indirect reasoning and the second, in direct reasoning.

Problem 1. (Suggest that your students draw a figure in which they assign the coordinate 0 to point X. The coordinates of R, S, T, U, and V are 1, 3, 6, 10, and 13, respectively, and the coordinate of Y is less than 25.) Point V cannot be the midpoint of XY, because, if it were, the coordinate of Y would be 26. Point T cannot be the midpoint of XY, because, if it were, the coordinate of Y would be 12; this contradicts the fact that V is on line segment XY. Points R and S are ruled out in the same way; so only U could be the midpoint of XY.

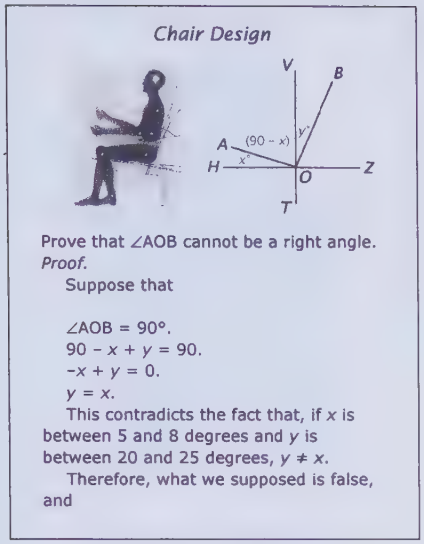
Problem 2. We are given that AC = BD = CE. It immediately follows that AC = CE, and so C is the midpoint of AE. If we imagine that segment BD was moved to the right along the line without changing its length, it is clear that B need not be the midpoint of AC. BC and DE must be equal in length, as the following proof establishes. BD = CE (given). BD = BC + CD and CE = CD + DE (Betweenness of Points Theorem). So BC + CD = CD + DE (substitution) and BC = DE (subtraction).

Second day

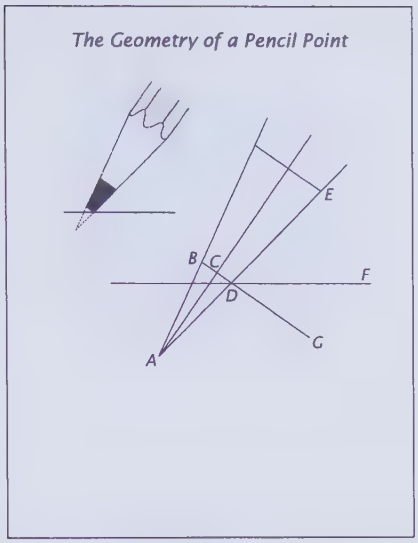
Introduction

Part of the pencil sharpening that goes on in every classroom is due to worn-down pencil leads and part is due to broken pencil points. Many years ago, Don Cronquist, an engineer in California, noticed that almost all broken pencil points have the same size and shape. In analyzing why this should be, Cronquist made drawings similar to the ones on

*More on the topic of angle perception can be found in an article titled "Why We See What We Do" in the May-June 2002 issue of *American Scientist* and in the book of the same name by Dale Purves, Michael A. Wieder, and R. Beau Lotto (Sinauer, 2003).



TRANSPARENCY 3-R-3

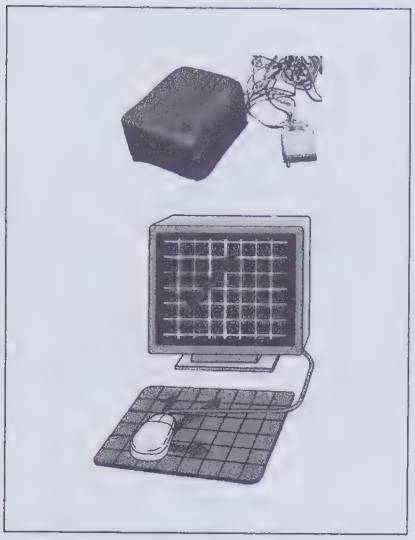


TRANSPARENCY 3-R-4

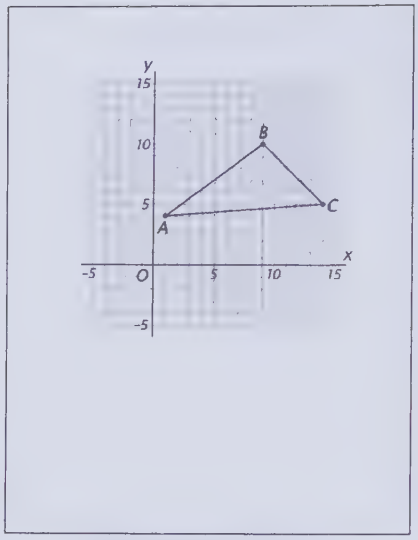
causing pencil points to break; we will look just at the angles in the figure. (This will be easier to do if you hand out duplicated copies of the transparency.) Although a scientist or engineer could use a protractor to measure all of the angles in a given figure, it is usually simpler to start with a few and use geometry to determine the rest. In the lettered figure, AC ⊥ BG and AC bisects ∠BAD. Use your protractor to measure ∠BAD, the angle of the cone of the pencil lead, and ∠EDF, the angle of the lead with the paper. (∠BAD = 20°, ∠EDF = 45°.) Now put your protractor aside and see how many of the following angles you can figure out.

1. ∠CAD. (10°, because AC bisects ∠BAD.)
2. ∠ACD. (90°, because AC ⊥ BG.)
3. ∠CDA. (80°, because it is the third angle in ΔCAD. We have just found the measures of the other two angles and so, from the Triangle Angle Sum Theorem, we have ∠CDA = 180° − 10° − 90°.)
4. What relation does ∠CDA have to ∠CAD? (They are complementary.)
5. ∠EDG. (80°, because ∠EDG and ∠CDA are vertical angles. Vertical angles are equal.)
6. ∠FDG. (35°. DE-DF-DG, and so ∠EDF + ∠FDG = ∠EDG; ∠EDF = 45° and ∠EDG = 80°.)
7. ∠ADG. (100°, because ∠ADG and ∠EDG form a linear pair. The angles in a linear pair are supplementary.)
8. ∠BDE. (100°, because ∠BDE and ∠ADG are vertical angles.)

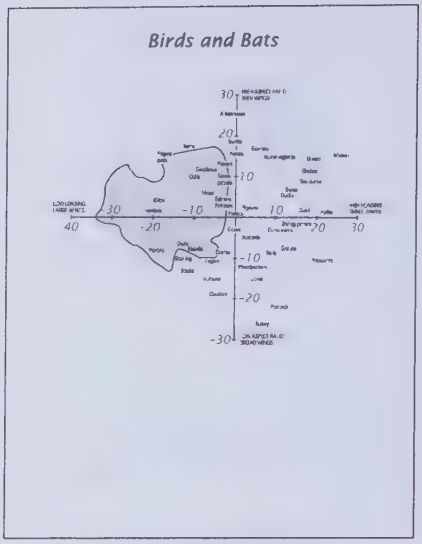
*Don Cronquist's study of broken pencil points was reported by Jearl Walker in his "Amateur Scientist" column in the February 1979 issue of Scientific American. Henry Petroski also includes it in chapter 3, "Pencil Points and Analysis," of his book titled Invention by Design—How Engineers Get from Thought to Thing (Harvard University Press, 1996).



TRANSPARENCY 4-1-1



TRANSPARENCY 4-1-2



TRANSPARENCY 4-1-3

Chapter 4
Congruence

Lesson 1
Coordinates and Distance

Lesson Outline

Terms: Coordinate systems (in one and two dimensions), origin, coordinates, axes, quadrants.

The Distance Formula: The distance between the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Transparencies

4-1-1 Computer mouse
4-1-2 Triangle and coordinate system
4-1-3 Bird and bats (Set I, Exs. 16–22)

Introduction

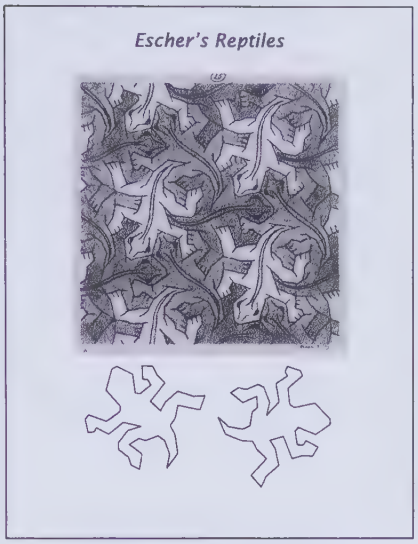
Keeping the rest of Transparency 4-1-1 covered, show the photograph at the top and ask if anyone can identify the object pictured. It is the first computer mouse, built and invented by Douglas Engelbart at the Stanford Research Institute in 1964. Dr. Engelbart called it a mouse "because the tail came out the end," and in his patent application described it as an "*x-y* position indicator for a display system."

Show the figure at the bottom of the transparency. A computer mouse contains a rubber ball that rotates as the mouse is moved. The ball turns two rollers, one to detect horizontal movement and the other to detect vertical movement. This information is sent by electric signals to the computer, which then moves the cursor on the screen accordingly. All of this is possible because the computer screen operates by a coordinate system.

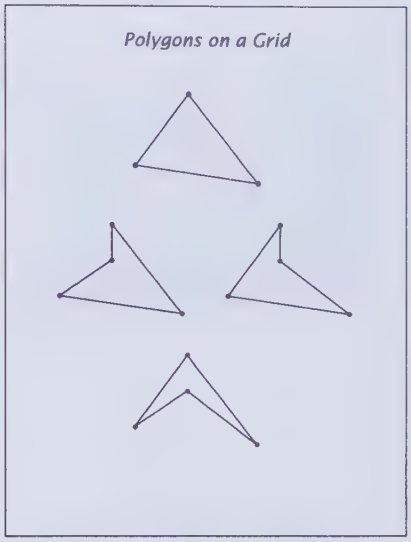
Show Transparency 4-1-2. In your algebra class, you used coordinate systems to picture linear equations in two variables. You learned that such equations are called *linear* because they can be graphed as *lines*. The solutions of a linear equation are pairs of numbers that can be used to locate geometric points. This connection between the numbers of algebra and the points of geometry works both ways. For example, the properties of this triangle can be studied by assigning pairs of numbers to its corners. Add the overlay to the transparency and guide your students in using the resulting coordinates to make an accurate copy of the figure on a sheet of graph paper. [*Note:* Tape the overlay to the transparency so that the coordinates of the vertices of the triangle are A(1, 4), B(9, 10), and C(14, 5).] The basic ideas of the lesson can be introduced as your students copy the figure. After developing the Distance Formula, apply it to finding the lengths of the sides of the triangle in terms of the unit used in the coordinate system. That the calculated lengths are reasonable can be verified by marking the edge of a file card with the units of the graph paper so that it can be used as a ruler to put in place along the sides of the triangle. (AB = 10, BC = $\sqrt{50} \approx 7.1$, AC = $\sqrt{130} \approx 11.4$.)



TRANSPARENCY 4-2-1



TRANSPARENCY 4-2-2



TRANSPARENCY 4-2-3

Lesson 2
Polygons and Congruence

Lesson Outline

Terms: Polygon, sides, vertices, congruent figures, congruence correspondence.

Definition: Congruent triangles.

Corollary to the definition of congruent triangles. Two triangles congruent to a third triangle are congruent to each other.

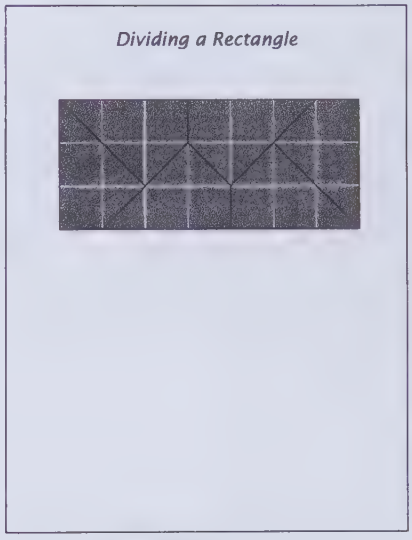
Transparencies

4-2-1 Gecko stones
4-2-2 Escher's reptiles
4-2-3 Polygons on a grid (Set II, Exs. 44–47)
4-2-4 Dividing a rectangle (Set II, Exs. 48–54)

Introduction

The photograph at the top of Transparency 4-2-1 shows a mold for making concrete paving stones in the shape of geckos, lizards found throughout the Hawaiian Islands. Created by John August, an architectural designer based in Maui, the stones can be interlocked with one another to pave walkways, as shown in the photograph at the bottom. (More on August's designs can be found on the following Web site: www.geckostone.com.)

August's paving stones are based on a drawing created by Maurits Escher. In contrast to the curved border of August's gecko figure, Escher's reptiles are bordered by *line segments*. For this reason, they are *polygons*. The lesson can now be developed as in the text. (Escher's reptile polygon is a 50-gon. The two copies of it at the bottom of Transparency 4-2-2 are for use in introducing the idea of a congruence correspondence.)



TRANSPARENCY 4-2-4

Lesson 3
ASA and SAS Congruence

Lesson Outline

Terms: Included and opposite sides and angles.

Postulate 5. The ASA Postulate. If two angles and the included side of one triangle are equal to two angles and the included side of another triangle, the triangles are congruent.

Postulate 6. The SAS Postulate. If two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle, the triangles are congruent.

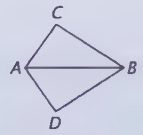
Transparencies

4-3-1 Artillery triangle
4-3-2 Example proof 1

Artillery Triangle

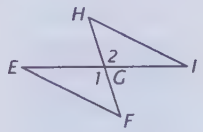


TRANSPARENCY 4-3-1



Given: AB bisects ∠CAD; AC = AD.
Prove: ΔABC ≅ ΔABD.

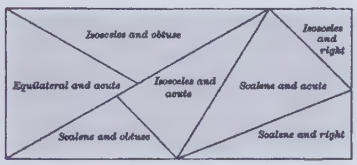
TRANSPARENCY 4-3-2



Given: ∠1 and ∠2 are vertical angles;
G is the midpoint of HF; ∠H = ∠F.
Prove: ΔGHI ≅ ΔGFE.

TRANSPARENCY 4-3-3

Triangle Names



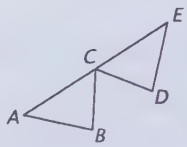
TRANSPARENCY 4-3-4

Trigon

"In astronomy, the aspect of two planets as seen from earth to be 120° apart in the night sky."

—*The Macmillan Dictionary of Measurement*

TRANSPARENCY 4-3-5



Given: C is the midpoint of AE;
∠A = ∠E and ∠ACB = ∠ECD.
Prove: AB = DE.

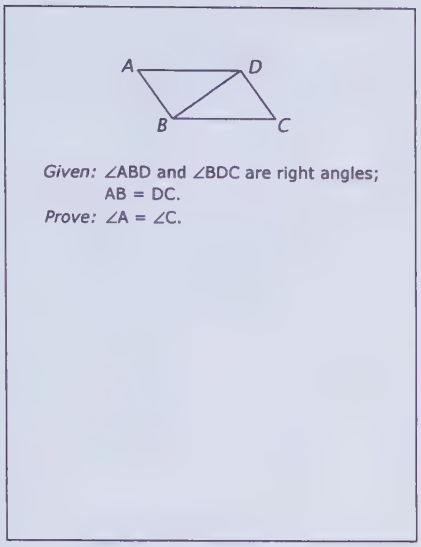
TRANSPARENCY 4-3-6

4-3-3 Example proof 2
4-3-4 Triangle names (Set I, Exs. 16–21)
4-3-5 Trigon
4-3-6 Example proof 3
4-3-7 Example proof 4

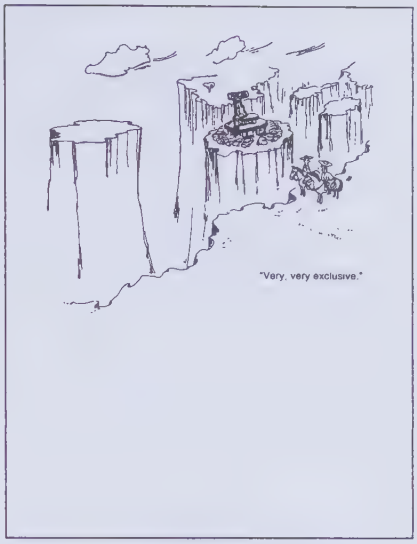
First day

Introduction

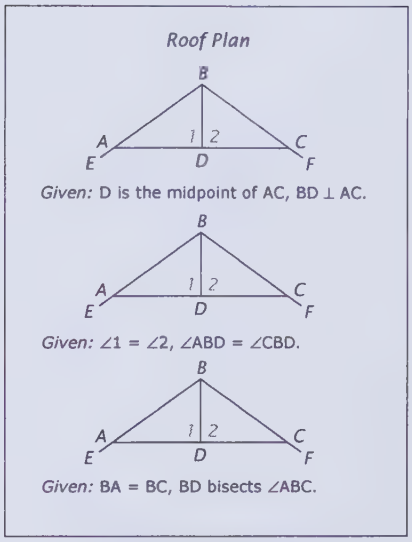
The figure on Transparency 4-3-1, from a book published in 1648, shows a cannon aimed at a target. Two people on either side of the cannon are shown measuring angles at the base of a large triangle to figure out the distance to the target. To see how the method works, we will draw a comparable figure. Suppose that the two angle measurers are 80 yards apart. Draw a horizontal line segment AB 4 inches long to represent this distance. How many yards does 1 inch along AB represent? (20.) Suppose that the person at A measures an angle of 60° between AB and the line of sight to the target. Center your protractor at A and draw a line forming a 60° angle with AB. Suppose that the person at B measures an angle of 75° between AB and the line of sight to the target. Center your protractor at B and draw a line forming a 75° angle with AB. The point in which the two lines intersect marks the location of the target; label it C. Notice that the shape of ΔABC is determined by the length of side AB and the measures of the two angles at its endpoints, ∠A and ∠B. Suppose that the cannon is located midway between A and B. Use your ruler to measure the distance from



TRANSPARENCY 4-3-7



TRANSPARENCY 4-4-1



TRANSPARENCY 4-4-2

the cannon to the target. (About 4.75 in.) Approximately how many yards is it from the cannon to the target? ($4.75 \times 20 = 95$ yd.)

Point out that, in drawing the artillery triangle, we did not need to know the lengths of all three of its sides and the measures of all three of its angles. The rest of the lesson can now be developed as in the text.

Transparencies 4-3-2 and 4-3-3 are for use in providing your students practice in proving triangles congruent.

Second day

Introduction

Show Transparency 4-3-5. The word "trigon" is another word for "triangle," although it is seldom used anymore. According to *The Macmillan Dictionary of Measurement* (Macmillan, 1994), "trigon" has also been used in astronomy to mean "the aspect of two planets as seen from earth to be 120° apart in the night sky."

Have your students use their protractors to draw a 120° angle. Label its vertex E to represent the position of the earth. Now suppose that Jupiter and Saturn are seen along the sides of this angle when their distances from earth are 400,000,000 miles and 950,000,000 miles, respectively. For convenience, we will let 1 cm represent 100,000,000 miles. Mark point J to represent Jupiter on one side of ∠E so that EJ = 4 cm. Mark point S to represent Saturn on the other side of ∠E so that ES = 9.5 cm. Draw and measure segment JS. How far apart are Jupiter and Saturn at this time? (JS ≈ 12 cm; so the distance between Jupiter and Saturn is about 1,200,000,000 miles.) Notice that the shape of ΔJES is determined by the measure of ∠E and the lengths of the two sides EJ and ES. Why is it reasonable to expect that all triangles drawn with these same measurements would be congruent? (They would be congruent by the SAS Congruence Postulate.)

Transparencies 4-3-6 and 4-3-7 are for use in providing your students practice in proving triangles congruent and that corresponding parts of them are equal.

Lesson 4
Congruence Proofs

Lesson Outline

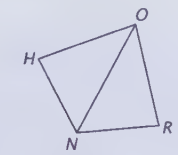
Definition (brief restatement for congruent triangles):
Corresponding parts of congruent triangles are equal.

Transparencies

4-4-1 The cowboys and the drive-in
4-4-2 Roof plan (Set I, Exs. 1–17)
4-4-3 Congruence proof (Set II, Ex. 30)
4-4-4 Congruence proof (Set II, Ex. 31)
4-4-5 Congruence proof (Set II, Ex. 32)
4-4-6 Congruence proof (Set II, Ex. 33)
4-4-7 Grid exercise (Set II, Exs. 34–46)

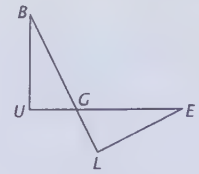
Introduction

The figure on Transparency 4-4-1 is for use in presenting Lesson 4 as in the text. The proof that the procedure works is another example of proving triangles congruent and using the definition of congruent triangles to conclude that their corresponding parts are equal. If you direct your students in



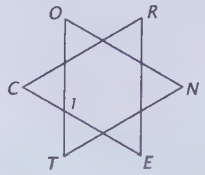
Given: HN = NR; NO bisects ∠HNR.
Prove: ΔHNO ≅ ΔRNO.

TRANSPARENCY 4-4-3



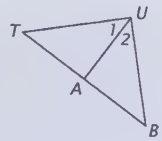
Given: ∠HNR and ∠HNR are vertical angles; BG = GE and UG = GL.
Prove: BU = LE.

TRANSPARENCY 4-4-4



Given: ∠C = ∠O; ∠R and ∠N are supplements of ∠1; CR = ON.
Prove: ΔCRE ≅ ΔONT.

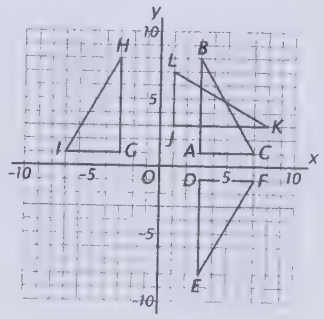
TRANSPARENCY 4-4-5



Given: ∠T and ∠2 are complements of ∠1; TA = AU and TU = UB.
Prove: AU = AB.

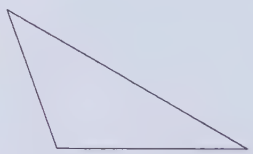
TRANSPARENCY 4-4-6

Grid Exercise



TRANSPARENCY 4-4-7

Triangle Challenge



TRANSPARENCY 4-5-1

writing this proof in their notes, it is best to state the "given" as in the text to ensure that the proof is reasonably straightforward. A method comparable to the one for finding the distance to the drive-in may have been used by the Babylonians and Egyptians for measuring the distances of ships from the shore.

Lesson 5
Isosceles and Equilateral Triangles

Lesson Outline

Terms: Types of triangles: scalene, isosceles, equilateral, obtuse, right, acute, equiangular.

Theorem 9. If two sides of a triangle are equal, the angles opposite them are equal.

Corollary to Theorem 9. An equilateral triangle is equiangular.

Theorem 10. If two angles of a triangle are equal, the sides opposite them are equal.

Corollary to Theorem 10. An equiangular triangle is equilateral.

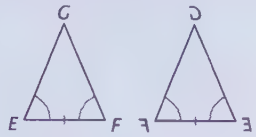
Transparencies

- 4-5-1 Triangle challenge
- 4-5-2 Theorem 10 (Set II, Ex. 33)
- 4-5-3 Corollary to Theorem 10 (Set II, Ex. 35)
- 4-5-4 Star puzzles (Set II, Exs. 43–47)
- 4-5-5 Drawing exercise (Set II, Exs. 49–53)

Introduction

Hand out duplicated copies of Transparency 4-5-1. This triangle once appeared in the "Mathematical

If two angles of a triangle are equal, the sides opposite them are equal.

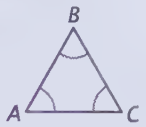


Given: In ΔEFG, ∠E = ∠F.
Prove: GE = GF.

Proof
1. In ΔEFG, ∠E = ∠F.
2. EF = FE.
3. ∠F = ∠E.
4. ΔEFG ≅ ΔFEG.
5. GE = GF.

TRANSPARENCY 4-5-2

An equiangular triangle is equilateral.

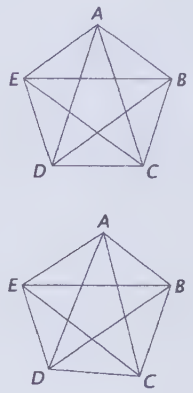


Given: ΔABC is equiangular.
Prove: ΔABC is equilateral.

Proof
1. ΔABC is equiangular.
2. ∠A = ∠B and ∠B = ∠C.
3. BC = AC and AC = AB.
4. BC = AB.
5. ΔABC is equilateral.

TRANSPARENCY 4-5-3

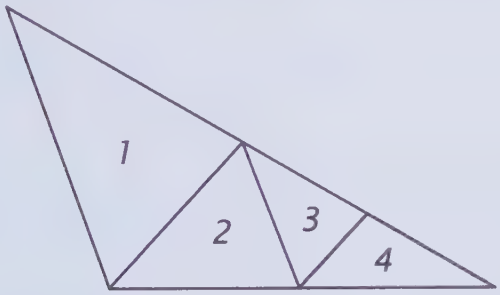
Star Puzzles



TRANSPARENCY 4-5-4

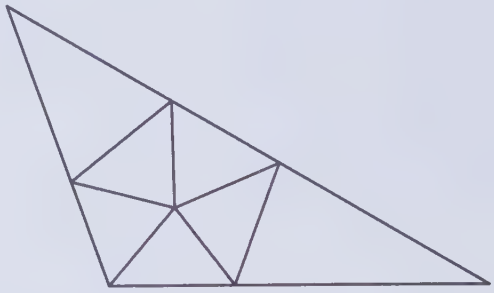
Games" column of *Scientific American* (February 1960). Ask your students to use their protractors to measure its largest angle. (110°.) Explain that this triangle is said to be "obtuse" because it has an obtuse angle, in contrast with a "right" triangle, which has a right angle, and an "acute" triangle, all of whose angles are acute.

The challenge in the magazine was to divide the obtuse triangle into smaller triangles, all of which are acute. As an example of an attempt to do so, add lines to the triangle on the transparency as shown in the following figure.

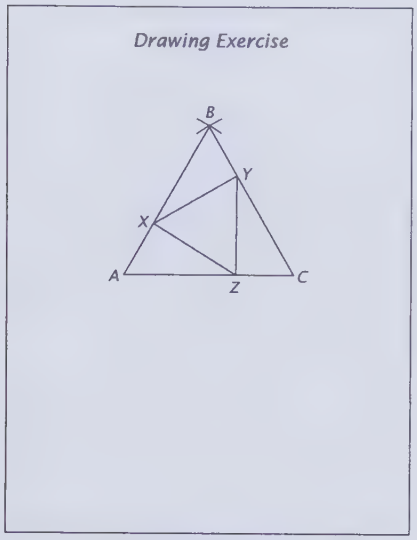


Explain that, although the first three triangles are acute, the fourth is obtuse, and so these lines do not solve the problem. Give your students a few minutes to try to find a solution. (You might even offer a small prize to the first student to do so.) It is unlikely that anyone will be able to solve the problem very quickly. You might want to leave the problem dangling for a few days. If so, you might suggest it as something to work on in other classes, if you get bored!

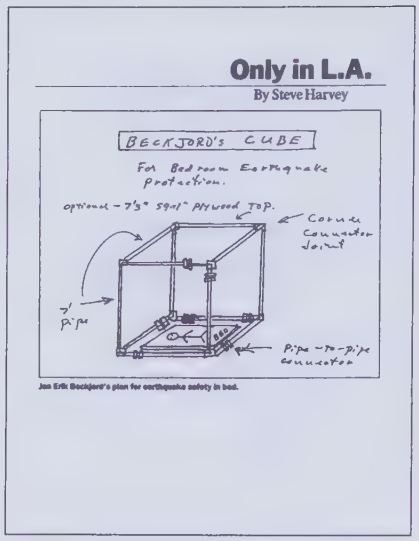
In the book in which the "Mathematical Games" column containing this problem was reprinted, Martin Gardner reported that "A number of readers [of *Scientific American*] sent 'proofs' that an obtuse triangle cannot be dissected into acute triangles, but of course it can. The following figure shows a seven-piece pattern that applies to any obtuse triangle."



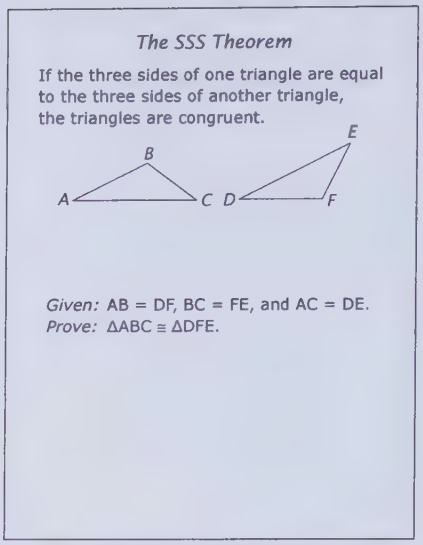
Gardner goes on to explain why at least seven triangles are necessary. The obtuse angle has to be divided by a line. If this line goes all the way to the other side, it will form either another obtuse angle or right angles, all of which would also have to be divided. So the line dividing the obtuse angle has to stop at some point inside the triangle. For all of the angles from this point to be acute, at least five lines have to be drawn. (Four lines would form four angles, but they cannot all be acute, because four numbers each less than 90 cannot add up to 360.) The result is an "inner pentagon" of five triangles, resulting in seven triangles altogether. Gardner also reported that it was a Brooklyn high-school student at the time, Wallace Manheimer, who figured out this proof. It was published in the *American Math-*



TRANSPARENCY 4-5-5



TRANSPARENCY 4-6-1



TRANSPARENCY 4-6-2

ematical Monthly.

In trying to find a solution to this problem, our attention is drawn to the *angles* of triangles. At the same time, we have seen that triangles are classified according to their angles. They are also classified according to their sides. The rest of the lesson can now be developed as in the text. Theorem 9, known as the Isosceles Triangle Theorem, has an interesting history. The faces of the pyramids have the shape of isosceles triangles, and the Egyptians undoubtedly knew as far back as 2700 B.C. that, if two sides of a triangle are equal, the angles opposite them also must be equal. The Greek mathematician Thales, born in 625 B.C., is thought to have learned geometry in his travels to Egypt and may have been the first to prove the Isosceles Triangle Theorem. Euclid, who lived about 300 years later, included the theorem in his *Elements*. The proof of the theorem considered in the text is simpler than Euclid's and is attributed to Pappus, a Greek geometer who lived in Alexandria in about 300 A.D. When computer programs were first developed to prove geometrical theorems in 1956, a computer came up with the very same proof! Some comments by Marvin Minsky about the computer proof can be found on the following Web site: www.math.niu.edu/~rusin/known-math/99/minsky.

Lesson 6
SSS Congruence

Lesson Outline

Theorem 11. The SSS Theorem. If the three sides of one triangle are equal to the three sides of another triangle, the triangles are congruent.

Transparencies

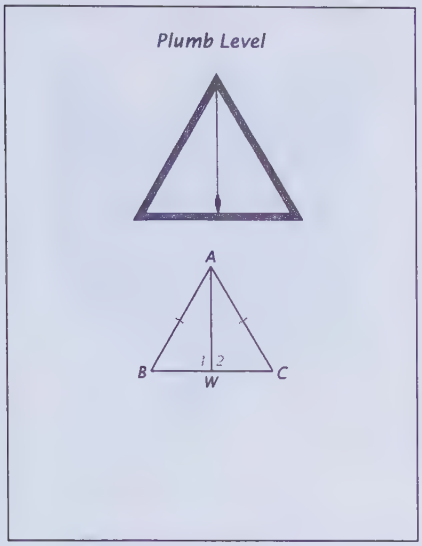
4-6-1 Beckjord's cube
4-6-2 The SSS theorem
4-6-3 Plumb level (Set I, Exs. 3–8)
4-6-4 Linkage problem 1 (Set II, Exs. 26–35)
4-6-5 Linkage problem 2 (Set II, Exs. 36–43)
4-6-6 Grid exercise (Set II, Exs. 44–51)
4-6-7 Bracing the square (Set III)

Introduction

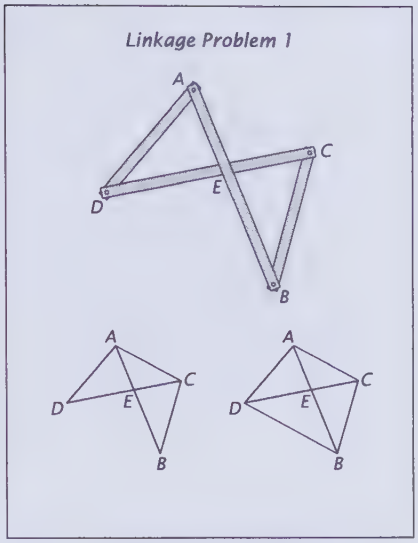
Not long after the big earthquake in Los Angeles in 1994, the figure on Transparency 4-6-1 was published in the *Los Angeles Times* (February 3, 1994). According to the article with which it appeared, actor Jerry Van Dyke had survived the quake because of a four-posted bed that propped up the falling ceiling of his bedroom. This gave Jon Erik Beckjord the idea of developing a cube made out of pipes for bedroom earthquake protection. Beckjord claimed that the thicker the pipe, the more secure the bed inside the cube would be.

A dramatic illustration that such a cubical structure, consisting only of squares, is not rigid would be to show one made of drinking straws threaded together. A second structure to which a diagonal has been added to each face is rigid because the shape of the resulting triangles, unlike that of the squares, is determined by the lengths of their sides.

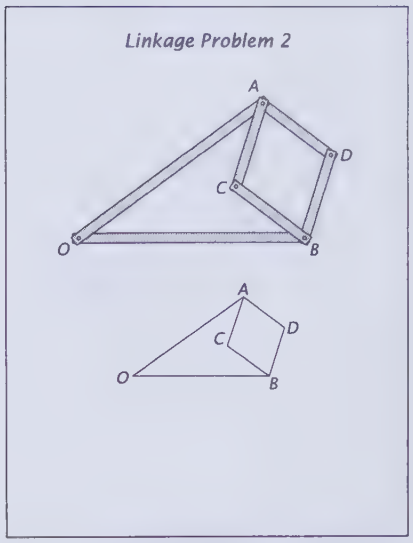
For convenience in developing the proof of the SSS Theorem, I suggest handing out duplicated copies of Transparency 4-6-2. Direct your students in using their protractor and ruler to copy ΔDEF below ΔABC as done in the text. There is room on the



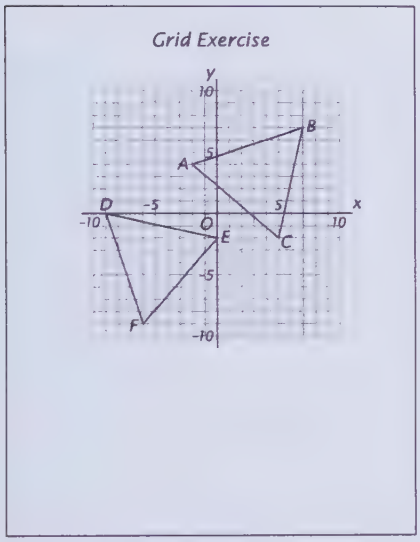
TRANSPARENCY 4-6-3



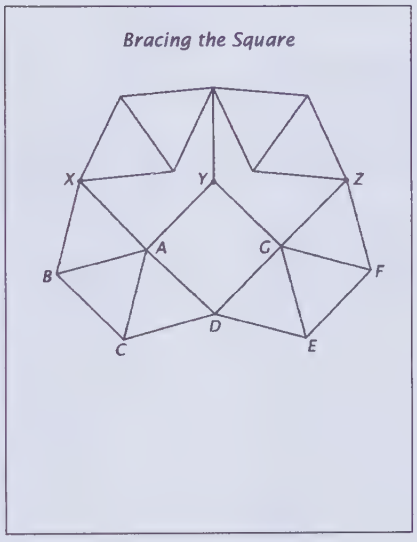
TRANSPARENCY 4-6-4



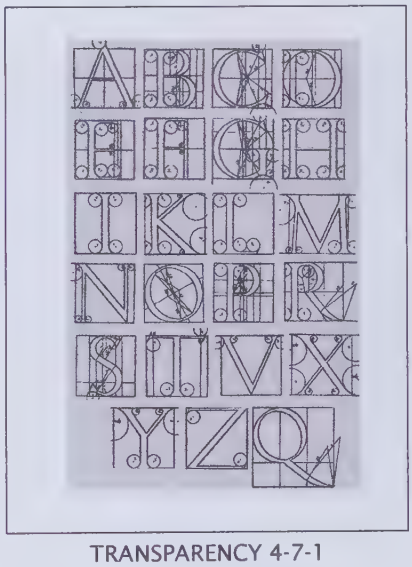
TRANSPARENCY 4-6-5



TRANSPARENCY 4-6-6



TRANSPARENCY 4-6-7



TRANSPARENCY 4-7-1

lower part of the transparency to outline the main ideas of the proof. When you have finished, be sure to let your students know that you hope they have understood how the proof works but that you do not expect them to remember all of the details. The goal here might be compared to understanding the structure of a poem rather than having to memorize it.

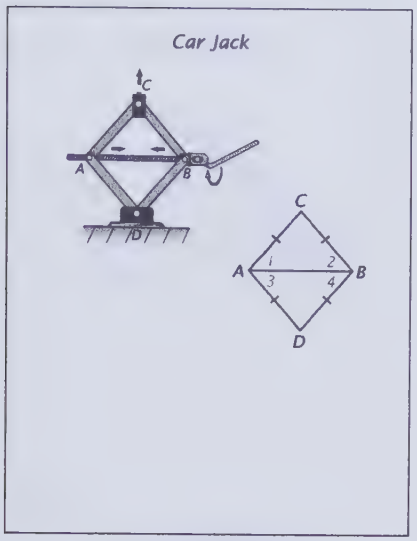
Lesson 7
Constructions

Lesson Outline
Construction 3. To copy a line segment.
Construction 4. To copy an angle.
Construction 5. To copy a triangle.

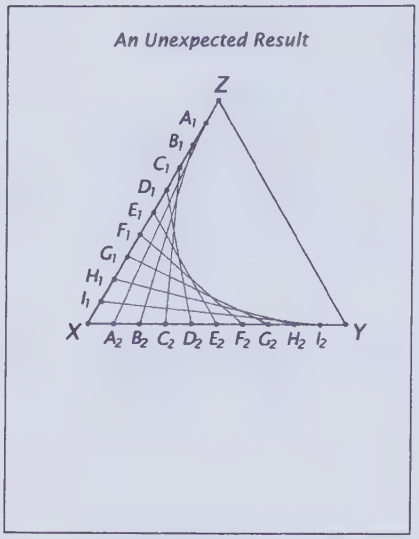
Transparencies
4-7-1 Torniello's alphabet
4-7-2 Car jack (Set I, Exs. 22–29)
4-7-3 An unexpected result (Set II, Exs. 30–33)

Introduction
In the Renaissance, several attempts were made to apply mathematics to designing the letters of the alphabet.* The Italian mathematician Luca Pacioli included an appendix on alphabets in a book on geometry published in 1509. Another Italian, the type designer Francesco Torniello, published a book titled *The Alphabet* in 1517 in which he showed how

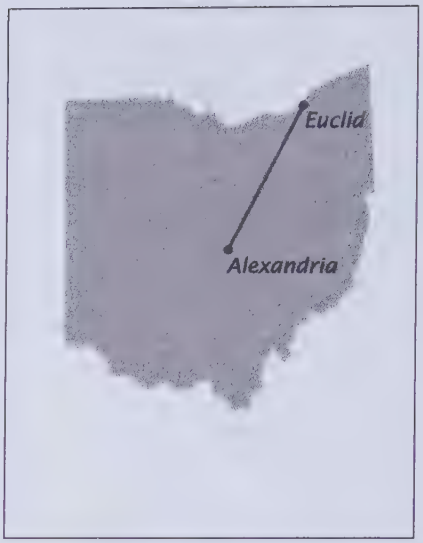
*TEX and METAFONT—New Directions in Typesetting, by Donald E. Knuth (Digital Press, 1979).



TRANSPARENCY 4-7-2



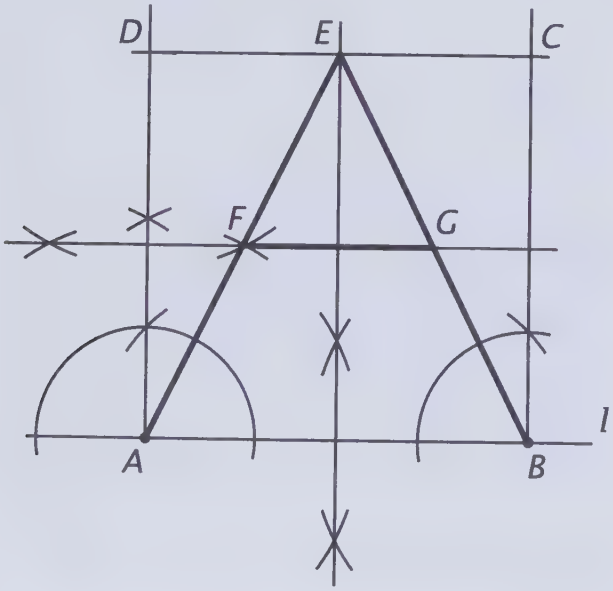
TRANSPARENCY 4-7-3



TRANSPARENCY 4-R-1

to construct capital letters with straightedge and compass. The figures on Transparency 4-7-1 are from Torniello's book. The outline of each letter consists of line segments, drawn with a straightedge, and arcs of circles, drawn with a compass.

Remind your students that it was Euclid who established the use of the straightedge and compass for making geometric constructions at the very beginning of his *Elements*. We constructed an equilateral triangle at the beginning of the course, and we have learned how to bisect both line segments and angles. In this lesson, we will learn how to make three more basic constructions. One way to introduce them would be to construct a simple version of a capital A as shown in the following figure and described in the steps that follow. (Note that the steps review the bisection constructions, as well as introducing the new ones; so demonstrate them at the overhead projector as you lead your students through them. There is no need to justify results such as the formation of the four squares.)



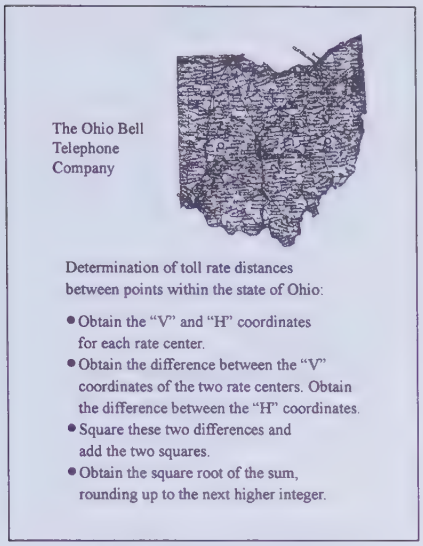
1. Draw a horizontal line l and choose a point A on it.
2. Bisect straight $\angle A$ to form two right angles.
3. Choose a second point B on l approximately 3 inches to the right of A.
4. Construct a right angle at B by copying one of the right angles at A.
5. Copy segment AB to get AD and BC.
6. Bisect AB and AD to divide the square into four smaller squares.
7. Label point E and draw EA and EB. Also draw the cross bar of the capital A and label its endpoints F and G.

Is ΔABE equilateral? (No.) Show your students how to copy the capital A in a different place on the paper by first copying ΔABE and then copying AF and BG to determine FG.

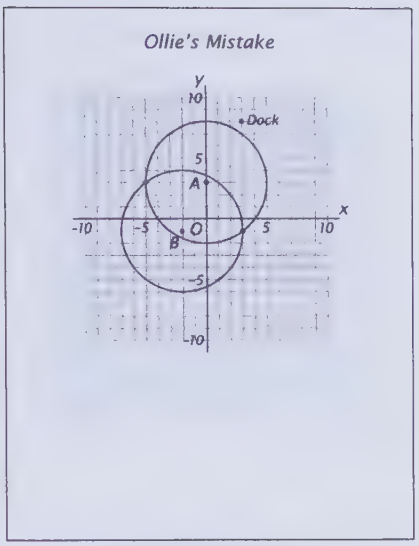
Review of Chapter 4

Transparencies

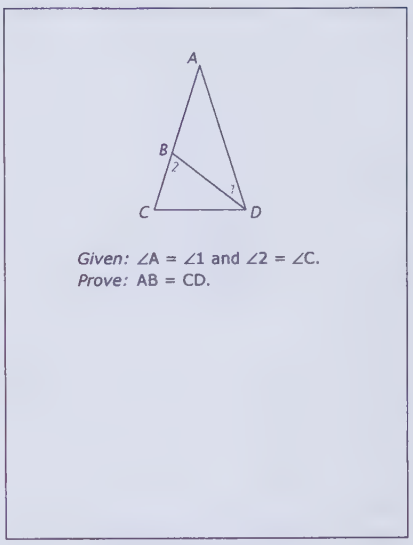
4-R-1 Euclid to Alexandria
4-R-2 Toll rate distances
4-R-3 Ollie's mistake (Set II, Exs. 37–42)
4-R-4 Proof (Set II, Ex. 51)
4-R-5 Proof (Set II, Ex. 52)
4-R-6 Euclid carved in stone
4-R-7 Euclid's *Elements*



TRANSPARENCY 4-R-2



TRANSPARENCY 4-R-3



TRANSPARENCY 4-R-4

First day

Introduction

It might seem impossible for Euclid to place a call to Alexandria, but it could happen in Ohio! Euclid is a city on the shore of Lake Erie, northeast of Cleveland (Transparency 4-R-1). It was named, as you might suspect, for the Greek mathematician. Alexandria, a small town located near the center of Ohio, is named for the city in Egypt where Euclid lived.

To calculate the toll rate for a call between two points in Ohio, the Ohio Bell Telephone Company applies a series of rules, part of which are listed on Transparency 4-R-2. Even though you have never seen these rules before, they may seem familiar. Why? (They sound like the Distance Formula.) As the Web site of the Telco Exchange explains, "The Telecom industry has its own unique system of geography. Pricing for many services is based on the 'airline mileage' or special distance calculations between points. A unique vertical and horizontal (V&H) coordinate system was developed by AT&T Long Lines in 1960 for the North American continent that is used to define telecom locations. The conventional system of longitude and latitude for describing geographic locations is not used. Although based on the well-known Pythagorean Theorem for calculating distances between points on an X-Y coordinate system, the telecom calculations must be made using a litany of special rules and rounding conventions. . . ."*

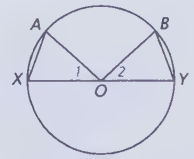
As a review of the derivation of the Distance Formula from the Pythagorean Theorem, have your students mark two points, A and E, on paper to represent Alexandria and Euclid. Label the coordinates of $A(V_1, H_1)$ and the coordinates of $E(V_2, H_2)$. Draw a horizontal line through A and a vertical line through E to form right ΔABE. In the distance calculation, what do the differences between the respective coordinates represent? (The lengths of the legs of the triangle.) The V&H coordinates for Euclid and Alexandria are (5536, 2527) and (5915, 2513), respectively. How does the V-H coordinate system seem to differ from our *x*-*y* coordinate system? (In our *x*-*y* coordinate system, the horizontal coordinate is listed first, whereas in the V-H system, the vertical coordinate is listed first. Also, in the V-H system, the vertical coordinates *decrease* as you proceed upward.) Use the coordinates of A and B to find the "lengths" of AB and BE. ($AB = 2527 - 2513 = 14$; $BE = 5915 - 5536 = 379$.) Use the Pythagorean Theorem to find the "length" of AE. ($AE^2 = AB^2 + BE^2$; so $AE^2 = 379^2 + 14^2 = 143{,}641 + 196 = 143{,}837$ and $AE = \sqrt{143{,}837} \approx 380$.) The distance between Alexandria and Euclid is 120 miles. Roughly how does the length of one unit in the V-H system compare with 1 mile? (It is about one-third of a mile.)

Second day

Introduction

The photographs on Transparency 4-R-6 are of windows in a monastery in Switzerland. The windows date at about 1320 and are examples of geometric tracery, ornamental stone work of interlaced lines and arcs. Their designer was clearly influenced by the constructions of geometry. In an article titled

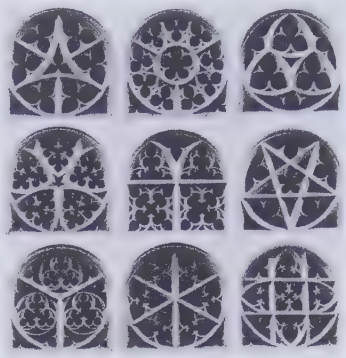
*www.telcoexchange.com/resources/carriers/distance.shtml



Given: O is the midpoint of XY;
$\angle 1 = \angle 2$ and OA = OB.
Prove: $\angle A = \angle B$.

TRANSPARENCY 4-R-5

"Euclid carved in stone"



TRANSPARENCY 4-R-6

Euclid's Elements

Book I

1. *Construct an equilateral triangle.*
2. *Copy a line segment.*
4. (SAS Congruence.)
5. Isosceles Triangle Theorem.
6. Converse of Isosceles Triangle Theorem.
8. SSS Congruence Theorem.
9. *Bisect an angle.*
10. *Bisect a line segment.*
13. Linear Pair Theorem.
15. Vertical Angles Theorem.
22. *Copy a triangle.*
23. *Copy an angle.*
26. (ASA Congruence.)
47. The Pythagorean Theorem.

Book IV

11. *Construct a regular pentagon.*

TRANSPARENCY 4-R-7

"The Cloisters of Hauterive" (pronounced *Oh-ter-eef*) in *The Mathematical Intelligencer* (Spring 1991), Benno Artmann describes the theme of the windows as "geometry itself" and described the windows themselves as "a commentary to Euclid . . . carved in stone."* Several windows contain designs that are related to the equilateral triangle; the first and third windows of the first row are especially obvious examples. Which windows seem to contain squares? (The second window of the second row; the last window of the third row.) Which window seems to contain the most *polygons?* (The last window of the second row.) What polygons do you see in it? (Five small acute isosceles triangles and a small pentagon, five overlapping obtuse isosceles triangles, five overlapping kite-shaped quadrilaterals. There are also five overlapping concave pentagons as well as a concave decagon, but I suggest that you mention these polygons only if your students happen to notice them.) We will look at one way to construct the basis pattern of this window, but first we will see where it fits into Euclid's *Elements*.

Some of the statements in the first and fourth books (chapters) of the *Elements* are shown on Transparency 4-R-7 (and listed here). Euclid called the statements "propositions"; we call them "constructions" and "theorems."

Book I

1. *Construct an equilateral triangle.*
2. *Copy a line segment.*
4. (SAS Congruence.)
5. Isosceles Triangle Theorem.
6. Converse of Isosceles Triangle Theorem.
8. SSS Congruence Theorem.
9. *Bisect an angle.*
10. *Bisect a line segment.*
13. Linear Pair Theorem.
15. Vertical Angles Theorem.
22. *Copy a triangle.*
23. *Copy an angle.*
26. (ASA Congruence.)
47. The Pythagorean Theorem.

Book IV

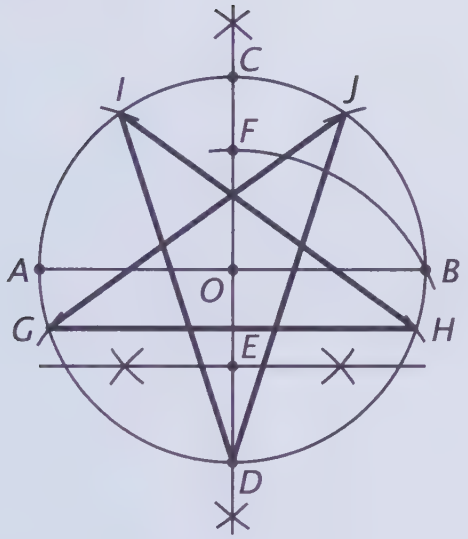
11. *Construct a regular pentagon.*

This list might provide a good basis for review. The italicized propositions are the constructions with which your students are now acquainted. The parenthetical propositions indicate that Euclid proved both SAS and ASA congruence as theorems; we treated these statements as postulates. The designer of the windows went well beyond the first book of the *Elements*. It was not until Book IV that Euclid showed how to construct a regular pentagon. (Explain that a regular pentagon, like an equilateral triangle, has equal sides and angles.) If the lines of the sides of a regular pentagon are extended until they intersect, they form a five-pointed star called a *pentagram*. This figure is the pattern of the window that we will now construct.

Give your students the following directions so that they can construct the figure as you demonstrate the steps either at the blackboard or on the

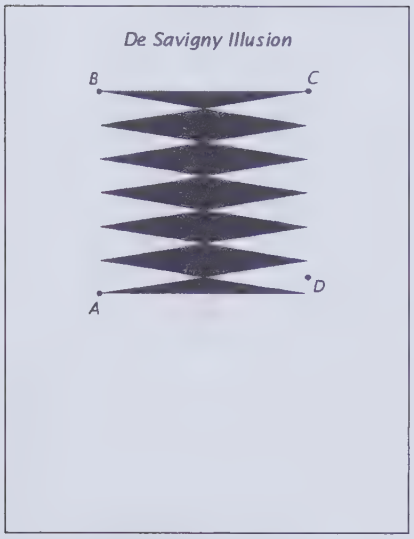
*Material on the mathematics of the Cloisters of Hauterive can be found on the following Web site:
www.maths.adelaide.edu.au/pure/pscott/place/pm13/pm13

overhead projector.

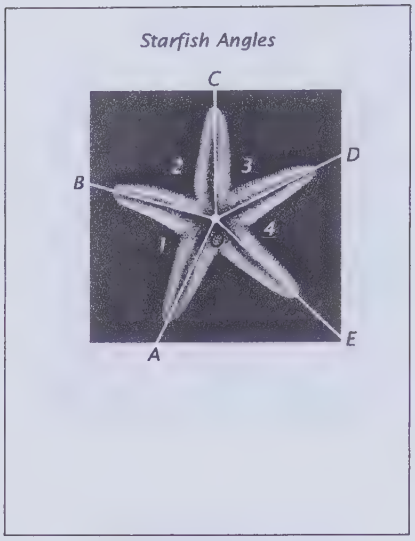


1. Draw a line segment AB about two inches long.
2. Bisect AB and label its midpoint O.
3. Mark points C and D on the bisecting line so that OC = OD = OA.
4. Bisect OD and label its midpoint E.
5. With E as center and EB as radius, draw an arc intersecting CD at F.
6. Draw circle O with radius OA.
7. With D as center and BF as radius, draw arcs intersecting circle O at G and H.
8. With G and H as centers and BF as radius, draw arcs intersecting circle O at I and J.
9. Draw the pentagram by drawing the segments DI, IH, HG, GJ, and JD.

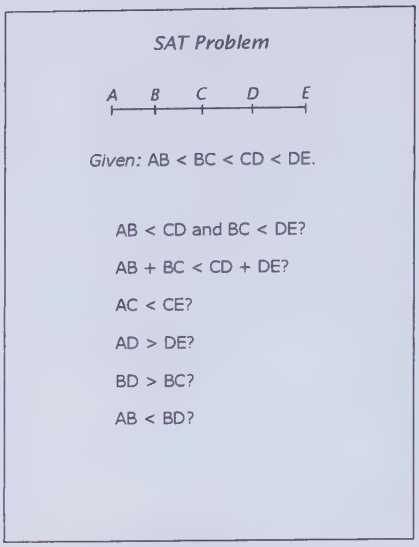
After your students have completed the construction, show Transparency 4-R-6 so that they can see the pentagram window again. Explain that, although we have carried out the steps in constructing the pentagram correctly, we are not yet in position to know *why* they work. This part of our understanding of geometry will come later in the course.



TRANSPARENCY 5-1-1



TRANSPARENCY 5-1-2



TRANSPARENCY 5-1-3

Chapter 5
Inequalities

Lesson 1
Properties of Inequality

Lesson Outline

Properties of inequality: "Three possibilities," transitive, addition, subtraction, multiplication, division, addition theorem, "whole greater than part" theorem.

Transparencies

5-1-1 De Savigny illusion
5-1-2 Starfish angles
5-1-3 SAT problem (Set II, Exs. 26–31)
5-1-4 Proof (Set II, Ex. 48)

Introduction

The figure on Transparency 5-1-1 is based on one from a book published in France in 1905 titled *The Entertainments of Science.** Four points have been added to the figure for reference. How do you think the distance from A to B compares with the distance from B to C? (You might poll your class on this question by a show of hands. After doing so, use a transparent ruler to measure the two distances. AB appears to be about 4 inches and BC almost a quarter inch longer.)

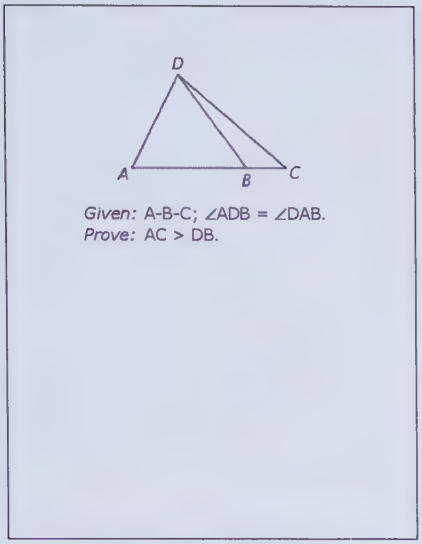
So far in our study of geometry, we have paid most of our attention to line segments and angles that are equal. We now turn to comparisons of line segments and angles that are not equal. To do so, we need to make some new assumptions. For example, in comparing the distances AB and BC in the illusion of Transparency 5-1-1, we took for granted that there were only three possibilities. Either AB > BC, AB = BC, or AB < BC. It is reasonable to think that no other possible relations between the two lengths might exist. We will assume that this is so and call it the *three possibilities* property of inequality.

Here is an example of another property of inequality. It is easy to see from the figure that AB > CD. From this observation and the fact that BC > AB, we can figure out how BC and CD compare in length without doing any more measuring. What conclusion can you draw? (Because BC > AB and AB > CD, BC > CD.) This conclusion is based on the *transitive* property of inequality.

We need to also include the operations of addition, subtraction, multiplication, and division in our list of properties of inequality. You are already familiar with these properties from your study of algebra. (As a quick review of them, you might ask your students to verbally solve each of the following equation-and-inequality pairs.)

1. $x - 3 = 7$. ($x = 10$.)
2. $x - 3 < 7$. ($x < 10$.)
3. $4 + x = 9$. ($x = 5$.)
4. $4 + x > 9$. ($x > 5$.)

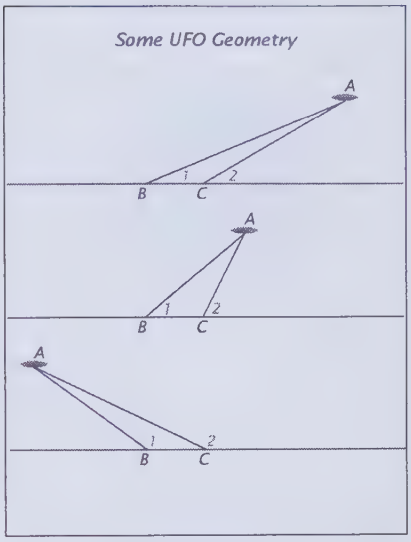
*In the book, by G. B. De Savigny, the figure was drawn so that AB = BC. The figure is a variation of the "Oppel illusion" and is discussed on page 140 of Jacques Ninio's book titled *The Science of Illusions* (Cornell University Press, 2001).



TRANSPARENCY 5-1-4



TRANSPARENCY 5-2-1



TRANSPARENCY 5-2-2

5. $\frac{x}{2} = 8$. ($x = 16$.)
6. $\frac{x}{2} < 8$. ($x < 16$.)
7. $5x = 30$. ($x = 6$.)
8. $5x > 30$. ($x > 6$.)

Transparency 5-1-2 is for possible use in introducing the two properties of inequality that are proved in this lesson: the Addition Theorem of Inequality and the "Whole Greater than Part" Theorem.

The angles between the arms of a starfish vary from one starfish to another. In the starfish shown in the photograph, $\angle 1 > \angle 3$ and $\angle 2 > \angle 4$. Given this information, how do you think $\angle AOC$ and $\angle COE$ compare in size? ($\angle AOC > \angle COE$.) This conclusion is based on the Addition Theorem of Inequality and substitution:

$\angle 1 > \angle 3$ and $\angle 2 > \angle 4$ (Given).

$\angle 1 + \angle 2 > \angle 3 + \angle 4$ (Addition Theorem of Inequality).

$\angle AOC = \angle 1 + \angle 2$ and $\angle COE = \angle 3 + \angle 4$ (Betweenness of Rays Theorem).

$\angle AOC > \angle COE$ (Substitution).

Point out to your students that we will ordinarily assume betweenness relations from the figures and sometimes even skip the betweenness-theorems steps in our proofs.

How do $\angle AOC$ and $\angle 1$ compare in size? ($\angle AOC > \angle 1$.) This conclusion is based on the "Whole Greater than Part" Theorem. $\angle AOC$, the "whole angle," is greater than either of its "parts," $\angle 1$ and $\angle 2$.

Your students are now prepared to summarize the eight properties of inequality by stating them symbolically.

Lesson 2
The Exterior Angle Theorem

Lesson Outline

Definition: Exterior angle of a triangle.

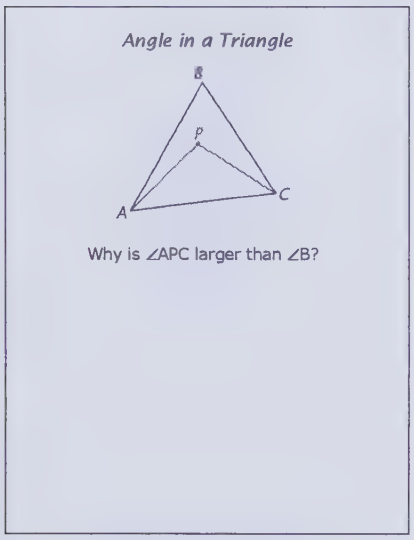
Theorem 12. The Exterior Angle Theorem. An exterior angle of a triangle is greater than either remote interior angle.

Transparencies

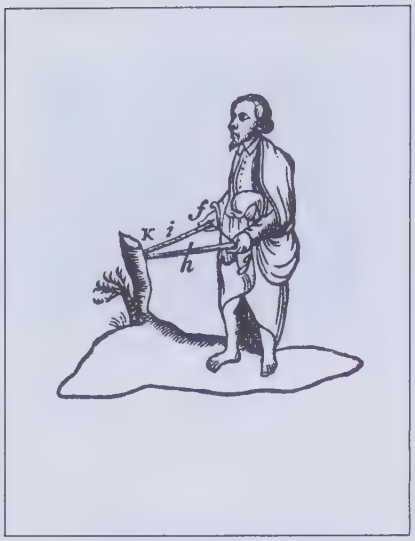
5-2-1 A photograph of a UFO?
5-2-2 Some UFO geometry
5-2-3 Angle in a triangle (Set II, Exs. 46 and 47)

Introduction

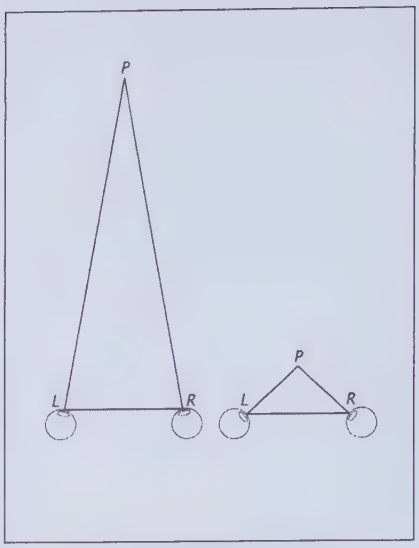
Show Transparency 5-2-1. This photograph was taken in São Paulo, Brazil. What do you suppose it shows? (Although it might appear to show some UFOs, those objects are actually lenticular clouds.) Reports of sightings of unidentified flying objects first took place in the summer of 1947, and UFOs have been a topic of controversy ever since. Transparency 5-2-2 can be used to introduce the lesson on the Exterior Angle Theorem as in the text. First explain how the elevation of the UFO can be found from a scale drawing on the basis of the fact that ΔABC is determined by ASA. The shape of ΔABC doesn't matter. Before getting into the Exterior Angle Theorem itself, ask your students how the two numbered angles in the second figure on the transparency appear to compare in measure. ($\angle 2 > \angle 1$.) The relation between these angles is less obvious in the



TRANSPARENCY 5-2-3



TRANSPARENCY 5-3-1



TRANSPARENCY 5-3-2

other two figures. You might also ask your students where the UFO would be located when $\angle 1 = \angle 2$. They may be surprised to find that there is no such location, because $\angle 2$ is *always* greater than $\angle 1$.

The proof of the theorem given in the text implicitly assumes that ray CP lies between rays CA and CD. The authors of the S.M.S.G. *Geometry* wrote in defense of this assumption:

"It is probably true that no kind of mathematics can be effectively presented in a completely rigorous form to a tenth-grade class. We should not feel guilty about teaching tenth-grade students merely as much as they can learn. The betweenness problem here will probably go unnoticed by most students. It should be called to the attention only of very capable and critical students."*

Lesson 3
Triangle Side and Angle Inequalities

Lesson Outline

Theorem 13. If two sides of a triangle are unequal, the angles opposite them are unequal in the same order.

Theorem 14. If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.

Transparencies

5-3-1 Blind man judging distance to object
5-3-2 Convergence when the object is equidistant
5-3-3 Convergence when the object is not equidistant
5-3-4 Triangle drawings (Set I, Exs. 14–22)
5-3-5 Triangle in perspective (Set I, Exs. 23–28)
5-3-6 Not quite equilateral (Set II, Exs. 36–44)

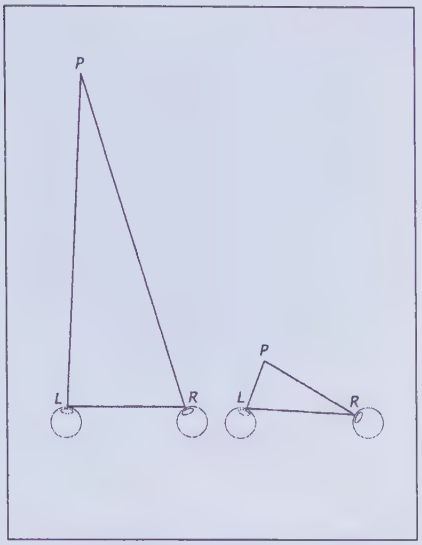
Introduction

The figure on Transparency 5-3-1 shows how a blind man using two sticks might judge the distance to an object. The more that he has to turn the sticks toward each other, the closer the object. The two eyes of a person who is not blind do the same thing. Have your students do the following experiment: Hold a pencil at arm's length and slowly bring it toward your face while focusing both eyes on it. As your finger comes closer, your eyes turn toward each other, to the extent that you might even appear to be cross-eyed. This turning, called *convergence*, produces tension in the muscles turning your eyeballs and helps to give your brain information about how close an object is. Try this experiment again, but this time keep the pencil closer to one eye than the other. It is harder to keep focused on your finger this time because one eye has to turn through a bigger angle than the other.

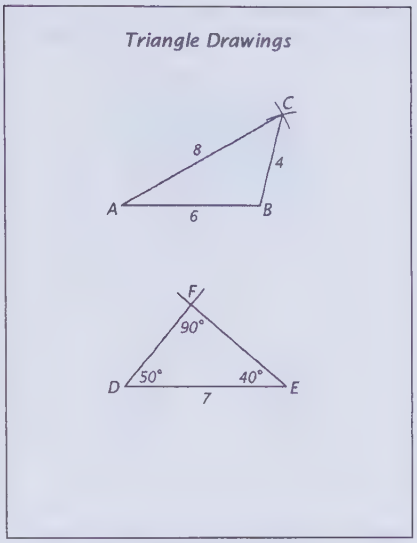
The figures on Transparency 5-3-2 show overhead views of your two eyes converging on the pencil at two different distances. If the pencil is held equidistant from your eyes in each case so that PL = PR, it follows that $\angle L = \angle R$. Why? (If two sides of a triangle are equal, the angles opposite them are equal.)

The figures on Transparency 5-3-3 also show overhead views of your eyes and the pencil, but here the pencil is held closer to your right eye than your

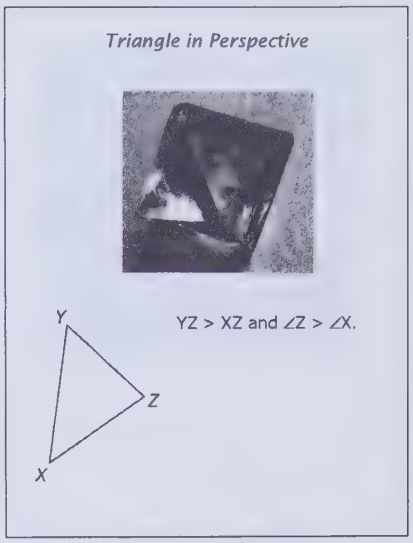
*School Mathematics Study Group, *Geometry—Teacher's Commentary*, Part 1 (Yale University Press, 1961), page 150.



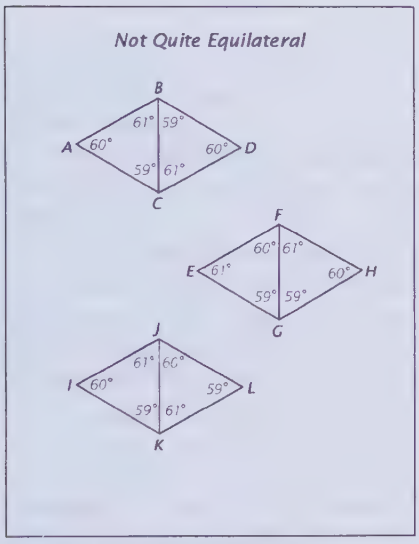
TRANSPARENCY 5-3-3



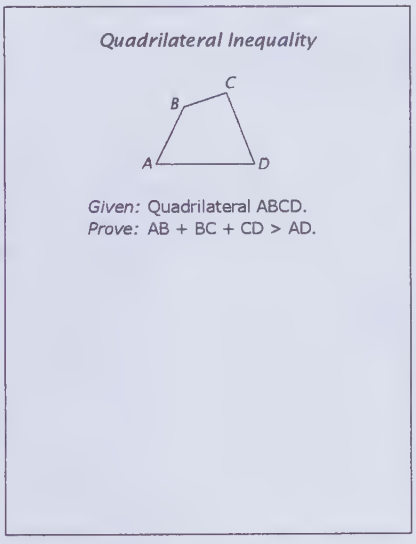
TRANSPARENCY 5-3-4



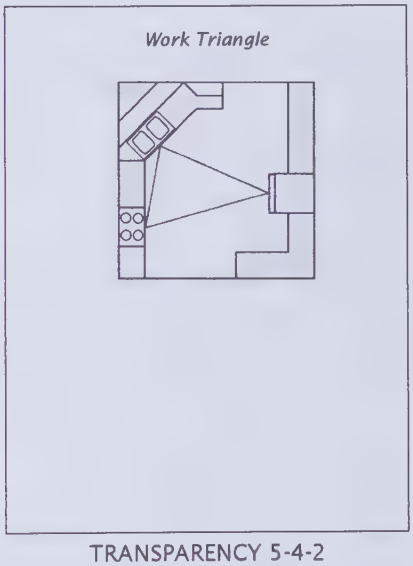
TRANSPARENCY 5-3-5



TRANSPARENCY 5-3-6



TRANSPARENCY 5-4-1



TRANSPARENCY 5-4-2

left as it is moved closer. In this case, PL > PR. What has happened to the relative sizes of the angles at L and R? ($\angle L < \angle R$; or, $\angle R > \angle L$.) Having put this observation into words, your students are now ready for Theorem 13 and its proof.

Lesson 4
The Triangle Inequality Theorem

Lesson Outline

Theorem 15. The Triangle Inequality Theorem. The sum of any two sides of a triangle is greater than the third side.

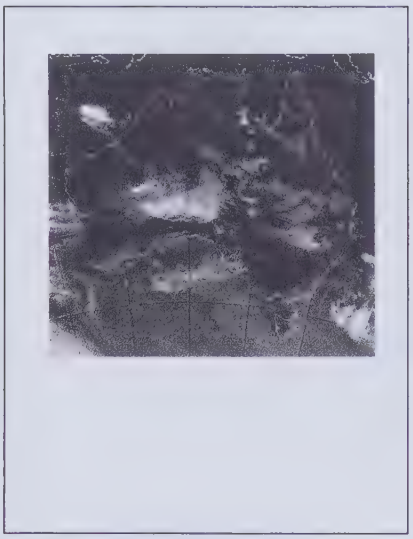
Transparencies

5-4-1 Quadrilateral inequality (Set II, Ex. 32)
5-4-2 Work triangle (Set II, Exs. 44–48)

Introduction

Take a stick of uncooked spaghetti, break it into three pieces of approximately equal length, and place them on the overhead projector. Move them together to form the sides of a triangle. Repeat this process with a second stick of spaghetti, but break it into three pieces whose lengths are not quite so close to being equal. Then ask your students if they think that a triangle can *always* be obtained from the three pieces formed by breaking a stick of spaghetti in this way.

Let a student who realizes that the answer is "no" break a stick into pieces to demonstrate why the answer is correct. This example raises the question of what it is about the lengths of the pieces that determines whether they can or cannot be made into



TRANSPARENCY 5-R-1



TRANSPARENCY 5-R-2


TRATADO ELEMENTAL
DE MATEMÁTICAS,
ESCRITO DE ÓRDEN DE S. M.
para uso de los caballeros seminaristas del Real Seminario de Nobles de Madrid y demas casas de educacion del reino.
POR D. JOSÉ MARIANO VALLEJO,
catedrático que fué de Matemáticas, Fortificacion, Ataque y Defensa de las plazas en dicho Real Seminario, &c. &c.
TOMO PRIMERO.
PARTE SEGUNDA,
LA GEOMETRÍA, TRIGONOMETRÍA RECTILÍNEA Y GEOMETRÍA PRÁCTICA.
TERCERA EDICION
CON LICENCIA.
Madrid: imprenta que fué de García.
1825.

TRANSPARENCY 5-R-3

a triangle. Your students are now ready for the Triangle Inequality Theorem and its proof.

Review of Chapter 5
First day

Transparencies

5-R-1 Satellite photo of Australia
5-R-2 The length of Australia's coastline
5-R-3 Title page of Spanish mathematics book
5-R-4 Theorem proof from Spanish mathematics book
5-R-5 Gateway arch (Set I, Exs. 5–8)
5-R-6 Proof (Set II, Ex. 49)
5-R-7 Proof (Set II, Ex. 50)

Introduction

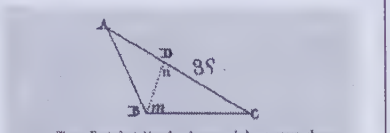
Show Transparency 5-R-1. Do you recognize what is shown in this photograph? (It is a satellite photograph of Australia.*) Two things have been added to the photograph: a grid showing circles of latitude and longitude and the border, or coastline, of Australia, shown in white. Australia is the largest country in the world that is completely surrounded by water. It is easy to measure Australia's width. It is almost 4,000 kilometers from east to west. To measure the length of Australia's coastline is another matter. Show Transparency 5-R-2. One way to try to do it is illustrated by the figure at the top. It shows what might happen if a giant walked along the coast. Suppose that each point represents one of the giant's footsteps, that the distance between each pair of footsteps is 300 km, and that there are 56 footsteps in all. How far would the giant walk? (56×300 km = 16,800 km.) This number, the perimeter of the red polygon, is a rough approximation of the length of the coastline. Now suppose that *you* walked along the coast. What do you suppose would happen to the apparent length of the coastline if you were to do so? (It would become longer.) This observation suggests something quite strange about the length of Australia's coastline. What do you suppose it is? (The "length" depends on how you measure it.)

An article in *The Australian Geographer*† answered the question, "What is the length of the Australian coast?" with "It all depends." The article went on to state that, when different divider lengths ("distances between footsteps") are used, the apparent length of the coast steadily increases as the divider length decreases. Steps of 1,000 km suggest a coastline of about 11,000 km; steps of just 1 mm suggest a coastline more than 10 times as long!

The second figure on Transparency 5-R-2 can be used to suggest why the apparent length of the coastline increases as the divider length decreases. Suppose that A, B, and C are three consecutive footsteps along the coast. Why is $AB + BC > AC$? (The Triangle Inequality Theorem.) Could A, B, and C be situated so that $AB + BC = AC$? (Yes. If A-B-C, then $AB + BC = AC$.) Could A, B, and C be situated so that $AB + BC < AC$? (No.)

*Current photographs of Australia, produced by the Japan Meteorological Agency, can be seen on the website: www.bom.gov.au/weather/satellite/

†"What is the length of the Australian coast?" by Robert W. Galloway and Margo E. Bahr (*The Australian Geographer*, 1979, vol. 3, pp. 244–247).

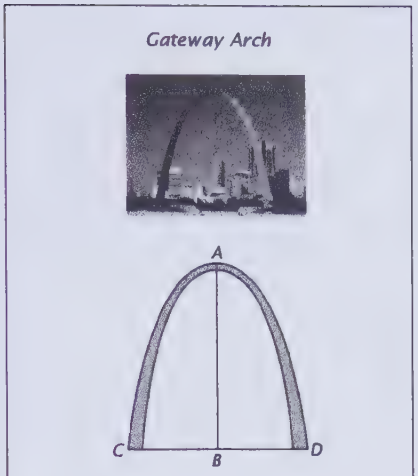


109. Teor. *En todo triángulo al mayor lado se opone el mayor ángulo.*

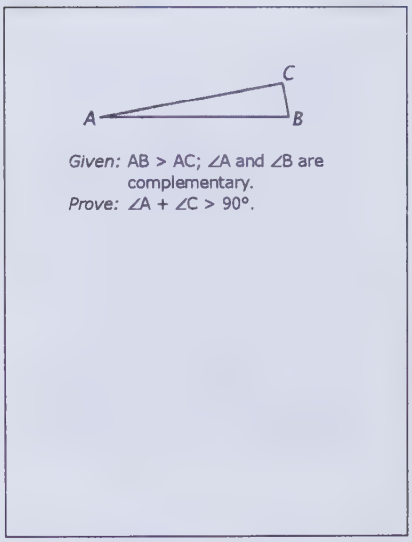
Sea el triángulo BAC (fig. 34): voy á demostrar que si el lado AC es mayor que BC, el ángulo ABC será mayor que BAC.

Dem. Tómese en el lado mayor AC una parte CD igual con BC, y tírese la BD; con lo cual tendremos el triángulo BDC que nos dará los ángulos e y m iguales; pero n > A por ser éstemo en el triángulo ABD, luego también será n > A; y como ABC es mayor que m, con más razon será mayor que el ángulo en A, o que BAC, que era L. Q. D. D.

TRANSPARENCY 5-R-4



TRANSPARENCY 5-R-5



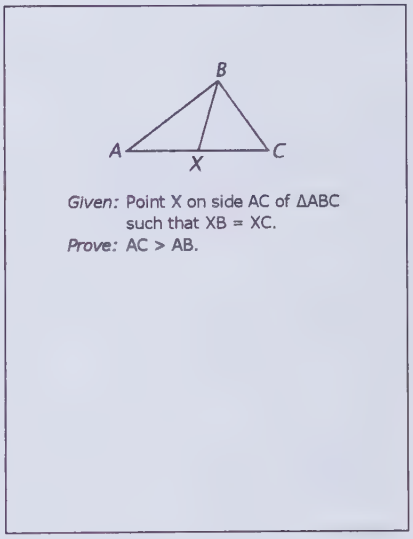
TRANSPARENCY 5-R-6

Second day

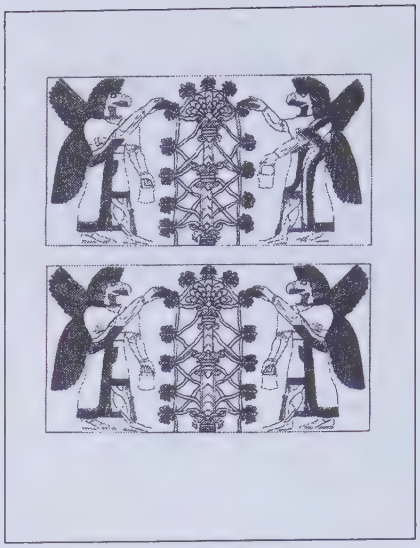
Introduction

Show Transparency 5-R-3. Here is the title page of a mathematics book of long ago. Can you tell where and when it was published? (Madrid, 1825.) Show Transparency 5-R-4. Here is one of the proofs from this book. Can you translate the theorem into English? (In every triangle, the greater side is opposite the greater angle.) The "given" and "prove" are stated below the theorem. Can you identify them? (Given: In ΔBAC, $AC > BC$. Prove: $\angle ABC > \angle BAC$.)

At this point you might ask your students to draw the figure and write below it our translations of the theorem, "given," and "prove." The proof, like the one of page 196 of the text, is Euclid's. Point out to your students that, although we first encountered this proof in two-column form, it is written here in paragraph form. Although the two-column form is somewhat helpful when a person first learns to write proofs, mathematicians, including Euclid, use paragraph form. Guide your students in listing the main details of the proof as they help you in translating it.



TRANSPARENCY 5-R-7



TRANSPARENCY 6-1-1

Chapter 6
Parallel Lines

Lesson 1
Line Symmetry

Lesson Outline
Terms. Perpendicular bisector, equidistant.
Definition. Two points symmetric with respect to a line.
Theorem 16. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.
Construction 6. To construct a line perpendicular to a given line through a given point.
Transparencies
6-1-1 Assyrian sculpture

Introduction
Show the upper figure on Transparency 6-1-1, keeping the lower figure covered. This drawing is of a sculpture created in the seventh century B.C. for the palace in the city of Nimrud (nim-rood) in ancient Assyria (now Iraq). It shows two winged creatures with heads like eagles standing on either side of a tree. They are holding pails and appear to be using sponge-like objects to wash, or possibly fertilize, the tree.

At first glance, the right side of the sculpture might appear to be a mirror image of the left side. Can you see why this is not the case? (One pail is higher than the other. More than that, the arms of the creatures are not mirror images of each other: both creatures are using their right hands to hold the sponges(?) and their left hands to hold the pails.)

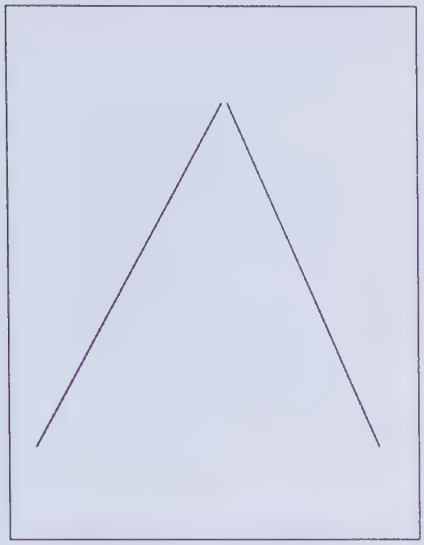
Show the lower figure on the transparency. Here the sculpture has been redrawn so that each side is now the mirror image of the other. The figure now has *line symmetry*. The two figures are from the book *Right Hand, Left Hand* by Chris McManus (Harvard University Press, 2002). About them, McManus writes:

> Technically, it would have been easier for the artist to draw the bottom picture, and so presumably there were strong reasons . . . for the additional effort put into drawing the top picture. Not the least of those was probably that the artist knew full well that most people are right-handed, and it was reasonable to presume that eagle-headed spirits would be also. The asymmetry at the heart of the original version of the picture adds to its interest, subtlety and realism, as the eye and the mind flip back and forth between the left and right halves, which are simultaneously similar and different.

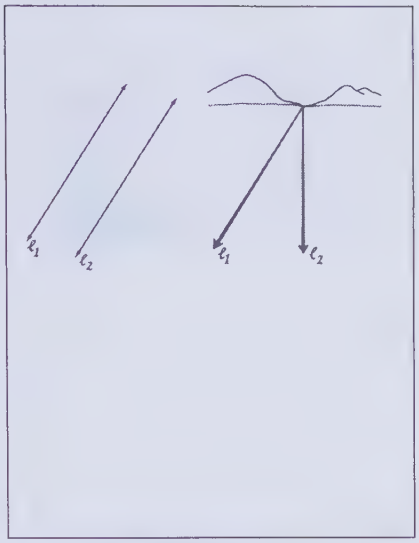
Following this introduction, the rest of the lesson can be presented as in the text.

Lesson 2
Proving Lines Parallel

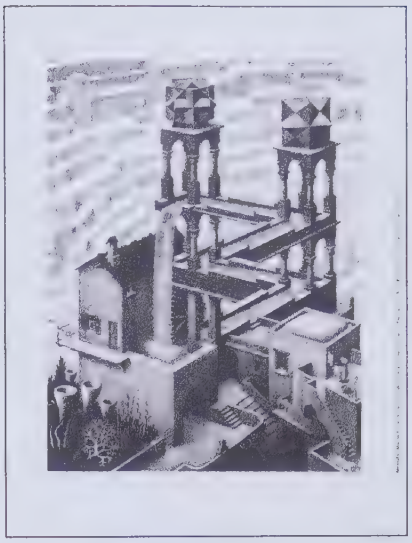
Lesson Outline
Terms: Parallel lines, transversal, corresponding angles, alternate interior angles, interior angles on the same side of a transversal.
Theorem 17. Equal corresponding angles mean that lines are parallel.
Corollary 1. Equal interior angles mean that lines are parallel.
Corollary 2. Supplementary interior angles on the



TRANSPARENCY 6-2-1



TRANSPARENCY 6-2-2



TRANSPARENCY 6-3-1

same side of a transversal mean that lines are parallel.

Corollary 3. In a plane, two lines perpendicular to a third line are parallel.

Transparencies

6-2-1 Parallel lines and train tracks
6-2-2 Parallel lines in perspective

Introduction

Show Transparency 6-2-1 and ask your students if the lines are parallel. After someone has explained why he or she thinks that the lines are not parallel, add the overlay. The photograph of the train rails suggests that the lines are parallel after all, but that when they are seen in perspective, they seem to intersect.

Show Transparency 6-2-2. These figures illustrate two different views of parallel lines. The first view shows the lines as seen from overhead and the second view shows them as seen in perspective. Draw a transversal in each figure, label the figures appropriately (see the figures on page 220 of the text), and develop the rest of the lesson as in the text.

Note that the angle relations introduced in this lesson—corresponding angles, alternate interior angles, and interior angles on the same side of the transversal—are not defined, but merely illustrated.

Lesson 3
The Parallel Postulate

Lesson Outline

Construction 7. To construct a line parallel to a given line through a given point.

Postulate 7. The Parallel Postulate. Through a point not on a line, there is exactly one line parallel to the given line.

Theorem 18. In a plane, two lines parallel to a third line are parallel to each other.

Transparencies

6-3-1 Escher's *Waterfall*

Introduction

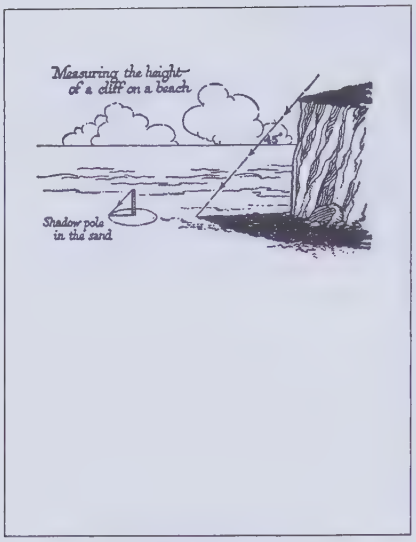
To introduce this lesson, you might show Escher's *Waterfall* (Transparency 6-3-1). Escher wrote about it:

> Falling water keeps a millwheel in motion and subsequently flows along a sloping channel between two towers, zigzagging down to the point where the waterfall begins again. The miller simply needs to add a bucketful of water from time to time, in order to compensate for loss through evaporation.*

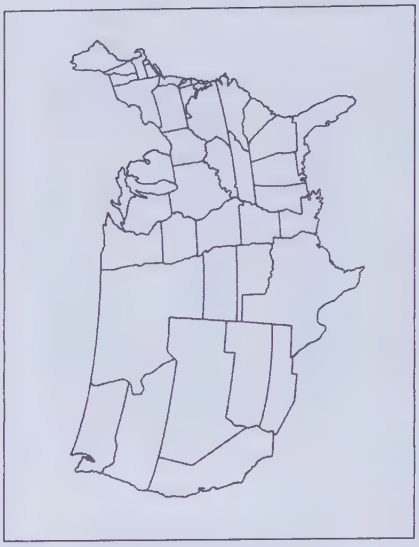
Waterfall was inspired by the Penrose triangle. It appears three times in the picture, which you can show by placing the overlay for Transparency 6-3-1 in the appropriate positions.

Point out that the picture contains several examples of parallel lines and present the rest of the lesson as in the text. It is important to emphasize that "at least one" and "no more than one" are distinct ideas and that proving one does not prove the other. An example such as the following one may be useful. If we wanted to find out how many stu-

**The Graphic Work of M. C. Escher* (Meredith Press, 1967; Ballantine Books, 1971).



TRANSPARENCY 6-4-1



TRANSPARENCY 6-5-1

dents at our school might have their birthday today, could we find out if there is *at least one* by asking everyone in our class? Could we find out if there is *no more than one* by asking everyone in our class?

Lesson 4
Parallel Lines and Angles

Lesson Outline

Theorem 19. Parallel lines form equal corresponding angles.

Corollary 1. Parallel lines form equal alternate interior angles.

Corollary 2. Parallel lines form supplementary interior angles on the same side of a transversal.

Corollary 3. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

Transparencies

6-4-1 Measuring the height of a cliff

Introduction

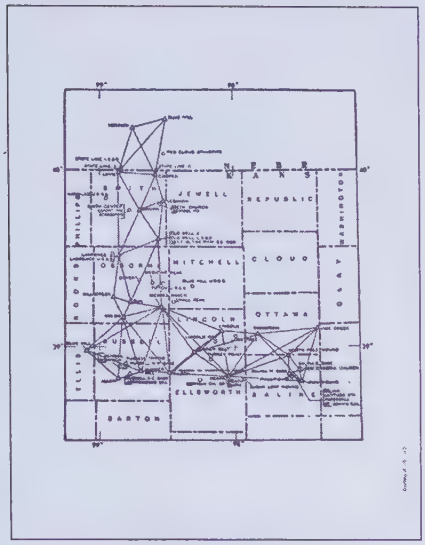
The upper figure on Transparency 6-4-1 illustrates a method for indirectly measuring the height of a cliff and is from the first edition of Lancelot Hogben's *Mathematics for the Million* (Norton, 1937). In his book *Geometry Civilized* (Clarendon Press, 1998), J. L. Heilbron describes it:

> You want to measure the height of the sheer cliff without climbing it. Put a stick vertically into the sand and draw a circle around its base with a radius equal to the stick's height. Then wait for a time on a sunny day when the stick's shadow just reaches the circle. At that instant, measure the cliff's shadow, which will equal its height.

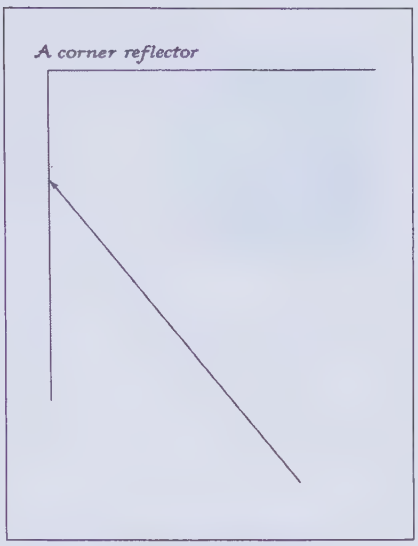
The lower figure suggests one way to show why the method works. AB represents the stick and DE represents the cliff. When the stick's shadow touches the circle at C, we can conclude that AB = BC. Why? (The circle was drawn so that its radius is equal to the stick's height.) What kind of triangle is ΔABC? (An isosceles right triangle; ∠B is a right angle since the stick was put vertically into the sand.) What can you conclude about the angles at A and C? (∠A = ∠C = 45°.) Since the sun is so far away, it is assumed that the lines represents the rays of light, AC and DF, are parallel. What does this suggest about ∠F? (That it also has a measure of 45°.) Why would it follow from this that the cliff's shadow is equal to its height? (Assuming that ΔDEF is also a right triangle, it follows that ∠D = 45° from which we can conclude that the triangle is also isosceles.)

The main assumption in this argument is that parallel lines form equal corresponding angles with a transversal. You might think that we have already proved this as a theorem, but the theorem we have proved is the *converse* of this. Remember that a statement and its converse do not say the same thing. If *a* implies *b*, it does not necessarily follow that *b* implies *a*.

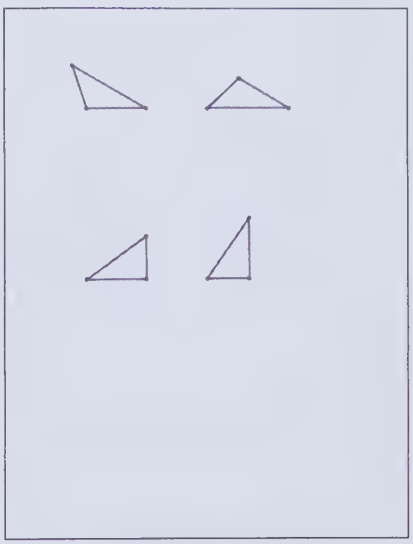
Following this introduction, the rest of the lesson can be presented as in the text.



TRANSPARENCY 6-5-2



TRANSPARENCY 6-6-1



TRANSPARENCY 6-6-2

Lesson 5
The Angles of a Triangle

Lesson Outline

Theorem 20. The Angle Sum Theorem. The sum of the angles of a triangle is 180°.

Corollary 1. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.

Corollary 2. The acute angles of a right triangle are complementary.

Corollary 3. Each angle of an equilateral triangle is 60°.

Theorem 21. An exterior angle of a triangle is equal to the sum of the remote interior angles.

Transparencies

6-5-1 Map: United States, 1858
6-5-2 Triangulation

Introduction

The map on Transparency 6-5-1 shows the United States as it appeared in 1858. The strange-looking regions are labeled, from the upper left, Washington, Nebraska, Oregon, Utah, New Mexico, and "Arrizonia." Oklahoma is labeled "Indian Territory." This map was copied from a map of railroad surveys undertaken by the U. S. Government to determine the best route from the Mississippi to the Pacific Coast.*

The figure on Transparency 6-5-2 also appears on page 236 of the text and can be used to lead into the rest of the lesson. In talking about triangulation, it would be interesting to tell your students where the triangulation station mark closest to your school is located. This formation is available from the U. S. Coast and Geodetic Survey.

*Many interesting maps of the United States are included in *The Mapping of America* by Seymour I. Schwartz and Ralph E. Ehrenberg (Abrams, 1980).

Lesson 6
AAS and HL Congruence

Lesson Outline

Theorem 22. The AAS Theorem. If two angles and the side opposite one of them in one triangle are equal to the corresponding parts of another triangle, the triangles are congruent.

Theorem 23. The HL Theorem. If the hypotenuse and a leg of one right triangle are equal to the corresponding parts of another right triangle, the triangles are congruent.

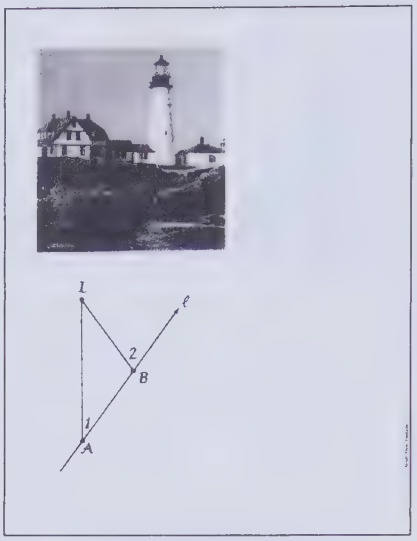
Transparencies

6-6-1 A corner reflector
6-6-2 AAS and HL congruence

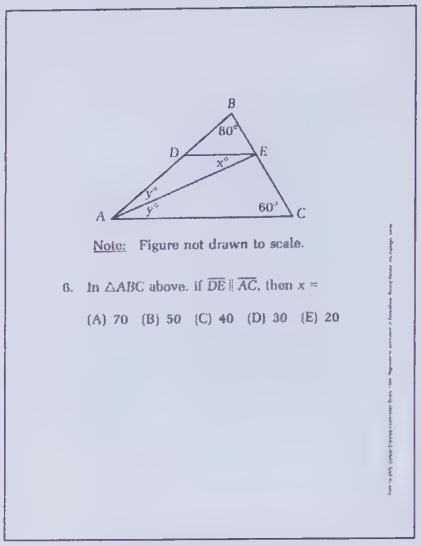
Introduction

An interesting model that can be used for several purposes consists of two hinged mirrors arranged so that they form a dihedral right angle. Set up such a model in a front corner of the room before class. The model should be approximately at the eye level of your students when they are seated at their desks.

The model is a simple example of a corner reflector: it reflects light back to its source regardless of the direction from which the light comes. This means that all of your students should be able to see their reflections, regardless of where they hap-



TRANSPARENCY 6-R-1



TRANSPARENCY 6-R-2

pen to be seated in the room.

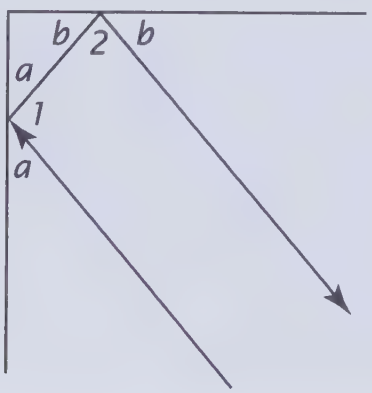
An array of corner reflectors was left on the surface of the moon at the time of the first landing in 1969. The reflectors make it possible to use laser light to measure the distance from the earth to the moon to within an inch.

Hand out duplicated copies of Transparency 6-6-1. This figure shows an overhead view of the two mirrors in the corner reflector and a ray of light hitting one of them. Note that the mirrors are perpendicular to each other. The angle at which a ray of light strikes a mirror is equal to the angle at which it is reflected. Your students can discover how the reflector works by using their protractors and rulers to draw the path of the ray of light after it hits the first mirror. The figure that results suggests that the ray of light is reflected back in the same direction from which it came—that is, the incoming and outgoing paths appear to be parallel.

To prove this, label the measures of the angles as shown in the figure **below**. The proof makes a good review of several facts from earlier lessons. Guide your students in writing it by observing that $a + b = 90$, $\angle 1 = 180 - 2a$, and $\angle 2 = 180 - 2b$, so that

$$\begin{aligned} \angle 1 + \angle 2 &= (180 - 2a) + (180 - 2b) \\ &= 360 - 2a - 2b \\ &= 360 - 2(a + b) \\ &= 360 - 2(90) \\ &= 180. \end{aligned}$$

Because we have shown that $\angle 1$ and $\angle 2$ are supplementary, it follows that l_1 and l_2 are parallel.



I suggest providing your students with duplicated copies of Transparency 6-6-2 for use in taking notes on the proofs of the AAS and HL congruence theorems.

Review of Chapter 6

Transparencies

6-R-1 Doubling the angle on the bow
6-R-2 S.A.T. problem

First day

Introduction

One of the seven wonders of the ancient world was a lighthouse at Alexandria in Egypt. Built at about the time of Euclid, it guided ships until as late as the twelfth century. Today more than 10,000 lighthouses mark the shorelines of the world's continents for ships traveling at night.

An example of how the navigator of a ship uses geometry to determine how far the ship is from a lighthouse sighted on the shore is illustrated on Transparency 6-R-1. This procedure is called "doubling the angle on the bow." Suppose that the ship

is moving along line l when a lighthouse at point L is sighted. As the ship travels along the line, the angle between its course and the line of sight to the lighthouse gets larger and larger. If the navigator measures this angle when the ship is at A and then waits until the ship has moved to a point where the angle has doubled, point B in the figure, the navigator can conclude that the distance from the ship to the lighthouse is equal to the distance the ship has traveled in going from A to B. In other words, if $\angle 2 = 2\angle 1$, then LB = AB.

Give your students time to try to write a proof of this procedure, taking as "given" that $\angle 2 = 2\angle 1$ and that $\angle 2$ is an exterior angle of ΔABL.

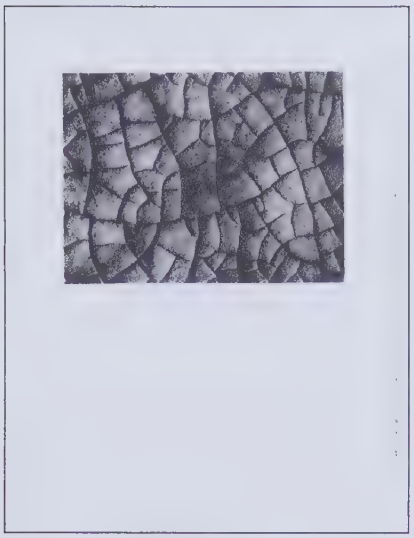
Proof.

1. $\angle 2 = 2\angle 1$ and $\angle 2$ is an exterior angle of ΔABL. (Given.)
2. $\angle 2 = \angle 1 + \angle L$. (An exterior angle of a triangle is equal to the sum of the remote interior angles.)
3. $2\angle 1 = \angle 1 + \angle L$. (Substitution.)
4. $\angle 1 = \angle L$. (Subtraction.)
5. LB = AB . (If two angles of a triangle are equal, the sides opposite them are equal.)

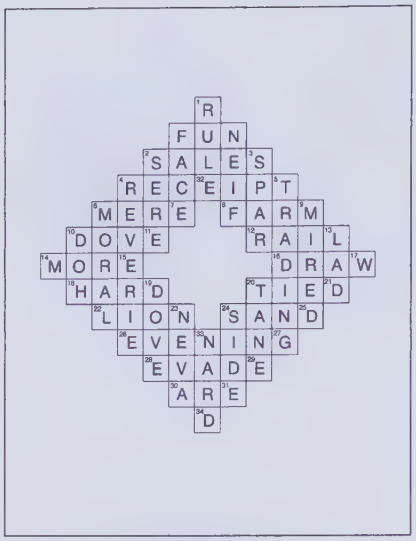
Second day

Introduction

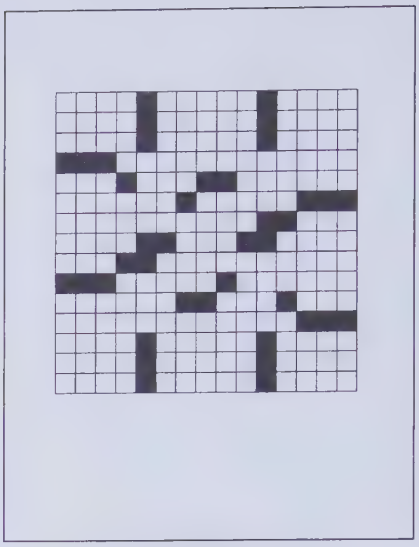
The S.A.T. problem on Transparency 6-R-2 provides further review of ideas in Chapter 6.



TRANSPARENCY 7-1-1



TRANSPARENCY 7-2-1



TRANSPARENCY 7-2-2

Chapter 7
Quadrilaterals

Lesson 1
Quadrilaterals

Lesson Outline
Terms: Convex polygon, concave polygon, diagonal of a polygon, rectangle.
Theorem 24. The sum of the angles of a quadrilateral is 360°.
Corollary. A quadrilateral is equiangular iff it is a rectangle.
Transparencies
7-1-1 Photograph of mud cracks

Introduction
Show Transparency 7-1-1 and let your students guess what the photograph is of. It shows cracks in dried mud and is taken from a fascinating book that every mathematics teacher should own: *Mathematical Snapshots* by Hugo Steinhaus (Oxford University Press, 1969).

Can you guess which cracks occurred first? (The longest ones.) It is interesting to notice that almost all of the cracks meet at right angles. (There is a physical explanation for this in terms of the laws of mechanics.) Most of the figures formed by the cracks have four sides; they are examples of quadrilaterals, the subject of the next chapter of the course.

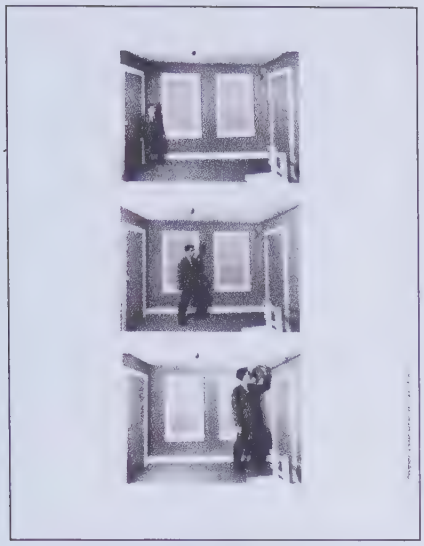
(Add the Overlay for Transparency 7-1-1.) Certain types of quadrilaterals are given specific names; some examples are shown here. How many of them can you identify? (1. square; 2. rectangle; 3. parallelogram; 4. trapezoid.)

Lesson 2
Parallelograms and Point Symmetry

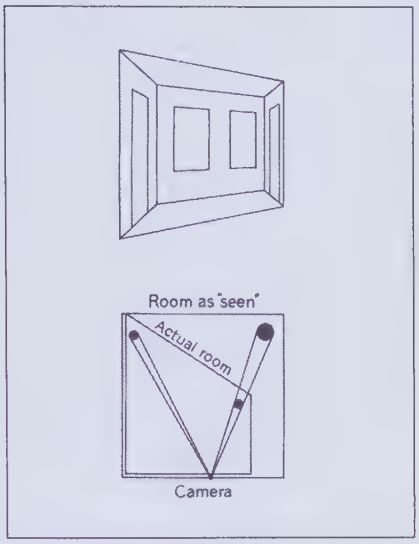
Lesson Outline
Terms: Parallelogram, point symmetry.
Theorem 25. The opposite sides and angles of a parallelogram are equal.
Theorem 26. The diagonals of a parallelogram bisect each other.
Transparencies
7-2-1 The first crossword puzzle
7-2-2 A crossword puzzle pattern

Introduction
The first crossword puzzle was published in the *New York World* on December 21, 1913 and is shown on Transparency 7-2-1. Unlike current crossword puzzles, it was diamond shaped and did not contain any black squares. Today, crossword puzzles appear daily in almost every American newspaper. They are almost always square and the black squares in them are usually arranged in symmetric patterns.

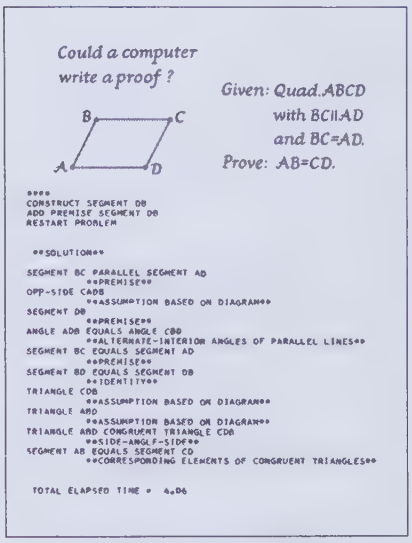
The arrangement shown on Transparency 7-2-2 is a typical example. The pattern does not have *line* symmetry because there is no way to fold it so that the two halves coincide. It has a different kind of symmetry. Can you describe what it is? (The pat-



TRANSPARENCY 7-3-1



TRANSPARENCY 7-3-2



TRANSPARENCY 7-4-1

tern would look the same if it were turned upside down.) A figure having this property is said to possess *point symmetry*. Mark the point that is the center of the pattern and show examples of pairs of points in the pattern that are symmetric with respect to this point. What relation does this point have to the line segments connecting such pairs of points? (It is their midpoint.)

The grid of a crossword puzzle contains squares and rectangles, both of which also possess point symmetry. It also contains *parallelograms;* squares and rectangles are also called parallelograms because their opposite sides are parallel. Following from this introduction, the rest of the lesson can be developed as in the text.

Lesson 3
More on Parallelograms

Lesson Outline

Theorem 27. A quadrilateral is a parallelogram if its opposite sides are equal.

Theorem 28. A quadrilateral is a parallelogram if its opposite angles are equal.

Theorem 29. A quadrilateral is a parallelogram if two opposite sides are both parallel and equal.

Theorem 30. A quadrilateral is a parallelogram if its diagonals bisect each other.

Transparencies

7-3-1 The Ames room

7-3-2 The Ames room explained

Introduction

Transparencies 7-3-1 and 7-3-2 can be used to introduce the new lesson as in the text. The Ames room was invented by the American psychologist Adelbert Ames. Some interesting experiments with an Ames room are described in the article "Visual Perception and Personality" by Warren J. Wittreich in the April 1959 issue of *Scientific American* (pages 56-60).

Lesson 4
Rectangles, Rhombuses, and Squares

Lesson Outline

Terms: Square, rhombus.

Theorem 31. All rectangles are parallelograms.

Theorem 32. All rhombuses are parallelograms.

Theorem 33. The diagonals of a rectangle are equal.

Theorem 34. The diagonals of a rhombus are perpendicular.

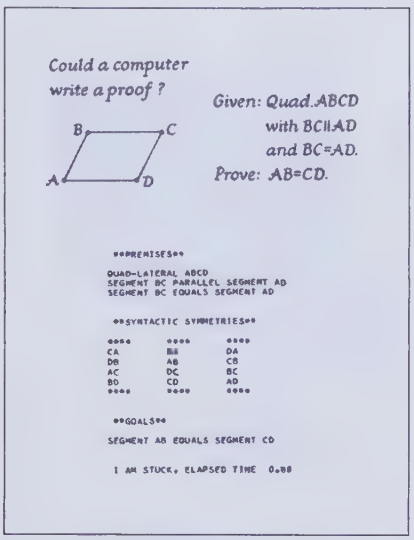
Transparencies

7-4-1 Could a computer write a proof?, 1

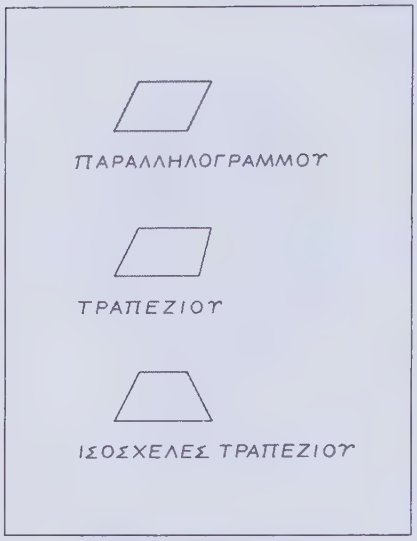
7-4-2 Could a computer write a proof?, 2

Introduction

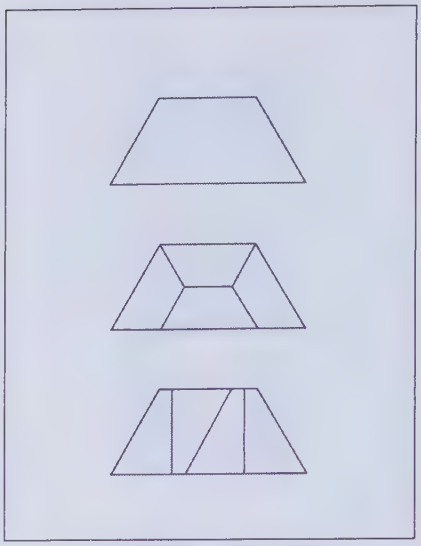
Not long after the invention of the computer, an experiment was conducted at the IBM Research Laboratory to see if a computer could be programmed to write geometric proofs. A number of basic postulates and theorems, along with the methods of deductive reasoning, were stored in the memory of an IBM 704 computer and then it was given some exercises to prove. One of them is similar to a theorem in yesterday's lesson. Show Transparency 7-4-1. The problem was to show that, if two sides of a quadrilateral are both parallel and equal, then the other two sides are equal. This is what the computer did. Show Transparency 7-4-2. Your stu-



TRANSPARENCY 7-4-2



TRANSPARENCY 7-5-1



TRANSPARENCY 7-5-2

dents should recognize the "premises" and "goals" as the "given" and "prove." The "syntactic symmetries" are a list of point relations that had to be included in the program because the computer could not see the figure.

After the computer had "thought" about the problem for 88 seconds, it reported that it was stuck (some of my students exhibit about the same amount of persistence, I am afraid). The computer was stuck because it did not know very much about quadrilaterals. To help out the computer, the person running it added a diagonal to the figure to form some triangles. After a few more minutes, the computer printed out the proof given. As you read it to the class, point out that we would take some of the steps for granted.

If the computer had been programmed with the theorems of yesterday's lesson, it might have come up with a much shorter proof. Can you think of what it is? (The quadrilateral is a parallelogram because two opposite sides are both parallel and equal. The opposite sides of a parallelogram are equal.)

Your students will be interested to know that in 1976, two mathematicians at the University of Illinois used a computer to finally prove a theorem that people had been trying to prove since its discovery in 1852.* Unfortunately, the proof is so long that it would be impossible for a human being to read it. It took 1,200 hours of computer time to write the proof and it would take more than a lifetime to read it!

*The theorem is the four-color map theorem referred to on page 6 of the text. An interesting book on this subject is *Four Colors Suffice: How the Map Problem was Solved* by Robin Wilson (Princeton University Press, 2002).

Lesson 5
Trapezoids

Lesson Outline

Terms: Trapezoid, bases, legs, base angles, isosceles trapezoid.

Theorem 35. The base angles of an isosceles trapezoid are equal.

Theorem 36. The diagonals of an isosceles trapezoid are equal.

Transparencies

7-5-1 Greek parallelogram and trapezoids

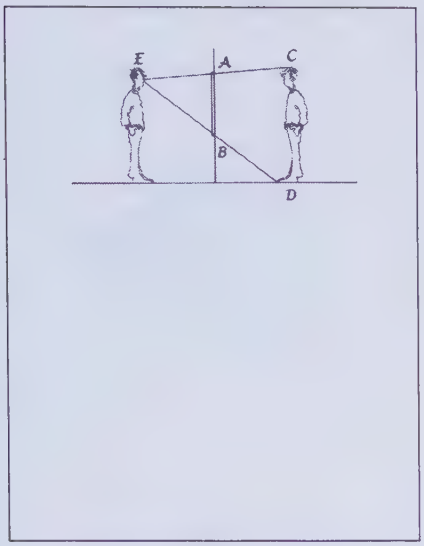
7-5-2 Congruence puzzle (Set III)

Introduction

Hand out duplicated copies of Transparency 7-5-1. It is interesting to realize that some of the words that we use in geometry would sound familiar to Euclid, even though he lived in Egypt more than two thousand years ago. The figures and words are from a geometry book written in Greek.

Have your students identify the figures and print their English names in capital letters below the Greek names. The Greek word "parallelogrammon" is derived from words meaning "bounded by parallel lines." "Trapezion" is related to trapezoid and the Greek word "isosceles" means "having equal legs."

Transparency 7-5-2 is for use in discussing the Set III exercise of Lesson 5. The figure in the center was originally thought to be the only solution to



TRANSPARENCY 7-6-1

this puzzle but Andrew Miller, the high school student of Richard Brady in Washington, D.C. mentioned in the text, discovered the solution pictured at the bottom. This solution was sufficiently surprising that it was reported in *Scientific American* (Martin Gardner's "Mathematical Games," February 1979).

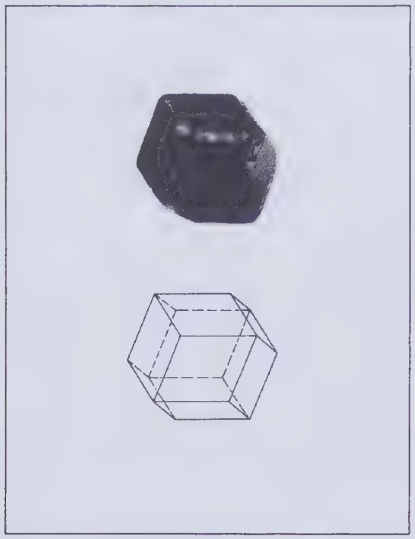
Lesson 6
The Midsegment Theorem

Lesson Outline
Terms: Midsegment
Theorem 37. The Midsegment Theorem. A midsegment of a triangle is parallel to the third side and half as long.
Transparencies
7-6-1 The mirror on the wall

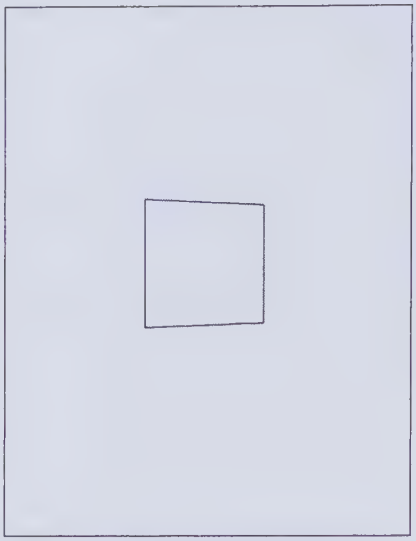
Introduction
If possible, place a large mirror on one of the walls of your room before class. Be sure that the mirror is vertical.

You have all looked at reflections of yourselves in large wall mirrors. How do you suppose your distance from the mirror affects the amount of yourself that you can see? (Because the farther you are from the mirror, the farther your reflection seems to be behind it, many people assume that you can see more of yourself as you move away from the mirror. Some people, realizing that, as you back away from a mirror you see less and less of the surrounding room reflected in it, think that you see less of yourself as well. Actually, your distance from the mirror makes no difference. The amount of yourself that you can see is controlled solely by the length of the mirror and not by how far away from it you happen to be standing. A student who walks from the back of the room toward the mirror at the front while looking at his or her reflection can confirm this.)

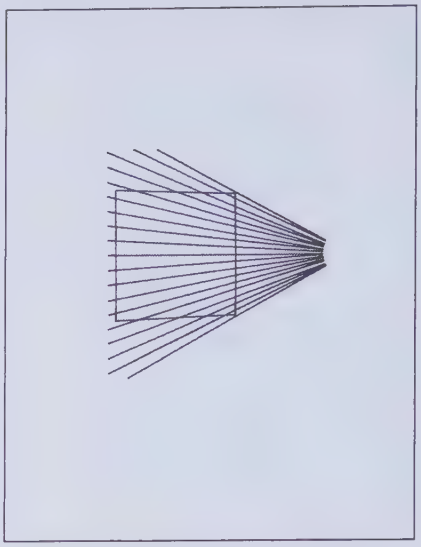
To see why this is so, look at this diagram (Transparency 7-6-1). The vertical segment in the center represents a side view of a wall and AB represents a mirror on it. The mirror is just long enough so that the person standing on the left can see his entire image on the right. (Add Overlay A.) If the person moves away from the mirror, so does his image. Notice, however, that his lines of sight to the reflections of the top of his head and bottom of his feet still pass through points A and B. (Replace Overlay A with Overlay B.) This figure shows what happens if the person moves toward the mirror instead. Again, his lines of sight pass through points A and B. (Remove Overlay B.) How long do you think the mirror would need to be for the person to see his entire image if he is six feet tall? In other words, if CD = 6, how long do you think AB is? It is not difficult to prove that A and B are the midpoints of sides EC and ED in ΔECD. We will call a segment that joins the midpoints of two sides of a triangle a *midsegment* of the triangle, and we will prove that such a segment is always parallel to the third side of the triangle and exactly half as long. The rest of the new lesson can be developed as in the text.



TRANSPARENCY 7-R-1



TRANSPARENCY 7-R-2



WORKSHEET FOR 7-R

Review of Chapter 7

Transparencies

7-R-1 Garnet crystal and polyhedron
7-R-2 Square illusion
7-R-3 Quadrilateral quiz
Square illusion (worksheet)

First day

Introduction

Transparency 7-R-1 contains a photograph of a crystal of garnet and a diagram of its structure.* Every edge of the crystal has the same length, and so its faces are equilateral. What kind of faces does the crystal have? (Rhombuses.) What other names for them can you think of? (Parallelograms, quadrilaterals, polygons.)

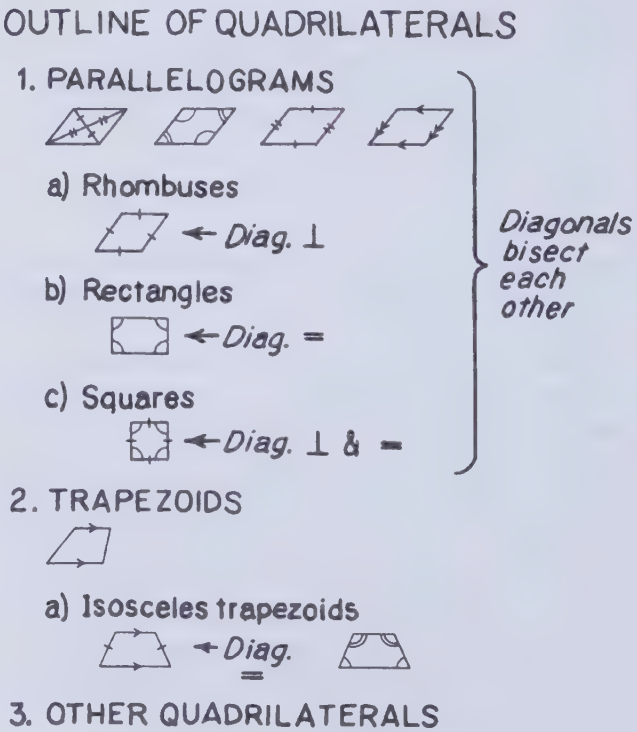
As a review of the various quadrilaterals studied in this chapter, you might direct your students in making an outline such as the one shown at the right.

Second day

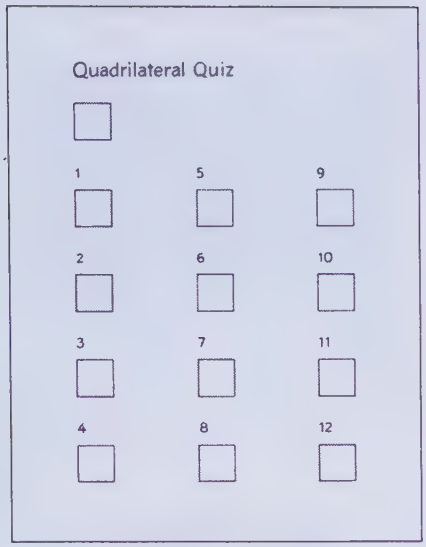
Introduction

Show Transparency 7-R-2 with the overlay in place and ask your students what kind of quadrilateral they think is pictured. (Square and rectangle are popular guesses.) Then hand out duplicated copies of Worksheet for Chapter 7 Review and give everyone a chance to measure the sides of the quadrilateral with their rulers. After they have discovered that the left and right sides are not equal, remove the overlay from the transparency to reveal the quadrilateral as an isosceles trapezoid. This is quite a remarkable optical illusion!

Next, hand out duplicated copies of Transparency 7-R-3 and tell your students that, although these figures look like the quadrilateral in the optical illusion, they are not to make any assumptions about the figures other than what you tell them. The



*This solid, called a rhombic dodecahedron, has 12 identical faces: each face is a rhombus having acute angles of 70°32'. An interested student might enjoy making a model of it for you to use.



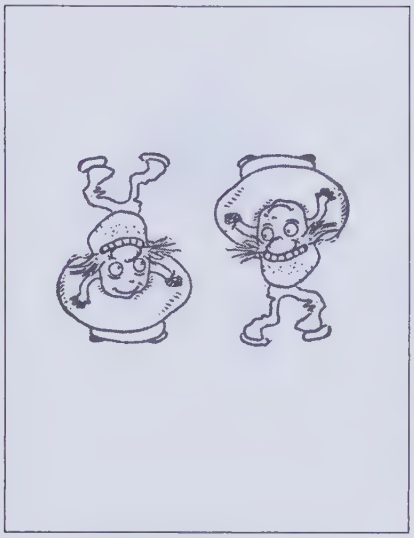
TRANSPARENCY 7-R-3

figure at the upper left is for use as an example. Use arrowheads to mark the top and bottom parallel and explain that the problem, if it is possible, is to identify the type of quadrilateral as specifically as they can. If it is not possible, they are to answer with a question mark. In the case of the example, we do not know anything about the other two sides. The figure could be a parallelogram or a trapezoid, and so the correct answer is a question mark.

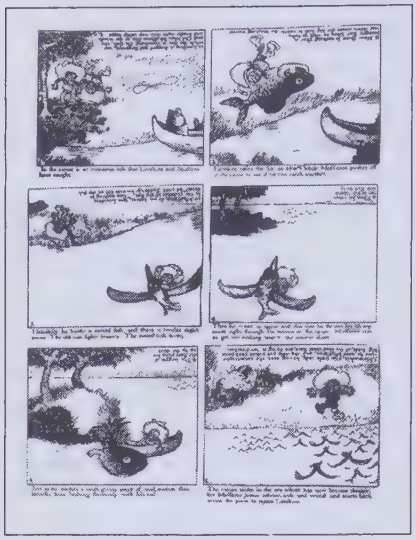
Before beginning the quiz, you might mention that the questions vary in difficulty from very easy to very tricky. I tell my students that anyone who is able to answer all twelve questions correctly deserves an A on the chapter. Ask all twelve questions before checking the answers.

What you draw on the figure	The answer
1. Mark the vertical sides parallel and equal.	Parallelogram
2. Mark all four angles equal.	Rectangle
3. Mark both pairs of opposite sides parallel.	Parallelogram
4. Mark the vertical sides equal.	?
5. Mark all four sides equal.	Rhombus
6. Mark both pairs of opposite angles equal.	Parallelogram
7. Mark all four sides equal and all four angles equal.	Square
8. Mark the top and bottom parallel and the vertical sides equal.	?
9. Draw the diagonals and indicate that they bisect each other.	Parallelogram
10. Mark both pairs of opposite sides equal.	Parallelogram
11. Mark one pair of opposite angles equal.	?
12. Draw the diagonals and mark them perpendicular.	?

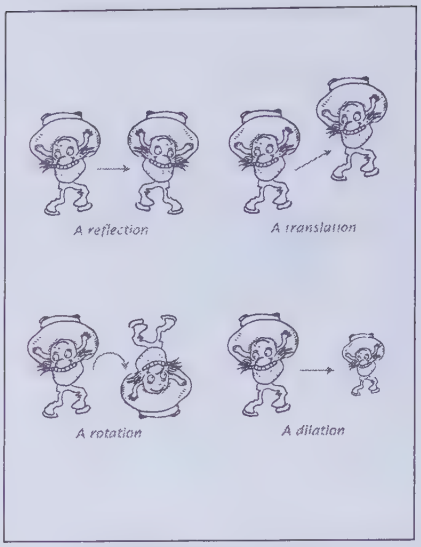
The last question is particularly treacherous because it is so easy for students who know that the diagonals of a rhombus are perpendicular to make the converse error.



TRANSPARENCY 8-1-1



TRANSPARENCY 8-1-2



TRANSPARENCY 8-1-3

Chapter 8
Transformations

Lesson 1
Transformations

Lesson Outline
Terms: Transformation, translation, reflection, rotation, dilation, isometry.
Transparencies
8-1-1 Upside-downs cartoon
8-1-2 Upside-downs cartoon strip
8-1-3 More transformations

Introduction
One of the earliest comic strips to appear in a newspaper was also one of the most remarkable. Its title was "The Upside-Downs of Little Lady Lovekins and Old Man Muffaroo" and it was published in the *New York Herald* from 1903 to 1905. The two characters of the cartoon are shown on Transparency 8-1-1. They were drawn so that each character became the other when turned upside down.

One of the comic strips in which these characters appeared is on Transparency 8-1-2. After reading it to your class, point out the amazing aspect of the strip. Not only does each panel make sense upside down, but the story also makes sense even though the first six panels of the strip become the last six panels in reverse order. The artist, Gustave Verbeek, drew a comic strip like this every week for a total of 64 weeks! His work is surely the most amazing application that the rotation transformation has ever had to the field of storytelling and art.

Transparency 8-1-3 can be used to develop the rest of the lesson as in the text.

Lesson 2
Reflections

Lesson Outline
Terms: Reflection of a point through a line, composite, translation, rotation, magnitude.
Construction 8. To reflect a point through a line.
Transparencies
8-2-1 Reflection in a lake

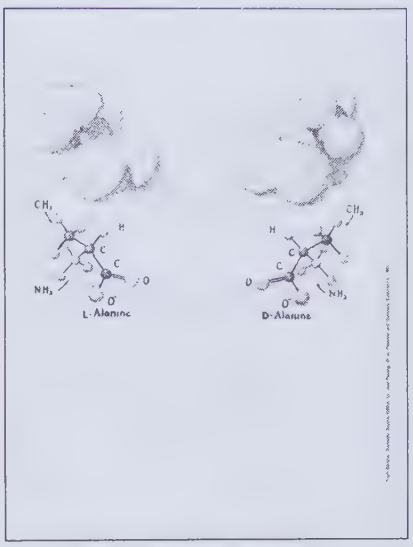
Introduction
Show Transparency 8-2-1, which is a photograph of Lake Mathieson in New Zealand. There is something peculiar about this picture. Can you guess what it is? The photograph is upside down. We were fooled by the almost perfect reflection of the landscape and sky in the water into mistaking their images for them. (Turn the picture rightside up.) The edge of the lake divides the picture into two parts in which most of the points above the edge have image points below it. (Draw a line along the lake's edge and mark several such pairs of points on the picture at various distances from the line. Include one point on the line.) Your students should note that the reflection line is the perpendicular bisector of the segments joining the points to their image points. On the basis of these observations, we will build a "mathematical model" of reflection.



TRANSPARENCY 8-2-1



TRANSPARENCY 8-3-1



TRANSPARENCY 8-3-2

Develop the rest of the lesson, including the method for locating the reflection of a point through a line by means of straightedge and compass, as in the text.

Lesson 3
Isometries and Congruence

Lesson Outline
Terms: Congruence, glide reflection, mirror image.
Transparencies
8-3-1 "Separate checks, please".
8-3-2 Alanine stereoisomers

Introduction
From what the fellow in the cartoon on Transparency 8-3-1 is saying to the waiter, it is apparent that he does not realize that the man sitting at the next table is his own reflection. Perhaps he doesn't want to pay for all eight drinks because he was irritated by observing that the other person copied everything he did during the meal!

Here is something interesting to think about. Suppose that one of the glasses in front of the man were still full; its image in the mirror, then, would also be full. If it were possible for the man to reach through the mirror and take a drink from the reflected glass, would it taste the same as the drink in the glass in front of him? The answer to this question is "probably not"!*

*Some of your students would enjoy reading a fascinating book by Martin Gardner on this subject: *The Ambidextrous Universe: Mirror Asymmetry and Time-Reversed Worlds*, Second Edition (Scribners, 1979). In this book, Gardner quotes a passage from W. H. Auden's poem *The Age of Anxiety* in which a right-handed man sitting in a bar and observing his reflection in a mirror says:

> My deuce, my double, my dear image,
> . . . What flavor has
> That liquor you lift with your left hand . . . ?

The figures on Transparency 8-3-2 illustrate two molecules, each of which is the mirror image of the other. Molecules such as these are called stereoisomers. In his book *The Architecture of Molecules* (W. H. Freeman and Company, 1964), Linus Pauling states that only "left-handed" molecules of the amino acid alanine occur in the proteins of living organisms. The other amino acids in proteins also are left-handed. Some of the right-handed amino acids cannot serve as nutrients and they may, in fact, be poisonous.

Lesson 4
Transformations and Symmetry

Lesson Outline
Terms: Rotation symmetry, reflection (line) symmetry, translation symmetry.
Transparencies
8-4-1 Upside-down names by Scott Kim
8-4-2 Inversions by Scott Kim
8-4-3 Symmetry examples

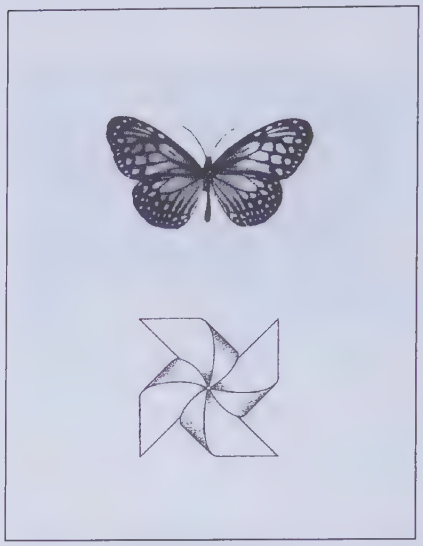
Introduction
The names of Transparency 8-4-1 are from a clever book by Scott Kim, who might be called the "Maurits Escher of calligraphy." Mr. Kim letters words and phrases in amazingly symmetrical pat-



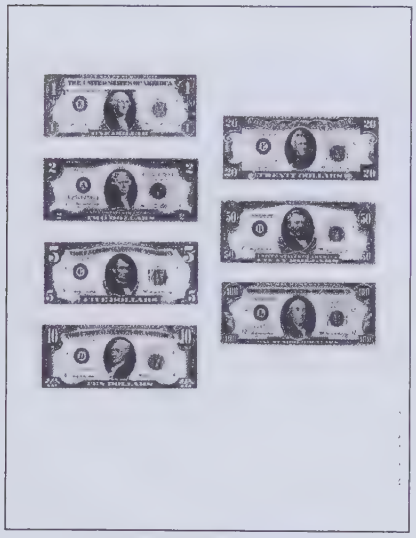
TRANSPARENCY 8-4-1



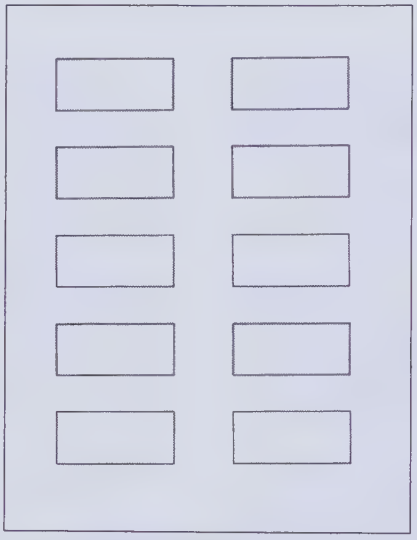
TRANSPARENCY 8-4-2



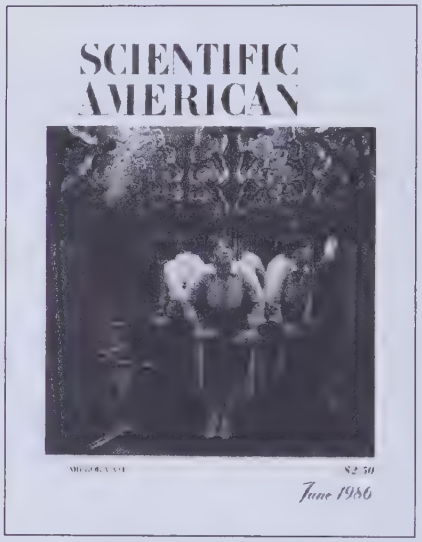
TRANSPARENCY 8-4-3



TRANSPARENCY 8-R-1



TRANSPARENCY 8-R-2



TRANSPARENCY 8-R-3

terns. Rotating the transparency 180° reveals that each name looks the same upside down. The names have been lettered so that they have *rotation symmetry*. A figure has rotation symmetry with respect to a point if it coincides with its rotation image about the point. The figure on Transparency 8-4-2 does not have rotation symmetry, but it is remarkable in that it is the name of Scott Kim's book, *Inversions*, and, upside down, becomes Scott Kim's name.

Show the picture of a butterfly on Transparency 8-4-3, keeping the figure below covered, as an example of *reflection symmetry*. A second copy of the paper-windmill figure to use with the figure on Transparency 8-4-3 is helpful to show that it does not have reflection symmetry but that it does have rotation symmetry. Show how the windmill can be rotated 90°, 180°, or 270° to coincide with itself.

Review of Chapter 8

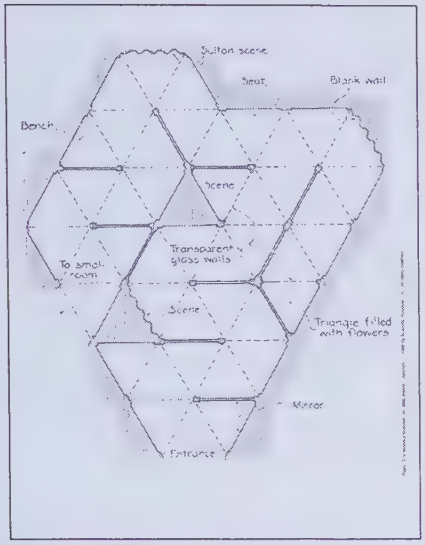
Transparencies

8-R-1 U. S. currency
8-R-2 Cutting corners
8-R-3 Hall of mirrors photograph
8-R-4 Hall of mirrors diagram
8-R-5 Big Dipper

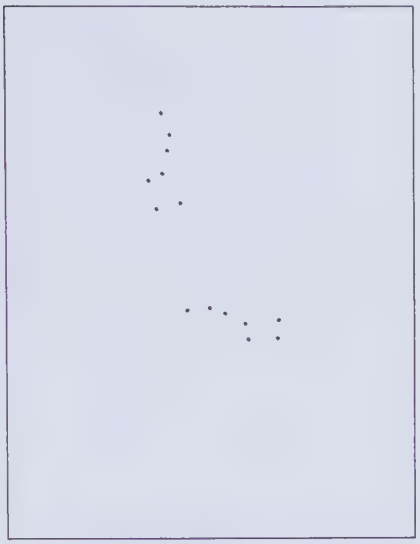
First day

Introduction

Show Transparency 8-R-1. A bill was once submitted to Congress proposing that U.S. currency be printed in a way that enables a person who is blind to determine the denomination of each note. It was



TRANSPARENCY 8-R-4



TRANSPARENCY 8-R-5

suggested that some of the corners of the notes be trimmed so that the values of the notes could be figured out by their shapes.

The seven denominations of bills currently in circulation are shown on the transparency. How many ways are there to cut (or not cut) the corners of a rectangle so that each shape is different? Assume that the cuts are angled at 45° and identical in size.

Hand out duplicated copies of Transparency 8-R-2 and give your students some time to try to discover all of the possibilities. Remind them that a blind person feeling a bill does not know what position the bill is in; considerations of symmetry are important in finding ways that are really different.

It is somewhat of a coincidence that there happen to be exactly seven different ways to cut (or not cut) the corners. They are:

1. No corners.
2. One corner.
3. Two adjacent corners along a short edge.
4. Two adjacent corners along a long edge.
5. Two opposite corners.
6. Three corners.
7. Four corners.

After the solutions have been found, it is interesting to think about how each particular denomination of bill should be cut. Which bill, for example, should have *no* corners cut and which should have *all four* corners cut? (The corners of the bills should be cut so that a person cannot deceive someone who is blind by cutting off additional corners to make them seem more valuable. Therefore, the $1 bill should have all four of its corners cut. The $2 bill should have three corners cut; the $5, $10, and $20 bills should have two corners cut; the $50 bill should have one corner cut; and the $100 bill should have no corners cut.)

Second day

Introduction

There is an elaborate maze of mirrors in Lucerne, Switzerland, that was originally built for the Swiss National Exhibition of 1896. A photograph showing part of the maze appeared on the cover of the June 1986 issue of *Scientific American* (Transparency 8-R-3). The floor plan of the maze, shown on Transparency 8-R-4, consists of an array of equilateral triangles. Mirror walls stand on one or two sides of most of these triangles and several scenes, shown on the floor plan, are inside the maze.

Hand out duplicated copies of Transparency 8-R-4 and explain that, if you stood at the entrance of the maze, you would see what looks like a long hallway. The problem is to use reflections to figure out what you would see at the end of the hallway. Help your students in getting started by drawing a line from the center of the entrance to represent a ray of light and showing its first two reflections from the walls. It may be helpful to think of the path as that of a ball rolled into the maze and bouncing around until it hits a surface that is not a mirror. A person standing at the entrance to the maze would see the scene enclosed in the upper triangle.

A second problem might be to figure out to what point in the maze you would have to walk in order to see the triangle filled with flowers. (You would

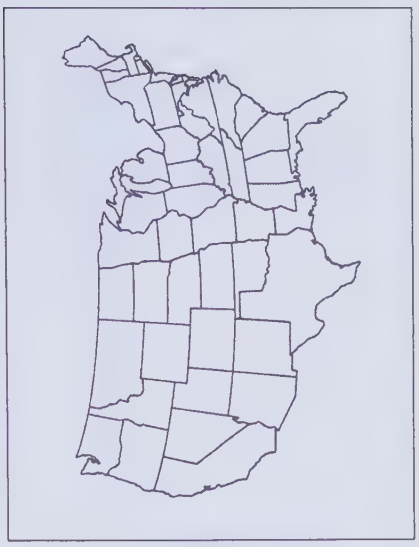
have to walk into the triangle adjacent to the first scene.)

Students who have enjoyed solving these reflection problems will find "The Amateur Scientist" column in the June 1986 issue of *Scientific American* very interesting. The column, by Jearl Walker, includes another color photograph of the Hall of Mirrors in Lucerne and suggests a variety of experiments to perform to learn more about mirror mazes.

An interesting class exercise for additional review is illustrated on Transparency 8-R-5, which should be duplicated and given to your students. Explain that during the night, the stars of the Big Dipper appear to rotate counterclockwise about Polaris, the north star. In the figure, two positions of the Big Dipper are shown. The first problem is to find Polaris, the center of the rotation. The second problem is to find the magnitude of the rotation. (75°.) The final problem is to estimate the length of time that it takes the Big Dipper to move in the sky from one of these positions to the other. (Five hours.)



TRANSPARENCY 9-1-1



TRANSPARENCY 9-1-2

Chapter 9
Area

Lesson 1
Area

Lesson Outline

Terms: Polygonal region.

Postulate 8. The Area Postulate. Every polygonal region has a positive number called its area such that
(1) congruent triangles have equal areas and
(2) the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts.

Transparencies

9-1-1 Map: Kansas and surrounding states
9-1-2 Map: United States

Introduction

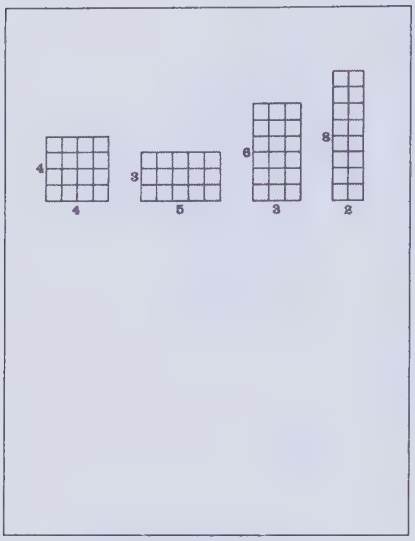
Show Transparency 9-1-1. The earth's present (2003) population is about 6,300,000,000 people.* Suppose that everyone moved to the state of Kansas: how crowded would it be? (Add the overlay to the transparency.) At the center of the United States, Kansas has an area of about 82,000 square miles. If all 6,300,000,000 people lived in Kansas, what would be the average number of people per square mile? (Approximately 77,000.)

Let's see how this figure compares with the "population density" of our school. (Give your class the approximate area of your campus as a fraction of a square mile and the number of people on it so that they can find out. For example, the campus of the school at which I teach has an area of about $\frac{1}{20}$ square mile and about 3,600 people on it during the school day; hence its "population density" is about 72,000 people per square mile.) According to the *Guinness Book of World Records*, the most densely populated region in the world is in a section of Hong Kong, with more than 650,000 people per square mile!

Point out that in each of our calculations, we used a number called the *area* of the region. The idea of "area" is a very fundamental one in geometry and it is the subject of the next chapter of the course.

Show the map of the continental United States on Transparency 9-1-2. The fifty states vary in size from about 591,000 square miles (Alaska) to slightly more than 1,200 square miles (Rhode Island). Note that the areas of states such as Wyoming, Colorado, and Utah are relatively simple to determine, whereas the area of a state such as Florida is not as easily found. Have your students draw a figure representing Utah and use it to introduce the idea of a polygonal region and the Area Postulate.

*Give the appropriate figure for the current year.



TRANSPARENCY 9-2-1



TRANSPARENCY 9-2-2

Lesson 2
Squares and Rectangles

Lesson Outline

Terms: Dimensions of a rectangle: base and altitude.

Postulate 9. The area of a rectangle is the product of its base and altitude.

Corollary. The area of a square is the square of its side.

Transparencies

9-2-1 National Assessment problem on area
9-2-2 World's longest mural

Introduction

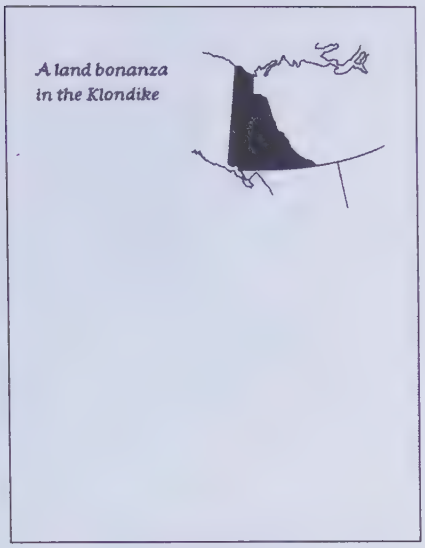
The figures on Transparency 9-2-1 once appeared in a question on a mathematics test (The National Assessment of Educational Progress) given to nine-year-olds all over the United States. The question was: Which rectangle has the same area as the 4×4 square?

After a student gives the correct answer, ask how he or she would explain it to a student who missed it. Only 44% of the students chose the correct answer. Almost as many chose the 3×5 rectangle. Can you explain why? (Your students may have a variety of answers for this.) After discussing the difference between perimeter and area, add the overlay to the transparency. We measure area in terms of square units. From these examples, it would seem that the area of a rectangle can always be found by dividing it into these square units and counting them. A shortcut for finding the number of squares would be to multiply the lengths of the "base" and "altitude."

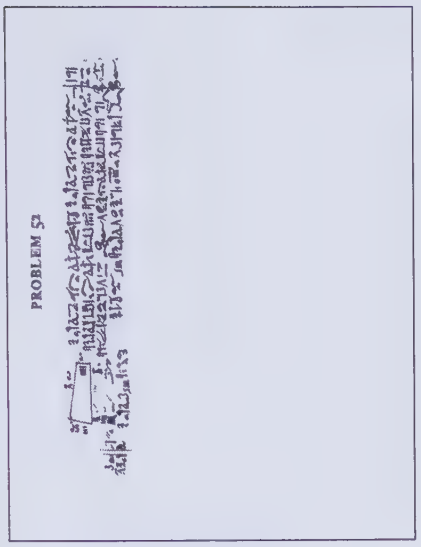
Develop the rest of the lesson as in the text, emphasizing that, because the dimensions of a rectangle are not always integers, we *assume* that the shortcut gives the area for every rectangle.

The following exercise is appropriate for the conclusion of this lesson. The longest mural in the world is located next to the school at which I teach (Grant High School, Van Nuys, California). It is called "The Great Wall of Los Angeles"; the photograph on Transparency 9-2-2 shows part of it as well as part of our campus. The mural is 2,000 feet long and 13.5 feet high. A gallon of the paint used covered an area of 150 square feet. On the basis of these figures, how many gallons of paint were needed to produce the mural? (180.)

Transparency 9-2-3 is for use in demonstrating the solution of the Set III exercise. More information about dividing rectangles into squares all having different areas can be found in *The Second Scientific American Book of Mathematical Puzzles and Diversions* by Martin Gardner (Simon & Schuster, 1961), pages 186-209 and *Ingenuity in Mathematics* by Ross Honsberger, in the New Mathematical Library (M.A.A., 1970), pages 46-60.



TRANSPARENCY 9-3-1



TRANSPARENCY 9-4-1

Lesson 3
Triangles

Lesson Outline
Terms: Base and altitude of a triangle.
Theorem 38. The area of a right triangle is half the product of its legs.
Theorem 39. The area of a triangle is half the product of any base and corresponding altitude.
Corollary. Triangles with equal bases and equal altitudes have equal areas.
Transparencies
9-3-1 A land bonanza in the Klondike

Introduction
Show Transparency 9-3-1. In the 1950s, the Quaker Oats Company sponsored a radio program called "Sergeant Preston of the Yukon." They once made an offer on their program to sell anyone who sent in 25¢ and a coupon from a box of their cereal some land in the Klondike. For your money, you got a deed to your new property and a small bag of dirt from the general vicinity. Of course, you couldn't expect a very big lot for only 25¢ and, in fact, each lot was only one square inch in area!

At this rate, how much was one acre of the land being sold for?* One acre is equal to 43,560 square feet. (One acre would have sold for $1,568,160!) The company didn't make any significant profit, however, because most of the 25¢ went toward the costs of preparing and mailing the deeds and samples of dirt.

*This is adapted from a problem in *The Lore of Large Numbers* by Philip J. Davis, in the New Mathematical Library (M.A.A., 1961).

Lesson 4
Parallelograms and Trapezoids

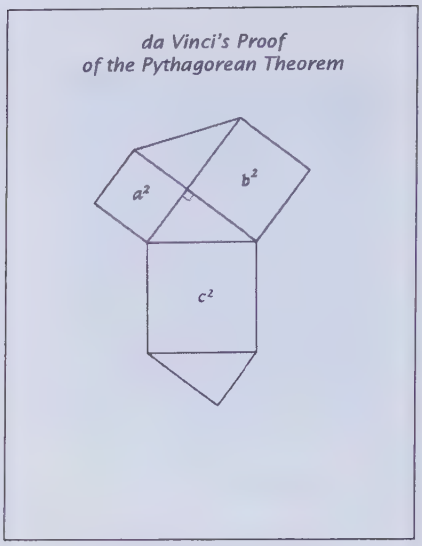
Lesson Outline
Terms: Altitude of a quadrilateral.
Theorem 40. The area of a parallelogram is the product of any base and corresponding altitude.
Theorem 41. The area of a trapezoid is half the product of its altitude and the sum of its bases.
Transparencies
9-4-1 Problem 52 from the *Rhind Papyrus*

Introduction
In 1858, a tourist named Rhind bought an old scroll in a shop in Egypt. The scroll, which is now in the British Museum, turned out to have been written in about 1650 B.C. Now called the Rhind Mathematical Papyrus, this scroll is the major source of our knowledge of early Egyptian mathematics.† It was copied by the scribe Ahmes from earlier writings and contains 87 problems in arithmetic and geometry.

Problem 52 of the Papyrus is shown on Transparency 9-4-1. The problem says:

A cut-off triangle of land is 20 khet on one side, 6 khet on its base, and 4 khet on its cut-off line. What

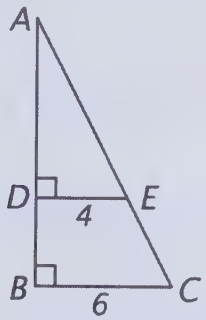
†The translation of *The Rhind Mathematical Papyrus* by Arnold Buffum Chace was published by the N.C.T.M. as volume 8 of the "Classics in Mathematics Education" series.



TRANSPARENCY 9-5-1

is its area? [A khet is equal to 100 cubits, which is roughly 150 feet.]

Although the statement of this problem is rather vague and the figure illustrating it is not accurately drawn, the way in which it is solved makes clear that the problem is about a trapezoid cut off from a right triangle. Have your students draw the following figure to illustrate it.



Although we have not yet derived a formula for the area of a trapezoid, we can find the area of this trapezoid by dividing it into a rectangle and a right triangle. Give your students time to do this before discussing the solution. Drawing a perpendicular from E to BC, we have a rectangle with area 80 and a right triangle with area 20; the area of the trapezoid is 100 square khets.

Following this introduction, the formulas for the areas of a parallelogram and a trapezoid can be derived. Now the Egyptian problem can be reconsidered in terms of the trapezoid area theorem.

Lesson 5
The Pythagorean Theorem

Lesson Outline

Theorem 42. The Pythagorean Theorem. The square of the hypotenuse of a right triangle is equal to the sum of the squares of its legs.

Theorem 43. Converse of the Pythagorean Theorem. If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle.

Transparencies

9-5-1 Da Vinci's proof of the Pythagorean Theorem

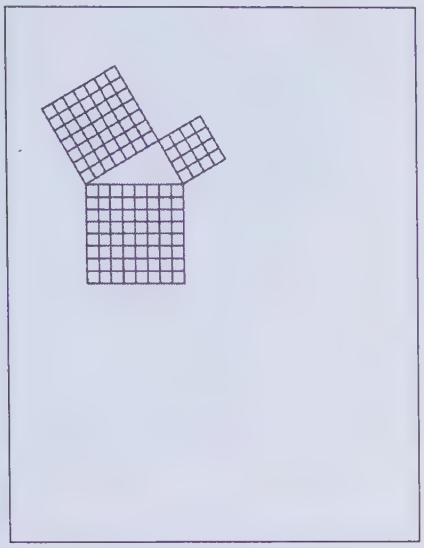
9-5-2 Converse of the Pythagorean Theorem

First day

Introduction

In this lesson, we come to the Pythagorean Theorem, the most famous theorem in geometry and, indeed, in all of mathematics. It has been proved in many different ways. Leonardo da Vinci invented a proof for it based on the figure on Transparency 9-5-1. Your students should find the following informal treatment of it interesting.

First, notice that the figure contains a right triangle with squares drawn on its three sides. To this da Vinci added some line segments to make two copies of the triangle. Look at the hexagon that contains the squares a^2 and b^2 and the two triangles at the top. What kind of symmetry does it seem to have? (Reflection.) Draw the line of symmetry and place, on both sides of it, two pieces that you have cut from colored plastic to fit the two quadrilateral



TRANSPARENCY 9-5-2

halves of the hexagon.

Next, move the two quadrilateral pieces to the side of the figure and call attention to the hexagon that contains the square c^2 and the two triangles above and below it. What kind of symmetry does it seem to have? (Rotation.) Draw a line through the vertices of its two right angles and place the two plastic pieces to fit the two quadrilateral halves of this hexagon.

It is possible to prove what the two arrangements of the plastic pieces suggest—that the two hexagons have equal areas. Remove the plastic pieces and call attention again to the regions contained in each hexagon. How does it follow from the fact that the hexagons havae equal areas that $a^2 + b^2 = c^2$? (The extra triangles are copies of the original triangle, and so they have equal areas. Letting the area of one of the triangles be x, we have

$$a^2 + b^2 + 2x = c^2 + 2x$$

and so

$$a^2 + b^2 = c^2.)$$

To introduce the converse of the Pythagorean Theorem, show Transparency 9-5-2 and explain that this figure is a triangle in which squares have been drawn on the three sides. Then ask if it is a right triangle. If $c^2 \neq a^2 + b^2$, does it follow that a triangle is not a right triangle? (Yes. Suppose that the triangle is a right triangle. Then $c^2 = a^2 + b^2$, which contradicts the hypothesis. So it is not a right triangle.) Now suppose that for the triangle on Transparency 9-5-2 we have found that $c^2 = a^2 + b^2$. Could we conclude from the Pythagorean Theorem that the triangle is a right triangle? (No. We would be making the converse error.) This raises the question of whether the converse of the Pythagorean Theorem is true. Remark that it is and that the proof is left as one of the exercises.

I recommend giving two assignments on this lesson, limiting the first assignment to some or all of the exercises of Set I.

Second day

Introduction

Every year the Nile River in Egypt overflows its banks in late summer and floods the surrounding land. Herodotus, a Greek historian of the fifth century B.C., wrote:

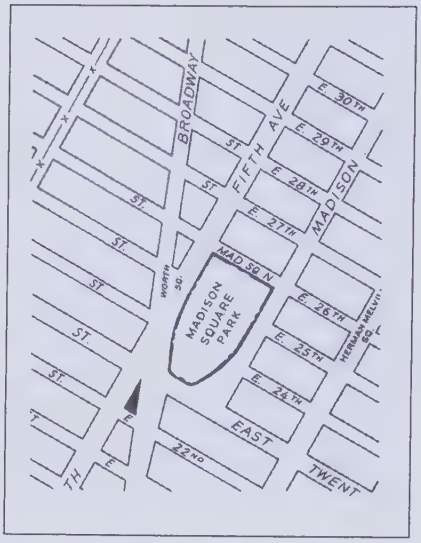
> When the Nile inundates the land all of Egypt becomes a sea, and only the towns remain above water, looking rather like the islands of the Aegean. At such times shipping no longer follows the stream, but goes straight across the country.

Because the flooding wiped out many boundary lines, the ancient Egyptians developed some of the basic principles of surveying to reestablish them. The picture on Transparency 9-5-3 is part of a mural more than 3,000 years old. It shows surveyors holding a rope with knots equally spaced along it. By stretching the rope out to form a triangle, the surveyors made use of something we considered in yesterday's lesson.

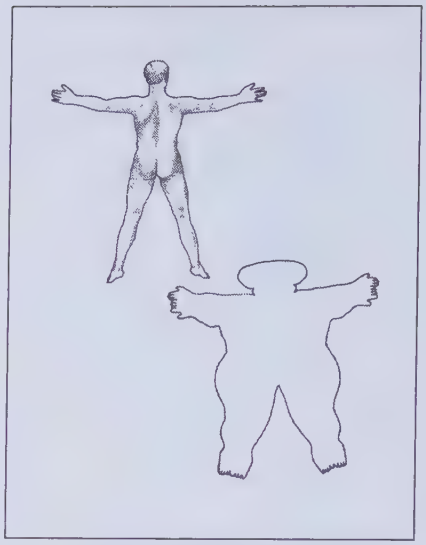
The following makes a nice demonstration. Take a rope about 25 feet long and put 13 knots in it at 2-foot intervals. Give a knot at one end to a student to hold and form, in the middle of your class, a 3-4-5 triangle by having other students hold the ap-



TRANSPARENCY 9-R-1



TRANSPARENCY 9-R-2



TRANSPARENCY 9-R-3

propriate knots. Do not tell the class what lengths the sides of the triangle are, but let the students observe them for themselves. The sides should be stretched out tight and the students outside the triangle should stand up and look at the result. Let a student explain what he or she observes. Note that we know that a 3-4-5 triangle is a right triangle not because of the Pythagorean Theorem but because of its converse. A nice transparency could be made from the photograph of an Egyptian field divided into squares from *Ancient Egypt* by Lionel Casson, in the Great Ages of Man series (Time-Life Books, 1965), pages 48-49. The caption to this photograph is especially appropriate:

> A panorama of applied geometry, this Egyptian field is a patchwork of small squares criss-crossed with irrigation canals and ditches. To sustain agriculture in an arid land, the people had to become engineers and learn how to construct complex irrigation works. To cope with the confusion caused by the annual flood of the Nile, Egyptians worked out the rudiments of geometry. "Rope-stretchers," or surveyors, applied the methods of geometry to redraw the property lines obscured during the flood. Thus the needs of agriculture led the ancient Egyptians to become more than good farmers: mastering both the desert and the river, they not only achieved new skills but expanded their intellectual horizons in the process.*

**Ancient Egypt* by Lionel Casson (Time-Life Books, 1965), pages 48-49.

Review of Chapter 9
First day

Transparencies

9-R-1 The Flatiron building
9-R-2 Map: Manhattan
9-R-3 The area of a human being
9-R-2 Table: A person's area

Introduction

Most of the streets of Manhattan intersect each other at right angles, as they do in most cities. One street that does not, however, is Broadway. The photograph on Transparency 9-R-1 was taken in 1903 at the spot where Broadway (on the left) intersects Fifth Avenue. As the picture shows, the two streets meet at a rather sharp angle and, on the property at the corner, one of the city's first skyscrapers was built. The building, popularly known as the Flatiron Building, still stands at this intersection.

The map on Transparency 9-R-2 shows the plot of land on which the Flatiron Building was constructed. It is in the shape of a triangle whose sides have lengths 80 feet, 150 feet, and 170 feet. What is the area of the land on which the Flatiron Building stands? ($80^2 + 150^2 = 170^2$, so the triangle is a right triangle. Its area is $\frac{1}{2}(80)(150) = 6{,}000$ square feet.)

A Person's Area

Height	Area (sq ft)	Height	Area (sq ft)
4'8"	13	5'10"	20
4'9"	14	5'11"	21
4'10"	14	6'	22
4'11"	15	6'1"	22
5'	15	6'2"	23
5'1"	16	6'3"	23
5'2"	16	6'4"	24
5'3"	17	6'5"	25
5'4"	17	6'6"	25
5'5"	18	6'7"	26
5'6"	18	6'8"	27
5'7"	19	6'9"	27
5'8"	19	6'10"	28
5'9"	20	6'11"	29

TRANSPARENCY 9-R-4

Second day

Introduction

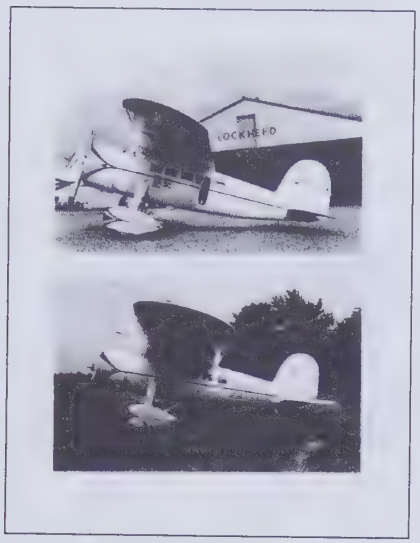
Almost everyone knows their height and weight quite accurately. Do you know, on the other hand, what your *area* is? (Show Transparency 9-R-3.) By this, I mean the total surface area of your skin. The lower figure represents the skin of the person in the upper figure stretched out flat like a pelt.

We have developed formulas for finding the areas of a variety of geometric figures, but they are of no use in finding the area of a figure as complex as this. The area of a person of average weight and body build depends on his or her height. Make an estimate of your own area and then I will give you a formula by means of which you can determine it more accurately. (It might be fun to have your students write their estimates on scraps of paper, and then collect the estimates and read some of the more extreme ones to the class before going any further.)

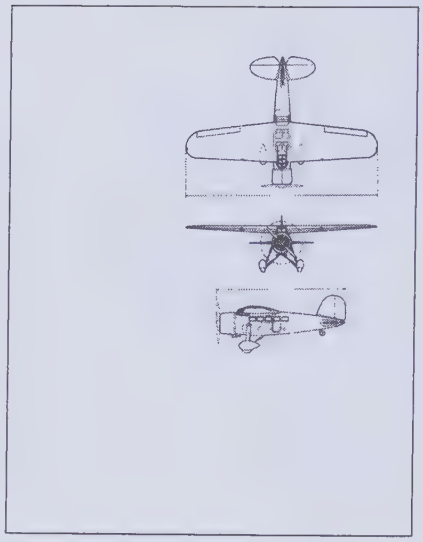
You can find your approximate area by means of the following formula:

$$A = \frac{3}{5}h^2$$

in which h is your height in feet. After your students have had time to compute their areas by this formula, show Transparency 9-R-4 so that they can check their answers. Each area is rounded to the nearest square foot.



TRANSPARENCY 10-1-1



TRANSPARENCY 10-1-2



TRANSPARENCY 10-2-1

Chapter 10
Similarity

Lesson 1
Ratio and Proportion

Lesson Outline

Terms: Ratio, proportion, terms, means, extremes, geometric mean.

Transparencies

10-1-1 Photographs of Winnie Mae and models
10-1-2 Diagrams of Winnie Mae

Introduction

One of the airplanes in the pictures on Transparency 10-1-1 is an exact replica of the first plane to be flown on a solo flight around the world. Its pilot was Wiley Post and the year was 1933. Although the other plane looks exactly like Post's plane, it could never be flown around the world because it is too small: it is a scale model of the real plane and weighs only 15 pounds. Can you guess which plane is which? (The one pictured at the top is the model.)

To be convincing, the model must have the same shape as the actual plane. Hand out duplicated copies of Transparency 10-1-2 and explain that the figures could represent views of either the plane or the model. Have your students write the letters w and l on the figures to represent wingspan and length. The plane is 27 ft long and has a wingspan of 42 ft. The model is 4.5 ft long. Can you find out what its wingspan is? (Give your students time to think about this and then let some of those who have the correct answer—7 ft—explain their methods.)

Use the proportion

$$\frac{27}{4.5} = \frac{42}{7}$$

to review the meanings of "ratio" and "proportion," and present the rest of the lesson as in the text.

Lesson 2
Similar Figures

Lesson Outline

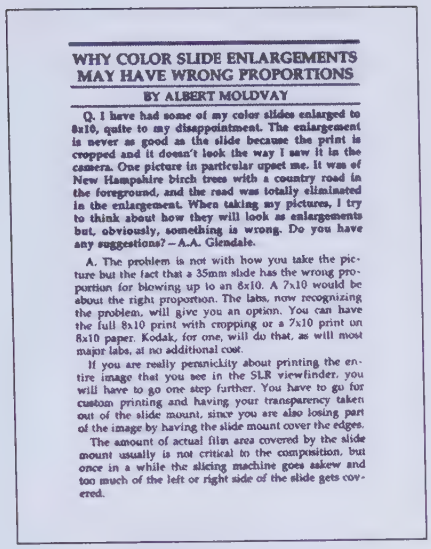
Terms: Similar triangles, similarity correspondence, center and magnitude of a dilation.

Transparencies

10-2-1 Betty Grable
10-2-2 Why slide enlargements have wrong proportions
10-2-3 Slide frame and an 8×10 photograph

Introduction

Show Transparency 10-2-1. (If your school has a photography class, you might ask a student to take a photograph of one of your school's well-known athletes—for example, a member of the basketball, baseball, football, gymnastic, soccer, or wrestling teams. Be sure that the photograph is clear enough so that your class can make the required measurements with accuracy.) The Renaissance artist Albrecht Dürer wrote several books on human proportions. Add the overlay. (Make a new overlay if you use a photograph of a student in your school.



WHY COLOR SLIDE ENLARGEMENTS MAY HAVE WRONG PROPORTIONS

BY ALBERT MOLDVAY

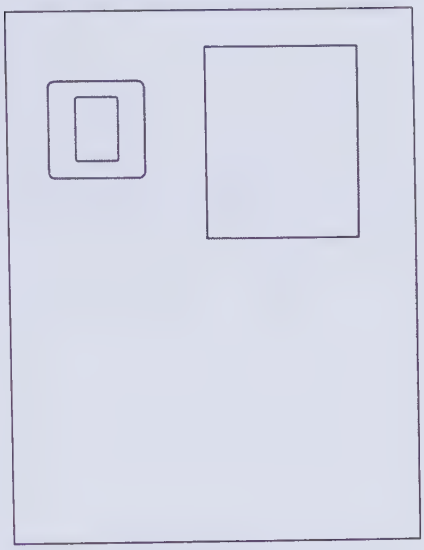
Q. I have had some of my color slides enlarged to 8x10, quite to my disappointment. The enlargement is never as good as the slide because the print is cropped and it doesn't look the way I saw it in the camera. One picture in particular upset me. It was of New Hampshire birch trees with a country road in the foreground, and the road was totally eliminated in the enlargement. When taking my pictures, I try to think about how they will look as enlargements but, obviously, something is wrong. Do you have any suggestions? – A.A. Glendale.

A. The problem is not with how you take the picture but the fact that a 35mm slide has the wrong proportion for blowing up to an 8x10. A 7x10 would be about the right proportion. The labs, now recognizing the problem, will give you an option. You can have the full 8x10 print with cropping or a 7x10 print on 8x10 paper. Kodak, for one, will do that, as will most major labs, at no additional cost.

If you are really persnickity about printing the entire image that you see in the SLR viewfinder, you will have to go one step further. You have to go for custom printing and having your transparency taken out of the slide mount, since you are also losing part of the image by having the slide mount cover the edges.

The amount of actual film area covered by the slide mount usually is not critical to the composition, but once in a while the slicing machine goes askew and too much of the left or right side of the slide gets covered.

TRANSPARENCY 10-2-2



TRANSPARENCY 10-2-3

The overlay should mark the positions of the neck, hip, knee, and ankle.) In one of his books, Dürer claimed that for most people, the distance from hip to knee is the geometric mean between the distance from neck to hip and the distance from knee to ankle.*

The lines on this photograph of the actress Betty Grable (the "pin-up" girl of World War II), mark the positions of these four points: neck, hip, knee, and ankle. Place an ordinary centimeter ruler on the picture and measure the distance between hip and knee and the distance between knee and ankle, each to the nearest tenth of a centimeter. (They are 5.6 cm and 4.7 cm, respectively.) Mark the distances on the picture and ask your students to use their calculators to figure out what the distance from knee to hip should be if Dürer's claim is correct. ($\frac{x}{5.6} = \frac{5.6}{4.7}$; $x \approx 6.7$.) Measuring this distance on the picture will reveal that Dürer's statement seems to be true.

Alternative Introduction

Show Transparency 10-2-2, keeping the answer to the question covered. This question once appeared in a photography column in the *Los Angeles Times*. Read it to your class and, still keeping the answer covered, remark that the answer to the question is the subject of the new lesson.

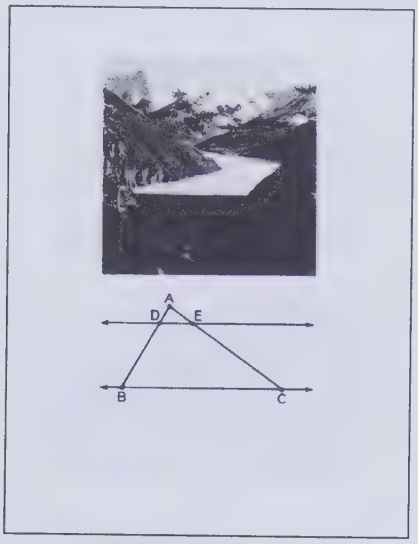
Hand out duplicated copies of Transparency 10-2-3, on which a 35-mm slide in a frame and an 8×10 print are represented. The print is drawn in centimeters rather than inches so that it will fit on the transparency. When a picture is enlarged, it and its enlargement have the same shape. Figures that have the same shape are *similar*. Define similar triangles and then explain that the definition can be extended to rectangles. Point out that the problem with the slide enlargement is obviously not with the angles. Have your students measure the inner dimensions of the frame in millimeters. (They are about 23 mm and 33 mm, the frame being slightly smaller than the picture on the 35-mm film.)

If the slide is enlarged to a width of 80 mm (8 cm), what will its length be? ($\frac{23}{80} = \frac{33}{x}$; $x \approx 115$.) The picture will be approximately 11.5 cm long and this is the cause of the problem. Have your students extend the rectangle representing the print downward another 1.5 cm to show the part of the picture that got eliminated.

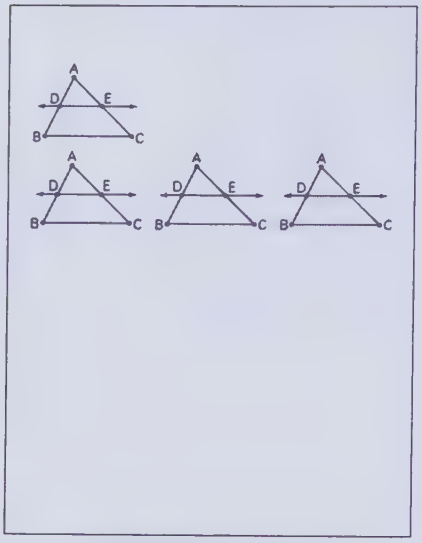
To avoid losing part of the picture, the slide could be enlarged to a length of 100 mm instead. In this case, what would its width be? ($\frac{23}{x} = \frac{33}{100}$; $x \approx 70$.) The picture would be approximately 7 cm wide.

Put Transparency 10-2-2 back on the screen and read the first paragraph of the answer, whose meaning will now be very clear.

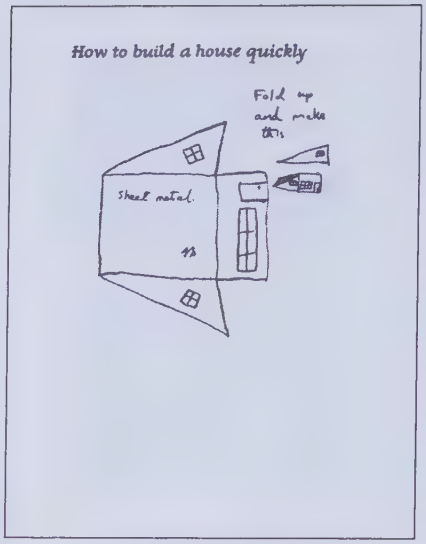
*Dan Pedoe devotes an entire chapter of his book *Geometry and the Liberal Arts* (St. Martin's Press, 1976) to the geometry of Dürer.



TRANSPARENCY 10-3-1



TRANSPARENCY 10-3-2



TRANSPARENCY 10-4-1

Lesson 3
The Side-Splitter Theorem

Lesson Outline

Theorem 44. The Side-Splitter Theorem. If a line parallel to one side of a triangle intersects the other two sides in different points, it divides the sides in the same ratio.

Corollary. If a line parallel to one side of a triangle intersects the other two sides in different points, it cuts off segments proportional to the sides.

Transparencies

10-3-1 Side view of the Grande Dixence dam
10-3-2 Side-splitter theorem figures

Introduction

Show Transparency 10-3-1. The photograph shows the Grande Dixence dam in the Swiss Alps, one of the highest dams in the world. The dam was completed in 1961 at a cost of 372 million dollars and holds back more than ten million tons of water.

Explain the meaning of the figure below the photograph (see page 392 of the text). Then present the problem of finding the length of EC indirectly from the lengths of AD, DB, and AE as is done in the text. The figure on the transparency is drawn on a somewhat different scale from that of the one shown in the text. Tell your students that 1 cm on the figure represents a distance of 70 m and measure AD, DB, and AE, expressing each length in meters. (They are 70 m, 280 m, and 105 m, respectively.) Ask your students if they can figure out the length of EC before you measure it and have someone who finds the length explain how he or she did it.

Hand out duplicated copies of Transparency 10-3-2 for your students to use in developing the proof of the Side-Splitter Theorem.

Lesson 4
The AA Similarity Theorem

Lesson Outline

Theorem 45. The AA Theorem. If two angles of one triangle are equal to two angles of another triangle, the triangles are similar.

Corollary. Two triangles similar to a third triangle are similar to each other.

Transparencies

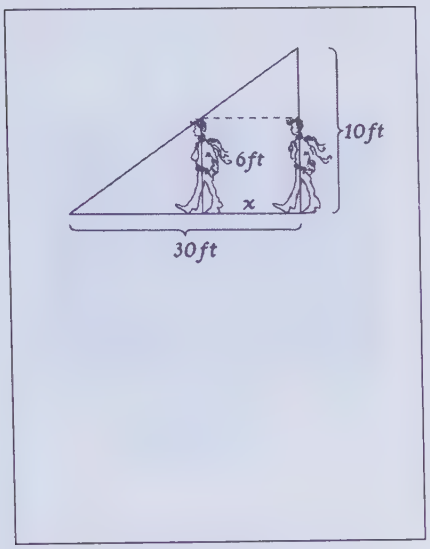
10-4-1 "How to build a house quickly"
10-4-2 Walking in the house
10-4-3 AA Similarity Theorem figures
10-4-4 How to measure up by looking down

First day

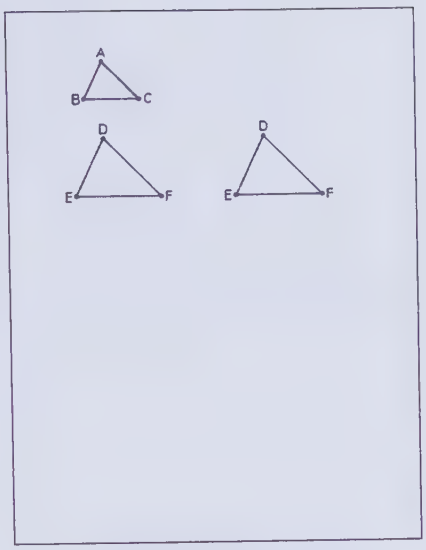
Introduction

The following exercise is based on a problem in a fascinating book titled *Children Solve Problems* by Edward de Bono (Harper and Row, 1974). Children of various ages were asked to solve problems such as how to stop a cat and dog from fighting, how to weigh an elephant, and how to build a house quickly.

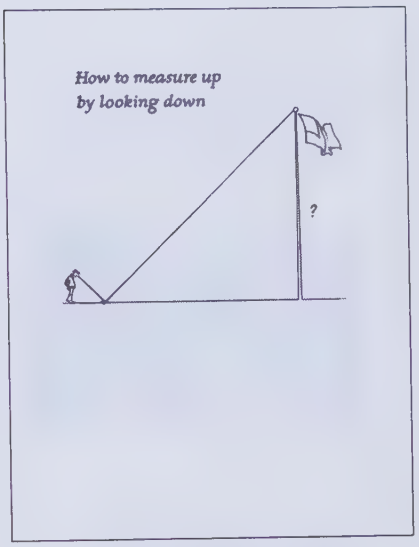
One boy, when given the problem of how to build a house quickly, drew the diagram on Transparency 10-4-1. He figured that, because toy houses are sometimes made by cutting and folding cardboard, the same method might be applied to constructing a house from sheet metal. One problem with his solution is the design that he proposes.



TRANSPARENCY 10-4-2



TRANSPARENCY 10-4-3



TRANSPARENCY 10-4-4

Show Transparency 10-4-2. The roof of the house slants down to one side so that someone walking in the front door would soon bump his head on the ceiling.

Suppose that the right-hand wall of the house as shown here is 10 feet high and that the house is 30 feet wide. (The figure is not drawn to scale.) How far do you think a person 6-feet tall could walk before his head would touch the ceiling?

There are three triangles in the figure, all of which look as if they might be similar. If they are, we can write a proportion between their sides, such as

$$\frac{6}{10} = \frac{30-x}{30} \quad \text{or} \quad \frac{4}{10} = \frac{x}{30},$$

to find that the person could walk 12 feet.

This raises the question of whether the triangles in the figure are really similar. Remind your students that we never bother to show that all six pairs of parts of two triangles are equal to prove that they are congruent. Certain sets of three pairs of parts are sufficient. (Ask someone to name them.) Then tell your students that it is even easier to prove that two triangles are similar. All we have to show is that two pairs of angles are equal.

Hand out duplicated copies of Transparency 10-4-3 for use in proving the AA Similarity Theorem as in the text. The corollary to this theorem should also be considered before returning to the figure on Transparency 10-4-2 to see why the three triangles are similar. The two smaller triangles are similar to the large one because of the AA Similarity Theorem (the triangles have equal right angles and common angles); they are similar to each other because of the corollary to the AA Theorem.

If you plan to do the height-measuring experiment on the next day, you might limit the assignment to selected exercises from Set I of Lesson 4.

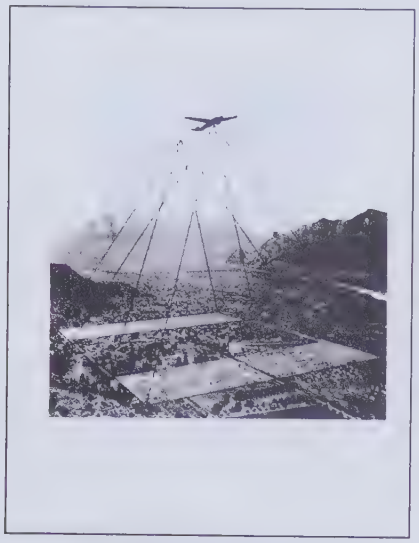
Second day

Introduction

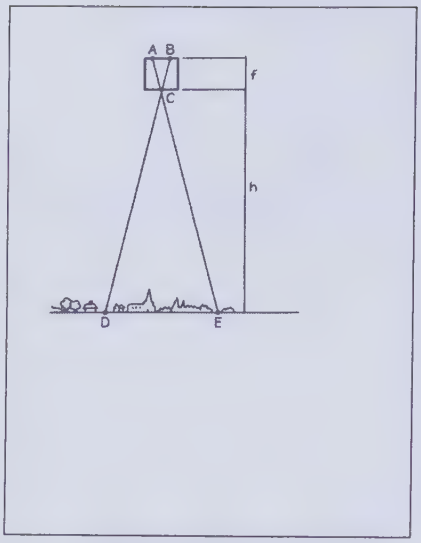
If possible, take your class outside and let your students try to measure the height of your school flagpole or a tree by means of the mirror method illustrated on Transparency 10-4-4. You need some small mirrors (one for every three students), at least one 50-ft tape measure, and some shorter tape measures. I assign my students to groups of three each: one student in each group acts as the observer, one as the measurer, and one as the recorder. Note that the method assumes that the feet of the observer, the mirror, and the base of the pole or tree are in a *horizontal* plane and also that the mirror is level. You might ask your students why these things are important. (They are necessary if the angles in the two triangles are to be equal; this, of course, is the basis for our treatment of them as similar when we write a proportion between their pairs of corresponding sides.)

Although the students work in small groups, I hold each student responsible for making and turning in his or her own calculations.

Some more exercises from Lesson 4 might be assigned for homework.



TRANSPARENCY 10-5-1



TRANSPARENCY 10-5-2



TRANSPARENCY 10-6-1

Lesson 5
Proportions and Dilations

Lesson Outline

Terms: Corresponding altitudes.

Theorem 46. Corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides.

Transparencies

10-5-1 Aerial photography
10-5-2 Aerial-photography scale diagram

Introduction

Show Transparency 10-5-1. Aerial photography plays an important part in modern map-making. The plane carrying the camera flies on a straight course parallel to the ground and the camera takes a continuous series of overlapping pictures. The figure on the transparency shows the overlap of two successive photographs taken in flight.

It is important to know what the scale of the pictures produced is—that is, how the dimensions of the photograph compare with the dimensions of the region being photographed. The diagram on Transparency 10-5-2 shows how this scale is determined. The figure at the top represents the camera: C is the lens, AB is the film, and f is the focal length of the lens. The altitude of the camera is h and the part of the ground picked up on the film is DE.

Given that AB ∥ DE, why does it follow that $\Delta ABC \sim \Delta EDC$? Although f and h are not lengths of sides of these triangles, they are lengths of *corresponding altitudes*. We can prove that corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides.

After developing this proof, go back to Transparency 10-5-2. The scale of the photograph illustrated in this diagram is $\frac{AB}{DE}$. Because we have proved that corresponding altitudes of similar triangles have the same ratio as that of the corresponding sides, we can conclude that

$$\frac{f}{h} = \frac{AB}{DE}.$$

This provides the way by which the scale of aerial photographs is determined: it is found by dividing the focal length of the lens by the altitude of the camera. For example, a camera with a focal length of 6 inches and an altitude of 2 miles produces photographs with a scale of

$$\frac{6 \text{ in}}{2 \text{ mi}} = \frac{0.5 \text{ ft}}{2(5{,}280) \text{ ft}} = \frac{1}{4(5{,}280)} = \frac{1}{21{,}120}.$$

Lesson 6
Perimeters and Areas of Similar Figures

Lesson Outline

Theorem 47. The ratio of the perimeters of two similar polygons is equal to the ratio of the corresponding sides.

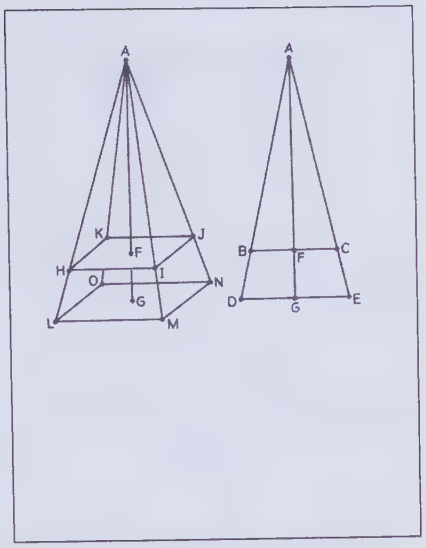
Theorem 48. The ratio of the areas of two similar polygons is equal to the square of the ratio of the corresponding sides.

Transparencies

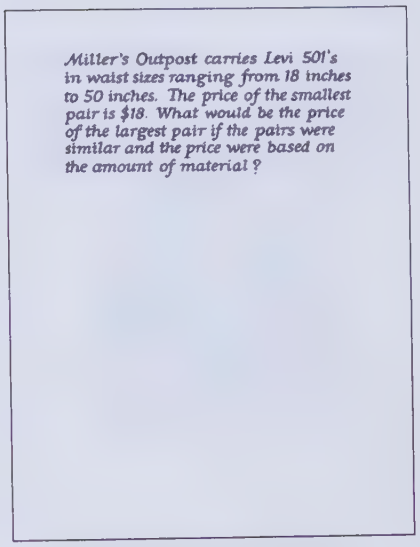
10-6-1 An early medical x-ray
10-6-2 X-ray of bird shot in hand
10-6-3 X-ray diagram



TRANSPARENCY 10-6-2



TRANSPARENCY 10-6-3



TRANSPARENCY 10-R-1

Introduction

The photograph on Transparency 10-6-1 was taken at Dartmouth College in 1896 and shows one of the medical uses of x-rays. The patient, sitting with his back toward the camera, is having his broken arm x-rayed. The doctors taking the x-ray were not yet aware of the dangers of prolonged radiation.

The x-ray photograph on Transparency 10-6-2 was taken in the same year and shows bird shot that became embedded in the patient's hand in a hunting accident.

When an x-ray photograph is taken, it is always larger than the object it pictures. The reason for this can be seen from the figures on Transparency 10-6-3. In the first figure, a three-dimensional view, the source of the x-rays is at A, the object being photographed is shown as the upper rectangle, and its image on the photographic plate is shown as the lower rectangle. The second figure is a two-dimensional view.

To emphasize the connection between the two figures, have your students add segments BC and DE to the first figure. How does it follow from the fact that BC ∥ DE that $\Delta ABC \sim \Delta ADE$ in the second figure? Given that AF ⊥ BC, AG ⊥ DE, AF = 16 cm, and FG = 4 cm, find $\frac{BC}{DE}$. (Because corresponding altitudes of similar triangles have the same ratio as the corresponding sides, $\frac{BC}{DE} = \frac{AF}{AG} = \frac{16}{20} = \frac{4}{5}$.) This means that the dimensions of the object being x-rayed are four-fifths the corresponding dimensions of the image in the photograph.

If the area of the image in the photograph is 50 cm^2, what is the area of the object x-rayed? To answer this question, we have to know something about the areas of similar polygons.

Use triangles as on pages 415-416 of the text to prove the perimeter and area relations for similar polygons and then repeat the question about the area of the x-ray. It can be proved that HIJK ~ LMNO. Therefore, $\frac{\alpha HIJK}{\alpha LMNO} = \left(\frac{HI}{LM}\right)^2$. Because BC = HI and DE = LM, we can conclude that $\frac{\alpha HIJK}{\alpha LMNO} = \left(\frac{BC}{DE}\right)^2$. If LMNO = 50 cm^2, find αHIJK. [Letting αHIJK = x, we have $\frac{x}{50} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$, from which it follows that $x = 32$.] The area of the object x-rayed is 32 cm^2.

Review of Chapter 10

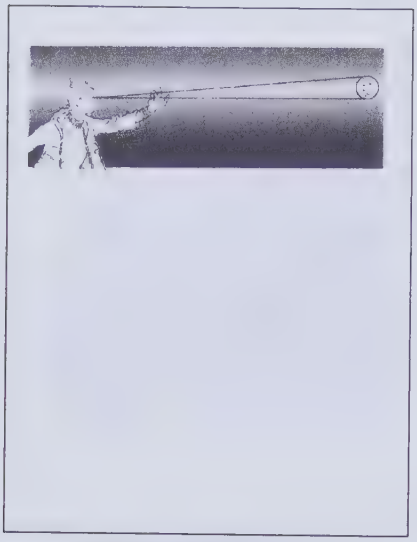
Transparencies

10-R-1 Levi 501's
10-R-2 Eclipsing the moon with a dime

First day

Introduction

The problem on Transparency 10-R-1 provides a review of the theorem about the areas of similar polygons. Because some students are likely to immediately conclude that the answer is $50, you may want to warn them that this is wrong. Let a student who finds the correct answer explain his or her method. (If the price is based on the amount of ma-



TRANSPARENCY 10-R-2

terial, we need to compare the areas of the cloth used in making each pair of pants. If they are similar and x represents the price of the largest pair, $\frac{18}{x} = \left(\frac{18}{50}\right)^2$; $x \approx 138.89$.) The largest pair would cost about \$139. After finishing the problem, you might point out that, although it seems reasonable to assume that clothing patterns over a small range of sizes are similar to each other, the assumption in the problem that the smallest and largest pairs of Levi's are similar is not very plausible.

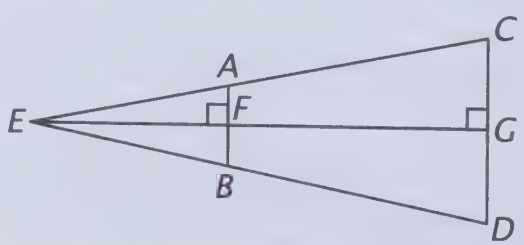
Second day

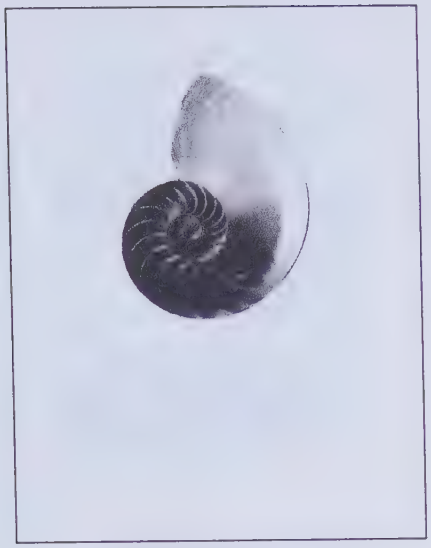
Introduction

Show Transparency 10-R-2. Imagine that you are standing outside on a night in which there is a full moon. If you closed one eye and held a dime directly in front of your other eye, you could easily "cover" the moon with it. In other words, held close to your eye, the dime would appear larger than the moon. Now suppose that you move the dime away from your eye and toward the moon. How far away do you think you would have to hold the dime in order for it to appear to be the same size as the moon? Six inches? A foot? Students who have dimes with them might hold the coins at various distances from their eyes before guessing the answer to this problem.

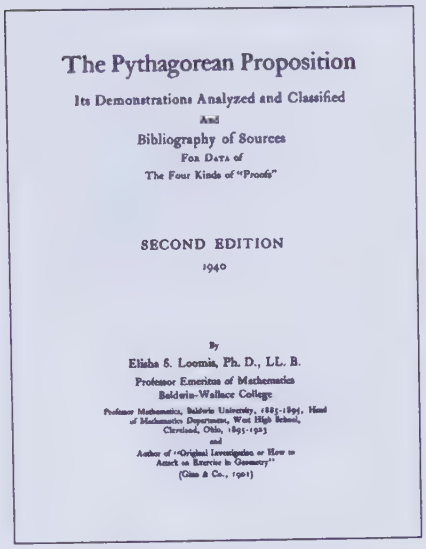
Have your students draw the figure shown at the right and explain that point E represents your eye; AB, the edge of the dime; and CD, the edge of the "disk of the moon." If the dime is held so that AB || CD, then $\Delta EAB \sim \Delta ECD$. Because the distances from your eye to the dime and moon are the lengths of the corresponding altitudes (EF and EG) of these triangles, we know that $\frac{EF}{EG} = \frac{AB}{CD}$. Which of the four lengths in this proportion is the answer to the question? What do the other lengths represent? EG, the distance to the moon, is about 385,000 km and CD, the diameter of the moon, is about 3,480 km. Students with dimes can measure AB, which is about 1.8 cm. By substituting, we get $\frac{EF}{385{,}000{,}000\text{ m}} = \frac{0.018\text{ m}}{3{,}480{,}000\text{ m}}$, and so $EF \approx 2$ m.

The distance from your eye that you would have to hold the dime in order that it would look equal in size to the moon is about two meters—much farther than anyone would be capable of actually holding it!

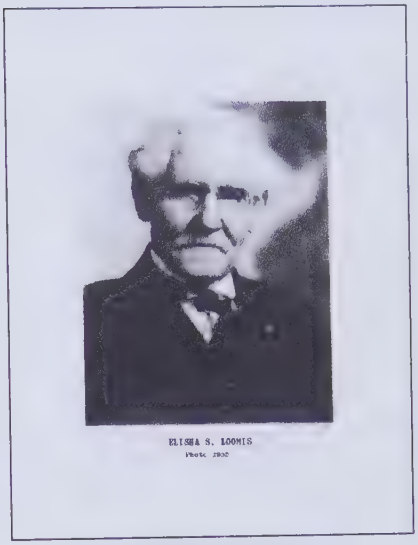




TRANSPARENCY 11-1-1



TRANSPARENCY 11-2-1



TRANSPARENCY 11-2-2

Chapter 11

The Right Triangle

Lesson 1
Proportions in a Right Triangle

Lesson Outline

Terms: Projection.

Theorem 49. The altitude to the hypotenuse of a right triangle forms two triangles similar to it and to each other.

Corollary 1. The altitude to the hypotenuse of a right triangle is the geometric mean between the segments into which it divides the hypotenuse.

Corollary 2. Each leg of a right triangle is the geometric mean between the hypotenuse and its projection on the hypotenuse.

Transparencies

11-1-1 Photograph of chambered nautilus

Introduction

A chambered nautilus shell cut in half so that the compartments can be seen is an attractive visual aid to have for the mathematics classroom and is worth looking for. The photograph on page 428 of the text is of a shell that can be bought for just a few dollars at an oceanside store.

The chambered nautilus is most abundant in the southwest Pacific where it swims on the ocean floor in search of food. The nautilus lives in the outermost chamber of its shell and can let air into or out of the other chambers to help in moving upward or downward through the water.

Show Transparency 11-1-1 and mark and label the center point of the shell point P. Have a student draw a line through point P in any direction and a second line through P that is perpendicular to the first. Then, label some of the consecutive points in which the curve of the shell intersects these lines and develop the rest of the lesson as presented on pages 428-429 of the text.

Lesson 2
The Pythagorean Theorem Revisited

Lesson Outline

Theorem. The Pythagorean Theorem. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

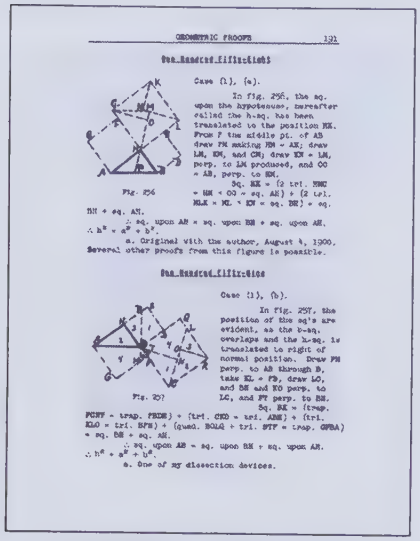
Transparencies

11-2-1 The Pythagorean Proposition
11-2-2 Photograph of Elisha S. Loomis
11-2-3 Page from *The Pythagorean Proposition*

Introduction

Dr. Elisha Loomis taught mathematics at West High School in Cleveland, Ohio, from 1895 to 1923. As one of his hobbies, he collected proofs of the Pythagorean Theorem. Three hundred and seventy of these proofs were published in a book whose title page is shown on Transparency 11-2-1.* A picture

**The Pythagorean Proposition* by Elisha Scott Loomis (privately printed in 1927 and 1940; reprinted in 1968 as volume 1 of N.C.T.M.'s *Classics in Mathematics Education* series).


GEOMETRIC PROOFS 131

One Hundred Fifty-Eight

Case (1), (a).

In fig. 256, the sq. upon the hypotenuse, hereafter called the h-sq, has been translated to the position HK. From F the middle pt. of AB draw FM making FM = AK, draw LM, KM, and CM; draw KN = LM, perp. to LM produced, and OO = AB, perp. to KM.

Fig. 256

Sq. HK = (2 tri. MHC = HK = OO = sq. AK) + (2 tri. MLK = ML = KM = sq. BK) = sq. AH + sq. AN.

∴ sq. upon AH = sq. upon AN + sq. upon AK.

∴ $h^2 = a^2 + b^2$.

a. Original with the author, August 4, 1900. Several other proofs from this figure is possible.

One Hundred Fifty-Nine

Case (1), (b).

In fig. 257, the position of the sq's are evident, as the h-sq, overlaps and the h-sq. is translated to right of normal position. Draw FH perp. to AB through B, take KL = FB, draw LO, and BE and KO perp. to LC, and FT perp. to EH.

Fig. 257

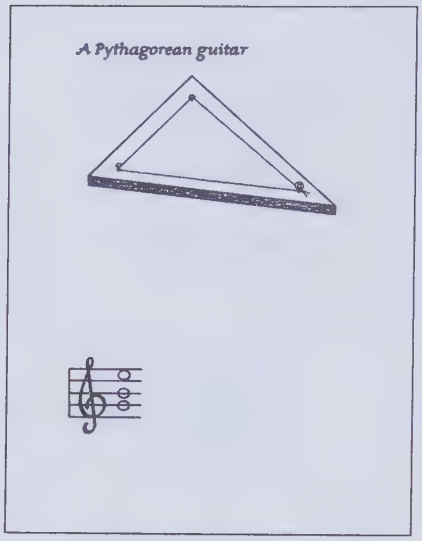
Sq. BK = (trap. FCHB) + (tri. CED = tri. AHE) + (tri. KLO = tri. BFA) + (quad. BOLQ + tri. BTF = trap. GFBA) = sq. BE + sq. AE.

∴ sq. upon AB = sq. upon BK + sq. upon AK.

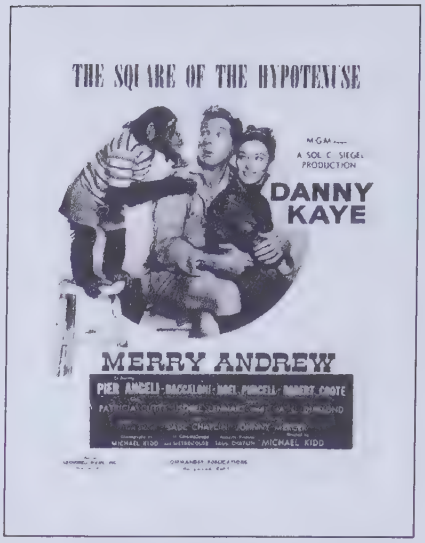
∴ $h^2 = a^2 + b^2$.

a. One of my dissection devices.

TRANSPARENCY 11-2-3



TRANSPARENCY 11-3-1



TRANSPARENCY 11-3-2

of Dr. Loomis is on Transparency 11-2-2 and a typical page from the book is shown on Transparency 11-2-3.

In our previous encounter with the Pythagorean Theorem, we considered several proofs based on area. The exercises also emphasized the area aspect of the theorem. In this lesson, we concentrate more on how the Pythagorean Theorem is used to find lengths. We begin with an unusually simple proof of it based on a theorem of yesterday's lesson. Guide your students in writing the proof presented on page 435 of the text.

Lesson 3
Isosceles and 30°-60° Right Triangles

Lesson Outline

Theorem 50. The Isosceles Right Triangle Theorem. In an isosceles right triangle, the hypotenuse is $\sqrt{2}$ times the length of a leg.

Corollary. Each diagonal of a square is $\sqrt{2}$ times the length of one side.

Theorem 51. The 30°-60° Right Triangle Theorem. In a 30°-60° right triangle, the hypotenuse is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.

Corollary. An altitude of an equilateral triangle having side s is $\frac{\sqrt{3}}{2}s$ and its area is $\frac{\sqrt{3}}{4}s^2$.

Transparencies

11-3-1 A Pythagorean guitar
11-3-2 Cover of *The Square of the Hypotenuse*
11-3-3 First page from *The Square of the Hypotenuse*

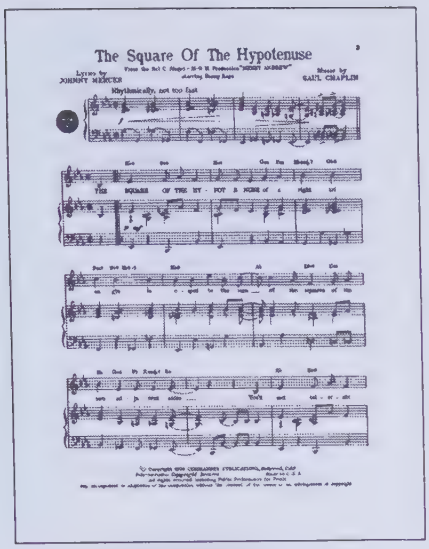
Introduction

In his book, *The Number of Things: Pythagoras, Geometry and Humming Strings* (Dutton, 1964), Evans G. Valens describes a "Pythagorean guitar" that your students will find especially interesting. The guitar, pictured on Transparency 11-3-1, is easy to build. All you need is a guitar string, a piece of wood, two nails, two small pulleys (to equalize the tension), and a set-screw.

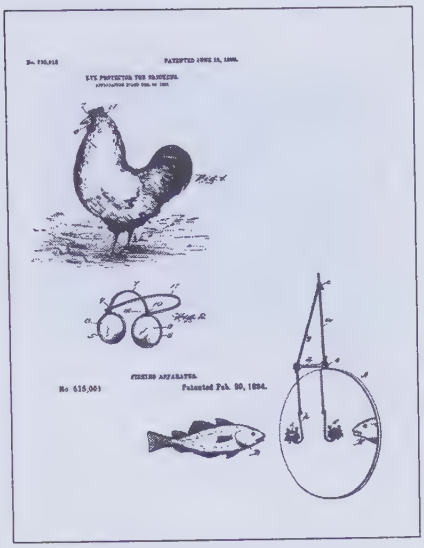
If a guitar string is stretched around three nails at the vertices of a 3-4-5 right triangle, the three sides of the string triangle, when plucked, will sound a musical chord. If the hypotenuse is tuned to G, the longer and shorter legs will sound B and E, respectively. Valens notes that these are the top three strings of a guitar and that the chord sounded is a minor triad. If a randomly shaped "triangle guitar" is played, it will not ordinarily produce a musical chord. Why does the 3-4-5 right triangle do so? Because Pythagoras, as well as being a great mathematician, was also the founder of the science of music. His belief in the importance of whole numbers led him to define the intervals in the musical scale in terms of them. The scales of Pythagoras are discussed in great detail in chapters 14 through 16 of Valens's book.

A song that would be especially appropriate to play on a Pythagorean guitar was once featured in a Hollywood movie (*Merry Andrew*, M-G-M Pictures, 1958). Its title is "The Square of the Hypotenuse" and the cover and first page of the sheet music are pictured on Transparencies 11-3-2 and 11-3-3.

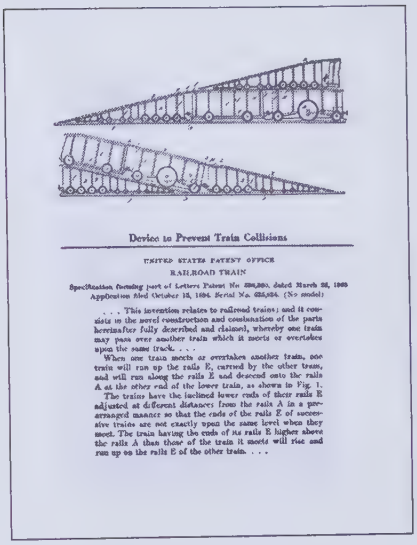
In presenting the new lesson on the relations of



TRANSPARENCY 11-3-3



TRANSPARENCY 11-4-1



TRANSPARENCY 11-4-2

the lengths of the sides of isosceles and 30°-60° right triangles, you should mention that these theorems are very important in trigonometry as well as geometry.

Lesson 4
The Tangent Ratio

Lesson Outline
Terms: Tangent ratio
Transparencies
11-4-1 Crazy inventions
11-4-2 A device for preventing train collisions

Introduction
Some extraordinary inventions have been patented in the U.S. Patent Office. A book titled *Absolutely Mad Inventions* by A. E. Brown and H. A. Jeffcott, Jr. (Dover, 1970) contains reproductions of the original drawings and descriptions filed for some of these patents.

Two of them are shown on Transparency 11-4-1. The first is a pair of goggles for a chicken, to protect its eyes in case other chickens peck it in the head. The second, a device for catching fish, is rather quaintly described by its inventor in these words:

> The fish B, when approaching the bait s, will see the reflection B′, of himself in the mirror, also coming for the reflection of the bait s′, and will be made bolder by the supposed companionship, and more eager to take the bait before his competitor seizes it. He will lose his caution, and take the bait with a recklessness that greatly increases the chances of his being caught on the hook.

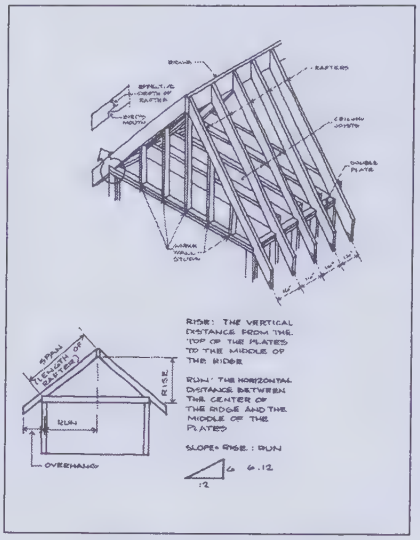
Another remarkable invention is one for preventing train collisions. It was registered in 1895 and is pictured on Transparency 11-4-2. If a train meets another one approaching on the same track, it can run up the slanted rails of a special car, along rails on top of the rest of the cars, and down the other end car. Instead of colliding, one train merely passes over the other! If two trains are traveling in the same direction, one can overtake and pass the other by the same procedure.

The figures reveal that the end cars are approximately right triangular in shape. If the box cars on the train are 14 ft high and the track on one of these end cars rises at a 15° angle with the horizontal, how long must the car be?

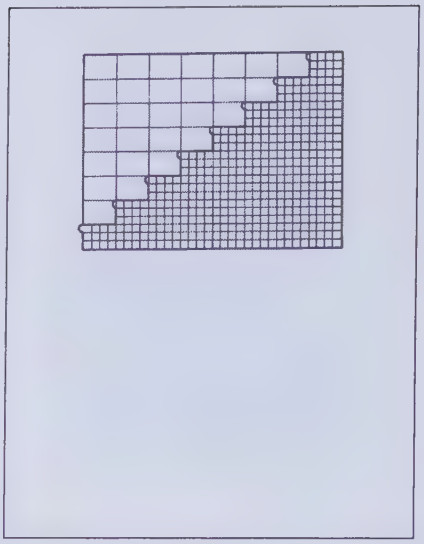
If the angle were 30° or 45°, we could use one of the theorems of the last lesson to find the answer. Because it is not, we need to use some other method. (Following this introduction, you can introduce the tangent ratio and show how it can be used to find the length of one of these end cars. Your students will need some guidance, of course, in learning how to find tangents and inverse tangents on their calculators. Letting x = the length, we have $\tan 15° = \frac{14}{x}$, from which it follows that $x \approx 52.2$ ft. For a rise of 15°, the end car would be about 52 feet long.)



TRANSPARENCY 11-5-1



TRANSPARENCY 11-6-1



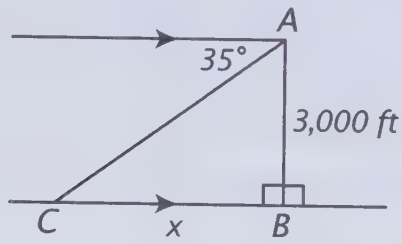
TRANSPARENCY 11-6-2

Lesson 5
The Sine and Cosine Ratios

Lesson Outline
Terms: Sine ratio, cosine ratio
Transparencies
11-5-1 Overhanging-rock photograph

Introduction
The two women in the photograph on Transparency 11-5-1 are dancing on Overhanging Rock at Glacier Point in Yosemite. The picture was taken in 1900, many years before the National Park Service decided that such stunts were too dangerous and barred access to the rock.

Overhanging Rock is 3,000 feet above the valley floor. Suppose that you are standing on the rock and looking down at a point in the valley such that the angle of depression of that point from your position is 35°. (Have your students draw a figure like the one shown here to illustrate this.) How far is



the point you are looking at from the base of the rock? After your students have had some time to work on this, have someone explain his or her solution. Note that the problem can be solved by either of the following equations:

$$\tan 55° = \frac{x}{3{,}000} \quad \text{or} \quad \tan 35° = \frac{3{,}000}{x}.$$

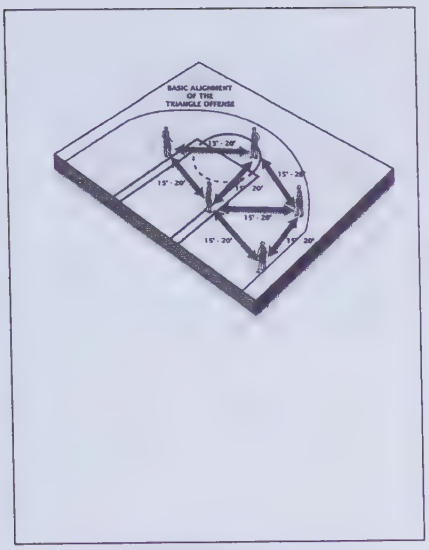
The answer is approximately 4,300 feet.

To introduce the new lesson, ask how far the point you are looking at is from you (label this distance y in the diagram). To find the answer, we can use either the Pythagorean Theorem or another trigonometric ratio. Define the sine and cosine ratios and show how the problem can be solved by using either ratio. The answer is approximately 5,200 feet.

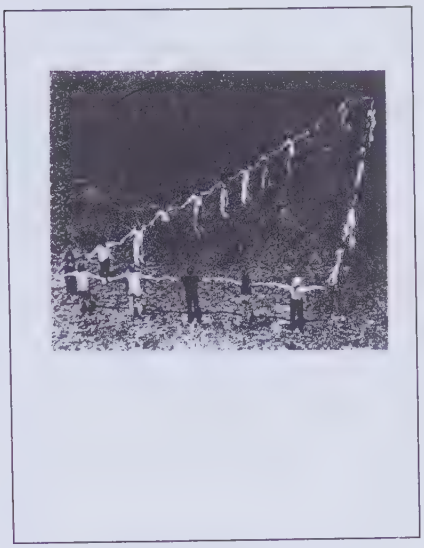
Lesson 6
Slope

Lesson Outline
Terms: Run, rise, slope, angle of inclination, slope.
Theorem 52. Two nonvertical lines are parallel iff their slopes are equal.
Theorem 53. Two nonvertical lines are perpendicular iff the product of their slopes is –1.
Transparencies
11-6-1 Rafters and slope
11-6-2 Staircase

Introduction
In designing the roof of a building, the architect has to take into account the forces of wind, rain, and snow. If the roof is very steep, for example, snow will tend to slide off it. If the slope is more gentle, snow will remain where it is, so that the roof must be able to support the resulting weight. The figures on Transparency 11-6-1 are from the chapter on roof

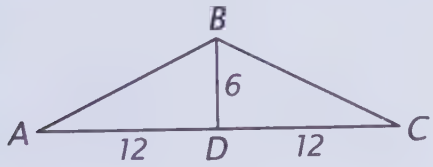


TRANSPARENCY 11-7-1



TRANSPARENCY 11-7-2

design in the book *How to Design and Build Your Own House* by Lupe DiDonno and Phyllis Sperling (Knopf, 1981). It is interesting to note that the architect and the mathematician use the same terms in discussing slope. With this introduction, you might have your students make a scale drawing on graph paper to represent a roof whose rafters match the 6:12 diagram at the bottom of the transparency. This figure can be used as a basis for developing the rest of the lesson.



Another example of the appearance of slope in architecture is in the design of stairways. The figure on Transparency 11-6-2 shows the steepest stairway allowed by most building codes in the United States. What is its slope? It is interesting to note that it is based on the 3-4-5 right triangle, so that its slope is $\frac{3}{4}$.

Lesson 7
The Laws of Sines and Cosines

Lesson Outline

Theorem 54. The Law of Sines. If the sides opposite $\angle A$, $\angle B$ and $\angle C$ of ΔABC have lengths a, b, and c, then $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Theorem 55. The Law of Cosines. If the sides opposite $\angle A$, $\angle B$ and $\angle C$ of ΔABC have lengths a, b, and c, then $c^2 = a^2 + b^2 - 2ab \cos C$.

Transparencies

11-7-1 The triangle offense
11-7-2 Human triangle in Japan

First day

Introduction

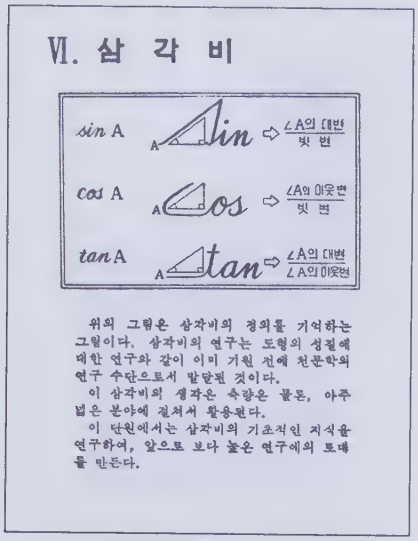
Because there is a lot to cover in this lesson, I recommend that you begin class by using Transparency 11-7-1 to introduce and present the lesson as in the text.

Second day

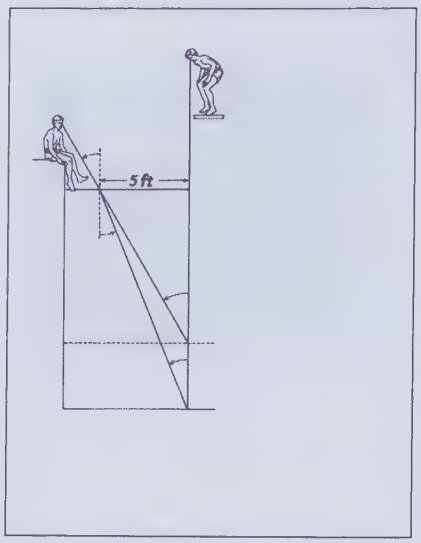
Introduction

The photograph on Transparency 11-7-2 was taken by Ronald L. Dirkse of the American School in Tokyo, Japan. It shows a group of students standing on the football field with their arms outstretched to form the sides of a large triangle. The side of the triangle closest to the camera is formed by five students; 12 students and 13 students stand along the other two sides. If all of the students are the same size, what kind of triangle do you suppose is formed? (Students who recognize that "5-12-13" is a Pythagorean triple will immediately recognize that the triangle is a right triangle. Those who do not can verify that $5^2 + 12^2 = 13^2$, from which we can draw the same conclusion from the converse of the Pythagorean Theorem.)

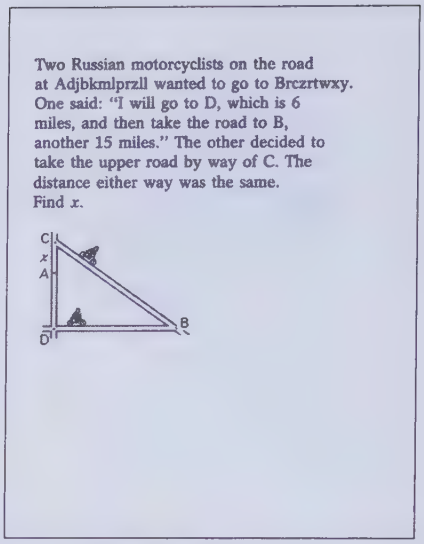
Now suppose that one more student joins the group, so that the shortest side of the triangle increases from 5 to 6. What kind of triangle would



TRANSPARENCY 11-R-1



TRANSPARENCY 11-R-2



TRANSPARENCY 11-R-3

result? (You might take a poll of your students to find how many think that the triangle would become acute and how many think it would become obtuse.) Which would you use, the Law of Sines or the Law of Cosines, to find the measures of the angles of a triangle, given its sides? (The Law of Cosines.) Why? (To write a proportion from the Law of Sines requires that we know at least one angle of the triangle, but we do not.) Which angle of the triangle do we need to determine to be able to tell whether the triangle is acute or obtuse? (The largest angle.) Letting C be the vertex of the largest angle, write an equation for the triangle based on the Law of Cosines. [$13^2 = 6^2 + 12^2 - 2(6)(12)\cos C$.] Solve the equation to find out if you guessed correctly as to what type a 6-12-13 triangle is. (Solving the equation for cos C, we get $\cos C = \frac{11}{144}$, from which $\angle C \approx 85.6°$. The triangle is acute.)

Having done this, you might have your students construct a triangle with sides of 6 cm, 12 cm, and 13 cm, so that they can measure its largest angle with their protractor to confirm that their calculations are reasonable.

Review of Chapter 11
First day

Transparencies

11-R-1 Trigonometric ratios from a Korean geometry book
11-R-2 Swimming-pool refraction problem
11-R-3 Russian motorcyclists problem
11-R-4 Gurus problem

Introduction

Transparency 11-R-1 shows a page from a Korean mathematics book. Your students will be interested in seeing the device used to help in remembering the definitions of the trigonometric ratios. It is clever, but surprising, because it is based on the *script* forms of letters of the *Latin* alphabet! Note that our knowledge of the definitions makes it possible to translate the Korean symbols inside the large box.

The figure on Transparency 11-R-2 illustrates refraction of light in a swimming pool.* The two observers are looking at a stone at the bottom of the pool. The observer on the diving board is looking directly down at the stone. The observer at the left sees the stone from the side. Light from the stone is bent at the surface of the water, producing the illusion that the bottom of the pool is higher than it actually is.

Hand out duplicated copies of the transparency and have your students label the angle above the surface of the water 30° and the angle just below the surface 22°. The problem is to use this information to find the actual depth of the water in the swimming pool (label this x on the figure) and the apparent depth of the water as seen by the observer sitting on the side of the pool (label this y).

Applying the tangent ratio to the right triangles in the figure, we get $\tan 22° = \frac{5}{x}$, $x \approx 12.4$ and

*The figure and the description given in this lesson are taken from *Introductory Physics* by Mashuri L. Warren (W. H. Freeman and Company, 1979).

Two gurus lived at the top of a cliff of height 100, whose base was distant 200 from a neighboring village. One descended the cliff and walked to the village. The other, being a wizard, flew up a height x and then flew in a straight line to the village. The distance traversed by each was the same. Find x.



TRANSPARENCY 11-R-4

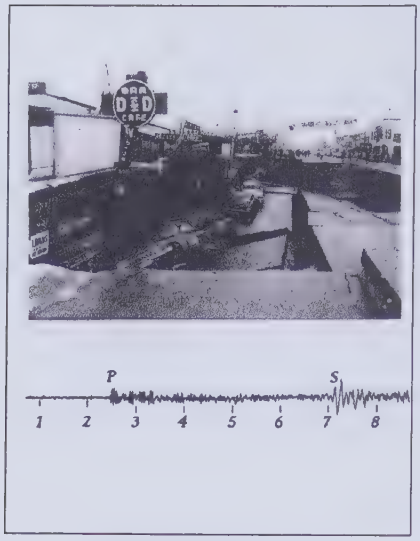
$\tan 30° = \frac{5}{y}$, $y \approx 8.7$. The water is actually 12.4 feet deep, whereas to the observer on the left, it appears to be 8.7 feet deep.

Second day

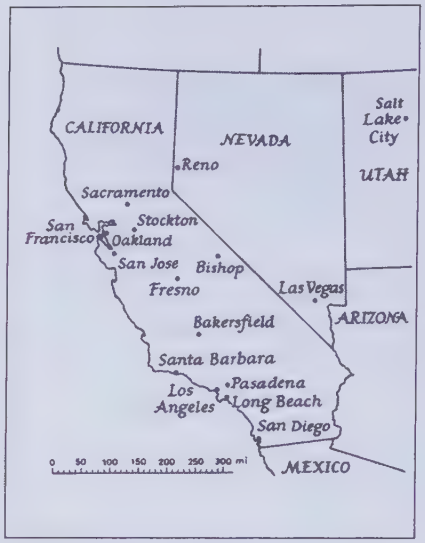
Introduction

The puzzle of the Russian motorcyclists on Transparency 11-R-3 appeared in a book published in England in 1926 (*Modern Puzzles* by Henry Ernest Dudeney). Your students may need some help in writing and solving the equation on which the solution is based.

After discussing the solution, show Transparency 11-R-4. The puzzle of the motorcyclists bears an interesting resemblance to this puzzle, which was invented by the Hindu mathematician Brahmagupta in about 630 A.D. Having just considered the solution of the other puzzle, most of your students should be successful in solving this one.



TRANSPARENCY 12-1-1



TRANSPARENCY 12-1-2

Chapter 12

Circles

Lesson 1
Circles, Radii, and Chords

Lesson Outline

Terms: Circle, concentric, radius, chord, diameter.

Corollary to the Definition of a Circle. All radii of a circle are equal.

Theorem 56. If a line through the center of a circle is perpendicular to a chord, it also bisects the chord.

Theorem 57. If a line through the center of a circle bisects a chord that is not a diameter, it is also perpendicular to the chord.

Theorem 58. The perpendicular bisector of a chord of a circle contains the center of the circle.

Transparencies

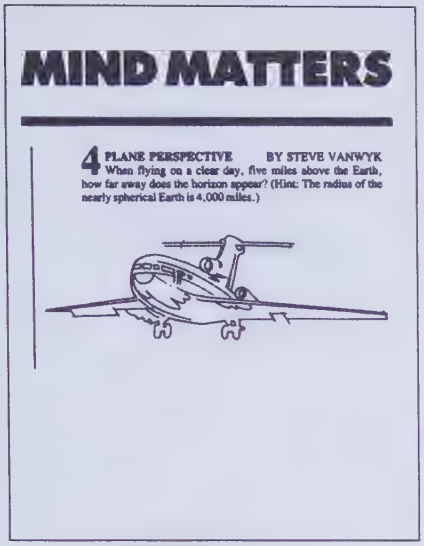
12-1-1 Photograph of Anchorage earthquake
12-1-2 Map: California

Introduction

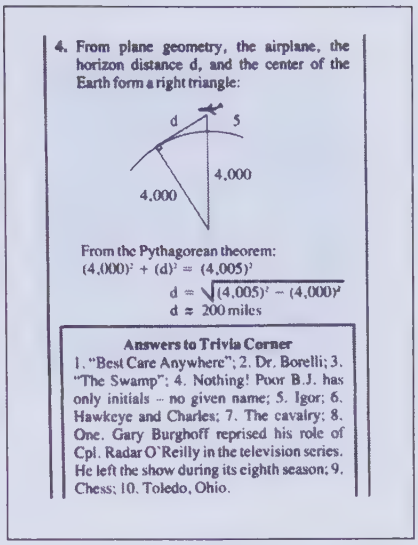
The photograph on Transparency 12-1-1 was taken in downtown Anchorage, Alaska, after an earthquake in 1964. The street in the picture was not very far from the earthquake's epicenter, or point of origin.

Scientists are now able to locate the epicenter of an earthquake quite accurately, even though it may be hundreds of miles from any populated area. When an earthquake occurs, its vibrations are recorded by seismographs. The graph on Transparency 12-1-1 was made on a rotating drum in a seismograph and shows the shock waves of an earthquake. The vertical segments mark off minutes. A little after two minutes from the beginning of the hour, the primary, or P, wave appeared; about five minutes later, it was followed by a secondary, or S, wave. By measuring the time interval between these two waves, seismologists can determine the distance the waves traveled and, hence, how far away the epicenter is.

Hand out duplicated copies of Transparency 12-1-2. Suppose that the waves of an earthquake are picked up by a seismograph at Cal Tech in Pasadena and that they show the epicenter is 230 miles away. Where could the epicenter be? (Somewhere on a circle with a radius of 230 miles and Pasadena as its center.) Have your students set their compasses to this radius, using the scale at the bottom of the map, and draw the circle. Suppose that a seismograph at the University of California at Berkeley, on the other hand, shows that it is 210 miles away from the epicenter of the same earthquake. Draw a second circle having this radius and with its center just above Oakland. What does this tell us about the location of the epicenter? (It could be in one of two places.) Suppose that the University of Nevada at Reno reports that the earthquake is 170 miles away. Draw a circle to illustrate this. What can you conclude from the three circles? (That the epicenter is the point in which they intersect, near Bishop, California.)



TRANSPARENCY 12-2-1



TRANSPARENCY 12-2-2



TRANSPARENCY 12-3-1

After remarking that the circle is of great importance in geometry, both because of its many interesting properties and because of its wide variety of practical applications, you might either present the lesson as in the text or assign it for individual study.

Lesson 2
Tangents

Lesson Outline

Terms: Tangent line.

Theorem 59. If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

Theorem 60. If a line is perpendicular to a radius at its outer endpoint, it is tangent to the circle.

Transparencies

12-2-1 Plane-perspective problem
12-2-2 Plane-perspective solution

Introduction

The problem on Transparency 12-2-1 appears in an issue of the in-flight magazine published by Delta Airlines. Help your students draw a figure to illustrate it, introducing the term *tangent* as you do so. As you draw the radius to the point of contact, ask how it seems to be related to the tangent. Explain that this can be proved, so that it is correct to assume that the triangle with its vertices at the airplane, the horizon point, and the center of the earth is a right triangle.

In discussing the solution, it is worth noting that the answer can be estimated without a calculator by factoring rather than squaring:

$$\begin{aligned} d &= \sqrt{(4{,}005)^2 - (4{,}000)^2} \\ &= \sqrt{(4{,}005+4{,}000)(4{,}005-4{,}000)} \\ &= \sqrt{(8{,}005)(5)} \approx \sqrt{40{,}000} \\ &\approx 200. \end{aligned}$$

The horizon appears about 200 miles away.

After finding the solution, your students will be interested in seeing Transparency 12-2-2, on which the answer given in the airline magazine appears. Obviously only passengers who had taken geometry would have been able to solve this problem—or even understand the solution given.

Remind your students that, in finding the solution, we assumed that the tangent is perpendicular to the radius drawn to the point of contact. This proof of this theorem and its converse can then be presented as in the text.

Lesson 3
Central Angles and Arcs

Lesson Outline

Terms: Semicircle, minor arc, major arc, central angle, reflex angle, degree measure of an arc.

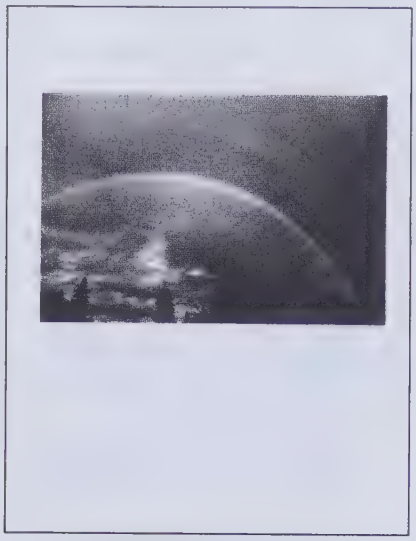
Postulate 10. The Arc Addition Postulate. If C is on AB, then $m\overset{\frown}{AC} + m\overset{\frown}{CB} = m\overset{\frown}{ACB}$.

Theorem 61. In a circle, equal chords have equal arcs.

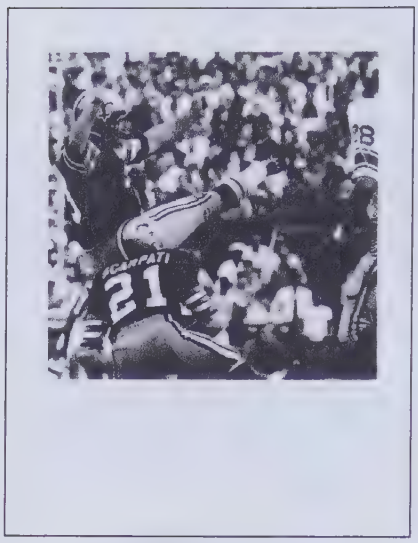
Theorem 62. In a circle, equal arcs have equal chords.

Transparencies

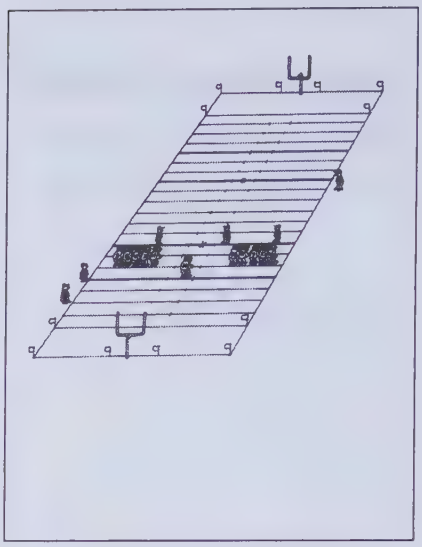
12-3-1 "The Theory of the Rainbow"
12-3-2 Photograph of rainbow



TRANSPARENCY 12-3-2



TRANSPARENCY 12-4-1



TRANSPARENCY 12-4-2

Introduction
The Greek philosopher Aristotle is thought to have been the first person to try to explain the rainbow. A lengthy article about the modern theory of the rainbow appeared in the April 1977 issue of *Scientific American;* the first page of this article is reproduced on Transparency 12-3-1. It, together with the rainbow photograph that appears on Transparency 12-3-2, would make a nice introduction to the new lesson. The lesson can then be presented as in the text.

Lesson 4
Inscribed Angles

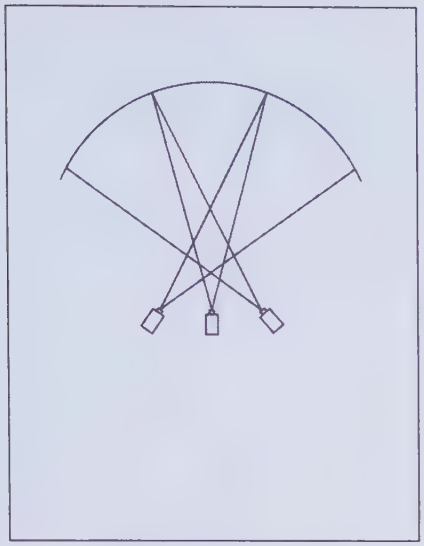
Lesson Outline
Terms: Inscribed angle.
Theorem 63. An inscribed angle is equal in measure to half its intercepted arc.
Corollary 1. Inscribed angles that intercept the same arc are equal.
Corollary 2. An angle inscribed in a semicircle is a right angle.
Transparencies
12-4-1 Longest field goal
12-4-2 Football field diagram

Introduction
The longest field goal in professional football was kicked by Tom Dempsey in a game between the New Orleans Saints and the Detroit Lions in 1970. The kick, shown on Transparency 12-4-1, covered a distance of 63 yards and is especially amazing because Dempsey has had only half a right foot since birth.

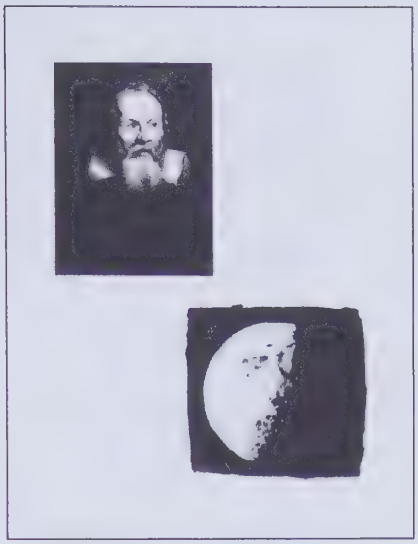
Mark a point on Transparency 12-4-2 to represent a spot 53 yards from the upper goal line (the distance from the goal line to the goal post is 10 yards) and draw rays from that point to the lower corners of the goal post. To make the goal, Dempsey had to kick the ball within this angle, which has a measurte of less than 6°! At what other spots on the field would the angle be the same? Explain that the answer to this question comes from a theorem of the new lesson and present the rest of the lesson as in the text. After the lesson has been completed, show Transparency 12-4-2 again. Explain that the other points on the field for which the goal post angle is the same lie on a circle that contains the kicking point and the corners of the goal post. We know that the angles are equal because of the corollary that inscribe angles that intercept the same arc are equal.

Lesson 5
Secant Angles

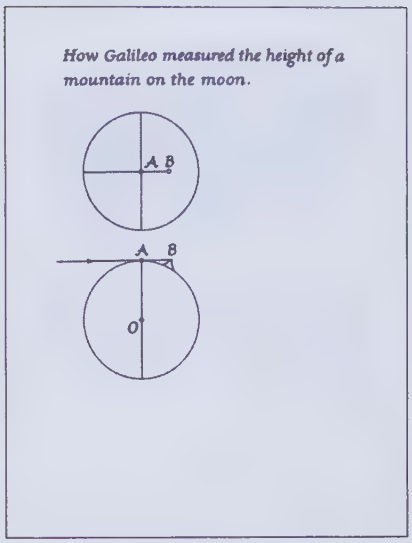
Lesson Outline
Terms: Secant, secant angle.
Theorem 64. A secant angle whose vertex is inside a circle is equal in measure to half the sum of the arcs intercepted by it and its vertical angles.
Theorem 65. A secant angle whose vertex is outside a circle is equal in measure to half the difference of its larger and smaller intercepted arcs.
Transparencies
12-5-1 Cinerama projector system



TRANSPARENCY 12-5-1



TRANSPARENCY 12-6-1



TRANSPARENCY 12-6-2

Introduction

In the 1950s, a motion-picture process using three projectors and a curved screen was introduced. Called "Cinerama," this process produced a stronger sense of reality than the ordinary flat screen, but it was so expensive to operate that it was given up in the 1960s.

The figure on Transparency 12-5-1 is an overhead view of a Cinerama auditorium, showing the screen and projectors. Add Overlay A and explain that the circle has been divided into 10° arcs. Each projector, then, covers 40° of the screen. Place Overlay B so that the vertex of the projector angle falls on the center of the circle. From the intercepted arc, we see that the projector angle has a measure of 30°. Now, place Overlay B in positions to illustrate an inscribed angle and a secant angle whose vertex is inside a circle. Overlay C can be used to illustrate secant angles whose vertices are outside a circle. After your students have discovered the measure relations between secant angles whose vertices are inside and outside a circle and their intercepted aracs, the proofs of the two secant angle theorems can be developed as in the text.

Lesson 6
Tangent Segments and Intersecting Chords

Lesson Outline

Terms: Tangent segment.

Theorem 66. The Tangent Segments Theorem. The tangent segments to a circle from an external point are equal.

Theorem 67. The Intersecting Chords Theorem. If two chords intersect in a circle, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.

Transparencies

12-6-1 Galileo and the moon

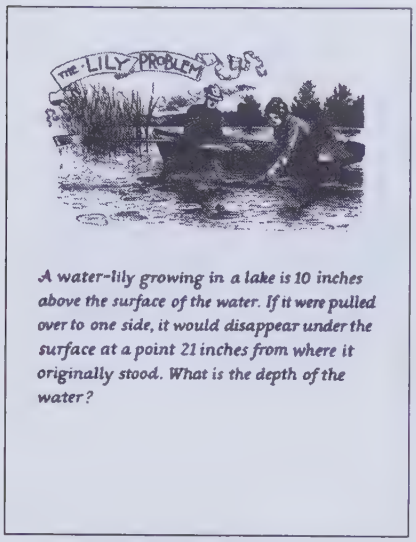
12-6-2 How Galileo measured the height of a mountain on the moon

Introduction

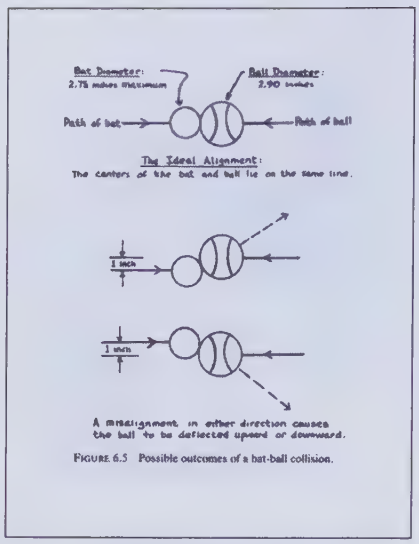
Before Galileo first looked at the night sky through his telescope in the fall of 1609, it was generally believed that the surface of the moon, unlike that of the earth, was perfectly smooth. Galileo noticed that the boundary between the dark and light sides of the moon was very ragged and concluded from this observation that the moon must have mountains and valleys. He made the painting of the moon shown on Transparency 12-6-1 from observations made through his telescope.

Notice that there are several light spots on the dark side, not far from the boundary of the lighted side. Galileo figured that these spots must be mountain peaks high enough to catch some light from the rising or setting sun. With this as the basis of his reasoning, he used some simple geometry to measure the height of a mountain on the moon.

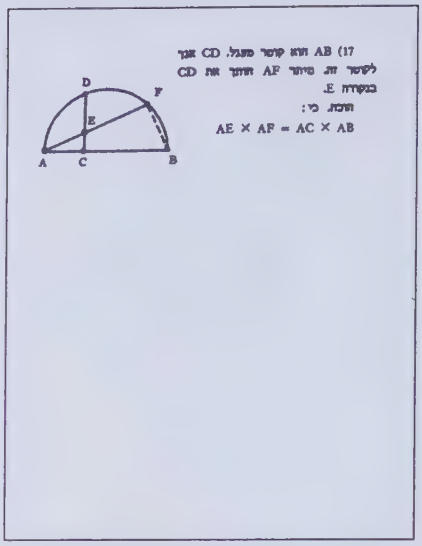
Hand out duplicated copies of Transparency 12-6-2. The first figure represents the moon and one of the light spots, at B, as seen through Galileo's telescope. The second figure represents a "side view" of the same thing: a ray of light from the sun is tangent to the moon at A and hits the top of the mountain peak at B. Galileo estimated the distance from



TRANSPARENCY 12-R-1



TRANSPARENCY 12-R-2



TRANSPARENCY 12-R-3

A to B by comparing it with the radius of the moon. In these figures, AB appears to be about $\frac{1}{2}$ the length of the radius; but in his painting, Galileo figured that AB was about $\frac{1}{12}$ as long as the radius.

Galileo knew that the radius of the moon was 1,080 miles, and so he estimated AB as $\frac{1}{12}(1{,}080) = 90$ miles. Write the lengths of OA and AB on the second figure, draw OB, and give your students some time to figure out the height of the mountain at B.

Because AB is tangent to the circle at A, $AB \perp OA$, and so ΔAOB is a right triangle. Applying the Pythagorean Theorem,

$$OB^2 = 1{,}080^2 + 90^2 = 1{,}174{,}500$$

And so

$$OB = \sqrt{1{,}174{,}500} \approx 1083.7.$$

Therefore, the mountain at B is about 3.7 miles high.

Review of Chapter 12

Transparencies

12-R-1 Longfellow's water-lily problem
12-R-2 Baseball misalignment
12-R-3 A proof from a Hebrew geometry book

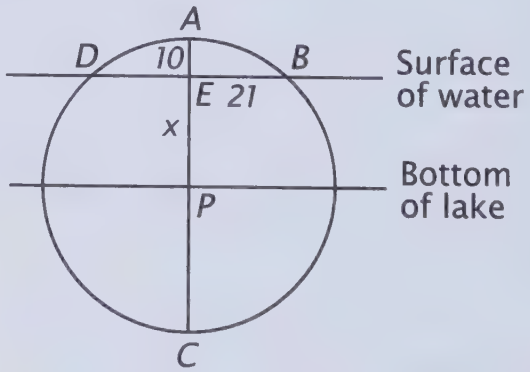
First day

Introduction

The most popular American poet of the nineteenth century was Henry Wadsworth Longfellow. Longfellow was also an amateur mathematician and among the problems that he created was one about a water lily in a lake.* (Show Transparency 12-R-1 with the overlay in place.)

> A water lily growing in a lake is 10 inches above the surface of the water. If it were pulled over to one side, it would disappear under the surface at a point 21 inches from where it originally stood. What is the depth of the water?

To solve this problem, we need a diagram. Remove the overlay and guide your students in drawing the figure shown below. First, draw just the two horizontal lines and PA. As the water lily is pulled over to one side, what kind of path does the top of it follow? Draw AB and then the entire circle; also, draw the entire diameter AC.



*This problem was originally in Sam Loyd's *Cyclopedia of Puzzles*, privately published in 1914. It is included in *Mathematical Puzzles of Sam Loyd*, volume 1, selected and edited by Martin Gardner (Dover, 1959).

Label EA, EB, and EP as indicated. Then ask your students how they would label ED and PC. (We are assuming that the water lily is originally perpendicular to the surface of the water so that AC bisects DB. ED = 21 and PC = x + 10.) Point out that, without adding anything more to the figure, the Intersecting Chords Theorem can be used to solve for x. Let a student demonstrate his or her solution to the class. Because DE · EB = AE · EC, $21^2 = 10(2x + 10)$. Solving for x, $x = 17.05$. The lengths in the problem are each given as a whole number of inches, and so it is appropriate to round this answer and to say simply that the lake is about 17 inches deep.

Second day

Introduction

The figures on Transparency 12-R-2, which show some possible outcomes when a baseball is struck by a bat, are from *Sport Science* by Peter J. Brancazio (Simon & Schuster, 1984). The first figure illustrates the ideal situation, in which the bat hits the ball head-on with the greatest possible force. The other figures show situations in which the bat and ball are not perfectly aligned so that the ball is popped up or hit toward the ground.

Have your students draw and label the second figure as shown at the right. On the assumption that the paths of the bat and ball are parallel and a misalignment of 1 inch, at what angle with the path of the bat is the ball deflected? In making the calculation, use the diameters of the bat (2.75 in) and ball (2.90 in) given.

In right ΔABC,

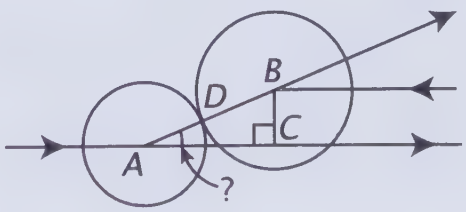
$$BC = 1 \quad \text{and} \quad AB = \frac{2.75 + 2.90}{2} = 2.825.$$

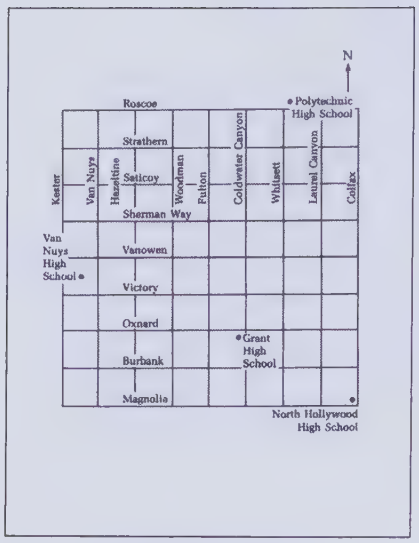
By the definition of the sine ratio,

$$\sin A = \frac{BC}{AB} = \frac{1}{2{,}825} \approx 0.354$$

And so $\angle A \approx 21°$. The ball is deflected at an angle of approximately 21°.

The proof exercise on Transparency 12-R-3, taken from a Hebrew geometry book, might be used for further review. The "given" seems to be that AB is a diameter of the circle of which the arc is a part, that CD $\perp$ AB, and that AF intersects CD at E. FB is evidently an auxiliary line, added as a clue to the proof. The proof is based on showing that the two triangles in the figure are similar.





TRANSPARENCY 13-1-1



TRANSPARENCY 13-2-1

Chapter 13
The Concurrence Theorems

Lesson 1
Triangles and Circles

Lesson Outline

Terms: Cylic, circumcircle, circumcenter, polygon inscribed in a circle, circle circumscribed about a polygon.

Theorem 68. Every triangle is cyclic.

Corollary. The perpendicular bisectors of the sides of a triangle are concurrent.

Construction 9. To circumscribe a circle about a triangle.

Transparencies

13-1-1 Locations of schools

Introduction

The following problem provides a nice introduction to the ideas of Lesson 1. I will describe it exactly as I present it to my own class; if it is not possible to recast the problem in terms of your own neighboring schools, you might state it in terms of fire stations or some other local landmarks.

Before Grant opened in 1959, the three high schools in this area were North Hollywood High, Van Nuys High, and Polytechnic High. Hand out a map comparable to the one on Transparency 13-1-1 to each student. This map shows the locations of these schools and of Grant. For simplicity, each school has been represented by a single point. As you can see, Grant is closer to North Hollywood High that to the other two schools. It might have been better to have located Grant so that its distances to the three schools were the same. For convenience, we will refer to the points representing the schools by the first letters of their names. Guess, without doing any measuring, where you think the point that is equidistant from V, P, and N is located and mark it with a small x.

If the point that you have guessed is really equidistant from V, P, and N, it should be possible to draw a circle with it as center that passes through all three points. See if you can do this.

Can you think of a way to use your straightedge and compass to find the point equidistant from V, P, and N that does not involve any guessing? (If necessary, ask where the points that are equidistant from just two of these points are located.) After someone has suggested constructing the perpendicular bisectors of the segments joining the points, have your students draw ΔVPN and construct the perpendicular bisectors of two of its sides.

How close is the point in which the two lines intersect to the point you marked x? Try to draw a circle with the point in which the two lines intersect that passes through all three vertices of the triangle.

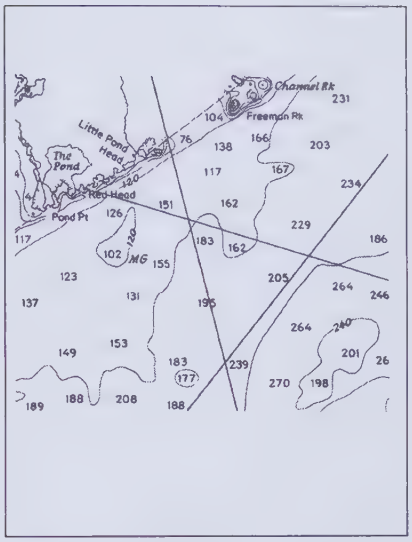
Explain that ΔVPN is called "cyclic." The rest of the lesson can be developed as in the text.



TRANSPARENCY 13-2-2



TRANSPARENCY 13-3-1



TRANSPARENCY 13-3-2

Lesson 2
Cyclic Quadrilaterals

Lesson Outline

Theorem 69. A quadrilateral is cyclic iff a pair of its opposite angles are supplementary.

Transparencies

13-2-1 Oak Island in the news
13-2-2 Map: Oak Island

Introduction

Ever since 1795, people have been searching for buried treasure on a small island off the coast of Nova Scotia. The island, named Oak Island, is pictured on Transparency 13-2-1. It has been speculated that as much as $30 million worth of treasure may be buried on the island, perhaps by Blackbeard the Pirate or Captain Kidd.

There is an elaborate maze of tunnels below the island's surface, which engineers say must have taken several years to construct. Unfortunately for people looking for the treasure, these tunnels seem to have been designed to flood any holes that may be dug on the island. Even though little evidence of buried treasure has been discovered, the tunnel system may very well have been built to protect it from discovery.

Hand out duplicated copies of Transparency 13-2-2. This map shows the part of Oak Island where the treasure is thought to be buried. Can you find the exact spot, given that it is equidistant from the four lettered points on the map?

After your students have found the spot, observe that any three of the four lettered points are sufficient to locate it. Draw the circle that contains all four points and the quadrilateral having those points as its vertices. Observe that this quadrilateral is cyclic because its vertices lie on a circle. After pointing out that, unlike triangles, not every quadrilateral is cyclic, present the rest of the lesson as in the text.

Lesson 3
Incircles

Lesson Outline

Terms: Circle inscribed in a polygon, polygon circumscribed about a circle, incircle, incenter.

Theorem 70. Every triangle has an incircle.

Corollary. The angle bisectors of a triangle are concurrent.

Construction 10. To inscribe a circle in a triangle.

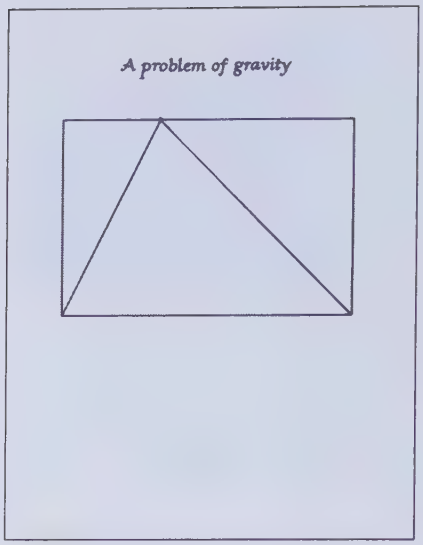
Transparencies

13-3-1 Map: Voyage of Columbus
13-2-2 Navigation chart

Introduction

The map on Transparency 13-3-1 shows the paths of Columbus's ships in his historic voyage from Spain to the New World and back again. Although pilots of modern ships have several methods for determining their positions at sea, Columbus used celestial navigation. He kept track of where he was by observing the sun and moon, the planets, and the stars.

A navigator records each sighting as a line on a chart. Three lines are usually used, and, if there are no inaccuracies, they intersect in a point that is the ship's position. Hand out duplicated copies of Transparency 13-3-2. This figure shows what usually hap-



TRANSPARENCY 13-4-1

pens: the three lines do not intersect in a common point, but form a triangle. The navigator takes the point that is equidistant from the sides of the triangle as the position of the ship. According to the standard reference on navigation, this point "can be determined geometrically or by computation, but in normal practice the navigator estimates it by eye."*

Have your students estimate the point that is equidistant from the sides of the triangle by eye. Remark that if the point is correct, it should be possible to draw a circle with this point as center that is tangent to each side of the triangle. A discussion of how to do this leads into the rest of the ideas of the lesson.

Lesson 4
The Centroid of a Triangle

Lesson Outline
Terms: Median, centroid, orthocenter.
Theorem 71. The medians of a triangle are concurrent.
Theorem 72. The lines containing the altitudes of a triangle are concurrent.
Transparencies
13-4-1 A problem of gravity

Introduction
Give each student an unlined 4-in × 6-in file card and a pair of scissors. Have your students mark the point on the upper edge 2 inches from the left corner as shown on Transparency 13-4-1 and draw lines from this point to the lower left and right corners of the card. Cut along these lines and put the two smaller pieces aside.

It is easy to balance the largest triangular piece on three fingers so that it is horizontal. (Illustrate this and have everyone do it.) It is not as easy to balance it on just two fingers. Can you do it? Can you balance the "triangle" on just one finger? (After a few attempts, most students should be able to do this.) Now a very challenging question. Can you balance the "triangle" on the metal point of your compass? You may poke a small hole in the card at whatever point you think will work. (It is unlikely that anyone will be able to guess the precise point on which the triangle may be balanced.)

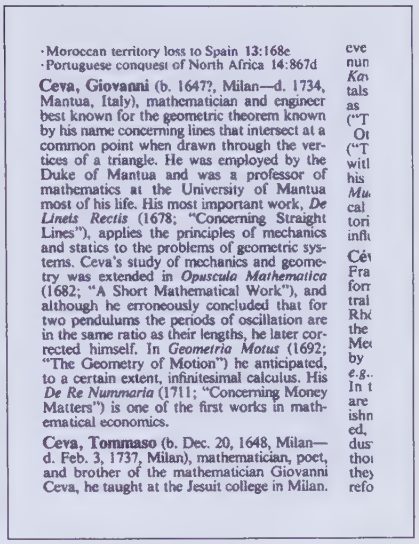
Let's see if we can find the balancing point by construction. Make a careful tracing of the triangular piece on a sheet of paper. What lines do you think we should draw in order to find the balancing point? (Have everyone try out any suggestions made. Someone will probably guess that the triangle will balance on its circumcenter and someone else its incenter. When each of these points has been located, it should be labeled on the card. After discovering that neither point will work, someone may think of the triangle's altitudes. If so, construct them, and, after the class has observed that they seem to be concurrent, label the point as the orthocenter. Because the idea of a median in a triangle has not been considered previously in the course, no one may think of drawing the medians. If this is the case, you might give the following clues.)

It seems that, if the triangle is to balance on one

**Dutton's Navigation and Piloting*, 14th edition, by Elbert S. Maloney (Naval Institute Press, 1985), page 417.



TRANSPARENCY 13-5-1



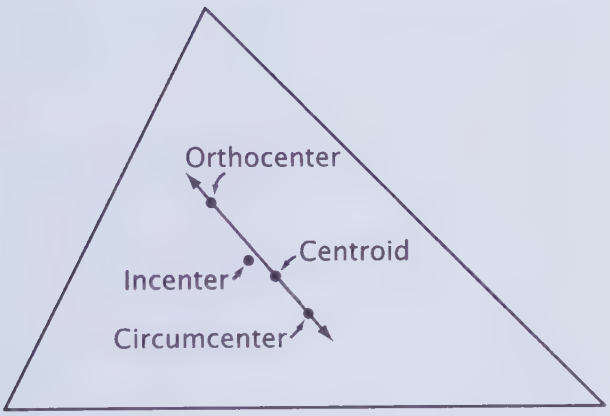
·Moroccan territory loss to Spain 13:168e
·Portuguese conquest of North Africa 14:867d

Ceva, Giovanni (b. 1647?, Milan—d. 1734, Mantua, Italy), mathematician and engineer best known for the geometric theorem known by his name concerning lines that intersect at a common point when drawn through the vertices of a triangle. He was employed by the Duke of Mantua and was a professor of mathematics at the University of Mantua most of his life. His most important work, *De Lineis Rectis* (1678; "Concerning Straight Lines"), applies the principles of mechanics and statics to the problems of geometric systems. Ceva's study of mechanics and geometry was extended in *Opuscula Mathematica* (1682; "A Short Mathematical Work"), and although he erroneously concluded that for two pendulums the periods of oscillation are in the same ratio as their lengths, he later corrected himself. In *Geometria Motus* (1692; "The Geometry of Motion") he anticipated, to a certain extent, infinitesimal calculus. His *De Re Nummaria* (1711; "Concerning Money Matters") is one of the first works in mathematical economics.

Ceva, Tommaso (b. Dec. 20, 1648, Milan—d. Feb. 3, 1737, Milan), mathematician, poet, and brother of the mathematician Giovanni Ceva, he taught at the Jesuit college in Milan.

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TRANSPARENCY 13-5-2



point, half of it should be on one "side" of the point and half on the other. In other words, each of the line determining this point should divide the triangle in half. Where do you suppose a line from one vertex of the triangle should be drawn in order to divide the triangle in half? (To the midpoint of the opposite side.) After everyone has drawn the medians, observe that they also seem to be concurrent and then let everyone see if the triangle *will* balance on this point.

Proofs that the medians of a triangle are concurrent and that the lines containing the altitudes are concurrent are considered in the exercises of the lesson.

Lesson 5
Ceva's Theorem

Lesson Outline

Terms: Cevian.

Theorem 73. Ceva's Theorem. Three cevians, AY, BZ, and CX, of ΔABC are concurrent iff

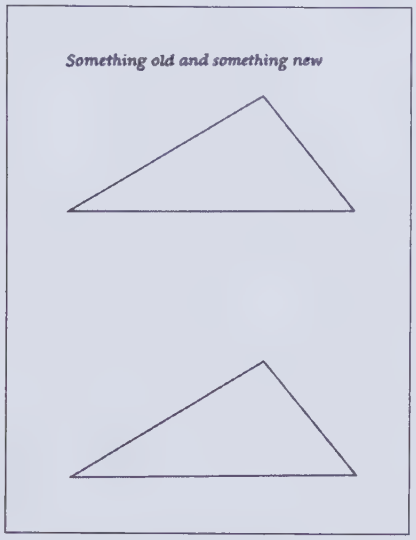
$$\frac{AX}{XB}\cdot\frac{BY}{YC}\cdot\frac{CZ}{ZA}=1.$$

Transparencies

13-5-1 Cartoon of geometers and triangle
13-5-2 *Encyclopedia Britannica* article on Ceva

Introduction

The cartoon on Transparency 13-5-1 was drawn by Kenneth J. Neal when he was a student at Penn Hills High School in Pittsburgh. It illustrates an imaginary discussion of the triangle between a Greek mathematician—perhaps Euclid or Pythagoras—and some European mathematicians of the seventeenth century. One of these mathematicians might be Ceva, whose theorem is the subject of the new lesson. Before introducing the theorem and its proof, show Transparency 13-5-2, on which the article on Ceva in the micropedia section of the *Encyclopedia Britannica* is reproduced. I recommend having your students draw the figures and write the key equations in the proof that if the cevians are concurrent, the product of the ratios is equal to one. The proof of the converse is probably best left to be read as part of the assignment.



TRANSPARENCY 13-6-1

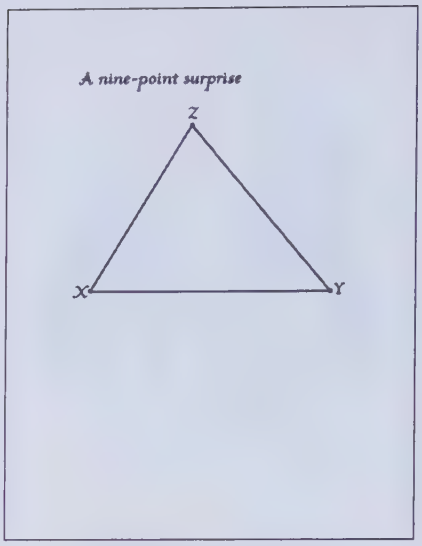
meo" and "De profundis clamavi," are considered among his best works.
·Italian forms as stylistic influence 12:708d

Morley's theorem, in Euclidean geometry, a theorem stating that if the angle trisectors are drawn at each vertex of a triangle, adjacent trisectors meet at the vertices of an equilateral triangle.
·model theory proof 11:1085b

mormaor (Gaelic *mor*, "great"; *maor*, "steward," or "bailiff"), also spelled MORMAER, a title of the rulers of the seven provinces into which Celtic Scotland (*i.e.*, the part of the country north of the Forth and the Clyde) was divided. These seven mormaorships, or original "earldoms," of Scotland, as they were afterward called, were Angus, Atholl with Gowrie, Caithness with Sutherland, Fife, Mar with Buchan, Moray with Ross, and Strathearn with Menteith.

Mormon, Book of, work accepted as holy scripture, in addition to the Bible, in the Church of Jesus Christ of Latter-day Saints. First published in 1830 in Palmyra, N.Y., it was thereafter widely reprinted and translated. Mormons hold that it is a divinely inspired work revealed to and translated by Joseph Smith.
It relates the history of a group of Hebrews

TRANSPARENCY 13-6-2



TRANSPARENCY 13-R-1

Lesson 6
Napoleon's Discovery and Other Surprises

Lesson Outline
Transparencies
13-6-1 Something old and something new
13-6-2 *Encyclopedia Britannica* article on Morley's Theorem

Introduction
Hand out duplicated copies of Transparency 13-6-1. If the calendar were suddenly turned back 2,000 years and you were still a high school (or college) student, you wouldn't be taking some of the courses that you take now because they wouldn't exist. Other courses that you are taking would probably be radically different from what they are now. Much of geometry, however, would be very much the same.

For example, the fact that the angle bisectors of every triangle are concurrent is established in Euclid's *Elements*. Use your straightedge and compass to construct, as accurately as you can, the angle bisectors of the first triangle on the sheet you have been given. After your students have done this, have them label the figure "The angle bisectors of a triangle are concurrent—proved by Euclid c. 300 B.C."

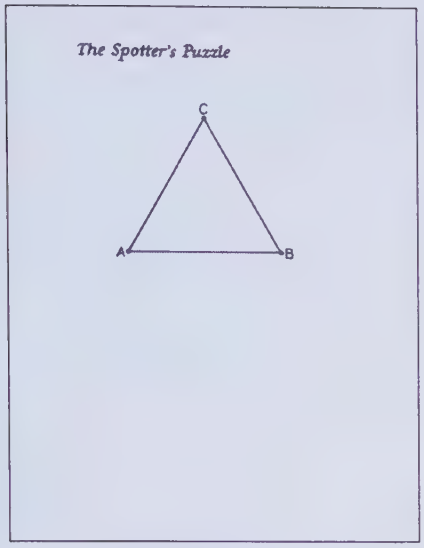
What would happen if, instead of *bisecting* the angles of a triangle, we *trisected* them? You may recall that it is impossible to trisect most angles with just a straightedge and compass. Because this is the case, we will use a protractor instead. Give your students time to draw the trisectors of the three angles as accurately as they can. To make this easier, the triangle has been drawn so that the measures of its angles are 30°, 51°, and 99°. Demonstrate the procedure on the overhead projector. If the three points of intersection of the adjacent trisectors of the angles are connected with line segments, an equilateral triangle is formed!

This is a relatively new theorem in elementary geometry because it was discovered at the end of the nineteenth century. Label the figure "The angle trisectors of a triangle determine the vertices of an equilateral triangle—discovered by Frank Morley in 1899." Frank Morley was a professor of mathematics at Johns Hopkins University and the father of the author Christopher Morley. It was not until ten years after Morley made this discovery that someone figured out how to prove it. The entry on Morley's Theorem in the *Encyclopaedia Britannica* is on Transparency 13-6-2. A proof of the theorem can be found in *Geometry Revisited* by H. S. M. Coxeter and S. L. Greitzer, a book in the New Mathematical Library (M.A.A., 1967) but it is by no means easy.

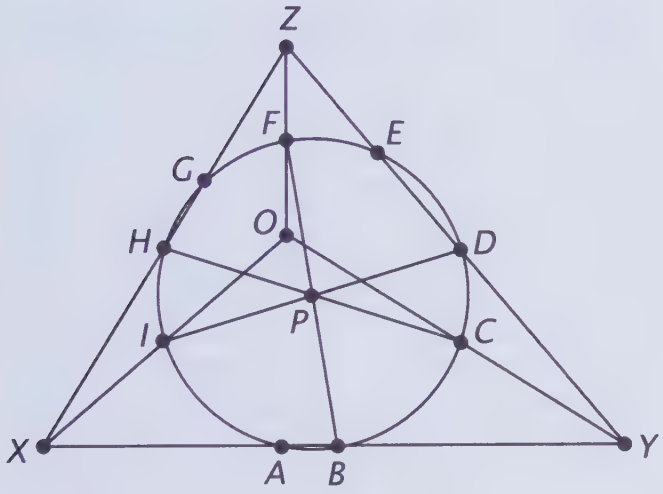
Review of Chapter 13

Transparencies
13-R-1 A nine-point surprise
13-R-2 Review: The spotter's puzzle

First day
Introduction
Hand out duplicated copies of Transparency 13-R-1. The following directions are for a rather elaborate construction that provides a good review of ideas as well as having an interesting result. First, construct the three altitudes of the triangle and name them XE, YG, and ZA. Label the orthocenter of the



TRANSPARENCY 13-R-2



triangle O.

Next, bisect the three line segments connecting the orthocenter of the vertices of the triangle. Label their midpoints as follows: OX, midpoint I; OY, midpoint C; OZ, midpoint F.

Next, find the midpoints of the sides of the triangle and label them as follows: XY, midpoint B; YZ, midpoint D; XZ, midpoint H.

What do you notice? (The points labeled A through I appear to lie on a circle.) That the nine points we have labeled A through I should lie on a circle is quite remarkable. It is not difficult to find the center of the circle because six of them are the endpoints of three of the circle's diameters. Can you guess which points these are? (The three diameters are FB, HC, and ID.) Draw the three diameters, label the point in which they are concurrent P, and draw the "nine-point" circle.

Two interesting properties of the "nine-point" circle are that its center is the midpoint of the line segment connecting the orthocenter and the circumcenter of the triangle and that its radius is half the radius of the circumcircle of the triangle.

Some of your better students might be interested in seeing a proof of why the nine points lie on a circle. An especially straightforward one is in *Excursions in Geometry* by C. Stanley Ogilvy (Oxford University Press, 1969), pages 119-120. Historians are not sure who first noticed the "nine-point" circle. The first proof of its existence seems to have been published in 1821.

Second day

Introduction

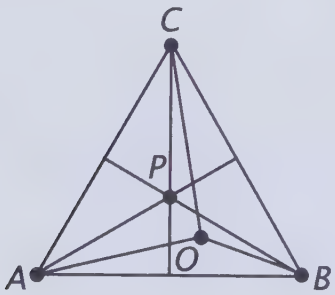
This is an appropriate point at which to reconsider the spotter's puzzle introduced at the beginning of the course. Begin by showing Transparency 0-4 and recalling the story of the surfer and the spotter. Your students may have already proved that it doesn't matter where the surfer builds his house on the island: the sum of the lengths of the perpendicular segments to the sides from every point on the island is the same.*

For the spotter, who wanted to build his house so that the sum of the lengths of the paths from it to the three *corners* of the island is a minimum, the situation is quite different. The sum depends on the position of the point. Show your students Transparency 0-5 and its overlay to remind them that the sum seems to be the smallest at the "center" of the

*See Chapter 9, Lesson 3, exercises 29-34, on page 355.

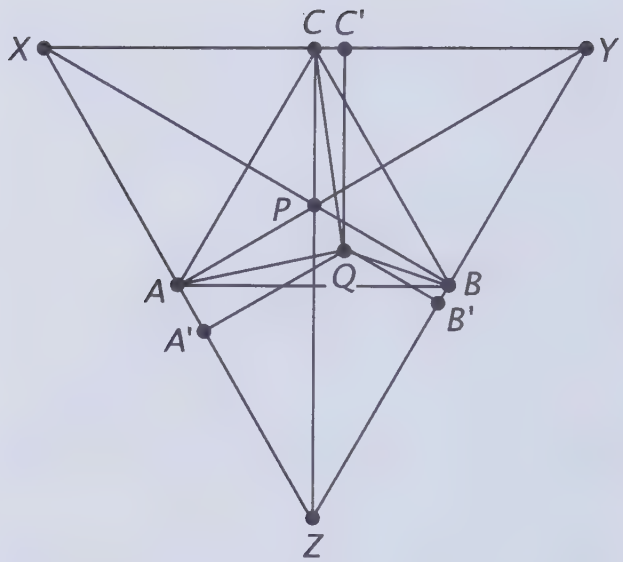
triangle.

Hand out duplicated copies of Transparency 13-R-2 and ask how the "center" of the triangle can be found. Using congruent triangles, it is easy to show that the angle bisectors, altitudes, medians, and hence, the perpendicular bisectors of the sides, of an equilateral triangle are all the same. So the four centers of the triangle that are determined by these lines are the same point. Find the point and label it P. Notice that $\angle APB = \angle BPC = \angle CPA = 120°$ because the six small triangles formed by the lines are 30°-60° right triangles.



Next, choose any other point in the triangle, label it Q, and draw QA, QB, and QC. We will show that PA + PB + PC is less than QA + QB + QC and thus prove that the sum of the lengths of the paths from point P to the three vertices is less than the sum from any other point, such as Q.

To do this, we draw lines through the vertices of ΔABC so that they are perpendicular to PA, PB, and PC. Label the triangle formed by these lines



ΔXYZ as shown in the figure above. Looking at quadrilateral XAPC, we see that $\angle XAP = 90°$, $\angle APC = 120°$, and $\angle PCX = 90°$, and so $\angle CXA = 60°$. By the same reasoning, we can conclude that the other two angles of ΔXYZ also have measures of 60°, and so ΔXYZ is equiangular and hence equilateral.

Now draw perpendicular segments from to the sides of ΔXYZ and label them as shown. Notice that $QA' < QA$, $QB' < QB$, and $QC' < QC$ because the perpendicular segment from a point to a line is the shortest segment connecting them.

Finally, observe that we have shown ΔXYZ to be equilateral and recall that, in reviewing the surfer's puzzle, we proved that for an equilateral triangle, the sum of the lengths of the perpendicu-

lar segments to the sides from every point in the triangle is the same. Therefore,

$$PA + PB + PC = QA' + QB' + QC'.$$

We have shown that $QA' < QA$, $QB' < QB$, and $QC' < QC$; so

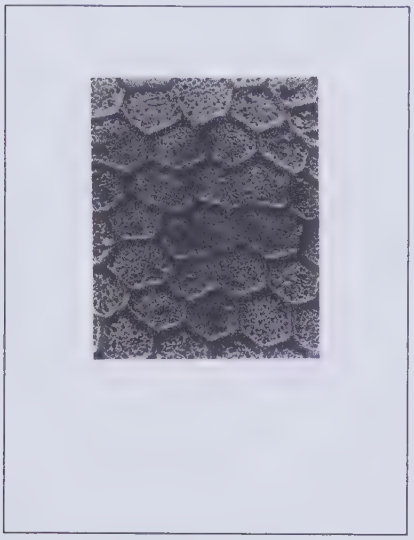
$$QA' + QB' + QC' = QA + QB + QC,$$

and hence,

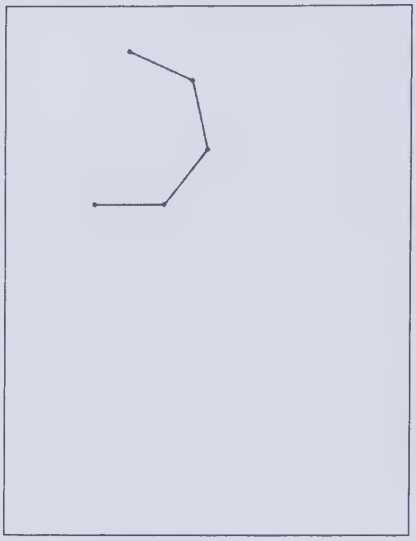
$$PA + PB + PC < QA + QB + QC.$$

This proof that P is the minimum point was invented by Evangelista Torricelli, an Italian mathematician who also invented the barometer. If the original triangle is not equilateral, then point P is not the circumcenter, incenter, orthocenter, or centroid, but a fifth point called the "Fermat point." This point is named for Pierre Fermat, who first proposed the general version of the spotter's problem in the seventeenth century. It is the point such that the segments joining it to the vertices of the triangle form equal angles of 120°.

More information on Fermat's problem can be found in *Mathematical Gems I* by Ross Honsberger (M.A.A., 1973), pages 24-34.



TRANSPARENCY 14-1-1



TRANSPARENCY 14-1-2



TRANSPARENCY 14-2-1

Chapter 14
Regular Polygons and the Circle

Lesson 1
Regular Polygons

Lesson Outline

Terms: Regular polygon, center, radius, central angle, apothem.

Theorem 74. Every regular polygon is cyclic.

Transparencies

14-1-1 Photograph of water-bug egg
14-1-2 Cyclic polygon figure

Introduction

The photograph on Transparency 14-1-1 was taken through an electron microscope by an entomologist. (A student who is familiar with the word might explain its meaning: a person who studies insects.) The photograph shows the surface of one of the eggs of a water bug. The extreme magnification (820×) reveals that the surface is covered with an elaborate network: this network seems to consist of polygons of various shapes. In what basic way are all of the polygons alike? (They are all convex.)

Add the overlay to the transparency and ask your students to identify the three polygons shown by name. The hexagon seems to be more symmetric than the other two. Why? (Because its sides and angles are close to being equal.) A convex polygon that is both equilateral and equiangular is called *regular*.

Following this introduction, hand out duplicated copies of Transparency 14-1-2 and use them to develop the proof that every regular polygon is cyclic. The figure shows only part of a regular polygon to emphasize that the proof is independent of the number of sides.

Lesson 2
The Perimeter of a Regular Polygon

Lesson Outline

Theorem 75. The perimeter of a regular polygon having n sides is $2Nr$, in which $N = n \sin \frac{180}{n}$ and r is its radius.

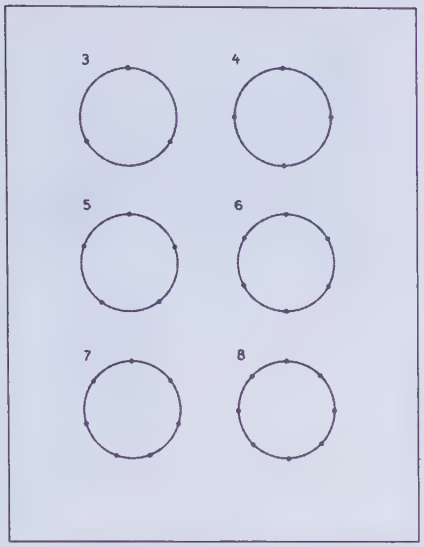
Transparencies

14-2-1 Picture of Gauss
14-2-2 Regular polygon constructions
14-2-3 Construction of a regular 17-gon
14-2-4 A regular pentagon

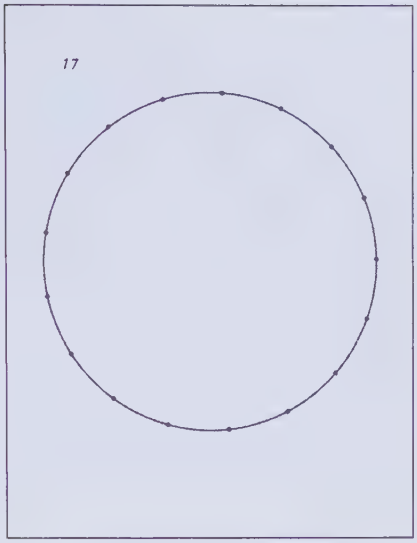
Introduction

Carl Friedrich Gauss, whose picture is on Transparency 14-2-1, was born in 1777 and died in 1855. Now considered one of the greatest mathematicians of all time, Gauss made an interesting discovery about regular polygons when he was nineteen years old.

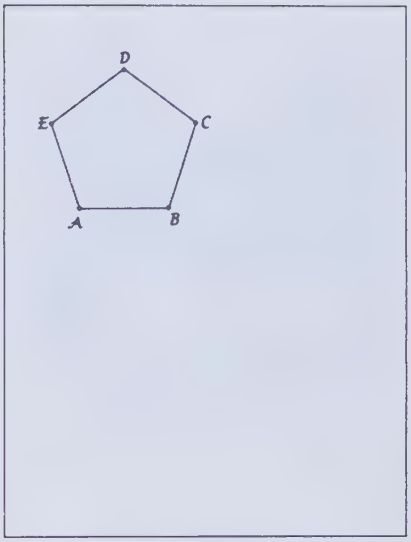
Show Transparency 14-2-2. In yesterday's lesson, we proved that every regular polygon is cyclic. Constructing a regular polygon with a given number of sides is equivalent to being able to locate that number of equally spaced points on a circle. Add the overlay to Transparency 14-2-2. Euclid be-



TRANSPARENCY 14-2-2



TRANSPARENCY 14-2-3



TRANSPARENCY 14-2-4

gan the *Elements* with an explanation of how to construct the simplest of the regular polygons—the equilateral triangle. Later he showed how to construct several other regular polygons by inscribing them in circles. These include the square, regular pentagon, and regular hexagon. By bisecting their central angles, other regular polygons having twice as many sides can be constructed.

Notice that the regular heptagon is missing. Euclid omitted it from the *Elements*, and for more than two thousand years it was unclear as to whether it is possible to construct a regular heptagon with just a straightedge and compass. Finally, in 1796, Gauss proved that the construction is impossible.

Show Transparency 14-2-3. At the same time that Gauss proved that seven equally spaced points cannot be located on a circle with just a straightedge and compass, he made a surprising discovery: it *is* possible to locate seventeen evenly spaced points using these tools. The method for doing so, as you might imagine, is quite complicated. (Add the overlay to Transparency 14-2-3.) Through a long series of steps, the vertical segments at the left and right are drawn. These segments cut off an arc of the circle that is exactly two-seventeenths of it. This arc is then divided in half to produce an arc that is exactly one-seventeenth of the circle. The method is described, together with a brief history of the constructability of the regular polygons, in *Introduction to Geometry*, Second Edition, by H. S. M. Coxeter (Wiley, 1969), pages 26-28.

To use in deriving the equation for the perimeter of a regular polygon in terms of its radius, I recommend providing your students with duplicated copies of Transparency 14-2-4. (Having a large, accurate figure to work with is important and this saves time that would be needed by your students to try to draw one.)

Lesson 3
The Area of a Regular Polygon

Lesson Outline

Theorem 76. The area of a regular polygon having n sides is Mr^2, in which $M = n \sin\frac{180}{n} \cos\frac{180}{n}$ and r is its radius.

Transparencies

14-3-1 Octagon Spa advertisement
14-3-2 Regular octagon

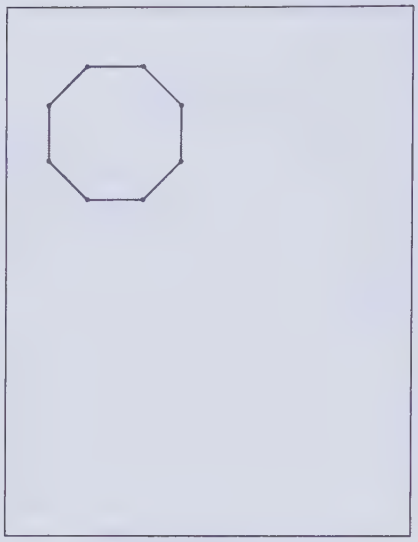
Introduction

The advertisement for the octagonal spa that is reproduced on Transparency 14-3-1 raises the question of how the area of a regular polygon might be determined. Explain that the central objective of this lesson is to derive an equation for the area of a regular polygon in terms of its radius and hand out duplicated copies of Transparency 14-3-2 for use in doing this.

After you have the equation, return to the octagon advertisement and show how its area can be determined. The dimension of the octagon given is its longest dimension: that is, the distance from a vertex to the opposite vertex. The radius of the octagon, then, is 3.5 feet. Guide your students in using their calculators to find its area. (It is approximately 34.7 square feet.)



TRANSPARENCY 14-3-1



TRANSPARENCY 14-3-2



TRANSPARENCY 14-4-1

Lesson 4
From Polygons to Pi

Lesson Outline
Terms: Circumference, limit, pi.
Theorem 77. If the radius of a circle is r, its circumference is $2\pi r$.
Corollary. If the diameter of a circle is d, its circumference is πd.
Transparencies
14-4-1 Russian dolls in a row
14-4-2 Russian dolls in a stack

Introduction
A nice way to introduce the idea of a limit is to show a set of Russian Matreshka dolls. The dolls are wooden and nest inside each other to represent a sequence of generations. The set that I use consists of twelve dolls ranging in height from almost 20 cm to less than 1 cm. If you have such a set, your students will enjoy seeing the actual dolls. If not, the picture on Transparency 14-4-1 shows what they look like.

Each successive doll is about 75% as tell as the one before it. Suppose that there were no end to the smaller dolls and that this size ratio continued. To what number do you think the heights of the successive dolls would get closer and closer? Even though we would never come to a doll whose height is actually zero, we say that the heights of the dolls *approach zero as a limit.*

Show Transparency 14-4-2, keeping the table covered. Now suppose that, as we come to each successive doll, we stack them one on top of another. Do you think that we could make the stack as tall as we please? For example, do you think if we had enough dolls, we could make a stack a mile high?

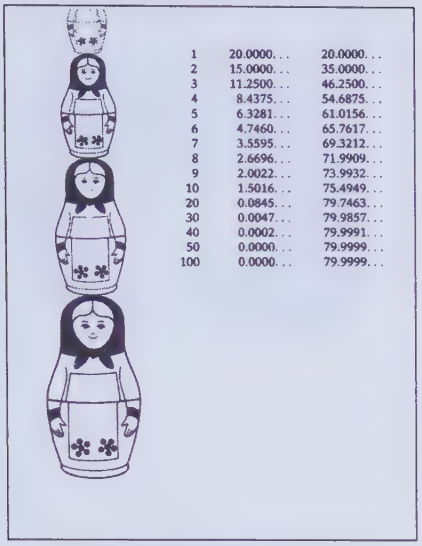
Show the table on Transparency 14-4-2 and explain that the first column represents the numbers of the dolls, the second column their heights, and the third column the heights of the stacks as more and more dolls are added. Beginning with the fifth line, the numbers have been rounded to four decimal places. After the tenth doll, the table jumps ahead to lines later in the sequence. What number does the height of the 100th doll seem very close to? What number does the height of the stack of the first 100 dolls seem very close to? If even more dolls were added to the stack, the height would continue to get closer and closer to 80 cm. For this reason, we say that it *approaches 80 as a limit.*

Following this introduction, the lesson on the circumference of a circle as the limit of the perimeters of the inscribed regular polygons can be presented as in the text.

Lesson 5
The Area of a Circle

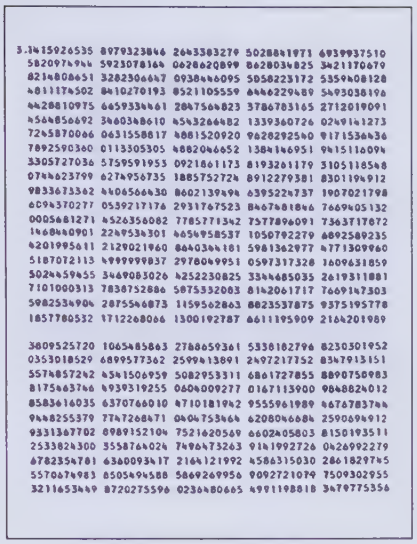
Lesson Outline
Terms: Area of a circle.
Theorem 78. If the radius of a circle is r, its area is πr^2.
Transparencies
14-5-1 Digits of pi

Introduction
Before class, I stretch adding machine tape on which I have written the value of π to several hundred decimal places across the front, a side, and back



1	20.0000...	20.0000...
2	15.0000...	35.0000...
3	11.2500...	46.2500...
4	8.4375...	54.6875...
5	6.3281...	61.0156...
6	4.7460...	65.7617...
7	3.5595...	69.3212...
8	2.6696...	71.9909...
9	2.0022...	73.9932...
10	1.5016...	75.4949...
20	0.0845...	79.7463...
30	0.0047...	79.9857...
40	0.0002...	79.9991...
50	0.0000...	79.9999...
100	0.0000...	79.9999...

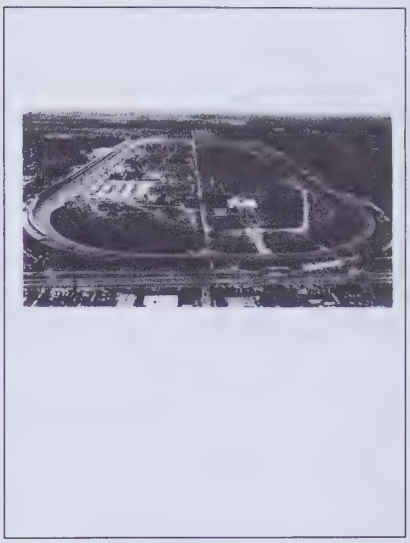
TRANSPARENCY 14-4-2



3.14159265358 97932384626 43383279 50288419716 93993751058
20974944 59230781640 62862089986 28034825 34211706798
82148086513 28230664709 38446095 50582231725 35940812848
11174502 84102701938 52110555964 46229489 54930381964
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68034532613 91766882355 37475942 78315649824 85290260395
73934116687 09231471605 66021386 38977680064 97534677306
78180699211 72079683843 69586506 76024824120 72458686643
57477161899 95809676271 00331739 72127458322 73819578544
34417906342 10710314163 85903759 98285358758 44094494901
33154135358 94884172053 59485345 99369439464 95806166252
79235466529 84004759709 50610570 77279210397 55967629862
17402943702 77069054953 58323106 05971490991 56048612180
71343326272 82965011863 56709144 22154764116 97351807176
07173267761 07646210339 98507888 16534673013 41670151182
74785771675 64275901505 99505446 30802534650 73185448620
23799362818 34568878849 66048422 07931033058 85727685882
15338182796 82303019520 35380751889 29109749440 28194810811
81615971891 13147497507 42443750 69291800705 52755361145
34222469851 56850422930 18024943 41215476516 50353896762
27022047587 53391743371 15325466 43690488160 75166788907
94958540586 72720355778 54205754 06430953646 31579429307
65989089141 20321705005 76405211 20143308643 77445539048
79626318538 79445411377 32124291 49630344859 42710367703
21713886763 34958881794 97713423 72961826810 98442474105
99368031725 88242196852 64332198 31393212937 67703915818
40441843650 61335931590 86164834 90342952430 80226336997
60682586377 24145183292 01420337 80248977298 93424376776
78393944369 38304676240 28935185 69624298517 67305675909
66954783835 15097566591 85986945 01621991067 44380461031
76927766999 36732030104 86978531 81666361476 43438662105
96277672114 07421823252 96980773 54825893850 57423454464
96284298489 86094333784 85438364 33074417368 67264054245
15259229094 30796488286 44248685 26345497041 86751461506
43089986550 12570963945 62166151 25891654571 48903771230
98478751464 44901048596 01153938 74384796051 43305727036
57595914337 67114287266 23594099 96591274117 40990721137
01586150307 96486032588 30375297 52957749158 99004331391
92127826545 20616678454 62778324 05126085251 08256150777
72468326368 94331617375 15101961 95353696431 37207178336
56702518932 41771369803 31734649 96029869329 91104113383
53878864657 34822374309 84997310 78477171951 40143024768
24294005841 24105015808 93439671 21704164748 37424243685
61456942431 04010326986 27499641 98199486072 48477966365
82452291846 67713239948 22911143 30147106276 49186483133
38494623224 16488637230 97699120 78446248127 72744266481
94673856208 38265659667 09575266 78333063159 17561293927
36067416217 54294589270 45695361 50279652068 51982872532
45821318886 46075462184 57929856 48395657284 55652708760
42683411837 31616878191 24866545 40906152209 28392184620
39022759354 11005937314 32216489 29759671777 60533473034
90428755468 31159562863 88235378759 37519578778
18577805321 71226086614 30019278787 66113959099 21842019892
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TRANSPARENCY 14-5-1



TRANSPARENCY 14-6-1

walls of my classroom. A table listing 1,650 decimal places of π is given on Transparency 14-5-1. I found a spacing of 2 inches per digit convenient. In lieu of the tape, you might, of course, simply show Transparency 14-5-1.

The history of π is very interesting. If you have a copy of the book *A History of* π by Petr Beckman (Golem Press, 1971), I recommend showing it to your class and then telling your students some of the story. The following outline lists some of the major events.

| π | | Decimal Places |
|---|---|---|
| 2000 B.C. | Babylonians: 3 1/8 | 1 |
| 300 B.C. | Euclid: 3.14 | 2 |
| 150 A.D. | Ptolemy: 3.1416 | 4 |
| 1610 | Van Ceulen | 35 |
| | *Calculus invented in 17th century* | |
| 1706* | Machin | 100 |
| 1873 | William Shanks 707, but wrong after 526 | |
| | *Computers invented in 20th century* | |
| 1949 | ENIAC | 2,000 |
| 1959 | IBM 704 (Paris) | >10,000 |
| 1961 | IBM 7090 (New York) | 100,000 |
| 1973 | (Paris) | 1,000,000 |
| 1981 | (New York) | 1,000,000,000 |
| 2002 | (Tokyo) | >1,000,000,000,000 |

Before the discovery of calculus, approximations of π were obtained from regular polygons. Ludolph van Ceulen, a sixteenth-century Dutch professor of mathematics and military science, carried out his calculation of π to 35 decimal places by starting with a regular polygon having 60 sides and doubling 29 times to get a polygon having more than 32 billion sides!

One of my students once learned the value of π to 50 decimal places, but according to the *Guinness Book of World Records*, in 1981 a 23-year-old man in India recited 31,811 places of π from memory, part of which was broadcast over the radio. It took him more than three hours, going at a rate of more than 150 digits per minute! (One wonders how many listeners stayed tuned through all of this!)

*Also named π in this year.

## Lesson 6
## Sectors and Arcs

*Lesson Outline*

*Terms:* Sector, area of sector, length of arc.

*Transparencies*

14-6-1 Photograph of the Indianapolis Speedway
14-6-2 Diagram of the Indianapolis Speedway

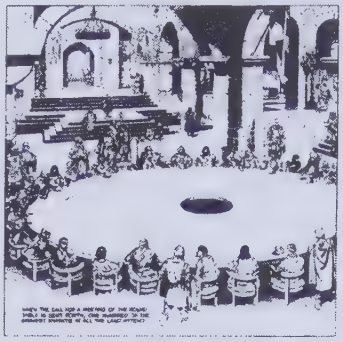
*Introduction*

The Indianapolis Motor Speedway, shown in an overhead view on Transparency 14-6-1, opened in 1909 and the Indianapolis 500 began two years later.

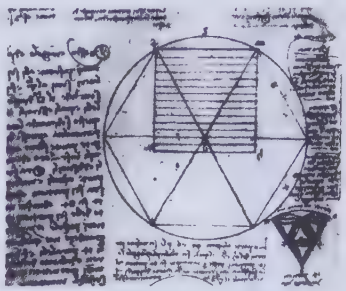
Hand out duplicated copies of Transparency 14-6-2. The track, shown in this diagram, consists of four straight segments and four circular arcs. The inside rectangle is 0.14 mile wide and 0.61 mile long and the circular arcs have radii of 0.16 mile. (Write these dimensions on the diagram.) What is the length of the track? (2.5 miles: the straight sections total 1.5 miles and the curved sections 1 mile.) Why
```



TRANSPARENCY 14-6-2



TRANSPARENCY 14-R-1



TRANSPARENCY 14-R-2

is the race called the Indianapolis 500? (Because it is 500 miles long.) How many times do the cars go around the track? (200.) How long does it take a car traveling 200 miles per hour to go once around the track? (45 seconds.)

What is the area of the region enclosed by the track? (Approximately 0.41 square mile.) The discussion of these calculations leads easily into a consideration of the topics of the new lesson: arc lengths, and sectors and their areas.

Review of Chapter 14

Transparencies

14-R-1 King Arthur's Round Table
14-R-2 Da Vinci's attempt to square the circle
14-R-3 Squaring a crescent

First day

Introduction

The drawing on Transparency 14-R-1 is of King Arthur's Round Table and is from the "Prince Valiant" comic strip. You might ask your students to imagine that King Arthur and one hundred of his knights are seated at a round table and each person takes up two feet of the table's circumference. How far is each man from the people seated farthest from him? (The men are at the vertices of a regular polygon having 101 sides. The problem is to find the length of one of its longest diagonals. The longest diagonals are slightly shorter than a diameter of the circle, but close enough to it that its length can be used for the calculation. $d = \frac{c}{\pi} = \frac{2{,}002}{\pi} \approx 64$. The distance is about 64 feet.)

Second day

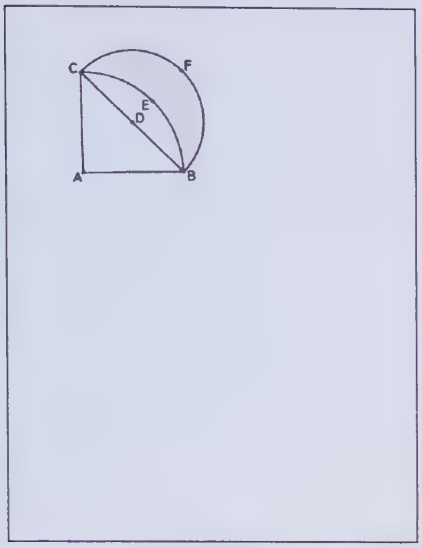
Introduction

One of the most famous problems in geometry is that of squaring the circle. To "square a circle" means to use only a straightedge and compass to construct a square that is equal in area to a given circle. Leonardo da Vinci devoted many hours to this problem. Part of a page of one of his notebooks in which he wrote about it is shown on Transparency 14-R-2. It was not until 1882 that it was proved that, like trisecting an angle, the problem of squaring the circle will never be solved.

Show Transparency 14-R-3. In the fifth century B.C., a Greek mathematician named Hippocrates discovered, in trying to square the circle, how to square the crescent-shaped region in this figure. Describe the figure as your students construct it. ΔABC is an isosceles right triangle and D is the midpoint of its hypotenuse. The center of CEB is point A and the center of CFB is point D. Let $DB = x$. The area of the "crescent" can be expressed in terms of x by noticing that it is the difference of the areas of the circular segments CBFC and CBEC. The area of the segment CBEC, in turn, is the difference of the areas of the sector ABEC and ΔABC. Using these clues, see if you can find an expression for the area of the "crescent" in terms of x.

Drawing DA forms isosceles right ΔABD, and so $AB = x\sqrt{2}$.

$$\begin{aligned} \alpha CEBFC &= \alpha CBFC - \alpha CBEC \\ &= \alpha CBFC - (\alpha ABEC - \alpha\Delta ABC) \\ &= \frac{1}{2}\pi x^2 - [\frac{1}{4}\pi(x\sqrt{2})^2 - \frac{1}{2}(x\sqrt{2})^2] \end{aligned}$$

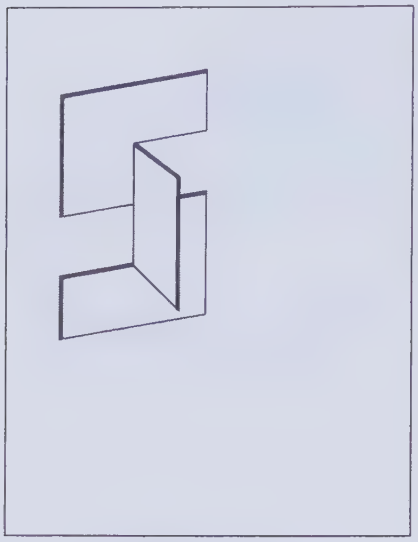


TRANSPARENCY 14-R-3

$$= \frac{1}{2}\pi x^2 - (\frac{1}{2}\pi x^2 - x^2)$$
$$= x^2.$$

Using this result, can you think of a simple way to "square the crescent" by adding a square to the figure? (Because the area of the square is x^2, its side is x. DA and DB might be used as two sides of the square.)

According to the Greek historian Proclus, Hippocrates, who first figured all of this out, wrote an "Elements of Geometry" more than a century before Euclid wrote his *Elements*, but Hippocrates' book, like most mathematical writing of that time, has been lost.



TRANSPARENCY 15-1-1



TRANSPARENCY 15-1-2

Chapter 15

Geometric Solids

Lesson 1
Lines and Planes in Space

Lesson Outline

Terms: Skew lines, parallel planes (or line and plane), perpendicular line and plane, perpendicular planes, oblique planes (or line and plane).

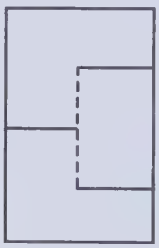
Transparencies

15-1-1 File-card puzzle
15-1-2 Church porch

Introduction

The figure on Transparency 15-1-1 once appeared in the "Mathematical Games" column of *Scientific American* and was said to have been in an entrance examination for the School of Architecture of the University of Leningrad. The problem was to make a model of the figure by taking an ordinary file card, cutting it with scissors, and folding it. In his column, Martin Gardner referred to it as a marvelous test of one's ability to think in three-dimensional space.*

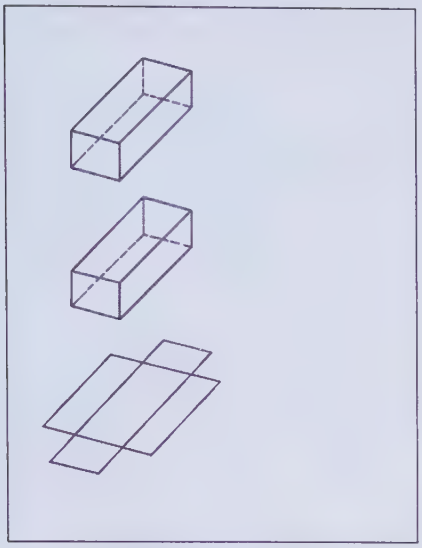
To introduce the lesson, give each of your students an unlined file card and a pair of scissors to try to create a three-dimensional model of the figure. After some of your students have succeeded at this, you might draw the following figure on the transparency as a clue on what to do.



The figure on Transparency 15-1-2 is another challenge to our ability to think in three-dimensional space. It is a design for a church porch.† An appropriate sign to put over the entrance would be "Abandon hope, ye who would enter here!" Why? (Bending a sheet of paper in the shape of the walls and ceiling over the door should reveal what is wrong.)

*"Mathematical Games," by Martin Gardner, *Scientific American*, November 1978.

†This figure is from *Perspective in Perspective* by Lawrence Wright (Routledge and Kegan Paul, 1983).



TRANSPARENCY 15-2-1

Lesson 2
Rectangular Solids

Lesson Outline

Terms: Polyhedron, rectangular solid, diagonal and dimensions of rectangular solid, cube.

Theorem 79. The length of a diagonal of a rectangular solid with dimensions l, w, and h is $\sqrt{l^2 + w^2 + h^2}$.

Corollary. The length of a diagonal of a cube with edges of length e is $e\sqrt{3}$.

Transparencies

15-2-1 Rectangular solid

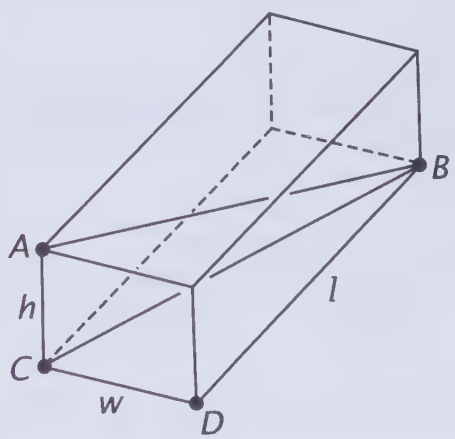
Introduction

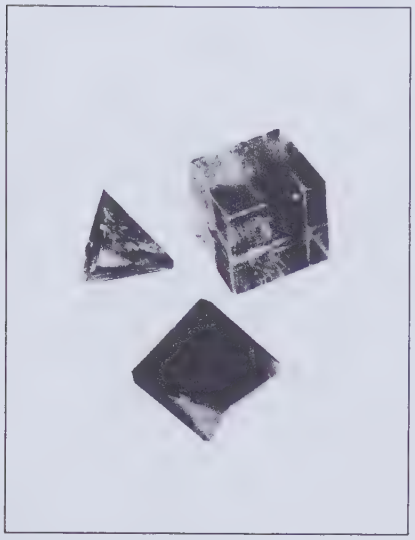
Before class, stretch a cord or a piece of brightly colored yarn diagonally across the room from one of the corners of the ceiling to the opposite corner of the floor. Begin class by handing out small pieces of paper. Remark that the subject of today's lesson is rectangular solids and that the classroom is a good example of a rectangular solid. Explain that the cord represents a diagonal of the room, and ask your students to guess its length in feet, writing their names and their guesses on the pieces of paper you have given them. Collect the guesses and set them aside.

Had out duplicated copies of Transparency 15-2-1 and have your students draw diagonal AB in the first figure, as shown at the right. Also label points C and D, and the dimensions of the room h, w, and l.

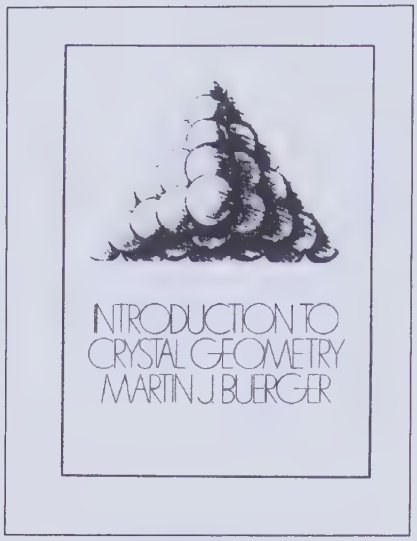
Explain that the length of AB is determined by the dimensions of the room. To see how this is done, draw CB to form a triangle with AB as one of its sides. After your students have noticed that ΔABC and ΔBCD are right triangles, guide them in applying the Pythagorean Theorem to the figure to derive the formula for the length of a diagonal of a rectangular solid. (This can be done at the right of the first figure.)

Then, give the dimensions of your classroom in feet and have your students write them on the second figure. Ask them to use the formula that they have just derived to calculate the length of the diagonal. (My classroom is 27 feet wide, 31 feet long, and 14 feet high, and so the length of the diagonal is $\sqrt{1{,}886} \approx 43.4$ feet.) While your students are doing this, sort out their guesses. After the calculation has been discussed, announce the name(s) of the students(s) who came closest. It might also be interesting to tell the smallest and largest guesses without telling the names of the students who guessed them. (The extreme guesses usually turn

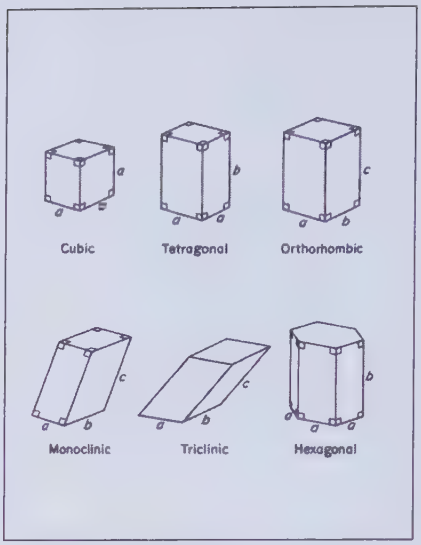




TRANSPARENCY 15-3-1



TRANSPARENCY 15-3-2

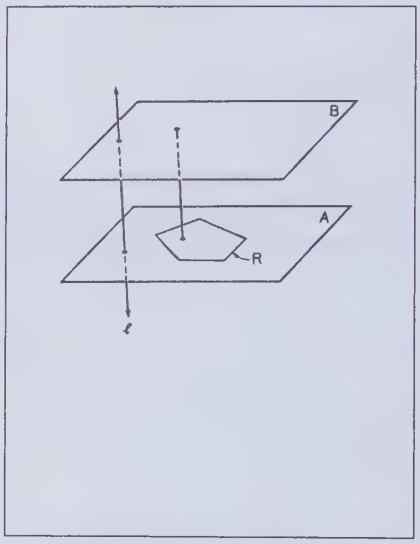


TRANSPARENCY 15-3-3

out to be quite outlandish.)

The third figure of the transparency might be used to derive the formula for the surface area of a rectangular solid. The figure can be thought of as the floor and walls of the room, collapsed outward. Have your students add a sixth rectangle at the right to represent the ceiling. Then label the dimensions of the rectangles in terms of l, w and h and use the figure to find the surface area formula. For practice, your students might calculator the total surface area of your classroom in square feet.

To close the lesson, you might guide your students in writing formulas for the length of a diagonal of a cube and its surface area. It is helpful to compare the formulas for the lengths of the diagonals of a square and a cube, emphasizing that both formulas contain square roots and asking your students how they might remember that the square formula has a 2 in it and the cube formula has a 3.



TRANSPARENCY 15-3-4

Lesson 3
Prisms

Lesson Outline

Terms: Prism, bases, lateral faces and edges, right prism, oblique prism, net, lateral area, total area of prism.

Transparencies

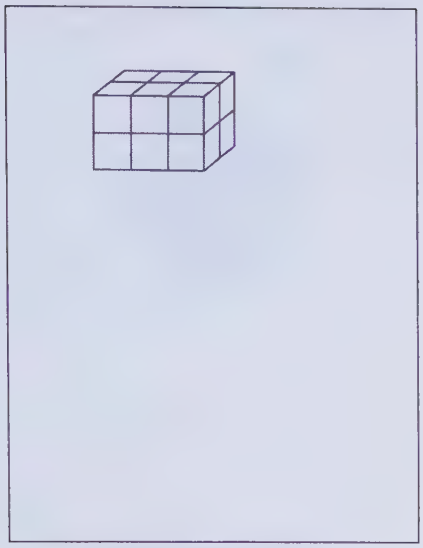
15-3-1 Photograph of crystals
15-3-2 Cover of *Crystal Geometry*
15-3-3 A page from *Crystal Geometry*
15-3-4 A prism

Introduction

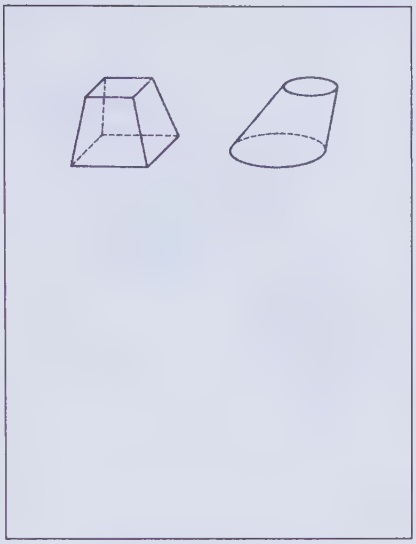
The pictures on Transparency 15-3-1 are photographs of crystals. The two solids at the top are sodium chlorate crystals and the solid at the bottom is a crystal of chrome alum. Many minerals have crystals in the shape of polyhedra. The figure on the dust jack of *Introduction to Crystal Geometry* (Transparency 15-3-2) suggests why.* Crystals are three-dimensional arrangements of atoms in which the atoms line up in rows that result in straight edges and flat surfaces.

Crystals can be imagined to consist of little building blocks. Each building block corresponds to an arrangement of atoms or molecules that is regularly repeated throughout the crystal.

**Introduction to Crystal Geometry* by Martin J. Buerger (McGraw-Hill, 1971).



TRANSPARENCY 15-4-1



TRANSPARENCY 15-4-2

Every crystal belongs to one of the six crystal systems illustrated on Transparency 15-3-3. Each system uses a different type of building block and it is the shapes of the building blocks that are shown. Notice that the first three shapes are rectangular solids. All six shapes are prisms.

Following this introduction, you can use Transparency 15-3-4 and its overlay to make the general definition of a prism clear. The rest of the lesson can be presented as in the text.

Lesson 4
The Volume of a Prism

Lesson Outline

Terms: Altitude of a prism, volume, cross section

Postulate 13. Cavalieri's Principle. Consider two geometric solids and a plane. If every plane parallel to this plane that intersects one of the solids also intersects the other so that the resulting cross sections have equal areas, then the two solids have equal volumes.

Postulate 14. The volume of any prism is the product of the area of its base and its altitude:

$$V = Bh.$$

Corollary 1. The volume of a rectangular solid is the product of its length, width, and height:

$$V = lwh.$$

Corollary 2. The volume of a cube is the cube of its edge:

$$V = e^3.$$

Transparencies

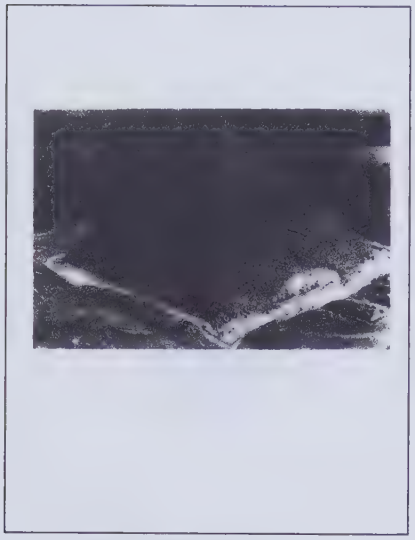
15-4-1 Cubes in box problem
15-4-2 Cavalieri's Principle

Introduction

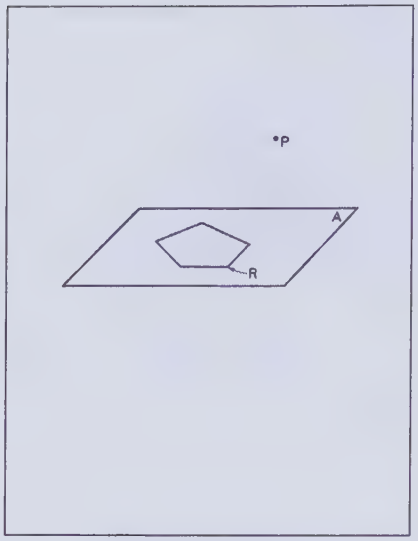
Transparency 15-4-1 can be used to introduce the new lesson. The figure on it appeared in a mathematics question in a test of the National Assessment of Educational Progress. The question was: How many cubes does the box contain?

After discussing the answer, your students might be interested to know that of the 9-year-olds asked this question, only 6% answered it correctly. Of the people 13 years old, 21% gave the correct answer, and of those 17 years old, only 43%! The most commonly given wrong answer was "16."

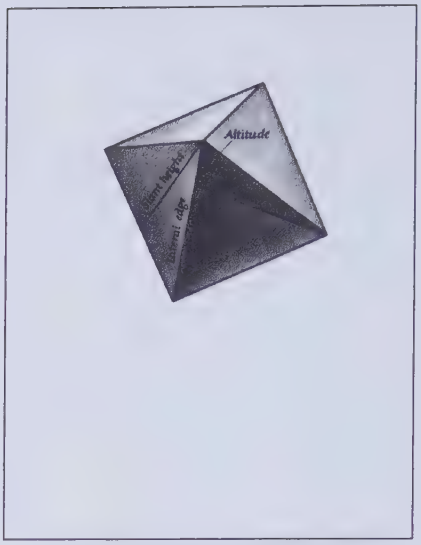
After observing that the volume of a rectangular solid that has been divided into unit cubes can be found by counting them, explain that not every solid can be divided into cubes. To find the volume of a general prism, we need to make some assumptions. First, introduce the $V = lwh$ and $V = Bh$ formulas for rectangular solids as in the text. Then, use a deck of playing cards or a ream of paper to suggest how the volume of an oblique rectangular prism might be found from a right rectangular prism. Transparency 15-4-2 and its overlay can be used to introduce Cavalieri's Principle and the rest of the lesson presented as in the text.



TRANSPARENCY 15-5-1



TRANSPARENCY 15-5-2



TRANSPARENCY 15-5-3

Lesson 5
Pyramids

Lesson Outline

Terms: Pyramid, base, lateral faces and edges, apex, altitude of pyramid

Theorem 80. The volume of any pyramid is one-third of the product of the area of its base and its altitude:

$$V = \frac{1}{3}Bh.$$

Transparencies

15-5-1 Photograph of the Great Pyramid
15-5-2 A pyramid
15-5-3 A square pyramid
Worksheet: Three pyramids that fold into a cube
15-5-4 Three pyramids that fold into a cube

Introduction

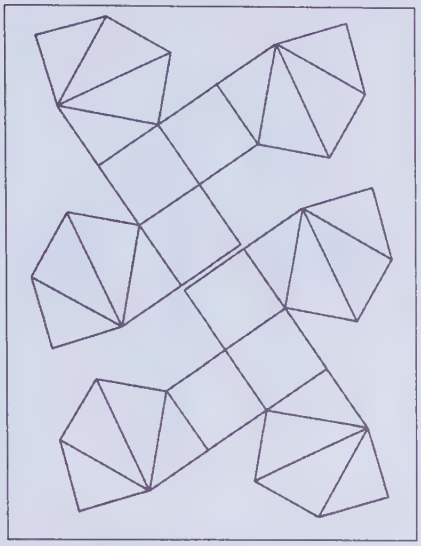
The word "pyramid" usually brings to mind the pyramids of Egypt. The Great Pyramid, shown in the photograph on Transparency 15-5-1, is certainly the most famous example. Its shape suggests that a pyramid has a square base and triangular faces that meet at a point. For pyramids in general, however, this is only partly true. Use Transparency 15-5-2 and its overlay to present the geometric definition of a pyramid.

Explain that the Great Pyramid of Egypt is an example of a *square* pyramid, hand out duplicated copies of Transparency 15-5-3, and introduce the terms pictured on it. Then have your students write on the figure the length of an edge of the base of the Great Pyramid, 760 ft, and its altitude, 484 ft. Guide them in using the appropriate right triangles to find the altitude of one of the faces (615 ft) and the length of a lateral edge (723 ft).

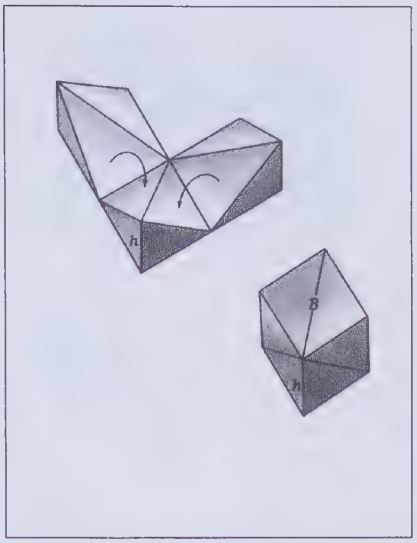
Hand out scissors and copies of the pattern on Worksheet for Lesson 15-5. (If possible, print the worksheet on cover stock; note that there are two patterns on each sheet.) Explain to your students that they are going to do an experiment that will enable them to guess a formula for the volume of a pyramid. Tell them to cut out the pattern, fold it along all the lines, and see what they notice. After they have had time to do this, ask someone to describe what they have noticed.

The pattern can be folded into three square pyramids (unlike the Great Pyramid, their lateral edges are not equal), which can, in turn, be folded toward each other to form a cube. Transparency 15-5-4 illustrates how this can be done. Note that the three pyramids seem to be identical, and so they have equal volumes. Because they can be folded together to form a cube, its volume is three times that of each pyramid. Hence, the volume of one of the pyramids is one-third that of the cube. Because the cube and pyramids have equal bases and altitudes, the volume of one of the pyramids is evidently one-third the product of the area of its base and its altitude. Explain that this conclusion is true for all pyramids and that we will consider it a theorem even though we have not proved it.

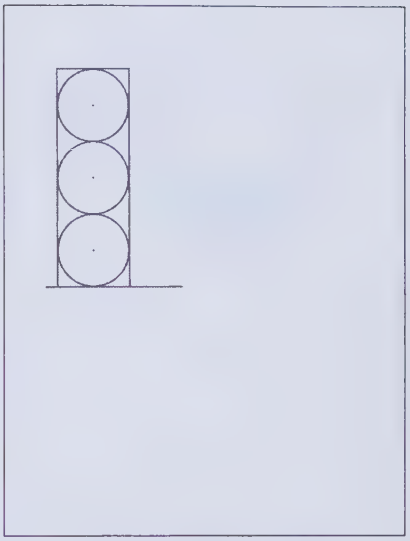
To conclude the new lesson, show Transparency 15-5-3 again and help your students, as needed, to find the lateral area and volume of the Great Pyramid. (They are approximately 935,000 square feet and 93,200,000 cubic feet, respectively.)



WORKSHEET FOR 15-5



TRANSPARENCY 15-5-4



TRANSPARENCY 15-6-1

Lesson 6
Cylinders and Cones

Lesson Outline

Terms: Cone, lateral surface, circular region, base, apex, altitude, axis, right and oblique cones, cylinder, bases, lateral surface, altitude, axis, right and oblique cylinders.

Theorem 81. The volume of a cylinder is the product of the area of its base and its altitude:

$$V = Bh = \pi r^2 h.$$

Theorem 82. The volume of a cone is one-third the product of the area of its base and its altitude:

$$V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h.$$

Transparencies

15-6-1 A can of tennis balls

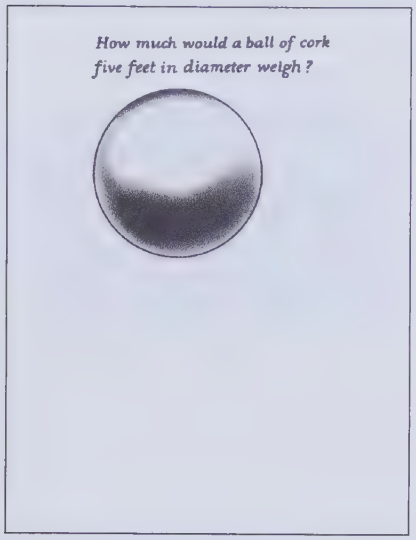
Introduction

Cut a strip of paper so that it will exactly fit when it is wrapped around a can of tennis balls. Tape the ends of the strip together so that it is held securely in place around the can. Start class by holding up the can and describing what you have done. Tell your students that you are going to cut the strip where you have taped it and stretch it beside the can, holding one end at the bottom. How high would the other end reach?

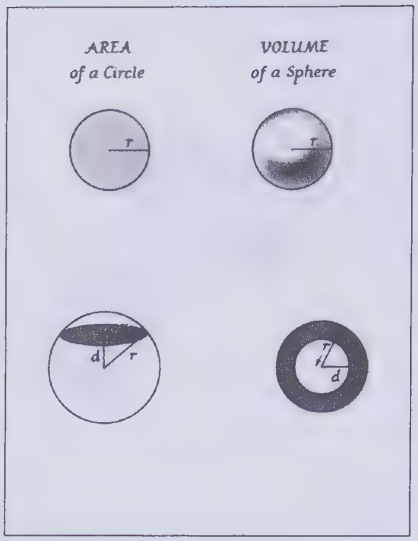
Hand out duplicated copies of Transparency 15-6-1 which contains a figure representing the can with the three balls inside it. Ask your students to draw a line segment upward from the bottom line to represent how far they think the strip of paper would stretch. While they are doing this, you might hold the can toward them so that they can see the size of its circular cross section.

Almost all of my students guess that the length of the strip is less than the height of the can. Without telling your students the answer, have them draw a segment through the centers of the balls from the bottom to the top of the can. Label the radii of the circles representing the balls r. What is the height of the can in terms of r? ($6r$.) What is the length of the strip of paper? (It is the circumference of the base of the can: $2\pi r$, or approximately $6.28r$.) So the length of the strip is *more* than the height of the can!

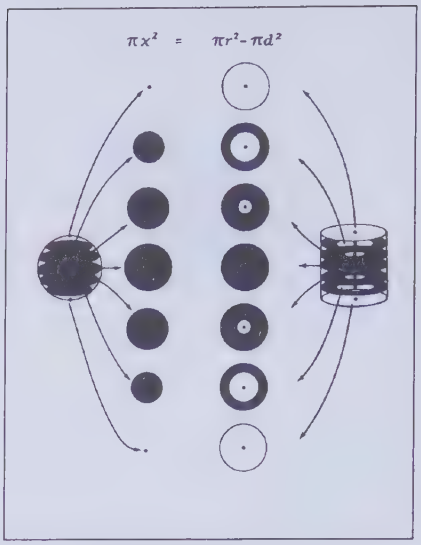
Following this introduction, the lesson on cylinders and cones can be presented as in the text.



TRANSPARENCY 15-7-1



TRANSPARENCY 15-7-2



TRANSPARENCY 15-7-3

Lesson 7
Spheres

Lesson Outline
Terms: Sphere, center, radius, diameter

Theorem 83. The volume of a sphere is $\frac{4}{3}\pi$ times the cube of its radius:

$$V = \frac{4}{3}\pi r^3.$$

Theorem 84. The surface area of a sphere is 4π times the square of its radius:

$$A = 4\pi r^2.$$

Transparencies
15-7-1 Cork ball
15-7-2 Circle and sphere
15-7-3 Cross sections of a sphere
15-7-4 A sphere and hollowed-out cylinder
15-7-5 Recreational sites on Mercury
15-7-6 The surface area of a sphere

First day

Introduction
Here is a good puzzle.*

You know how very light cork is. When you have a cork stopper in hand—just now taken out of a bottle—it weighs practically *nothing*. If thrown into a bowl of water it floats hardly submerged. The stuff is very, very light. So—quick now—we have a ball of cork—a sphere of cork—5 feet in diameter. Question: What does it weigh? Could you lift it? No calculations! Just give us a quick guess.

(Show Transparency 15-7-1 and ask your students to write their guesses before continuing.) In order to figure out what a ball of cork 5 feet in diameter would actually weigh, we need to know two things: the density of cork—that is, how much a given volume of it weighs—and, how to find the volume of a sphere.

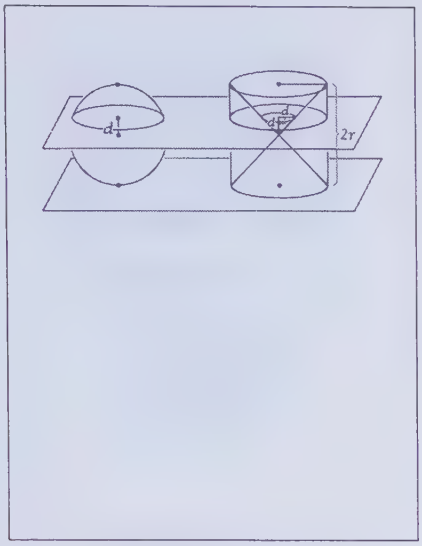
One cubic inch of cork weighs about 0.14 ounce. To find out how many cubic inches of cork there would be in a ball 5 feet in diameter, we need a formula for the volume of a sphere.

(Show the top half of Transparency 15-7-2.) We know that the area of a circle is a little more than three times the square of its radius: more precisely, πr^2. Would you expect the volume of a sphere to also be measured in terms of the square of its radius? (No. Since volume is measured in cubic units, it should be some number times the *cube* of the radius: that is, $?r^3$.) To find this number, we can use Cavalieri's Principle.

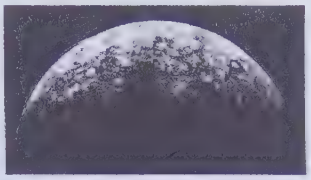
In deriving the formula for the volume of a sphere, it is helpful to have some models. I use two 4.5-in diameter Styrofoam spheres (available in art and crafts stores), one of which I have sliced into two pieces as in the figure on page 663 of the text. I also use a model of the hollowed-out cylinder (also shown on page 663) whose dimensions match those of the spheres. This model can be made from tag board and tape.

I have my students make a drawing showing a side view of the sphereas shown in the next column.

*From *Millergrams I*, by Julius Sumner Miller (Doubleday, 1970).

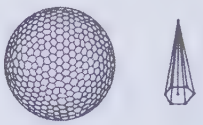


TRANSPARENCY 15-7-4

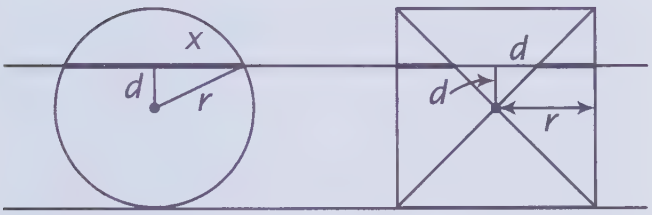


Recreational sites of at least 14,000 acres are available for $25 donations.

TRANSPARENCY 15-7-5



TRANSPARENCY 15-7-6



$$\pi x^2 = \pi(r^2 - d^2) = \pi r^2 - \pi d^2$$

We treat the discovery of the hollowed-out cylinder as a detective story, illustrating it first with a square and then adding the details step by step. The figures on Transparencies 15-7-3 and 15-7-4 should also be helpful in enabling your students to understand the volume derivation.

After the volume formula has been derived, show Transparency 15-7-2 again and note that we have found the number we were looking for: it is $\frac{4}{3}\pi$. Then return to Transparency 15-7-1 and find the volume and weight of the cork ball.

Since the diameter of the ball is 5 feet, its radius is 2.5 feet or 30 inches:

$$V_{\text{ball}} = \frac{4}{3}\pi(30)^3 = 36{,}000\pi \approx 113{,}000 \text{ cubic inches.}$$

Since one cubic inch of cork weighs about 0.14 ounce, the ball would weigh 113,000(0.14) = 15,820 ounces, or almost half a ton!

Since this lesson covers a lot of material, I suggest not assigning any problems for homework until the next day.

Second day

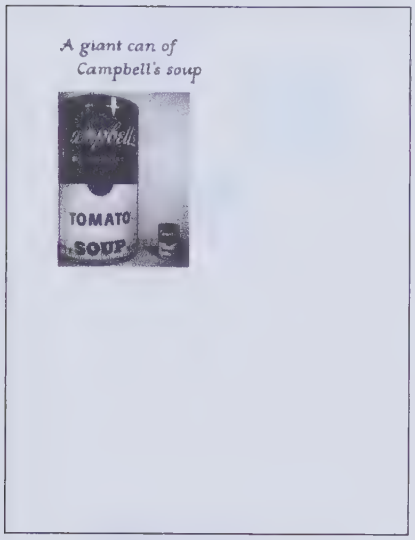
Introduction

Earlier in the course, we calculated the cost of an acre of land in the Yukon Territory that was being sold for 25¢ per square inch. *Omni* magazine once reported that the Astronomical Society of the Pacific was selling recreational sites on the planet Mercury (Transparency 15-7-5). The sites are advertised as "pollution-free and also atmosphere-free," but buyers are also warned that the temperatures are somewhat balmy (actually, hot enough to melt lead). At the rate of $25 for 14,000 acres, how much would it cost to buy the entire planet?

To be able to answer this question, we need to know how to find the surface area of a sphere. Use Transparency 15-7-6 to derive the formula for the surface area of a sphere as in the text and then return to the problem about Mercury as a class exercise.

The diameter of Mercury is 3,100 miles and 1 square mile is equal to 640 acres. The surface area of the planet is $4\pi(1{,}550)^2 \approx 30{,}000{,}000$ square miles. At $25 for 14,000 acres, the entire planet would cost approximately 35 million dollars.

The Society that made the land offer is a nonprofit organization dedicated to increasing public understanding of astronomy and admits that it does not actually own any land on Mercury.

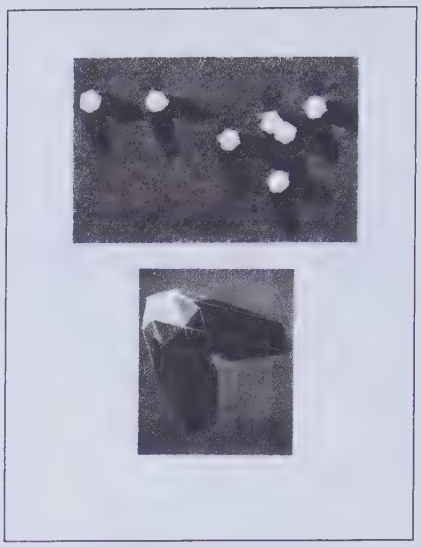


TRANSPARENCY 15-8-1

Theorem.
The ratio of the surface areas of two similar solids is equal to the square of the ratio of any pair of corresponding segments.

Theorem.
The ratio of the volumes of two similar solids is equal to the cube of the ratio of any pair of corresponding segments.

TRANSPARENCY 15-8-2



TRANSPARENCY 15-9-1

Lesson 8
Similar Solids

Lesson Outline
Terms: Similar solids.
Theorem 85. If the ratio of a pair of corresponding dimensions of two similar solids is r, then the ratio of their surface areas is r^2.
Theorem 86. If the ratio of a pair of corresponding dimensions of two similar solids is r, then the ratio of their volumes is r^3.
Transparencies
15-8-1 A giant can of Campbell's soup
15-8-2 Similar solids theorems

Introduction
An amusing way to introduce this lesson is with a large wastebasket that has been painted to look like a can of soup or soda. The photograph on Transparency 15-8-1 shows a large "Campbell's soup can" that I bought at a novelty store. An actual can of soup stands beside it.

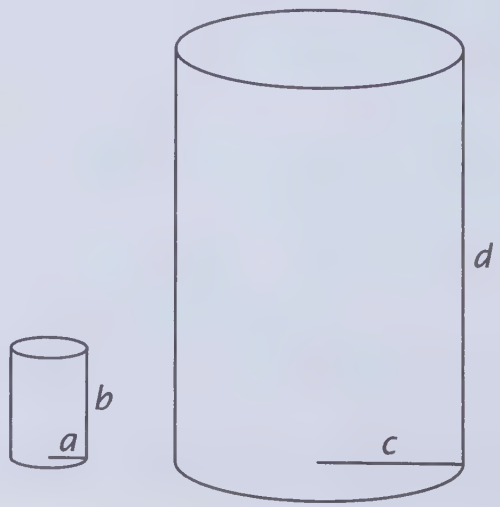
Although this giant can is a wastebasket, it looks very much like a can of Campbell's soup. It is about five times as tell as an actual can and has about five times its diameter. As a result, the two cans, giant and genuine, have the same shape.

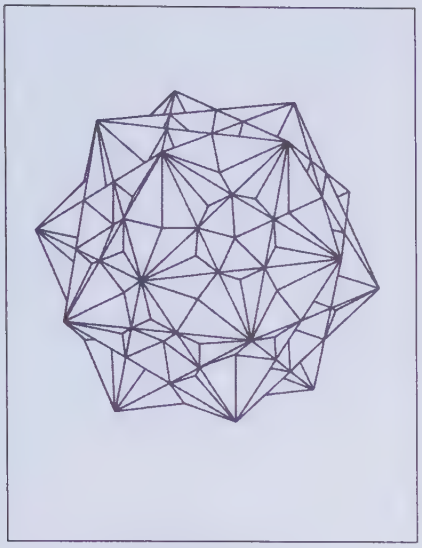
Have your students draw pictures of two cylinders to represent the cans and label their dimensions as indicated in the figure at the right. If two geometric solids are similar, every pair of distances between pairs of corresponding points in the two solids is proportional. If the cans are similar, $\frac{c}{a} = \frac{d}{b} = r$. (Explain that r represents the ratio of the corresponding dimensions and that in the case of the cans, $r = 5$.) Note that $c = ra$ and $d = rb$.

How do the lateral areas of the two cans compare? For the small can, $A = 2\pi ab$, and for the giant can, $A = 2\pi cd = 2\pi(ra)(rb) = r^2(2\pi ab)$. The ratio of the lateral areas is r^2. Because $r = 5$, the lateral area of the giant can is $5^2 = 25$ times that of the small can. The "label" on the giant can, then, is 25 times as large as the label on the small can.

How do the volumes of the two cans compare? For the small can, $V = \pi a^2 b$, and for the giant can, $V = \pi c^2 d = \pi(ra)^2(rb) = r^3(\pi a^2 b)$. The ratio of the volumes is r^3. The volume of the giant can is $5^3 = 125$ times that of the small can.

The actual can of soup weighs $10\frac{3}{4}$ oz and costs 99¢ (at the time of writing). If the giant can were





TRANSPARENCY 15-R-1



THE STEREOCHEMICAL THEORY OF ODOR

by John E. Amoore, James W. Johnston, Jr. and Martin Rubin

TRANSPARENCY 15-R-2

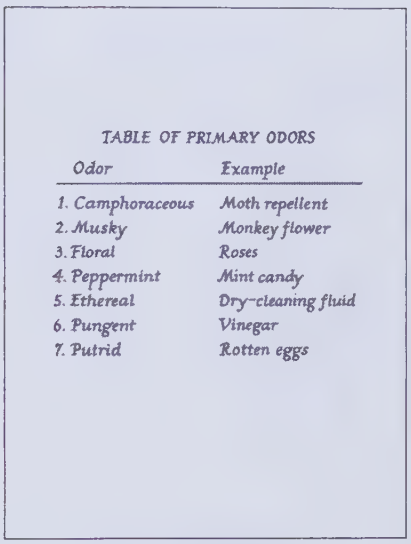


TABLE OF PRIMARY ODORS

Odor	Example
1. Camphoraceous	Moth repellent
2. Musky	Monkey flower
3. Floral	Roses
4. Peppermint	Mint candy
5. Ethereal	Dry-cleaning fluid
6. Pungent	Vinegar
7. Putrid	Rotten eggs

TRANSPARENCY 15-R-3

filled with soup, how much do you think it would weigh and cost? (Having 125 times the volume of the small can, it is reasonable to think that it would weigh 125 times as much, or about 84 lb. If it also costs 125 times as much, it would cost $123.75.)

Suppose that an even larger can, still similar in shape to the small one, was *ten* times as tall. How many times bigger do you think its label, weight, and cost would be? (Because $r = 10$, $r^2 = 100$ and $r^3 = 1{,}000$. The large can would have a label 100 times as large and would weigh and cost 1,000 times as much.)

Show Transparency 15-8-2 and explain that these theorems hold true, not just for similar cylinders, but for *any* pair of similar solids. Point out that it is easy to remember that the area ratio is found by *squaring* and that the volume ratio is found by *cubing* because area and volume are measured in *square* and *cubic* units, respectively.

Lesson 9
The Regular Polyhedra

Lesson Outline

Terms: Regular polyhedron, tetrahedron, octahedron, icosahedron, dodecahedron.

Transparencies

15-9-1 Icosahedral virus

Introduction

The photograph at the top of Transparency 15-9-1 was taken through an electron microscope and shows particles of a virus enlarged about 58,000 times. Note the double shadows of the particles and the constancy of their shape. A geometric solid that casts shadows exactly like these particles is pictured at the bottom of the transparency. It is called an *icosahedron* and, because its shadows match those of the virus, biologists think that the particles of the virus are in the shape of tiny icosahedra. Models like those shown on page 678 of the text would be helpful in introducing the other regular polyhedra.

Review of Chapter 15

Transparencies

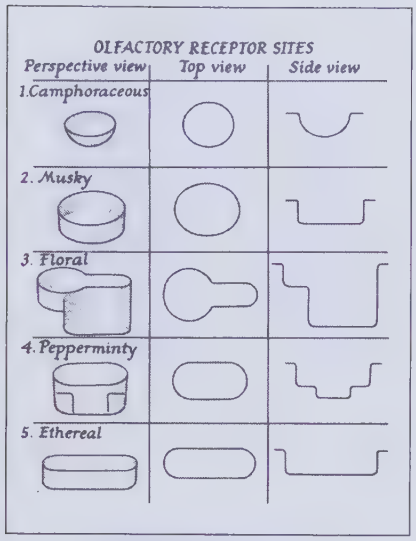
15-R-1 Five intersecting cubes
15-R-2 "The Stereochemical Theory of Odor"
15-R-3 Table: Primary odors
15-R-4 Olfactory receptor sites
15-R-5 An almond odor

First day

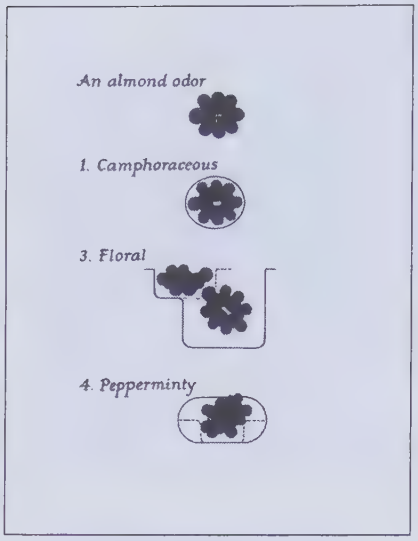
Introduction

The polyhedron pictured on Transparency 15-R-1 is very complex and yet very simple. Its edges form equilateral triangles, squares, regular pentagons, and five-pointed stars, and it contains 360 faces in all. The 360 faces do not lie in 360 different planes, however. They are contained in only 30 planes and this is what makes the polyhedron simple. Perhaps some of your students will be able to recognize that there are large cubes hidden in the polyhedron. The overlay can be put on the transparency to reveal one of these cubes.

I made the model in the photograph from cardboard and tape, with sticks of balsa wood inside to make it rigid. Information on the patterns needed is given on page 136 of *Mathematical Models*, Revised Edition, by H. Martyn Cundy and A. P. Rollett (Ox-



TRANSPARENCY 15-R-4



TRANSPARENCY 15-R-5

ford University Press, 1961), but its construction requires a lot of time and patience.

Second day

Introduction

Take one or two bottles of almond extract and either remove or tape over the labels so that the names are covered. Hand the bottle(s) out at the beginning of class and ask your students to try to identify the odor.

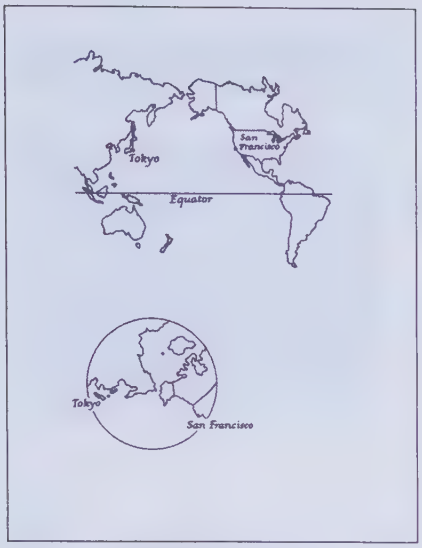
A Roman poet and philosopher named Lucretius speculated two thousand years ago that the sense of smell is based on geometry. He thought that every substance that has an odor gives off tiny particles of a specific shape. When these particles enter a person's nose, they fit into certain pores that match their shape. The pores into which the particles fit enable the person to identify the substance from which they came.

Modern scientists now think that Lucretius's theory is correct. Show Transparency 15-R-2, on which the first page of an article on odor that appeared in *Scientific American* is reproduced.* Note the sentence under the title stating that "there is evidence that the sense of smell is based on the geometry of molecules." There seem to be seven basic odors, as listed in the table enlarged from the article on Transparency 15-R-3. Each of the first five is associated with molecules of a particular geometric shape. Show Transparency 15-R-4. These figures show perspective, top, and side views of the receptor sites for these five odors. The camphoraceous site is hemispherical in shape. The site corresponding to the odor of musk is shaped like a cylinder. Those for floral and pepperminty odors are more complex in shape, whereas the ethereal site is long and rod-shaped. Add the overlay to the transparency. These diagrams represent molecules that fit each of the five sites.

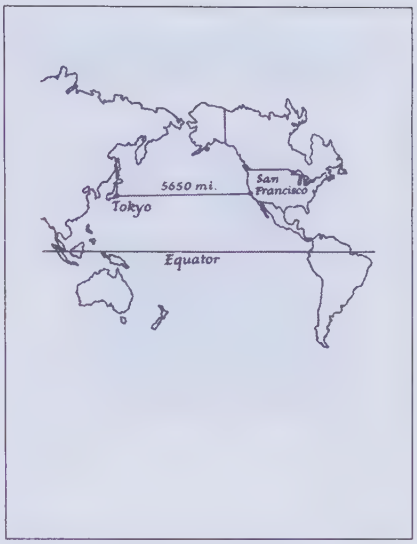
The two basic odors not shown here, pungent and putrid, are determined not by the shape of the molecule but rather by its charge. Pungent odors, such as that of vinegar, are caused by positively charged molecules, whereas putrid odors are caused by molecules that are negatively charged.

By this time, everyone should have had a chance to smell the almond extract. After someone has identified it, show Transparency 15-R-5. It appears that all odors can be explained in terms of the seven basic ones. For example, the odor of almonds is a combination of camphoraceous, floral, and pepperminty odors. Molecules that possess this odor can fit any one of the three sites (actually, two of them in the floral site). A remarkable consequence of this theory is that, by putting their knowledge of three-dimensional geometry to use, chemists have been successful in synthesizing substances having various odors simply by producing molecules of the right shape.

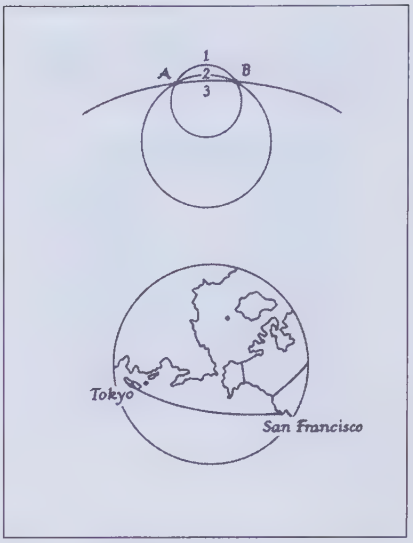
*This lesson and its illustrations are adapted from "The Stereochemical Theory of Odor" by John E. Amoore, James W. Johnston, Jr., and Martin Rubin, *Scientific American*, February 1964, pages 42-49.



WORKSHEET 16-1



TRANSPARENCY 16-1-1



TRANSPARENCY 16-1-2

Chapter 16
Non-Euclidean Geometries

Before beginning to teach this chapter, it may be helpful to read the following articles:

"Euclid's Parallel Postulate and Its Modern Offspring" by Martin Gardner, *Scientific American*, October 1981, pages 23–33.

Mathematics in Western Culture by Morris Kline (Oxford University Press, 1953), Chapter 26, "New Geometries, New Worlds."

Einstein's Legacy by Julian Schwinger (Scientific American Books, 1986), Chapter 5, "Geometry and Physics."

Another valuable reference is the book *Euclidean and Non-Euclidean Geometries*, Second Edition, by Marvin Jay Greenberg (W. H. Freeman and Company, 1980).

Lesson 1
Geometry on a Sphere

Lesson Outline
Terms: Great circle, antipodal points.
Transparencies
16-1-1 From San Francisco to Tokyo
16-1-2 The great circle route

Introduction
Hand out duplicated copies of Worksheet for Lesson 16-1. Here are two maps showing the cities of Tokyo and San Francisco. Suppose that a ship sails from one city to the other. Use your pencil to mark on the first map the path of the ship.

After your students have done this, show Transparency 16-1-1. Gerardus Mercator, the Flemish cartographer who devised the method of mapping the earth used here, wrote:

> If you wish to sail from one point to another, here is a chart, and a straight line on it, and if you follow this line carefully you will certainly arrive at your destination. But the length of the line may not be correct. You may get there sooner or may not get there as soon as you expected but you will certainly get there.*

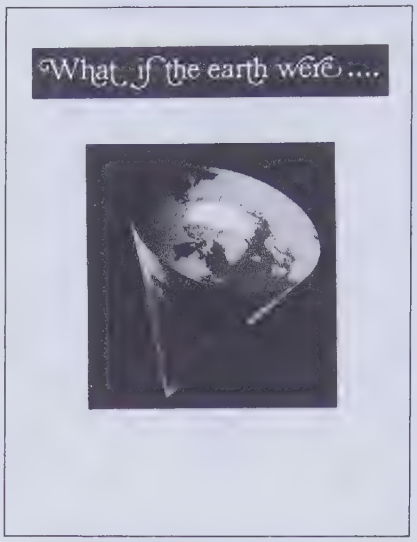
Mercator was referring to a problem that many people are not aware of when they look at a map such as his. It would seem that any other path between Tokyo and San Francisco would be longer than the one we have drawn, yet here is a path (add the overlay to Transparency 16-1) that is significantly shorter. Can you explain how this can be?

Because these paths are on the surface of a sphere, neither one is straight. Both, in fact, are arcs of circles. Show Transparency 16-1-2. In the top figure, two points, A and B, are joined by three arcs numbered 1, 2, and 3. Which arc is shortest? (Arc 3.) How does the circle of which it is a part compare

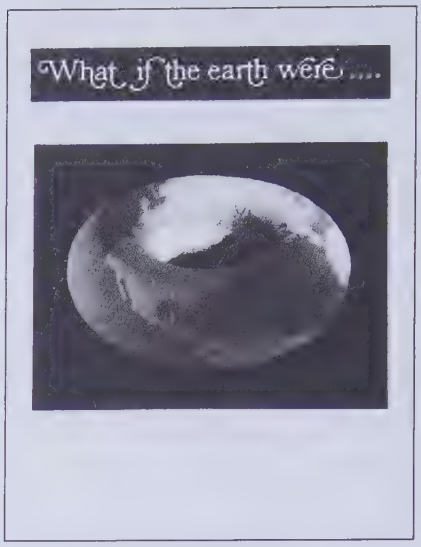
*From *Mapping* by David Greenhood (University of Chicago Press, 1964), page 128.



TRANSPARENCY 16-2-1



TRANSPARENCY 16-2-2



TRANSPARENCY 16-2-3

with the other circles? (It is the largest.) If you could not draw a straight line through points A and B, but could only draw a circle, what would you do to make the arc from A to B as short as possible? (Make the circle as large as possible.)

Look at the bottom figure of the transparency. Because Tokyo and San Francisco are on the surface of a sphere, we cannot draw a straight line along it from one point to the other. We will have to settle for a circle instead. How should we choose it to make the arc from A to B as short as possible? (Choose the largest possible circle through A and B that lies on the surface of the sphere; namely, the circle whose radius is the same as that of the sphere.) Add the overlay to Transparency 16-1-2 and explain the idea of a *great circle*. Now have your students mark the center of the earth on the second figure on their worksheets and sketch the great circle that goes through Tokyo and San Francisco.

The rest of the lesson can be developed as in the text.

Lesson 2
The Saccheri Quadrilateral

Lesson Outline

Terms: Birectangular quadrilateral: base, legs, summit, base angles, summit angles; Saccheri quadrilateral.

Theorem 87. The summit angles of a Saccheri quadrilateral are equal.

Theorem 88. The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.

Theorem 89. If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.

Theorem 90. If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.

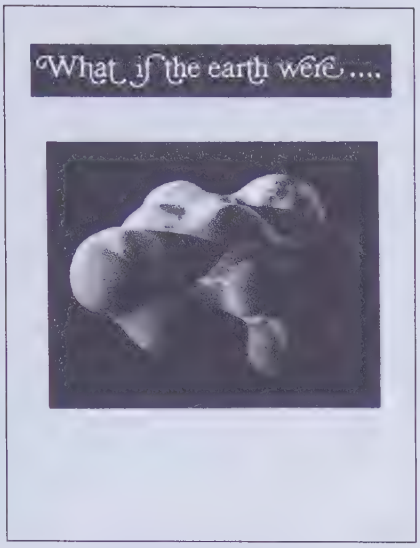
Transparencies

16-2-1 What if the world were cubical?
16-2-2 What if the world were conical?
16-2-3 What if the world were toroidal?
16-2-4 What if the world were cataclysmal?
16-2-5 Euclid's fifth postulate
16-2-6 Geometry books
16-2-7 A deductive system

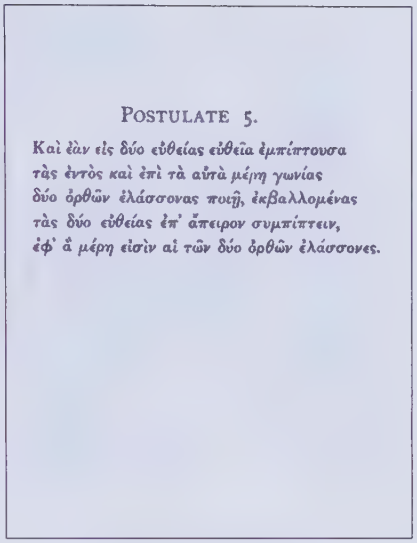
Introduction

Joseph N. Portney has designed a very attractive set of 9-in × 12-in posters on the theme "What if the world were . . .?" There are twelve posters in the set, showing photographs of models of an earth that is flat, cubical, dodecahedral, pyramidal, cylindrical, conical, toroidal, hyperboloidal, hyperbolic paraboloidal, oblate spheroidal, "Wegeneroidal," and "cataclysmal." The set can be seen on http://www.nrich.maths.org.uk/mathsf/journalf/dec99/art1/. The figures on Transparencies 16-2-1, 16-2-2, 16-2-3, and 16-2-4 are taken from these posters.

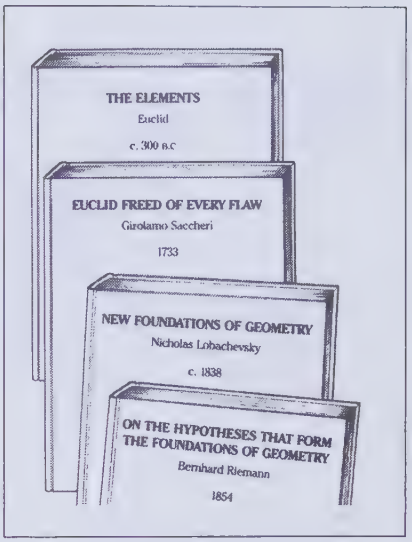
You know that the shortest path between two points in a plane is the line segment joining them. What is the shortest path between two points on a sphere? (An arc of a great circle.) What do you think the shortest path between two points on the sur-



TRANSPARENCY 16-2-4



TRANSPARENCY 16-2-5



TRANSPARENCY 16-2-6

face of a cube would be? (A set of one or more line segments.) The shortest paths on the cone and toroid are more complicated and on the "cataclysmal" surface, extremely complex.

To introduce the new lesson, you might show Transparency 16-2-5. Perhaps the most controversial statement to be made in all of the history of mathematics is Euclid's fifth postulate. Shown here in the original Greek, the postulate states, in effect, that through a point not on a line, there is exactly one line parallel to the line.*

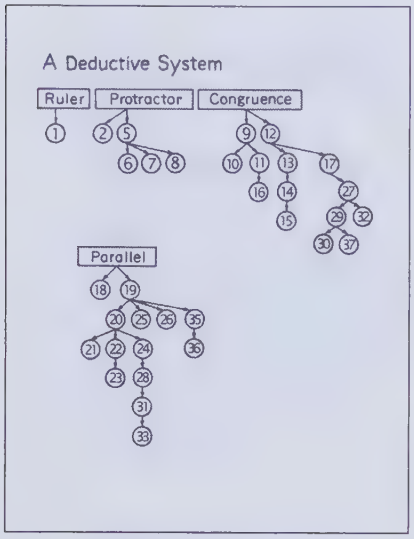
What made the postulate controversial is not that people did not believe it, but rather that others thought Euclid should have proved it as a theorem. Show Transparency 16-2-6. One person who thought this was an Italian priest named Saccheri and who went so far as to write a book titled *Euclid Freed Of Every Flaw* in which he tried to prove the Parallel Postulate. In this lesson, we will see how Saccheri set out to do this.† The rest of the lesson can now be presented as in the text.

Be sure to emphasize that because Saccheri was trying to prove the Parallel Postulate, he could not use any theorems whose proofs depend on it. The diagram on Transparency 16-2-7 should help clarify the situation. The boxes represent postulates and the circle identify by number some of the theorems that depend on these postulates. The arrows show only a few of the relations between the statements in the system. In essence, we cannot use any theorems in the bottom group as we prove the theorems in this lesson. You may want to show the transparency from time to time as questions arise. For example, some students will concluce that because the summit angles of a Saccheri quadrilateral are equal, it must be a rectangle. This follows from Theorem 32 (the sum of the measures of the angles of a quadrilateral is 360°). But the proof of Theorem 32 uses Theorem 28 (the sum of the measures of the angles of a triangle is 180°). Theorem 28, in turn, follows from Theorem 25, which follows from the Parallel Postulate. Therefore, if the purpose of the Saccheri quadrilateral is to prove the Parallel Postulate, we cannot assume that it is a rectangle.

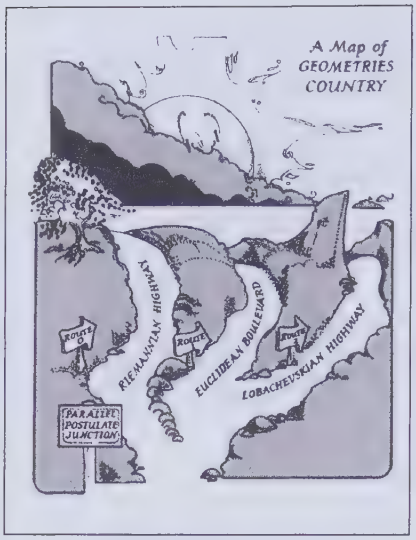
The proofs of the four theorems are outlined in the exercises. The proofs of the theorems not assigned should be demonstrated in the lesson.

*This is known as "Playfair's postulate" and is, of course, the version of the Parallel Postulate that we have used throughout the course. Euclid's version is: "That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles." I would not bring this up in the lesson.

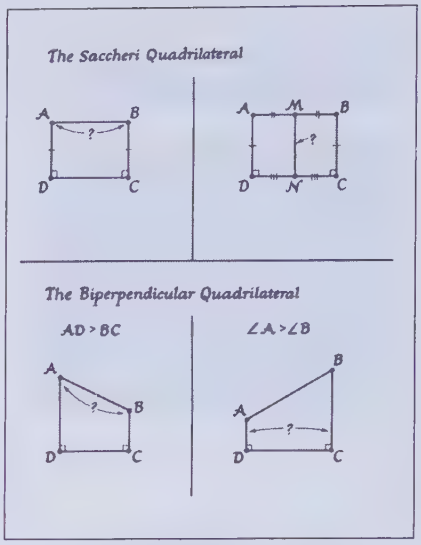
†The article "Saccheri, Forerunner of Non-Euclidean Geometry" by Sister Mary of Mercy Fitzpatrick in *The Mathematics Teacher* (May 1964), pages 323-332, is a valuable reference.



TRANSPARENCY 16-2-7



TRANSPARENCY 16-3-1



TRANSPARENCY 16-3-2

Lesson 3
The Geometries of Lobachevsky and Riemann

Lesson Outline

Terms: The Lobachevskian Postulate. The summit angles of a Saccheri quadrilateral are acute.

Lobachevskian Theorem 1. The summit of a Saccheri quadrilateral is longer than its base.

Lobachevskian Theorem 2. A midsegment of a triangle is less than half as long as the third side.

Transparencies

16-3-1 Map: Geometries country
16-3-2 Review of Theorems 87 through 90
16-3-3 Theorem 87 (Lesson 2: Set I, Exs. 7–13)
16-3-4 Theorem 88 (Lesson 2: Set II, Exs. 31–35)
16-3-5 Theorem 89 (Lesson 2: Set I, Exs. 14–21)
16-3-6 Theorem 90 (Lesson 2: Set I, Exs. 22–28)
16-3-7 Table: Comparing three geometries

Introduction

To open this lesson, show Transparency 16-3-1. This map compares our study of geometry to traveling through the country. Earlier in the course, we came to the junction pictured at the bottom of the map. When we assumed the Parallel Postulate—that is, that through a point not on a line, there is exactly one line parallel to the line—we chose Route 1. Since then, we have traveled on Euclidean Boulevard a long distance past the junction. Before the nineteenth century, it was assumed that Euclidean Boulevard was the "only way to go"—that no other roads existed. Now we are, in effect, returning to the junction so that we can travel a short way along each of the other highways in order to see what the scenery along them looks like. Route 0 corresponds to the assumption that through a point not on a line, there are *no* parallels to the line. We have an introduction to this in our study of geometry on the surface of a sphere. Route 2 corresponds to the assumption that through a point not on a line, there is *more* than one parallel to the line.

Transparency 16-3-2 is for use in reviewing the statements of the four theorems of Lesson 2. Transparencies 16-3-3, 16-3-4, 16-3-5, and 16-3-6 should be helpful in discussing the proofs of these theorems.

To introduce the new lesson, show Transparency 16-3-7, keeping the right half covered. Explain that either of these statements can be proved from the other. Next, reveal the alternative conclusions to the statements, still keeping the pictures and names of Lobachevsky and Riemann covered. Point out that Saccheri's plan was to prove Euclid's Parallel Postulate by proving that the summit angles of a Saccheri quadrilateral are right angles. Saccheri thought that he could do this indirectly by assuming that the summit angles were either acute or obtuse and showing that these assumptions lead to contradictions. (Reveal the rest of the transparency.) Later, other mathematicians, among them Lobachevsky and Riemann, showed that logically consistent geometries could be based on these other assumptions.

Show Transparency 16-2-6 again. The dates shown here reveal that more than one hundred years passed between Saccheri's pioneering work and the writings of Lobachevsky and Riemann. Show Transparency 16-3-7 again and explain that we will start with the assumption that the summit angles of a

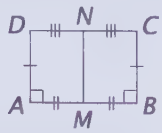
Theorem 87 The summit angles of a Saccheri quadrilateral are equal.



Given: Saccheri quadrilateral ABCD with base AB.
Prove: $\angle C = \angle D$.

TRANSPARENCY 16-3-3

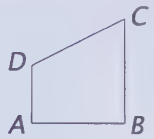
Theorem 88 The line segment connecting the midpoints of the base and summit of a Saccheri quadrilateral is perpendicular to both of them.



Given: Saccheri quadrilateral ABCD with MN connecting the midpoints of AB and CD.
Prove: MN $\perp$ AB and MN $\perp$ DC.

TRANSPARENCY 16-3-4

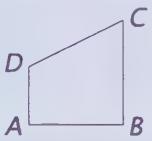
Theorem 89 If the legs of a birectangular quadrilateral are unequal, the summit angles opposite them are unequal in the same order.



Given: Birectangular quadrilateral ABCD with base AB, CB > DA.
Prove: $\angle D > \angle C$.

TRANSPARENCY 16-3-5

Theorem 90 If the summit angles of a birectangular quadrilateral are unequal, the legs opposite them are unequal in the same order.



Given: Birectangular quadrilateral ABCD with base AB, $\angle D > \angle C$.
Prove: CB > DA.

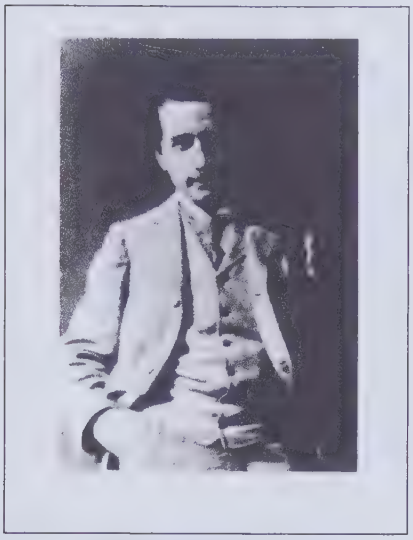
TRANSPARENCY 16-3-6



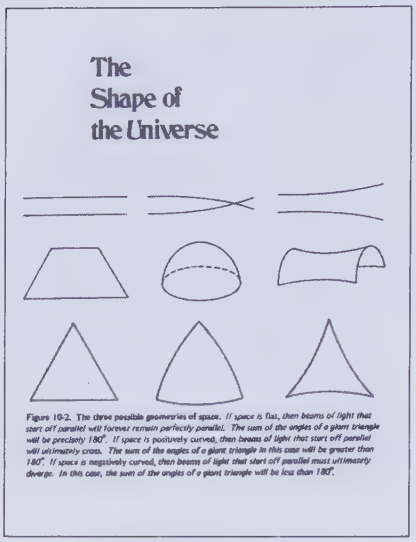

	Euclid	*Lobachevsky*	*Riemann*
Through a point not on a line, there is	*exactly one parallel to the line.*	*more than one parallel to the line.*	*no parallel to the line.*
The summit angles of a Saccheri quadrilateral are	*right.*	*acute.*	*obtuse.*

TRANSPARENCY 16-3-7

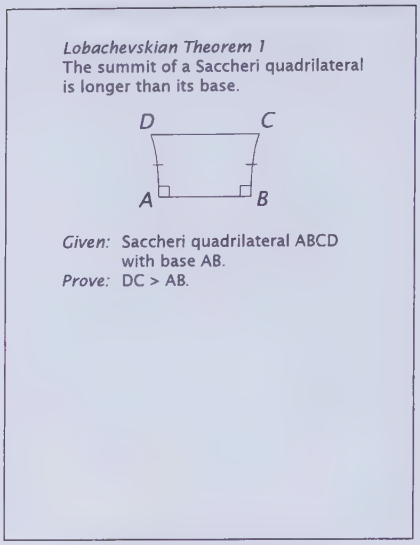
Saccheri quadrilateral are acute. We will refer to this as the "Lobachevskian Postulate" and we will use it to prove some theorems in Lobachevskian geometry. Although the proofs of the two theorems in this lesson are outlined in the exercises, I recommend considering the proof of Theorem 1 in class.



TRANSPARENCY 16-4-1



TRANSPARENCY 16-4-2



TRANSPARENCY 16-4-3

Lesson 4
The Triangle Angle Sum Theorem Revisited

Lesson Outline

Lobachevskian Theorem 3. The sum of the angles of a triangle is less than 180°.

Corollary. The sum of the angles of a quadrilateral is less than 360°.

Lobachevskian Theorem 4. If two triangles are similar, they must also be congruent.

Transparencies

16-4-1 Photograph of Einstein
16-4-2 The shape of the universe
16-4-3 Lobachevskian Theorem 1 (Lesson 3: Set I, Exs. 1–10)
16-4-4 Lobachevskian Theorem 2 (Lesson 3: Set I, Exs. 11–18)

Introduction

The photograph of Einstein on Transparency 16-4-1 was taken when he was in his early twenties, not long before he developed the theory of relativity for which he was become so famous. Einstein once wrote that if he had not been acquainted with non-Euclidean geometry, he never would have been able to develop the theory of relativity.

The figures on Transparency 16-4-2 are from a book titled *Relativity and Cosmology** and illustrate three possible geometries of space. (Read the caption to the figure.) From the standpoint of mathematics, all three geometries are equally valid. It makes no sense to say that one of them is true and the others are false. If we are interested in deciding what the universe is like, then it does make sense to ask which geometry fits it. The theory of relativity implies that it may be one of the non-Euclidean geometries.

Transparencies 16-4-3 and 16-4-4 are for use in reviewing the proofs of the Lobachevskian Theorems 1 and 2. Again, even though the proofs of the theorems in the new lesson are outlined in the exercises, it may be best to preview at least one of them in class.

Review of Chapter 16

Transparencies

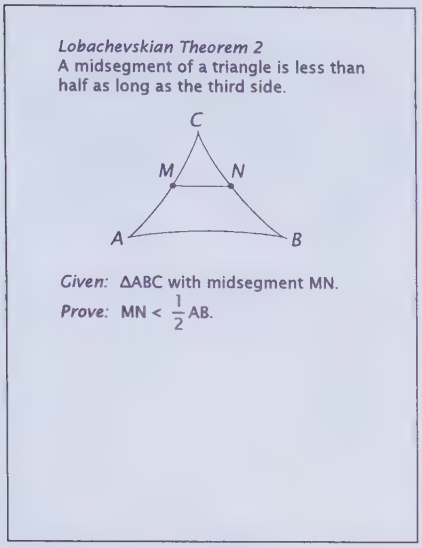
16-R-1 The orbit of Mercury
16-R-2 Lobachevskian Theorem 3 (Lesson 4: Set I, Exs. 1–9)
16-R-3 Corollary (Lesson 4: Set I, Exs. 10–13)
16-R-4 Lobachevskian Theorem 4 (Lesson 4: Set I, Exs. 14–20)
16-R-5 McDonald's restaurants around Kansas City
16-R-6 Point distributions on a plane and on a sphere
16-R-7 Escher's *Circle Limit III* and diagram

First day

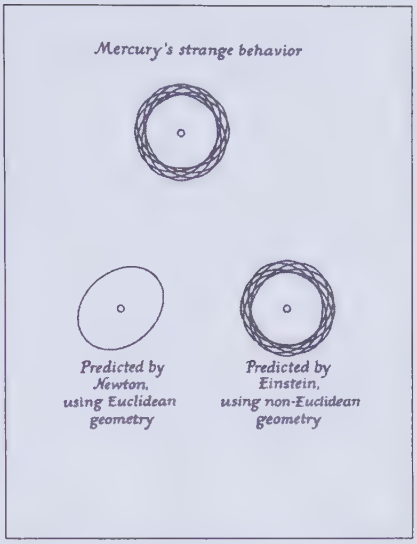
Introduction

Sir Isaac Newton, one of the greatest scientists of all time, was strongly influenced by Euclid. Of his books, the *Principia*, which is modeled after Euclid's *Elements*, the *Encyclopaedia Britannica* says:

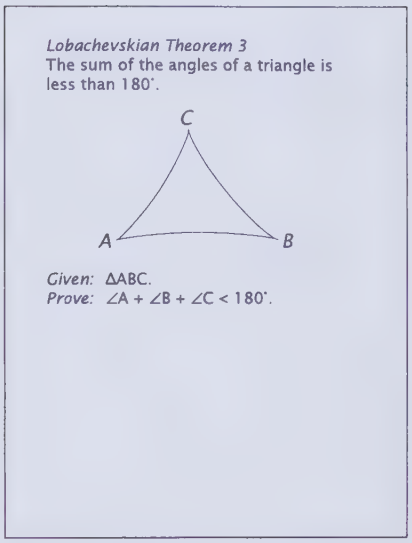
**Relativity and Cosmology* by William J. Kaufmann III (Harper and Row, 1973).



TRANSPARENCY 16-4-4



TRANSPARENCY 16-R-1



TRANSPARENCY 16-R-2

It is explained for the first time the way in which a single mathematical law could account for phenomena of the heavens, the tides, and the motions of objects on the earth. The whole development of modern science begins with this great book.

The *Principia* was published in 1687. One hundred fifty years later, a French astronomer made a surprising discovery about the planet Mercury. Show the top figure on Transparency 16-R-1. This figure, greatly exaggerated to make it easier to see, represents the orbit of Mercury about the sun. In the course of each revolution, the orbit shifts very slightly. This is strange because, according to Newton's laws of gravity, Mercury's orbit shouldn't do this. Show the lower-left figure on the transparency. The orbit should be a simple ellipse that remains fixed in space. Instead, it rotates very slowly, by about 43 seconds of arc each century.

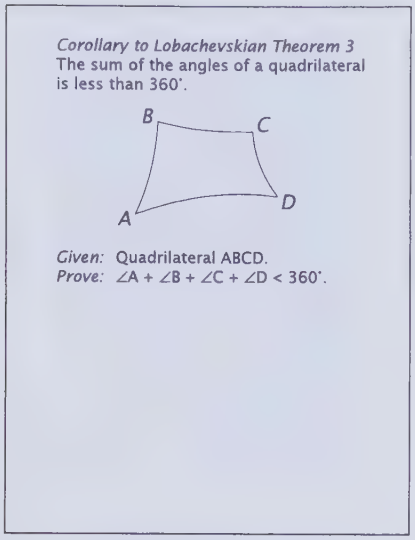
In 1915, Einstein figured out why. Read the following passages, which are from Chapter 8 of *Einstein: The Life and Times* by Ronald W. Clark:

> . . . Thus the universe which Newton has seen, and for which he had constructed his apparently impeccable mechanical laws, was not the real universe but only what he had seen through the misleading spectacles produced by gravity. The law which appeared to have worked out so well had been drawn up for a universe that did not exist, as though a tailor had made a suit for a man he had seen only in a distorting mirror . . .
>
> Einstein's paper gave not only a corrected picture of the universe but also a fresh set of mathematical laws by which its details could be described . . . . These laws utilized Riemannian geometry, the need for which had been the direct result of the assumption that light would be deflected by a gravitational field and that the shortest distance between two points in such a field would not, when viewed from outside it, be coincident with a straight line . . .
>
> Einstein had seen that his assumption of a curvature of light in a gravitational field meant that Euclidean geometry, satisfactory enough when coping with the small distances of everyday life, had to be replaced by something more sophisticated when dealing with the universe . . .
>
> Einstein used Riemannian geometry to create equations by which the movements of the stars in their courses and the structure of the universe itself could be described . . .*

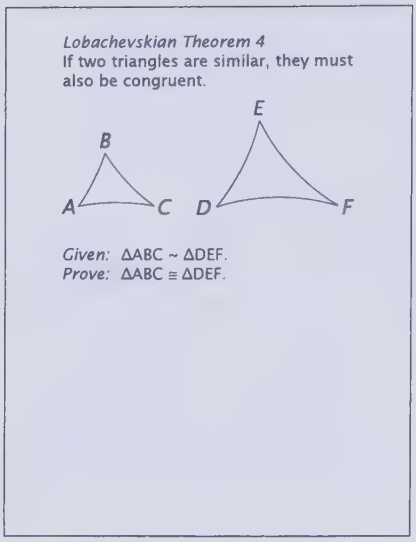
Show the lower-right figure on Transparency 16-R-1. Whereas Newton's theory of the universe, based on Euclidean geometry, predicts an orbit for Mercury that does not match its actual path, Einstein's theory, based on Riemannian geometry, predicts it perfectly. According to Einstein, the orbit should rotate 43 seconds of arc each century—the very amount that it does!

Because of this and other evidence, Einstein's theory of relativity is generally accepted as correctly

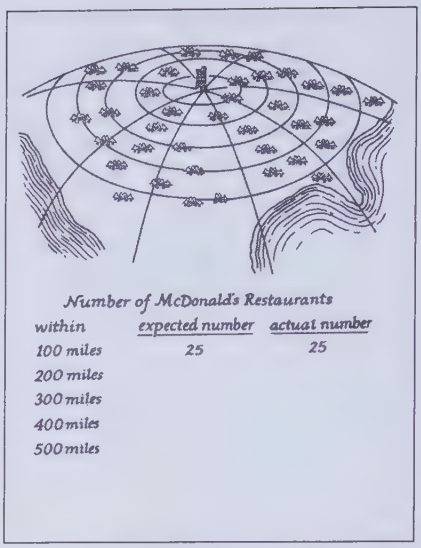
*From *Einstein: The Life and Times* by Ronald W. Clark. Copyright © 1971 by Ronald W. Clark, with the permission of Thomas Y. Crowell Company, Inc., publisher, and The Harold Matson Company, Inc.



TRANSPARENCY 16-R-3



TRANSPARENCY 16-R-4



TRANSPARENCY 16-R-5

explaining the nature of the universe. It is amazing to think that it is based on a geometry that is non-Euclidean.

Transparencies 16-R-2, 16-R-3, and 16-R-4 are for use in reviewing the proofs of the theorems in Lesson 4.

First day

Introduction

The first McDonald's hamburger stand opened in 1940 and since then thousands of McDonald's restaurants have opened in the United States.* Suppose that the president of McDonald's wanted to find out how the restaurants are distributed through the country. Show Transparency 16-R-5. He might choose a point in the center of the United States—say Kansas City—and decide to count the number of restaurants within 100 miles from this point, 200 miles from it, 300 miles from it, and so on.

Suppose that there are 25 McDonald's restaurants within 100 miles of Kansas City. If the restaurants are distributed uniformly throughout the country, how many would you expect to find within 200 miles of Kansas City? Because the area of a circle varies as the square of its radius, it would seem that there should be four times as many restaurants in the circle whose radius is twice as large. Within 300 miles of Kansas City, we might expect nine times as many restaurants, and so on. Complete the expected-number column at the bottom of the transparency. The numbers are: 25, 100, 225, 400, and 625.

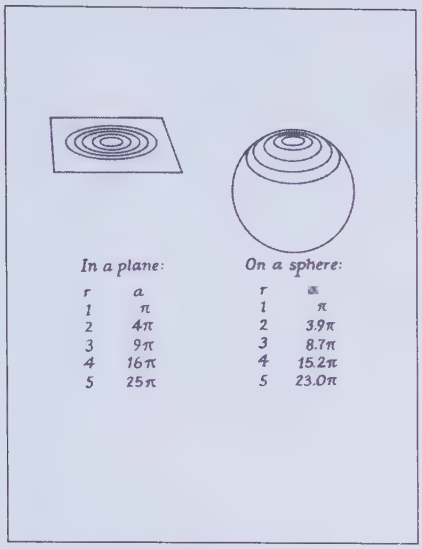
Now suppose that the actual numbers of restaurants turn out to be as shown here. (Add the overlay.) We see that, as we get further and further from Kansas City, the actual numbers of restaurants fall further and further behind what we had expected. Does this mean that the restaurants are not uniformly distributed but are concentrated around Kansas City?

The answer is no! Our calculations were based on the assumption that the earth's surface is flat, but it is not. It is curved. Show Transparency 16-R-6. These figures represent the sets of points within 1, 2, 3, 4, and 5 units from a given point in a plane and on a sphere. The relations of the areas of the circles on the plane are what we assumed in predicting the numbers of McDonald's restaurants at various distances from Kansas City. The areas of a comparable set of circle on the sphere have a different relation. As their radii increase, their areas do not grow quite as fast. This explains how the restaurants could be uniformly distributed and yet fall short of the numbers we predicted.†

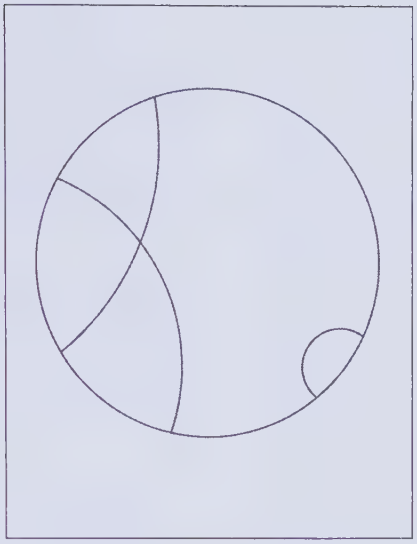
The point to be learned from this story is that, if we assume that something is uniformly distributed over a surface, we can determine whether the surface is flat (Euclidean) or curved (non-Euclidean) by measuring numbers of the thing at varying distances from a given point in the surface. This is one of the methods astronomers have used to try to de-

*This story is adapted from one by George Gamow in his book, *Mr. Tompkins in Paperback* (Cambridge University Press, 1965).

†The decline in "actual" numbers of McDonald's restaurants in the table on the overlay to Transparency 16-R-6 is actually highly exaggerated to make the point more obvious.



TRANSPARENCY 16-R-6



TRANSPARENCY 16-R-7

termine whether physical space is Euclidean or non-Euclidean. Although the problem is now three-dimensional rather than two-dimensional, the principle is the same.

In a survey of radio sources in space, astronomers at Ohio State University discovered that the farther out in space one looks, the more the number of observed radio sources falls short of what one would expect if space were Euclidean.* If we assume that radio sources are uniformly distributed throughout space, one implication that can be drawn is that space is non-Euclidean in nature. This, as you may remember, is in agreement with Einstein's theory of relativity.

It would be interesting to know, if there are other intelligent civilizations in the universe, whether they also began with Euclidean geometry as we did. Of course they would call it by a different name. Because, on a small scale, Euclidean geometry seems so obvious, this seems very probable.

It would also be interesting to know how long it took them to realize that there are other geometries which might fit physical space instead. It took mankind two thousand years to do so!

Transparency 16-R-7 and its overlay are for use in discussing Poincaré's model of Lobachevskian geometry. Show it first with the overlay in place and then remove the overlay to review the idea of orthogonal circles. Escher was inspired to create *Circle Limit III* in 1959 as the result of correspondence with the geometer H. S. M. Coxeter.

*From *Relativity and Cosmology* by William J. Kaufmann III (Harper and Row, 1973), pages 112-113.

About GEOMETRY by Harold R. Jacobs
Notes by Peter Renz

Geometry is intuitive and appealing with useful results but it can be demanding, and it has tricky twists and turns. The challenge is to build intuition and develop the curiosity and persistence needed to push through tough problems and avoid pitfalls. Working with Jacobs on this book was a chance to fit the pieces of geometry into a coherent whole.

Particular examples are the core of this book, especially those in its exercises, as Don Albers notes in his Foreword. Every chapter offers a new crop of puzzles and applications. By following the steps of the exercises, you can always see *what* is happening, but in seeing *why* may be a challenge.

As I struggled, I remembered the NCTM Standards recommendations about flexibility. Use the tools best suited to a particular problem; choose your approach — direct geometric arguments, transformations, matrices, graphs, technology, whatever. This is fine advice for me, but the students are just starting out. Many times I found an easy approach but sometimes there were difficulties. Perhaps you will find the Royal Road I missed.

Euclidean geometry can be regarded as a system developed from postulates or as the geometry of the coordinate plane with the usual distance formula. Either way, the same properties hold. These properties are built into geometric software; they underlie computer representations, three-dimensional graphics, etc. If students understand that things must fit together, they will be able to check ideas for correctness using whatever tools are at hand — starting with paper, pencil, and brain power.

My experience is that looking things up and asking others for ideas are excellent problem-solving techniques. But nothing beats knowing the answer from school. A reasonable repertoire of standard results and formulas is useful, which is one reason a Formulary was added in this edition.

These notes outline key and motivating ideas for students and provide background for teachers, as well as outlining how to crack some tough nuts for those who want to prove some of the conjectures arising from the exercises.

Reading *Geometry* or teaching from it, you are bound to find questions and answers not touched on in these notes or in Jacobs's *Teacher's Guide*. Harold Jacobs and I would be happy to read your comments and suggestions and to try and answer any questions you may have. We can be reached care of W. H. Freeman and Company by mail or electronically.

Peter Renz

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Chapter 1. An Introduction to Geometry.

To the Student. Your study of geometry builds on intuitive understandings: for example, of how rulers are used and of how braces can steady a framework. Geometry gives you tools for systematically solving problems such as those of the Surfer and Spotter. This chapter shows how problem-solving led to the development of geometry and how geometry, in turn, helps solve problems. Here we first meet the geometric terms and ideas that are used throughout the text.

The final lesson, We Can't Go on Like This, shows the need for a more systematic approach. In geometry examples let you see what is going on — what works and what doesn't. But you need tools to help you discover things, to avoid fooling yourself, and to back up your insights. These you will master as you work through this book. Revisit the Surfer, the Spotter, and other problems. You will find simpler approaches and new understandings as you move ahead.

Lesson 1. Lines in Designing a City. The terms *point, line, line segment*, and *plane* are informally introduced, followed by definitions of several terms relating these basic geometric concepts: *collinear* and *noncollinear, coplanar*, and *concurrent.*

Interesting configurations of lines are included in the exercises. Students are asked to look at figures, identify and count their components, and draw other figures that lead to results that are both puzzling and surprising. These teach the basics and will suggest a world of possibilities to the student whose curiosity is aroused.

Lesson 2. Angles in Measuring the Earth. The terms *ray* and *angle* are informally introduced and the use of a protractor to measure angles is reviewed.

The exercises ask the students to make visual judgments followed by practice in making accurate measurements to verify them.

Lesson 3. Polygons and Polyhedra: Pyramid Architecture. The terms *polygon* and *polyhedron* are informally introduced, together with the names of the most commonly encountered polygons. The concepts of *perimeter* and *area* are reviewed.

The exercises range from questions about pentominoes to building a model of the Great Pyramid to a remarkable dissection problem (for the latter see also under Set III of Lesson 1, Chapter 7).

Lesson 4. Constructions: Telling Time with Shadows. The idea of a construction is introduced and the methods for carrying out two basic constructions, bisecting a line segment and bisecting an angle, are introduced without proof.

In the exercises, constructions are applied to finding north with shadows, determining the hour spokes of

a simple sundial, and discovering something surprising about bisecting the angles of a rectangle.

Lesson 5. We Can't Go on Like This. The concluding lesson of this chapter emphasizes that our thus-far informal approach to geometry has led to interesting as well as practical results but is ultimately unsatisfactory because we have no convincing basis for knowing that the conclusions we have been drawing are necessarily correct.

The exercises show the need for critical thinking and careful analysis. They are also provocative: Why is it easy to trisect a line segment, yet impossible to trisect an angle with only straightedge and compass? Why do several consecutive figures showing regions formed in circles seem to establish a simple pattern, after which the pattern fails?

Chapter 2. Deductive Reasoning.

To the Student. Chapter 1 showed geometry being used in many ways over the ages but concluded "We can't go on like this." Then how to proceed? The answer is: "Deductively and carefully." Euclid showed the way around 300 BC, and there have been many additions and improvements since. He gave us geometry as a *deductive system*, with clear assumptions (the *postulates*), definitions, and methods for proving what is true, our *theorems*.

You already know how to make deductions. If you look out the window and see rain, you deduce that there are clouds in the sky, even if you can't see the sky from your window. Deductive reasoning is the stuff detectives use; it can help you solve puzzles and solve problems. We end this chapter with some famous theorems of geometry, some of which you probably know already. As we work through the course, you will see why these must be true and how they can be proved.

Lesson 1. Conditional Statements. These are the basic tools of deduction. Using Agatha Christie's detective Hercule Poirot, we introduce such statements and the ways they are represented symbolically and by Euler diagrams. The examples and exercises give students practice in interpreting these forms and moving between them.

Lesson 2. Definitions. In formal arguments, we often need to replace a reference to some property or thing with the definition of that property or thing, so this chapter covers definitions, the converses of conditionals, and logical equivalence. The symbolic forms $a \rightarrow b$, $a \leftarrow b$, and $a \leftrightarrow a$, as well as the abbreviation "a iff b" are covered. The examples and exercises give practice in recognizing and shifting between these various representations.

Lesson 3. Direct Proof. This workhorse of deduction is introduced by a Rube Goldberg cartoon. The symbolic and Euler diagram interpretations reinforce the idea behind this valid form of inference. Exercises show that direct proof is an effective tool for critical thinking and for solving puzzles and problems.

Lesson 4. Indirect Proof. If assuming the opposite (or logical negation) of your desired conclusion leads to a contradiction, then your desired conclusion must be true. Examples given of this basic method of proof range from concluding that an ammonia molecule cannot be planar, to analyzing possible poker hands, to coin-weighing problems.

Lesson 5. A Deductive System. To avoid circularity or infinite regress, some terms must be taken as basic and left undefined; certain observations must be taken as postulates. With *postulates*, *definitions*, and means for *direct* and *indirect* proof to derive our *theorems*, we have the basis for the *deductive system* we use in the study of geometry.

This lesson includes our first two postulates. Postulate 1: Two points determine a line. Postulate 2: Three noncollinear points determine a plane. Note that "two points" should be understood as meaning the same as the alternative "two distinct points." Either form is acceptable, but the former is simpler and agrees more closely with common usage.

Lesson 6. Some Famous Theorems of Geometry. An opening "B.C." cartoon makes the point that many students will already know results such as the $a^2 + b^2 = c^2$ of the Pythagorean Theorem and the fact that the sum of the angles in a triangle is 180°. Jacobs takes advantage of this head start and brings other students up to speed by introducing these results along with Circumference $= 2\pi r$ and Area $= \pi r^2$ for the circle. He gives plausibility arguments for the Pythagorean Theorem and the Angle Sum Formula for triangles and discusses the results for the circle, leaving proofs for later.

Using these results, students can do applied problems that provide practice with algebraic methods, as the exercises show.

Looking Ahead. We prove the Angle Sum Formula for triangles using the Parallel Postulate in Chapter 6. We offer two approaches to the Pythagorean Theorem: one via areas (the formal version of the plausibility argument given in this chapter) and the other by similar triangles (Chapter 11, Lesson 2). The later approach uses the Angle Sum Theorem. For the circumference and area of the circle we use a limiting argument, like the method of exhaustion used by Archimedes (Chapter 14 of *Geometry* and in these notes).

Chapter 3. Lines and Angles.

To the Student. Intersecting pairs of lines form angles, and pairs of points define segments. We measure angles in degrees and intervals by their lengths, using various units. When you compare equal segments using a ruler, you could as well have copied the first segment onto the edge of the ruler and moved that copy over the other segment, matching it exactly. This matching is geometric equality, called *congruence.* A protractor lets you compare angles in the same way.

We begin with the algebraic tools used in geometry. Next come the Ruler and Protractor Postulates and the ideas about measuring distances and angles sketched above. We will cover complementary and supplementary angles, linear pairs, vertical angles, perpendicular and parallel lines, among other ideas.

Lesson 1. Number Operations and Equality. This covers the transitive property of equality and "common notions of Euclid" such as "If equals be added to equals, the results will be equal." These are part of algebra, and students will have seen them, but possibly in different forms. Here they are, with examples of their geometric uses. This lesson is an anchor for the *algebra strand* in this book. An algebraic puzzle from Lewis Carroll wraps up the lesson.

Lesson 2. The Ruler and Distance. Numbers and algebra do much of the work in our geometry. To move ahead we need the Ruler Postulate and the betweenness of points properties of this lesson. We have the theorem: If B lies between A and C on line l, then $AB + BC = AC$. The converse of this theorem follows from the triangle inequality: $AB + BC > AC$, unless B lies in the interval from A to B. We will prove this Chapter 5, on inequalities.

Lesson 3. The Protractor and Angle Measure. The Protractor Postulate does for angles what the Ruler Postulate did for the lengths of segments. There is no natural scale for lengths, but a half-rotation and a right angle are natural units for angles. When Euclid stated his Fourth Postulate, "All right angles are equal," he was stating that angles that have the same measures in terms of halves or quarters of a circle are geometrically the same (congruent). **Set III** introduces the Tantalus problem. The key to the Tantalus problem lies in the figure for the Angles Problem on page 515.

Lesson 4. Bisection. Rulers and protractors allow us to divide segments or angles into any desired proportions. Such division can be done using algebra, with a calculator or computer software to carry out the details. Geometrical bisection is based on congruence, as we will see in Chapter 4. It also brings in ideas of reflection and symmetry that we will revisit in Chapter 5. A fold, not a ruler, gives you the quickest way to divide a piece of paper in half. Applications include origami, molecular structures, carpentry, and the shapes of flowers. This lesson and its exercises revisit earlier ideas, including measurement, the use of coordinates, and constructions.

Lesson 5. Complementary and Supplementary Angles. These ideas are *algebraic.* Angles A and B are *complementary* if $A + B = 90°$, they are *supplementary* if $A + B = 180°$. Angles supplementary (or complementary) to the same angle are equal. The exercises give practice in algebra, logical reasoning, and seeing geometric configurations.

Lesson 6. Linear Pairs and Vertical Angles. Linear pairs are a natural *geometric* setting for applying the ideas of supplementary angles. The Protractor Postulate lets us prove that vertical angles are equal using algebra. Similar reasoning is used in the exercises to work out angles in folding chairs and in garden shears. In **Set III**, students figure out a proof from a Chinese geometry book.

Lesson 7. Perpendicular and Parallel Lines. Here we encounter two key concepts. If one of the four angles formed by intersecting lines is a right angle, then the lines are *perpendicular* and all four angles are right angles. *Parallel lines* lines lie in the same plane but do not intersect. Already, students should have the idea that *parallel lines have the same direction.* Indeed, this is what the Parallel Postulate of Chapter 6 will tell us.

Chapter 4. Congruent Triangles.

To the Student. Congruent triangles have the same size and shape. We begin our study of shapes with them. You will see how their shapes are determined by lengths of sides and sizes of angles. Shapes can also be described using coordinates, as you will see in the first lesson of this chapter. The distance and angle approach of congruence and the coordinate approach work together. Surveyors work out coordinates using distances and angles, and you will see that the Pythagorean Theorem lets you find distances from coordinates.

Key concepts in this chapter are: coordinates and the distance formula (use of the Pythagorean Theorem), and congruence and the ways to prove that triangles are congruent. Key techniques are: using coordinates to describe shapes and find distances, using and marking diagrams to show geometric relationships, and showing that triangles are congruent so that various segments and angles must be equal.

Lesson 1. Coordinates and Distances. Maps motivate the idea of a rectangular coordinate system and its coordinate grid. The Ruler Postulate gives us the numbers on the axes. The Pythagorean Theorem (Chapter 2, Lesson 6) gives us the *distance formula.* Examples in the exercises include the Geoboard, map reading, optics, plots of characteristics of birds and bats, architecture, and computer displays. The vector picture is introduced using water velocities.

Lesson 2. Polygons and Congruence. Congruence is the key idea here. Trace a polygon, if the tracing can be set down on another polygon so the two coincide then they are *congruent.* The lengths of *corresponding* segments and the sizes of *corresponding* angles must be equal. For triangles, the equality of corresponding parts *defines* congruence. Later, we will relate congruence to isometries. Examples used in the exercises include: a connect-the-dot figure, a patent drawing, an "art gallery" problem, and a *fractal curve.* There are many discovery exercises, and there is much practice with coordinates.

Lesson 3. ASA and SAS Congruence. A dramatic application of the ASA method from World War II opens the lesson. The lesson shows why the ASA and SAS Congruence Postulates *should* be true. The exercises show how useful these postulates are. Key terms such as *included angles* and *opposite sides* are defined. An exercise anticipates the names of types of triangles that will be formally given in Lesson 5. Applications include: binocular vision, measuring the Earth, and surveying.

Lesson 4. Congruence Proofs. A cartoon introduces a use of the ASA Postulate. A paragraph proof leads students to a second two-column proof. A discussion follows of the various ways proofs can be written. Applications are given to roof construction and estimating distances. There is practice bringing coordinate and congruence ideas together. There are exercises involving overlying triangles by folding paper that bring home the physical meaning of congruence.

Lesson 5. Isosceles and Equilateral Triangles. *If two sides of a triangle are equal, the angles opposite them are equal.* Jacobs presents Pappus' beautiful proof of this result. He avoids "base angles" to let students think more easily of these triangles in various orientations. This lesson has a rich harvest of theorems and corollaries, and the exercises offer practice with proofs.

An application to the geometry of the hydrogen sulfide molecule is worth mention. This atom has a bond angle of 92.8°. The Pythagorean Theorem does not apply directly, but as an approximation it is good to within 2%. An Escher tiling is the **Set III** puzzler, where a mirror-image piece provides a final twist.

Lesson 6. SSS Congruence. Seeing that triangular braces make frameworks rigid motivates introduction of the powerful SSS congruence theorem. Jacobs gives Euclid's proof.

Exercises include showing *why* the bisection constructions from Chapter 1 work.

A cautionary exercise shows that two triangles can have five pairs of parts equal and not be congruent. The geometry of *linkages* and of the *plumb level* is examined in the exercises. *Vectors* appear as *forces* that are *in balance* if when arranged head to tail they form a triangle. Identifying types of triangles given in coordinate form gives practice with the use of coordinates and the distance formula.

Lesson 7. Constructions. Methods for copying segments, angles, and triangles complete our list of basic straightedge and compass constructions. Walk your compass down a line marking equal segments and you have the start of a ruler. Add in marks for halves, quarters, and so on by bisections, and you can pin down the coordinates of points as closely as you wish. Such constructions are background for the rulers, protractors, and coordinate systems of metric geometry — and they show how distances and shapes are determined by lengths and angles.

The exercises give a striking example of a construction discovered in 1743 in Sweden for determining where the frets go on a guitar, still a tricky problem. In making this construction, we use a ruler to shorten the work of making equal divisions on a base line. Other examples in the exercises include girders, jacks, and the reflection of light.

Chapter 5. Inequalities.

To the Student. Geometric inequalities spring from this fact: *The whole is greater than the part.* For example, we show that the angle opposite the smaller of two sides in a triangle is *congruent* to a *part* of the angle opposite the larger side. This gives a theorem: *If two sides are unequal in a triangle, the angles opposite to them are unequal in the same order.* Congruence, the whole-part relationship, and the algebraic properties that begin the chapter will be important tools throughout it. When direct comparison of A and Z is difficult, a convenient intermediate quantity, say X, can often be found so that you can show $A < X$ and $X < Z$. Then, you can conclude $A < Z$ using the transitive property of inequality. Watch for this technique.

The changes in appearance as a configuration shifts or is seen from different points of view are a theme in this chapter, starting with a description of experiments on an infant's sense relative distance and ending with Galileo's observations of the planet Venus. The Triangle Inequality lets us work out the possibilities for the distances Galileo wondered about. These are consistent with the range of apparent sizes for Venus in its various phases. Art and architecture are concerned with the effects of changes in point of view and many examples in this chapter are drawn from these fields. In addition, you will see applications to planning kitchens and designing mechanical devices.

Lesson 1. Properties of Inequality. This lesson opens discussing experiments that showed that depth perception is inborn. This leads to the algebraic properties of inequality, part of the algebra strand of this

book. The sum of positive numbers is greater than any of its summands. For Euclid this was again: *the whole is greater than the part.* The phrasing suggests a key idea of this chapter. Transitivity plays an equal role, as well. The other algebraic properties of inequalities are needed for working out the consequences of various geometric conditions.

Visual effects and illusions are a theme in this chapter. Students draw figures, measure, and discover relationships that will be proven later. Applications include art and architecture.

Lesson 2. The Exterior Angle Theorem. A hovering U. F. O. is sighted from two points on the ground giving a triangle whose geometry shifts as the U. F. O. moves. We see one relationship that seems to hold in this triangle whatever shape it has: *The exterior angle is larger than either remote interior angle.* Jacobs gives Euclid's proof using congruence and the relationship of the whole to a part.

Applications in the exercises include scale drawing, measurement, and mechanical devices. In the exercises, students extend the argument given in the text to deal with the other remote angle, and they see an alternative proof of the Exterior Angle Theorem devised by Proclus.

Lesson 3. Triangle Side and Angle Inequalities. The opening painting shows how a change in point of view can dramatically alter appearances. As in Lesson 2, we see that certain geometric relationships *must* hold no matter how the scene is viewed.

If a triangle has unequal angles the opposite sides will be unequal in the same order. This gives the qualitative relationship between angles of a triangle and their opposite sides. The quantitative relationship is given by the Law of Sines, which appears in the lessons on trigonometry in Chapter 10, The Right Triangle.

There is opportunity for a good deal of critical thinking in the exercises. **Set II** begins with a discovery exercise leading to an alternative proof of the main result of this lesson. The last exercise in Set II revisits the Surfer problem from the book's Prelude.

Lesson 4. The Triangle Inequality Theorem. The final lesson of this chapter, like the initial lesson is concerned with the perception of distance. This time we begin with Galileo's speculations on the distance to the planet Venus. This lesson opens with five pictures of the planet Venus at various phases all shown at the same scale. The smallest disk, the full phase, is about a sixth the diameter of the new phase, the largest of these images. The geometry of the Venus, Sun, Earth system gives a triangle and the variations of that triangle motivate our discussion of the Triangle Inequality: *The sum of two sides of a triangle exceeds the length of the third.*

Exercises are based on examples from planetary motion, optics, and kitchen design, with the Surfer of the Prelude appearing as well.

In Euclid's time people objected that the Triangle Inequality was obvious and did not require proof. But the interest in this theorem lies not in the fact *that* it is true, but in *why* it is true — and in the consequences we can derive from it. Using it and some basic distances, we see that Venus at its most distant point (in its full phase) is over six times the distance it was from Earth at its closest approach (in its new phase). This suggests why Venus in its full phase has an apparent size about a sixth of that of its full phase, as our pictures show.

The working out of the sizes of the orbits of the planets is a difficult technical matter. The underlying principle that we see here is that *all of the measurements and observations must form a consistent picture.* The variations in the apparent diameter of Venus in its various phases fit with the orbital sizes we give. Apparent diameters are measured in degrees or radians and the rough inverse proportionality to distance observed here should be expected. The reason is that, for small angles, the angle measure, the sine, and the tangent are all close to being proportional.

Lines Are Determined by Distances. A corollary of the Triangle Inequality Theorem is that for three noncollinear points AB + BC > AC, so B can not satisfy the metric condition for being between A and C. Consequently, if AB + BC = AC, then B must lie on the same line determined by A and C, and B must be *between* A and C on that line. If we know the distances between all pairs of points, we can tell which points are collinear and which points lie between any given pair of points. Euclidean geometry can be developed using ideas of distance alone, although this approach is neither as easy nor as intuitive as ours. See, for example, Leonard M. Blumenthal's book *A Modern View of Geometry*, W. H. Freeman and Company, San Francisco, 1961.

About Estimation. Inequalities are often used in estimation. For example, Archimedes trapped the circumference of a circle of diameter $d = 1$ between the perimeters of inscribed and circumscribed regular polygons of $6 \times 2^4 = 96$ sides. Using these perimeters, he showed that the circumference of this circle, $\pi d = \pi 1 = \pi$, lay between $3\frac{10}{71}$ and $3\frac{1}{7}$. In short, he proved that $3\frac{10}{71} < \pi < 3\frac{1}{7}$. This was a considerable achievement for Archimedes, although a student with a scientific calculator can whip out the equivalent result in decimal form in a few minutes. That's progress. We also have progress in estimation of π, which is now known to an accuracy of billions of decimal places, although not by finding the perimeters of regular polygons. (See Supplemental notes for Chapter 14 for Archimedes' calculation and see "to calculate better" in the supplemental material for Chapter 11 for high-precision π calculations.)

Precision calculations come with error estimates that are worked out using basic properties of inequalities.

For this reason, the study of inequalities is not a sideline, but is a central concern both for mathematics and for its applications.

Chapter 6. Parallel Lines.

To the Student. Lines are *parallel* if they lie in the same plane and don't intersect. Parallel lines should have the same direction. In particular, *through a point* P *not on a given line* ℓ *there should be exactly one line* ℓ' *parallel to the line* ℓ. Why? Because through the point P there should be exactly one line ℓ' having the *same* direction as line ℓ. The Parallel Postulate tells us there *is* exactly one such line ℓ'. With the Parallel Postulate we can characterize parallel lines by the angles formed by lines cutting across them, *transversal* lines. We will also prove the Angle Sum Formula for triangles and establish the Angle Angle Side (AAS) and Hypotenuse Leg (HL) congruence theorems.

Lesson 1. Line Symmetry. Paper folds, linkages, and symmetrical art and diagrams motivate the study of line symmetry in this lesson. Two points A and B are symmetric with respect to the line ℓ iff ℓ is the perpendicular bisector of the segment AB. The construction of perpendicular bisectors and the proof of the correctness of this procedure is background for the construction of parallels in the next lesson. It also sets the stage for the contrast made in Lesson 3 between the uniqueness of perpendiculars, which we can already prove, and the uniqueness of parallels, which can only be proved by introducing the Parallel Postulate. Line symmetry is treated as a property of figures. In Lesson 3 of Chapter 8, we will revisit line symmetry in terms of reflections and transformations.

The **Set III** exercise asks the students discover some properties of an amazing set of points in the plane. See the discussion given with Jacobs's answers.

Lesson 2. Proving Lines Parallel. Parallel lines were defined earlier in Lesson 7 of Chapter 3. Here we see that the Exterior Angle Theorem allows us to conclude that two lines in a plane that make equal corresponding angles with a transversal cannot intersect. This gives us Theorem 17: *Equal corresponding angles mean that lines are parallel.* This statement is given in a simple and general form. Each application of this result requires students to think through the geometry and recognize the corresponding angles so that the basic idea and its uses are gone over repeatedly. The same is true for the corollaries given in this lesson. The exercises drive home these ideas. The converse of Theorem 17 is true, but its proof comes in Lesson 4 after the Parallel Postulate has been given in Lesson 3.

Lesson 3. The Parallel Postulate. Given a line ℓ and a point P not on ℓ we know how to construct a line n through P parallel to ℓ. We also can construct an m which is perpendicular to ℓ and passes through P. For perpendiculars, we can prove that there is exactly one such line m. This lesson asks the question: Can we prove that there is only one such line n, when n line is to be parallel to ℓ rather than perpendicular to ℓ? The answer is: Not without a further assumption. This is further explored in Chapter 16, Non-Euclidean Geometries, but the assumption we need, the Parallel Postulate, is introduced here.

Set I. Exercises 11–18, Exactly One, focus on the meanings of "at least," "no more than," and "exactly." Gateway (37–41) raises questions about what we know about figures and what is yet to be proven. Windshield wipers, gateways and paper folding are among the examples used in the exercises.

About The Parallel Postulate. It took more than two thousand years for geometers to discover that the Parallel Postulate, or equivalent statements, could not be proven from the other axioms of Euclid's system. The discovery of non-Euclidean geometries shook the faith people had in fixed systems of knowledge. The words of Morris Kline at the beginning of his chapter on non-Euclidean Geometries in *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, gives a sense of this. Kline wrote:

"Amidst all the complex and technical creations of the nineteenth century the most profound one, non-Euclidean geometry, was technically the simplest. This creation gave rise to important new branches of mathematics but its most significant implication is that it obliged mathematicians to revise radically their understanding of the nature of mathematics and its relation to the physical world."

Where does this leave us? For one view, read the conclusion to Jan Gullberg's section on non-Euclidean geometries in *Mathematics: From the Birth of Numbers*: "... Euclidean geometry is still the basis for most practical applications of geometry — it has taken human beings to the Moon and beyond."

Lesson 4. Parallel Lines and Angles. The naturalist Jean Henri Fabre's analysis of the geometry of a spider's web leads in to this lesson on the converses of Theorem 17 and its corollaries. Now that we have the Parallel Postulate these converses can be proven. This lesson provides a natural opportunity to go over the distinction between a statement and its converse, both in the text and exercises.

The exercises include: proving that parallel lines are everywhere equidistant, an anticipation of the proof of the Angle Sum Theorem for triangles, and an exploration of the methods Kepler used in working out the path of Mars.

Lesson 5. The Angles of a Triangle. Over distances that are short compared to the earth's radius, the sum of the angles in a survey triangle will be 180°. This is used as a check for measurement errors. This

idea opens our lesson. We prove the Angle Sum Theorem for triangles — a result announced in Lesson 6 of Chapter 2. We also prove Theorem 21: *An exterior angle of a triangle is equal to the sum of the remote interior angles.* This is a strengthening of Theorem 12, the Exterior Angle Theorem found in Lesson 2 of Chapter 5. Useful corollaries include: *If two angles of a triangle are equal to two angles of another triangle, then angles of the third pair are also equal* and *The acute angles of a right triangle are complementary.*

The exercises include proofs of the corollaries in the lesson and an explanation of how the latitude of a town in the northern hemisphere can be determined by sighting the North Star.

Lesson 6. AAS and HL Congruence. A fractal construction using triangles introduces two new methods for proving that two triangles are congruent: the Angle Angle Side method (AAS, Theorem 22) and the Hypotenuse Leg method for right triangles (HL, Theorem 23). The proofs are immediate consequences of the Angle Sum Theorem of Lesson 5. This lesson gives an opportunity to review the methods of showing triangles congruent and to consider the questions of how many parts of two triangles can be equal without the triangles being congruent. This problem was looked at in the exercises for Lesson 6, SSS Congruence, of Chapter 4.

Set I. Triangle Problem (18–22) shows that a triangle is isosceles if its altitudes are equal. The **Set II** exercises begin by establishing the properties of mirror images, and **Set III** unveils the deceptions of the fallacious proof that *all* triangles are isosceles.

Chapter 7. Quadrilaterals.

To the Student. This page is rectangular, so are the doors you walk through and the floors you walk on. Rectangles, squares, parallelograms, and other quadrilaterals are found everywhere. In this chapter you will see that what you know about triangles and parallel lines will let you understand the geometry of quadrilaterals. Squares, rhombuses, parallelograms, and other quadrilaterals are used to design buildings and machines and in art. The chapter is organized around the types of quadrilaterals and their properties.

The dozen theorems in the body of this chapter catalog various sorts of quadrilaterals and the properties they *must* have. For your overview of this chapter, run through the types of quadrilaterals, drawing pictures of each type and reviewing its properties. The final lesson of the chapter gives you a new result about triangles that can be proven using what you have just learned. This result is the Midsegment Theorem, which tells you that the segment joining the midpoints of two sides of a triangle is parallel to and half the length of the third side.

Lesson 1. Quadrilaterals. The chapter opening is an unusual shape for a quadrilateral. Jacobs chose this to sharpen the student's appreciation of what is common and what is unusual in quadrilaterals.

In Chapter 1, Lesson 3, Polygons and Polyhedra, we defined quadrilaterals. We return to quadrilaterals with the machinery needed to develop their properties. Learning to see and identify the various sorts of quadrilaterals and to instinctively know their properties is the key to this chapter.

First, we distinguish between convex polygons and concave ones. Second, we observe that the Angle Sum Theorem for triangles lets us conclude that the angle sum for quadrilaterals in 360°. Hence, an equiangular quadrilateral has four right angles. This gives us the rectangle, the most useful quadrilateral.

Notice the discovery exercises in **Set I**: Rectangles (16–18), Rhombuses (19 and 20), Squares (20 and 21), and Parallelograms (23–25). Here students begin to learn the names of various special quadrilaterals, to recognize them, and to see what properties they have. See **Set II**, Gemstone Patterns (36–49) and Linkage Problems (52–55), which are also exploratory. The amazing dissection in the **Set III** exercise was seen before in Lesson 3 of Chapter 1. The dimensions and angles in this dissection can be worked out starting from the fact that the area of both figures must be the same. This might be suitable for motivated students: perhaps working as a team, using technology such as *Geometer's Sketchpad*™, or looking material up in the library.

Lesson 2. Parallelograms and Point Symmetry. We define *parallelograms* in this lesson and develop their properties, including their *point symmetry.* In the next lesson we will look at converse results, seeing which sets of these properties imply that a quadrilateral is a parallelogram. We return to symmetry from the point of view of transformations in the next chapter.

Set I. Optical Illusions (4–10) involving parallelograms are seen here. The point symmetry of parallelograms is proven in Exercises 16–23. **Set II** exercises begin with the parallelogram law for the addition of vectors. The **Set III** problem asks students to determine the symmetries of playing cards.

Lesson 3. More on Parallelograms. Here we see that quadrilaterals whose opposite angles are equal are parallelograms and so on. The lesson opens with a powerful visual illusion based on a room in which seeming rectangles turn out to be trapezoids. Note the way algebra and geometry work together in this lesson and in its exercises.

Set II. Rope Trick (36–43) gives a beautiful African method for laying out the rectangular base for a house. Flexible Grid (47–55) is intriguing. A surprising construction using equilateral triangles is the basis for the **Set III** exercise. See Alternating Triangles on page 565

of *Geometry*, and see these notes for hints on how to proceed.

Lesson 4. Rectangles, Rhombuses, and Squares. These are the figures you encounter most frequently in construction. We know a great deal about quadrilaterals, and the student sees how this knowledge can be put to use. See especially Exercises 26–39, Checking a Wall, which walk the student through the practical aspects of measuring the sides and diagonals of a wall to see whether it is rectangular.

Lesson 5. Trapezoids. These are less constrained: exactly one pair of sides of a trapezoid is parallel. Structure remains, particularly for *isosceles trapezoids*, whose nonparallel sides are of equal length. Trapezoids show up Stepladders (34–42) and in geometric Trademark (9–13). **Set III** features a surprising solution to a classic problem. This solution was found by high school student Andrew Miller, while he was was working from the First Edition of this book.

Lesson 6. The Midsegment Theorem. The segment connecting the midpoints of two sides of a triangle is called a *midsegment,* and it is parallel to the third side and half its length. This is a new fact about triangles that follows from methods developed in this chapter. The Midsegment Theorem anticipates some ideas related to similar triangles that will be developed in Chapter 10.

Set II. The three-dimensional models in Tetrahedra (37–41) are worth special attention. See the discussion of these in the Answer section. In Exercises 27–36, Midpoint Coordinates, students discover the relationship between the geometric and coordinate views of midpoints. Base Average (49–54) begins by asking the student what *seems* to be true. Then we see how the natural conjecture can be proved. **Set III** looks at a geometric progression of similar triangles and asks the student to estimate the sum of the lengths of all their sides. See the Answers for comments and links to other material in the book.

Chapter 8. Transformations.

To the Student. Trace a shape onto a transparent sheet and see whether you can move the sheet so that the tracing matches a second shape. If you can, the shapes are *congruent.* The motion of the sheet gives a *transformation* of the plane. If the motion is a sliding of the sheet without turning, the transformation is a *translation.* If you could pin the sheet at one point and turn it to make the match, the transformation is a *rotation* about the pinned point, its *center.*

To match a left handprint with a right one, you must flip the sheet over. You would reverse it left-to-right about the symmetry line between the two handprints. The handprints are *congruent*, but they look different because each is a *mirror image* of the other. *Flipping* the sheet *has the same effect as reflecting* one handprint through the line of symmetry between the two prints.

To show that a left footprint is congruent to the next right footprint, trace the left footprint, slide the sheet forward a pace, then flip it over. A flip is always needed when matching shapes that are mirror images. This slide and reflection gives a *glide reflection.*

Two shapes in the plane are *congruent* precisely if one can be carried onto the other by one of the transformations described above. These are the transformations that preserve size and shape in the plane, the *isometries* of the plane. This chapter shows you what these transformations are, how they are defined by reflections, and how they relate to symmetry.

On Transformations. The material in this chapter has only been studied systematically for about 150 years. The ideas are basic and can be grasped intuitively, but the details could fill a course by themselves. Jacobs's approach in the text is intuitive. This note sketches the origins of the transformational approach, how it relates to coordinates, and how one would prove the facts seen in this chapter.

Before geometry became a science, people recognized that some pairs of different things had the same or similar shapes. If two shapes are the same, then we could move one on top of the other so that they would coincide. Euclid used this notion for figures; it is called *superposition* in older texts. Postulates give us conditions for triangles to be congruent, meaning they are superposible. A full appreciation superposition came only with the systematic use of transformations in geometry. This was launched 1872 by Felix Klein in a lecture referred to as "Kline's *Erlanger Programm.*"

For Euclidean geometry, the key transformations are the *isometries*, the transformations of the plane onto itself that preserve Euclidean distances. A physical model these isometries is given by tracing figures onto sheets to see whether you can make one figure match another.

We can imagine our transparent sheet extends as far as we wish, so the *isometry* that shows that two figures are congruent can be applied to other figures traced onto the same sheet. Before there were flexible transparent sheets, geometers had already classified the isometries of the plane and shown how these transformations can be produced by single reflections or combinations of two or three reflections. Knowing what the isometries are allows us to make precise the Euclidean idea of superposition and to define *congruence* for any pair of figures.

In this chapter the properties of isometries are developed geometrically. In the exercises, students discover how transformations affect the coordinates of the vertices of figures. More elaborate coordinate methods are used for applications such as computer graphics or map making, but the link between transformations and coordinates is already clear from these exercises. (See

Matrices and Computer Graphics in the supplemental notes for Chapter 12 here.)

Jocobs lets students discover what transformations tell us about congruence and symmetry. He omitted proofs, but many facts about isometries are easy to establish. For example, if an isometry carries $\triangle ABC$ onto $\triangle A'B'C'$, then $\triangle ABC \cong \triangle A'B'C'$, by SSS, so $\angle ABC = \angle A'B'C'$, by CPCTE. This proves that isometries preserve angles *because* they preserve distances. For convenience, Jacobs made "preserving angles" part of his definition of *isometry*, but this property is a consequence of preserving distances.

The properties of reflections are also taken for granted here, but they are also easily proven. Intuition tells us that when A and B are reflected in line ℓ to give A′ and B′ we have equality of lengths: $AB = A'B'$. This follows from the definition of reflection and the congruence postulates. It is also intuitive that if two transformations both preserve distances, then their composition also preserves distances. Because of this, any composition of reflections is an isometry. Jacobs left this as an unstated assumption, but is easy to state and prove.

Isometries of the plane are either *reflections*, *translations*, *rotations*, or *glide reflections* — plus the unmentioned *identity transformation*. Each sort of isometry is characterized by certain properties. For instance, reflections leave just the points on their mirror line fixed and rotations leave only their centers fixed, except for rotations by multiples of 360°, which leave every point fixed. Translations by a positive distance move lines onto lines parallel to themselves and leave no point fixed.

Conversely, a 1:1 transformation of the plane onto itself that sends lines onto lines parallel to themselves and that leaves no point fixed must be a translation. Here is why: Pick a test point P, it moves to some other point P′. Look at the line PP′. The image of this line is parallel to the original line and contains P′, so this line is sent into itself. We say this line is *invariant*. The same argument applies to any other point Q, so the line QQ′ is also invariant. If these lines were to intersect, their point of intersection would have to be a fixed point. Because our transformation has no fixed points, $QQ' \parallel PP'$. Also, $PQ \parallel P'Q'$, because our transformation sends lines into lines parallel to themselves. So PQQ′P′ is a parallelogram, and it follows that an arbitrary point Q moves in the same direction and through the same distance as our test point P. This shows that our transformation is a translation.

A *dilation* about the *center* $(0,0)$ by *scale factor* λ sends point (x, y) to point $(\lambda x, \lambda y)$. I chose the center $(0,0)$ so that it would be easy to express this transformation in coordinate form. Dilations also send lines into lines that are parallel to themselves. Here I allow the possibility of a negative λ, so that reflections through the center of dilation are also considered dilations. I allow centers other than $(0,0)$; the geometry is the same, but the algebraic formulas are more complicated. A dilation with $\lambda = -1$ is the point symmetry about its center. This point symmetry is also a rotation by 180° about the same center. Dilations have unique fixed points, except when $\lambda = 1$ and every point stays fixed. If $\lambda = 1$, we have the *identity transformation*.

The composition of a translation and a dilation sends lines onto lines parallel to themselves. If $\lambda \neq 1$, this combination must be equivalent to a single dilation with the same λ, but with a new center. This shows that the composition of a translation and dilation with $\lambda \neq 1$ again gives us a dilation. This is a general result in the field of *transformational geometry*.

We could make *distance* the basis for our study of the geometry. This is done analytically when we look at the coordinate plane with the usual Pythagorean distance. A circle is the locus of points equidistant from its center, so distance alone lets us define circles. Every line is the locus of points equidistant from the end points of any segment that that line bisects, so lines can also be defined using only distance. Isometries preserve distance, and so they necessarily preserve lines and circles.

Furthermore, any isometry will send a pair of parallel lines, ℓ and m, into lines ℓ' and m' that must again be parallel. Using these properties of isometries we can see why an isometry must be of one of the types already described. This requires some care, but no new ideas are involved. I outline a possible approach following my lesson-by-lesson comments below.

A good general source on geometry and transformations is H. S. M. Coxeter's *Introduction to Geometry*, Second Edition, John Wiley & Sons, New York, 1961. You might also look at books having "transformational geometry" in their titles or descriptions.

Lesson 1. Transformations. Transformations and symmetries are essential tools in art, crafts, and technology. We begin with a comic strip that must be read both right-side-up and upside-down to be appreciated. This lesson defines *transformation*, and describes basic types of geometric transformations: translations, rotations, reflections, and dilations.

The exercises are concrete: drawing and observing, analyzing the operation of an escalator in terms of translations and rotations. We see transformations used in computer graphics, and we see how transformations and coordinates are related.

Lesson 2. Reflections. Points A and B are symmetric with respect to line ℓ if ℓ is the perpendicular bisector of the segment AB. This we saw at the start of Chapter 6. Here we define the *reflection* of P through the line ℓ to be the point P′ such that ℓ is the perpendicular bisector of PP′, if P is not on ℓ. If P is on ℓ, we define P′ = P. Reflections generate figures that are symmetric with respect to the mirror line.

In the chapter opener, two successive reflections in parallel mirrors gives an image moved in a direction perpendicular to the mirror planes by a distance twice the distance separating the mirrors. This double reflection is our definition of a *translation.* The *direction* of the translation is perpendicular to the mirrors, and the *magnitude* of the translation is the distance the image moves. Every point moves the same distance and in the same direction. See Triangle Reflections (44–47). After two parallel reflections a line returns to its original direction, so the translate of a lines is parallel to the original line. This is characteristic of translations, as noted above.

Rotations are defined as the compositions of reflections in intersecting lines. The point at the intersection of the lines of reflection remains fixed and is the *center* of the rotation. The *magnitude* of the rotation is the angle through which the image is rotated. This angle is twice the angle between the lines of reflection that define the rotation. (See Exercises 48–52.) A characteristic of rotations is that they have a single fixed point. The angle between a line ℓ and its image ℓ' under a rotation is always equal to the magnitude of the rotation.

Lesson 3. Isometries and Congruence. Two very different looking shapes turn out to be congruent in the chapter opener. In this lesson we give the general definition of *congruence* based on isometries. We see how shapes that are congruent can be made to coincide by transforming them by one, two, or three reflections. The three-reflection transformation introduced here is a *glide reflection.*

The exercises include: art, piano-moving, rowing, children's swings, and coordinate methods for transformation problems. The **Set III** exercise makes interesting use of the transparency of tracing paper.

Lesson 4. Transformations and Symmetry. A figure has a *symmetry* if it can be mapped onto itself by an isometry. We have seen line and point symmetries in Chapters 6 and 7. Now we return to these symmetries from point of view of transformations. We add *n-fold rotation symmetry* and *translation symmetry* to our list of symmetry types.

In the exercises we see how rotational symmetry shows that vertical angles are equal and how the fact that the angles opposite the equal sides of an isosceles triangle are equal also follows from symmetry. Water wheels, piano keyboards, orchards, and quantum wave functions are all seen to possess special sorts of symmetry. The beauty of symmetrical structures goes with their useful properties. Jacobs gives some sources relating to symmetry, and your students can find many others in libraries or on the Web.

Classification of the Types of Isometries. The idea is to systematically consider various properties that an isometry might have. We begin by looking at transforms that send some line ℓ into itself but do not leave any points fixed. We will see that these must be translations or glide reflections. We then work through all the other possibilities.

Case One: *Isometries that leave some line invariant.* Let ℓ be such an invariant line.
Subcase A: *isometries that leave no point fixed.* Putting a ruler along ℓ, we see that there must be some constant c such that points the points of ℓ are shifted from a position with coordinate x to one with coordinate $x + c$ along ℓ.

Next, establish Cartesian coordinates in the usual way using the ruler we have along ℓ and a ruler with the same scale on a perpendicular axis crossing ℓ at the point with coordinate 0. This point will be taken as the origin, O, of our coordinate system. Further analysis shows that this isometry must be either a translation along ℓ of magnitude c or a glide reflection, where the glide line is ℓ and the glide magnitude is c.

Subcase B: *Our isometry leaves a line ℓ invariant and fixes some point* P. Then our isometry must fix the point on ℓ closest point P. We make this point our origin and call it O. Again we look at what such an isometry can do by placing a ruler along ℓ with ruler's zero at our origin O. Either the coordinates of points on ℓ are unchanged by the isometry or they must be multiplied by -1. We introduce Cartesian coordinates with a second axis through O perpendicular to ℓ. This second axis is also invariant and the isometry leaves its zero point fixed. So the coordinates of transformed points on the second axis are also multiplied by 1 or by -1. Because isometries preserve distances and angles, this pattern holds for the coordinates of all points in the plane, and the transformation is either a reflection in one axis or in the other axis, a rotation by 180° about O, or the identity. All of these transformations are on our list.

These Subcases A and B exhaust all the possibilities for an isometry that leaves some line invariant.

Case One Conclusions: *Isometries that leave some line invariant must be either translations along that line, glide reflections along it, reflections in that line or in a line perpendicular to it, or rotations through* 180° *about some point on the invariant line*

Case Two: *Isometries that leave no lines invariant.*
Lemma. *Such isometries must leave some point fixed.* Why must this be so? Pick any point P. If P is fixed we are done, so we may suppose $P \neq P'$. If $P = P''$, then the midpoint of segment PP' would be fixed, and we would be done. Continuing, if necessary, we would look at the distinct points P, P', P''. We note that if they were collinear, then they would determine an invariant line, contrary to hypothesis. Hence, in this case, we may conclude $\angle PP'P'' < 180°$. Consequently, the perpendicular bisectors of segments PP' and $P'P''$

must meet at a point O interior to $\angle PP'P''$. Drawing the figure, we will show that O is a fixed point below.

We know that $\triangle OPP'$ is sent onto $\triangle O'P'P''$ by our isometry. But there would seem to be two possibilities for O', one the point O and the other the reflection of O in the line $P'P''$. However, the latter is impossible, because, if it were the case, then $\angle PP'P''$ and $\angle P'P''P'''$ would be equal alternate interior angles, and our isometry would send segment PP' onto the equal and parallel segment $P''P'''$. This would give us a parallelogram: $PP'P'''P''$. Let M be the midpoint of PP'. its successive images under our isometry are the midpoints of successive equal segments in the path: $PP'P''P'''$ From what we know about parallelograms and triangles, we can conclude that these midpoints are collinear, and it would follow that the line they determine is invariant, contrary to our supposition in Case Two. Hence, $O' = O$, and in Case Two there must always be some fixed point.

Case Two Concluded. We are looking at an isometry without invariant lines, and we have established that there must be a fixed point, which we take to be our origin O. We center a 360° protractor on O. Recall that our isometry must preserve angles and that we are looking at an isometry that leaves *no* lines invariant. Using these facts, you can show that this isometry must be a rotation about O by some angle that is not a multiple of 180°.

Case Two Conclusions: *An isometry that leaves no line invariant must leave some point fixed and it must be a rotation by an angle other than $n \times 180°$ about that point.*

Cases One and Two are exhaustive and they give transformations of the sorts we listed, so our list of isometries is complete.

There are other possible approaches to classifying isometries. We could have begun by looking at isometries with one or more fixed points. Whatever the approach, care is needed to make sure that all possibilities have been looked at. This sketch shows that methods needed are part of school geometry. The length of this sketch also suggests why a serious study of transformational geometry would be better left for a separate course.

Chapter 9. Area.

To the Student. An Olympic swimming pool is 50 meters long and 21 meters wide. Placing a meter-square grid on this rectangular pool, you can see that its area is $50 \times 21 = 1050$ square meters. For rectangles in general, Area = Base × Height, using consistent units. We make this fact our Postulate 9. To work out the areas of other shapes we use Postulate 8, which is a consistency principle. *Area Postulate: Every polygonal region has a number called its area such that: 1) congruent triangles have the same area, and 2) the area of a polygonal region is equal to the sum of the areas of its nonoverlapping parts.*

A rectangle with base b and height h can be split into two congruent right triangles each with the same base and height. Postulate 8 lets you conclude that the area each of these right triangles must be $\frac{1}{2}$Base × Height = $\frac{1}{2}bh$. This formula works for all triangles, once you understand of how the height, h, or *altitude* corresponding to the base, b, is measured. When you see an area formula, relate it to simpler formulas that you know. Linking algebra to geometry this way will let you work out areas whenever you need to.

Areas are practical, allowing you to figure out how much paint or wallpaper is needed for a room. They also are the key to proving essential geometric facts such as the Pythagorean Theorem in this chapter or the Side-Splitter Theorem in the chapter on similarity.

About Area. This chapter builds on three facts: *The area of a rectangle is the product of its length by its width. Congruent regions have the same area.* And the principle that *the area of a region should be the sum of the areas of its nonoverlapping parts* (consistency of areas). Given these, students can find the formulas for the areas of common polygonal shapes.

This material depends on linking the geometry of the diagrams to the algebra of the formulas. Each aspect reinforces the other and strengthens understanding.

This chapter draws on and strengthens the student's intuitive sense of geometric areas. Postulates 8 and 9 formalize these intuitions, allowing us prove the Pythagorean theorem using the area argument informally outlined in Lesson 6 of Chapter 1. Area is also used to establish the Side-Splitter Theorem in the next chapter.

Working in the Cartesian coordinate plane, it is possible to define an area function that satisfies Postulates 8 and 9. I outline this approach in a note at the end of my notes on this chapter's lessons. This approach starts from a deep understanding of coordinate geometry and works backward to justify the intuitive assumptions used in elementary geometry. It was developed by mathematicians who knew their Euclid and wanted a deeper understanding of area. For our purposes, the understanding given by Postulates 8 and 9 is sufficient and easier to grasp.

Appropriate Technology. The tools we have for calculating areas allow a wide range for the use of algebra. Many exercises involve passing back and forth between geometry and algebra. See, for instance, the Trapezoidal Rule (37–40), Lesson 4. These and other exercises can be worked using programmable calculators or

computers, although Jacobs has not made this a requirement. Many exercises invite use of mental calculation or of a scientific calculator. See Exercises 15–30 of Lesson 5, for instance. The text does not suggest a particular approach. Teachers can suggest the means they think appropriate or to leave this as a matter of student choice.

Lesson 1. Area. The areas of various states serve as reminders of our common ideas of area. We define a *polygonal region* to be a polygon and its interior. We will work out the areas of such regions using The Area Postulate. see above or *Geometry*. Part 2 of this postulate says that the areas of the pieces add up to give to give the area of the whole 9*consistency*). Part 1 is a special case of the more general principle that congruent regions have the same areas (again, *consistency*).

The exercises show how far the simple and intuitive Area Postulate can carry students. **Set I**. The early examples conceptual (1–6), then come those using algebraic notation (7–19). Contrast Exercises 1–15 with Exercises 16–24. **Set II**. Note Exercises 46–55, Comparing Parallelograms, which anticipate later results on areas. A dissection paradox from Jerry Slocum and Jack Botermans gives us our **Set III** exercise.

Lesson 2. Squares and Rectangles. The basic insight comes from laying out a square grid, illustrated by the regular pattern of western townships. This insight is the basis for the theory of area sketched in my note at the end of these lesson-by-lesson comments. This insight provides the key for our work on area, Postulate 9. *Area of a Rectangle: The area of a rectangle is the product of its base and altitude.* In symbols, $A = bh$. More commonly, "Area = Length × Width," but Jacobs uses "base $(= b)$" and "altitude $(= h)$" because these notations and ideas will carry through for the areas of triangles, parallelograms, and trapezoids.

Notice the importance of units in **Set I** Exercises 1–4 and 7–10. Notice also the use of algebra here. See Distributive Property (11–17). Notice the role of calculators and calculating in Two Squares (22–25). Jacobs shows us various units used by people in different fields. Students should learn to work with whatever units are required. There is a natural evolution of units, as older systems defined in easy practical units are replaced by systems defined more generally and abstractly. For instance, the foot and inch, related to the human frame, are now defined in metric or S.I. units. The international inch is 2.54 centimeters, *exactly*. The meter, intended to be (1/10 000 000)th of the distance from the North Pole to the Equator, is now defined as the distance light travels in a vacuum in (1/299 792 458)th of a second. Practical measurement may use rulers, meter sticks, or paces. Students must be at ease working with any of these.

Set III. Dividing into Squares, this problem of "Squaring the Square" was solved by a group of students at Trinity College, Cambridge. One of them, Willian T. Tutte, wrote a delightful account of their work for Martin Gardner's Mathematical Games column in Scientific American. This is reprinted in Gardner's *Second* Scientific American *Book of Mathematical Puzzles & Diversions*, 1961, with updates to be found in Gardner's *Fractal Music, Hypercards, and More*, 1992. A charming aspect of this work is that a crucial contribution was made by Mrs. Brooks, the mother of one of the four students working on this problem. She found an unexpected solution to a puzzle her son made out of a pattern for squaring a rectangle. This discovery changed the whole direction of their effort. An exposition of the methods used can be found in Sherman Stein's *Mathematics: The Man-made Universe*, available from Dover Publications in 2003. Students could use these sources for reports or projects.

Lesson 3. Triangles. The Rhind papyrus, which was written about 1650 BC by the Egyptian scribe Ahmose, gives instructions for computing the area of a triangle. Interpreted algebraically these amount to $\alpha\triangle ABC = \frac{1}{2}ba$. This is correct for a right triangle with legs of length a and b. A diagram shows that the required area is exactly half of that of a rectangle of base b and height or altitude a, so this formula follows from Postulates 8 and 9. Further diagrams show that $\alpha\triangle ABC = \frac{1}{2}bh$, where b is any *base* of $\triangle ABC$ and h is the *corresponding altitude*. The formula is part of this lesson, which includes learning to visualize this result using diagrams, express it in words and algebraically, and to use it to solve practical problems.

Set I. Geometric reasoning suffices to work out some of the exercises, for instance Math Puzzle (1–3). Algebra plays a role in others; see Isosceles Triangle (4–9). Enlargement (15–23) leads students to discover the scaling properties of perimeters and areas. We will return to scaling in Lesson 6 of Chapter 10 and in the similar solids lesson in Chapter 15.

Set II. Surfer Puzzle (29–34) returns us to the problem that opens the book. Now we see how areas can be used to show why every location on the island gives the same sum of distances for the Surfer. The area argument used here also establishes the constancy for the analogous sum for any convex, equilateral n-gon. In particular, it proves the constancy of such sums for a regular pentagon (Exercise 28, Lesson 1, of Chapter 6). For the equilateral triangle, the constancy can be proved using symmetry rather than area. You might challenge your students to find a proof for equilateral triangles that does not use areas.

Dividing The Cake (47–49) has a neat solution that students are asked to extend in Exercise 50. The Shingle Roof (51–54) shows a common practical use of geometry.

Set III asks students to explore Heron's Theorem for finding the area of a triangle using the lengths of its

sides. The algebraic proof of this formula is routine and gives no geometric insight. Jocobs omitted it, although it is found in earlier editions. Some students have access to CAS (Computer Algebra Systems) and for them setting up and checking Heron's formula could be instructive, especially seen in the light of working this out by hand, which is a project just within the capabilities of skilled and determined students. We better appreciate the value of aids such as CAS when we see how easily they deal with what is possible but difficult for us to do unaided.

Lesson 4. Parallelograms and Trapezoids. An example from D'Arcy Thompson's classic book *On Growth and Form*, shows how a mathematical transformation can change the outline of a fish of one species into that of another. This transformation turns rectangles into parallelograms. In this lesson we learn how to find the areas of parallelograms and trapezoids. Simple diagrams and Postulates 8 and 9 show us that for parallelograms αABCD $= hb$, where b is any base of the parallelogram and h is the corresponding height or altitude. For trapezoids, we must ust the lengths of the two parallel bases, here a and b, and the height, h, is the distance between the parallel lines that these bases lie on. The area of a trapezoid is then αABCD $= \frac{1}{2}h(a+b)$. As always, students need to understand these results visually (from the diagrams), in words, and algebraically.

Set I. From Above (1–5) is conceptual and visual. Tax Assessor Formula (6–16) returns students to the area formula of Lesson 5 in Chapter 1. In these exercises, students must see how to compute areas using the methods we have developed and compare these correct areas with the results of the Egyptian formula. This gives practice in recognizing areas that can be computed and in using formulas to compute these areas. There two themes run through the exercises of this lesson. See also the practical surveying problem of **Set II**, Inaccessible Field (30–36), and the problem of estimating the area under a curve in Trapezoidal Rule (37–42). This last group of exercises ties into ideas used in calculus. For students having access to programmable calculators or Computer Algebra Systems (CAS), the Trapezoidal Rule used here could be programmed to run automatically.

The Egyptian Assessor's Estimate. This formula is a puzzle, so giving out the following solution would spoil the fun. But it is interesting that there is a geometric and algebraic argument that shows that the assessor never loses income by using his formula. The assessor's formula comes down to estimating the area of a quadrilateral using $\frac{1}{4}(ab + bc + cd + da)$, where a, b, c, and d are the lengths of its sides, taken in cyclic order. Assume the quadrilateral is convex, since if it were concave we could turn it into a convex quadrilateral by popping one pair of sides out and this would increase its area. The sides a and b determine a triangle whose area is no larger than $\frac{1}{2}ab$. Likewise, the triangle determined by sides c and d has an area no larger than $\frac{1}{2}cd$. These triangles cover the quadrilateral, so Area of Quadrilateral $\leq \frac{1}{2}(ab + cd)$. A similar argument shows that Area of Quadrilateral $\leq \frac{1}{2}(bc + da)$. Hence $2 \times$ Area of quadrilateral $\leq \frac{1}{2}(ab+bc+cd+da)$. Dividing by 2 gives Area of Quadrilateral $\leq \frac{1}{4}(ab+bc+cd+da)$, so the Egyptian assessor's formula never underestimates the area. Moreover, this reasoning shows that his formula gives the "right" area only when the quadrilateral is a rectangle, and each of triangles we considered is a right triangle.

Lesson 5. The Pythagorean Theorem. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on its legs.* This is the most famous result in geometry. Its algebraic form, $a^2+b^2=c^2$, is familiar to many. The linking of algebra and geometry is what allows us to calculate distances and what makes this result so useful. If a triangle's sides satisfy $a^2 + b^2 = c^2$, it must be a right triangle, but this *converse* of the Pythagorean theorem needs a separate proof.

Set I. See Theorem 43 (4–14). The links between calculations and geometry are reinforced in Squares on the Sides (15–30). The proof using areas is intuitively appealing. This key property of right triangles has been studied down the ages and in many cultures.

Set II. See the ancient illustration used in Was Pythagoras Chinese? (33–40). Also see Garfield's Proof (41–47).

The Theory of Area. The Area Postulate, Postulate 8, is a weak form of facts about area that are proven in calculus and analysis courses. The approach in such courses is to look at the Euclidean plane as the coordinate plane with the distance between points as the square root of the sum of the squares of the differences of the coordinates. This distance is based on the Pythagorean Theorem, which is built into the assumptions of the theory of area outlined below.

The inspiration for this theory is simple and practical: get upper and lower estimates of the area by laying a square grid over the region. This is the most natural approach to estimating the area of an irregular region.

The unit interval gives us the linear unit and it also gives us the unit square, the basic unit of area. The area of a region is estimated by placing a grid over the whole plane and finding the area of the squares of the grid falling wholly within the region (a lower bound on the region's area) and the area of the squares of the grid needed to cover the entire region (an upper bound on the same area). If making the mesh finer and finer brings these upper and lower bounds as close as we please, then these bounds will converge to a common limit as the mesh size goes to zero. If so, then this common limit defines the region's area. This notion of

area is called *Jordan Content.* See, for instance, *Mathematical Analysis* by Tom M. Apostol, Addison-Wesley, Reading, Massachusetts, 1957.

It is not possible to assign an area to every conceivable region using this definition. The reason is that making the mesh finer and finer will not necessarily bring these upper and lower bounds for the area arbitrarily close together. Increasingly zigzagged boundaries of fractal regions can keep the upper and lower bounds apart. But all *polygonal* regions will have areas, as would all regions of finite extent bounded by smooth curves. These areas will satisfy our postulates.

We can extend these ideas by covering our region by a collection of squares of any sizes we wish to use, perhaps using an infinite number of such squares, and then adding up the areas of these squares. There is no need to avoid overlapping of the squares, if we simply want an upper bound for the area of the region. The set of upper bounds for the region's area obtained this way will be an interval of positive numbers, stretching to infinity, because we can increase the sizes of some of the squares in the covering and still get a covering. The lower end point of this interval is our best estimate of the area of the region, αR. If our region R sits within some rectangle T, consistency would require that $\alpha R + \alpha$(The points in T outside of R) = αT, where α gives the estimate of the area of any set gotten by looking at upper bounds. If this consistency principle holds, R is said to be *measurable*, and its area is αR. This notion of area satisfies our consistency conditions and it can be used to figure the areas of irregular plane sets such as fractals. The underlying ideas are simple, but technical difficulties make study of this notion of area, Lebesgue measure, a subject usually only treated in advanced courses. See, for instance, *A Primer of Lebesgue Integration*, H. S. Bear, Academic Press, San Diego, California, 1995.

Chapter 10. Similarity.

To the Student. A model car is the same shape as the original but at a smaller size. The original and the model are *similar* geometric shapes. You can transform the original shape so that it can be superimposed on the smaller model by first shrinking it to the size of the model. The shrunken car is then *congruent* to the model, and the shrunken car can be sent onto the model by an isometry. The end result of this shrinking followed by an isometry is a transformation from the original to the model which preserves angles between corresponding segments and reduces all distances by the same scale factor.

Model cars are made at scales such as 1/20th. At this scale, a car that was 15 feet long and 5 feet wide would have a model that was 15/20 = .75 feet long and 5/20 = .25 feet wide. In inches, the model would be .75 x 12 = 9 inches long and .25 x 12 = 3 inches wide, and all the other dimensions would be in proportion.

Scale models are not just toys, they are a big business that includes maps, which are schematic scale models of cities and countries. On the Web, you will find companies that make scale models for architects and for movie and television productions.

With a reduced scale model, you can hold something like the Empire State Building in your hands and examine it. With an enlarged model, you can see how a living cell works or how atoms are arranged in a crystal. The geometry of similar figures, maps, and scale models is explored in this chapter.

General Remarks about Similarity.

Scale Models and Similarity. Similarity is the basis for making scale models. You can't make a strictly correct scale model in spherical geometry, because as you scale down the lengths of the sides of a triangle, you change the sum of its angles. The distortions are negligible if you are making scale models following the surface of the earth in your backyard, but they become important on larger scales. Similarity and scale models are practical tools used by all of us, from casual readers of maps to architects planning cities or modeling new buildings.

Similarity, Dilations, & Congruence. Two shapes are *congruent* if they are *the same size and shape.* Using the transformational point of view, this is equivalent to saying that there is an *isometry* that carries one onto the other. In exactly the same way, two shapes are *similar* if they have *the same shape*, although they may differ in size. In terms of transformations, shapes are *similar* if there is a *similarity* that carries one onto the other. A *similarity* is a mapping in which the distance between transformed points is a constant multiple of the distance between the points before transformation. In terms of segment lengths, the transformation T is a similarity if T(P)T(Q) = λPQ, for all points P and Q. The number λ is the scale factor of the similarity. The SSS similarity theorem (page 412) tells us that similarities send angles onto equal angles, and thus, they *preserve the shapes of triangles.*

The AA similarity theorem is something new. It holds because the *shape* of a triangle in Euclidean geometry depends only on its angles, and because by the Angle Sum Theorem, any two of its angles determine the third angle. If the Parallel Postulate were false, we would not be able to construct scale models and there would be no AA similarity theorem for triangles.

A dilation with magnitude λ about O is a similarity. A dilation followed or preceded by an isometry multiplies the lengths of segments by the λ of the dilation, and so such compositions are also similarities. Indeed, every similarity may be written as an isometry followed by

dilation (or just as easily, a dilation followed by an isometry). We could restrict ourselves to dilations about O, the origin of our coordinate system. In terms of coordinates, dilations about the origin have the form $T((x, y)) = (\lambda x, \lambda y)$. This generalizes to three dimensions where $T((x, y, z)) = (\lambda x, \lambda y, \lambda z)$. *Dilations* allow us to *change size* leaving the shape the same.

Side-Splitter Theorem is Key. The study of *similar triangles* follows the pattern we used for studying *congruent triangles.* We can conclude triangles are congruent by SAS, ASA, or SSS, and after the Parallel Postulate is assumed, we have SAA as well. For $\triangle ABC$ and $\triangle A'B'C'$ to be *similar*, written $\triangle ABC \sim \triangle A'B'C'$, The corresponding angles must be equal (same *shape*), and the corresponding sides must be proportional (so $A'B' = \lambda AB$, $B'C' = \lambda BC$, and $C'A' = \lambda CA$). The key to establishing all of the properties of similar triangles is the Side-Splitter Theorem and its Corollary, Lesson 3.

Transformations & Similarity Proofs. The proofs given in this chapter follow the pattern used used in our study of congruent triangles. For example, we might copy $\triangle ABC$ onto $\triangle A'B'C'$. This static picture makes it easy to follow the details. For a more dynamic view, think of using a dilation to shrink or expand one of the triangles being studied so that you then have two triangles the same size, then show that these same-sized triangles are congruent. The composition of the dilation and isometry gives you a similarity between the original triangles. The static and dynamic arguments are essentially the same, but the first emphasizes construction while the second emphasizes transformations.

Lesson 1. Ratio and Proportion. We recognize many things by their *shape* without regard to their *size.* As the Lichtenstein mural thatthe opens this lesson shows, reference objects found in a picture let us estimate the true sizes of all the objects shown. This is why geologists include rulers, rock hammers, or their colleagues in their photographs of rock formations. The algebraic treatment of ratio and proportion in this lesson puts our intuitions about size and shape on a firm mathematical basis. These algebraic tools are used throughout this chapter. They are what let us work things out using scale drawings and scale models.

Exercises and Calculators. Calculators allow us to work out proportions even when the numbers are awkward. Most of the exercises involve ratios that can be handled easily by hand, but some students will want to use their calculators for all the exercises. **Set II**, Sinker or Floater (53 and 54) involves densities and a calculator is needed for it.

Applications include art, architecture, and music. A project for students interested in music might be to compare the Pythagorean Tuning (41–47) to the well-tempered scale. The underlying assumption in these exercises is that the strings and string tensions are constant, so that Frequency × String Length is a constant. A good source on musical notes and frequencies is *The Science of Musical Sound* by John R. Pierce, SCIENTIFIC AMERICAN Books, New York, 1983.

Lesson 2. Similar Figures. Similar figures have the same shape, but what is *shape*? Everyone who has watched a movie on both small and large screens knows that things are missing when you see the movie on a TV screen. Here we learn that the reason is because these two types of screens are rectangles of different shapes.

Two shapes are *similar* if there is a correspondence between their points such that corresponding segments are proportional. Such a correspondence is a *similarity.*

Note that *the definition of similarity for triangles* includes the fact that corresponding angles are equal. This allows us to freely use the properties of similar triangles without relying on the SSS similarity theorem (page 412). That theorem tells us that if the corresponding sides of two triangles are proportional then their corresponding angles are equal.

Dilation about a point O with magnitude r is a similarity. The illustration on page 387 shows how dilations act on triangles and circles, showing both magnification and reduction. Students will be familiar with the similarities you get by using the enlarging or reducing features of a copying machine. (As an aside, a curious fact is that copy machines set at 100% usually enlarge slightly to avoid any dark lines at the edges when a document is not precisely centered. After five generations of copies on my machine, a line that was 8.5 inches long had grown longer by a sixteenth of an inch. The magnitude of a single dilation in this series is $(8.5625/8.5)^{\frac{1}{5}} \approx 1.0015$. Knowing the enlargement at each stage, a detective can estimate how many generations of copying were needed to get to the degree of enlargement seen in a crime document from a given original.)

Set I. Plane Table (3–80) shows how accurate maps were made without calculations or numerical measurements. This uses Angle Side Angle similarity. We may think of this as AA similarity, but it is useful to keep the S in mind because the ratio of the baseline side of the map to the baseline on the ground determines the scale for the map. There is a nice section in *The Ring of Truth* by Philip and Phylis Morrison, Random House, New York, 1987, on making a plane table map of Thomas Jefferson's estate, Monticello. This book is based on a PBS series which is available on video. In Exercises 9 – 11 you see Kempe's Linkage that trisects angles. The black circles with white centers are pinned joints. Those at H and J bisect the rods they lie on. The solid black dots are sliders that can move along the rods they are on. The sliders keep the configuration in one plane. The linkage is built out of units like that made up of points OAHC in the diagram. These Kempe

called "contra-parallelograms" in his pamphlet *How to Draw a Straight Line.* This pamphlet was republished in *Squaring the Circle and Other Monographs*, Hobson et al., Chelsea Publishing Company, 1953, which is now available from Chelsea at the American Mathematical Society. There was also an NCTM reprint of Kempe's booklet.

A. B. Kempe (pronounced Kemp) is also famous for his contributions to the proof of the Four Color Map Conjecture. (Any map of the globe can be colored with four colors so that no two adjacent countries have the same color.) See page 6 or *Four Colors Suffice* by Robin Wilson, Penguin Books and Princeton University Press, 2002. In 1879 Kempe published his results on this problem, giving a proof that was seen as convincing until P. J. Heawood published an example in 1889 that showed that one of Kempe's methods failed for certain sorts of maps. Later, Kempe's techniques were used in work that culminated in a proof of the Four Color Map Conjecture by Kenneth Appel and Wolfgang Haken in 1976. Their proof required more than 1200 hours of computer time to check its technical details. See, for instance, Appel and Haken's October 1977 SCIENTIFIC AMERICAN article.

Dilation Problem (15–19) shows dilation at work. Practical measurements are used here.

Calculator Answers. In Billboards (12–14), we start by observing $\frac{5}{11} \approx \frac{11}{23}$. Then in Exercise 13 we solve the exact relationship $\frac{5}{11} = \frac{11}{x}$ for x. On my calculator if I find x by $x = 1/[5 \div (11 \times 11)]$, I get $x = 24.20000\ 00001$. But the more sensible $x = (11 \times 11) \div 5$ gives $x = 24.20000\ 00000$. This second calculator result correctly suggests that the exact answer $x = 24\frac{1}{5}$. Students should understand that when it comes to practical measurements, answers that differ by 1 in the twelfth significant figure may be considered the same. However, if they were asked for an exact algebraic solution, then the 1 in the tenth decimal place of my first answer should be counted as an error.

In connection with rounding for convenience, students might make a *project* of finding the International Standards Organization (ISO) paper sizes on the Web and looking at how these were worked out and comparing this with the numbers used in **Set II** Paper Sizes (41–47). This project shows how rigid mathematical requirements must be relaxed to give numbers that are easily dealt with in practice.

Violin Family (35–40) is about the scaling of similar instruments. Carleen Maley Hutchins worked this out for the violin family. Hutchins was a master of theory and practice, making many fine instruments herself. Her contributions are well documented on the Web. See www.newviolinfamily.org, her November 1962 SCIENTIFIC AMERICAN article, "The Physics of Violins," or her book *The Physics of Music*, W. H. Freeman and Company, San Francisco, 1978.

Similar-Triangles Proof (24–34) gives a special case of the SAS Similarity Theorem. See page 412 and its figures. Given $\triangle ABC$, if D lies on segment AB and E lies on segment AC so that $AD/AB = AE/AC = r = \frac{1}{2}$, then $\triangle ABC \sim \triangle ADE$. In this figure the angles that are between the two sets of sides that are proportional are equal angles — because they are both angle A. The conclusion is $\triangle ABC \sim \triangle ADE$. The first steps of the proof on page 412 show how we pass between $\triangle ADE$ and a general $\triangle A'B'C' \cong \triangle ADE$. See comments above on Transformations and Similarity Proofs.

Digression: Midsegments and SAS Similarity. If $\angle A' = \angle A$ and $A'B'/AB = A'C'/AC = r$ we have SAS conditions for $\triangle A'B'C' \sim \triangle ABC$ with a similarity ratio of r. Exercises 24 – 34 show that the Midsegment Theorem proves that these conditions imply similarity when the ratio $r = \frac{1}{2} = 2^{-1}$. Iterating this argument shows that $\triangle A'B'C' \sim \triangle ABC$ for similarity ratio $r = k2^{-n}$, where n is a positive integer and $1 \leq k \leq n$.

To see how this works you need to draw a diagram as follows: Use the midpoints of the sides to split $\triangle ABC$ into $(2^1)^2 = 4$ congruent subtriangles all similar to $\triangle ABC$. See the related figure for Exercises 1–9 in Lesson 5 of this chapter. Iterating this subdividing process through n stages gives $(2^n)^2 = 4^n$ congruent subtriangles, all similar to $\triangle ABC$ at ratio $r = 2^{-n}$. Start from vertex A and move down side AB k of these small triangles to locate point X. Along side AC move down k small triangles from A to locate point Y. Then $AX/AB = AY/AC = k2^{-n}$ so $\triangle AXY$ satisfies the SAS condition for similarity with $\triangle ABC$. Segment XY is made up of k segments that are sides of the smallest subtriangles in our subdivision, each parallel to BC and each 2^{-n} times the length of BC, the sides of $\triangle AXY$ are all proportional to the corresponding sides of $\triangle ABC$. Because XY is made up of segments all parallel to BC, it is parallel to BC, and the corresponding angles $\triangle AXY$ are equal to those of $\triangle ABC$. So $\triangle AXY \sim \triangle ABC$

Suppose that $\triangle A'B'C'$ satisfies SAS conditions for similarity to $\triangle ABC$ with similarity ratio r of the form considered above. Then $\triangle A'B'C' \cong \triangle AXY$, where $\triangle AXY$ is constructed as above. Then $\triangle A'B'C' \cong \triangle AXY \sim \triangle ABC$, so $\triangle A'B'C' \sim \triangle ABC$. Numbers of the form $k2^{-n}$, where k and n are integers are *dyadic rationals.* Every number can be approximated as closely as you please by dyadic rationals, so the fact that SAS conditions imply similarity for dyadic rational ratios of similarity plus the assumption that things behave continuously implies that SAS conditions imply similarity for all ratios.

The arguments above shows the power of the Midsegment Theorem. The proof of the Side-Splitter Theorem in the next lesson uses area, and it is much easier than the proof based on the Midsegment Theorem outlined above. Harold Jacobs tells me that he has seen proofs as described in geometry books that were around in his school days.

Set III. In the Villa Foscari plan notice that if a smaller similar rectangle is snuggled in the corner of a large rectangle and if these have the same orientation, then the diagonal of the larger rectangle lies on the diagonal of the smaller one. This observation is used by graphic designers in scaling drawings. By drawing the diagonal of an existing rectangle, they can use this diagonal to draw other similar rectangles at any given size.

The Villa Foscari exercise yields only to a *systematic search* of all the rectangles in the figure. *Systematic search* is an important problem-solving technique. Students should be encouraged to think about how they managed to check every possibility or how were missed. This exercise would be good for group work.

Lesson 3. The Side-Splitter Theorem. The dam pictured has a vertical upstream face to simplify the discussion. Most dams have slanted faces and are held in place largely by gravity. The trick is to make the dam so that it will be stable whether it is full or empty. Students may be interested in looking up how dams are constructed. J. E. Gordon's book *Structures: Or Why things Don't Fall Down*, a Penguin Pelican Book, 1978, reprinted by Da Capo Press 1988, pages 184–187 is a good source and has many examples that can be understood using simple geometric or algebraic reasoning.

The Side-Splitter Theorem and its Corollary (page 394) are more important and more intuitive that the proof of these results. Given $\triangle ABC$ we split its sides AB and AC with points D and E so that $DE \parallel BC$. By what we know about parallels, the corresponding angles of $\triangle ABC$ and $\triangle ADE$ are equal. The corollary tells us that $AD/AB = AE/AC$. We would like to know that $\triangle ABC \sim \triangle ADE$. For this we need to show that $AE/AC = DE/BC$ also. This follows by applying the Corollary to a congruent copy of $\triangle ADE$ along the lines used to prove the AA Similarity Theorem in the next lesson.

Set I. Turkish Theorem (1–3) and Errors of Omission (4–6) are review. Proportion is related to products in Picturing Products (7–12). Supply and Demand (13–17) relates geometric proportions and linear relations between two variables. Side-Splitter Practice (18–21) is presented visually. **Set II**. Note the *algebraic strand* in Corollary to Theorem 44 (29–33). Other applications are to the geometry of bridges and, in **Set III**, to the pantograph. Building a pantograph would be a good *project*.

Lesson 4. The AA Similarity Theorem. Indirect measurement is a great triumph of geometry. Thales lived from 624–546 BC, according to *Isaac Asimov's Biographical Encyclopedia of Science and Technology*. It is reported that Thales used similar triangles to determine the height of the Great Pyramid. Records of this feat can be traced to Hieronymus (4th century BC) as reported by Diogenes (2nd century AD). You might think that after some 2500 years nothing new could be said about this, but Lothar Redlin, Ngo Viet, and Saleem Watson's article, "Thales' Shadow," in the December 2000 issue of *Mathematics Magazine* has a number of interesting points to make, backed by computer-drawn pictures. The problem is that at many times and on many days the shadow of the apex of the Great Pyramid falls at inconvenient places, such as on the sides of the pyramid. So when *could* Thales have made his measurements? Thinking about this could be a big *project* for students.

Redlin, Ngo, and Watson conclude, "Did Thales actually measure the height of the pyramid at all? It is impossible to say for sure, but the *idea* of measuring the height of such a tall object using only its shadow is so beautiful and striking that it overshadows any of its practical applications. . . . "

Theorem 45, AA Similarity, is a key result for practical purposes, and it allows us to prove that *similarity is a transitive relation for triangles.* Intuitively we know that this *must* be true if similar figures have the *same* shape.

Set I. The exercises begin with Thales' Method (1–4). Triangle Average (5 and 6) explores a fascinating fact about directly similar triangles. The more general result can be established by understanding the algebraic nature of similarity transformations. The details are outlined below.

Digression: Similarities and Complex Numbers. Similarities in the plane can be handled algebraically using complex numbers. This digression draws on the lesson Proportion and Dilations, as well as on ideas about complex numbers, including de Moivre's Theorem. Students will see this result later in their second-level algebra course.

We start with a Cartesian coordinate system. The point A will have coordinates (a_1, a_2). To use complex numbers, we identify the point A with a complex number $A = a_1 + ia_2$, where $i^2 = -1$. Given any complex numbers $w \neq 0$ and v, the function $T(z) = wz + v$ of z defines a mapping of the complex plane to itself. We will see that T is a similarity.

We represent w in polar form using the de Moivre formula. Thus, $w = re^{i\theta}$, where r is positive and $0 \leq \theta < 2\pi$. Think of carrying z onto $T(z)$ in three steps:

$$z \xrightarrow{1} e^{i\theta}z \xrightarrow{2} r(e^{i\theta}z) = wz \xrightarrow{3} wz + v = T(z).$$

Step 1 is a counterclockwise rotation of magnitude θ about the origin, O, and it preserves lengths and angles. Step 2 is a dilation about O of magnitude r, and it preserves angles and multiplies the lengths of all segments by r. Step 3 is a translation, and it preserves lengths and angles. The combined result is T, which is a similarity that preserves angles (because each step preserves them), rotates lines through a clockwise angle of θ, and multiplies the lengths of segments by r.

Suppose that $\triangle ABC \sim \triangle DEF$, where the similarity is direct, meaning that when if go around $\triangle ABC$ in clockwise order, then the corresponding vertices of $\triangle DEF$ are also traversed in clockwise order. There is a positive ratio of similarity r such that $rAB = DE$ (equality of lengths) and an angle θ, $0 \leq \theta < 2\pi$ such that after counterclockwise rotation through θ radians AB becomes parallel to DE. Recalling that we have identified points with complex numbers, we may let $w = re^{i\theta}$ and define $v = D - wA$. Then the transformation T defined above, with these values of w and v, sends $\triangle ABC$ onto a triangle that is the same size as $\triangle DEF$ and whose sides are parallel to those of $\triangle DEF$, and whose clockwise sequence of vertices matches that of the target triangle, $\triangle DEF$. Since $T(A) = D$, the similarity T must send $\triangle ABC$ onto $\triangle DEF$. Informally, $T(\triangle ABC) = \triangle DEF$. This shows how we can use operations defined on complex numbers to construct a similarity mapping of the sort described above that sends $\triangle ABC$ onto $\triangle DEF$, if these triangles are directly similar.

If $\triangle ABC \sim \triangle DEF$ have opposite orientations of their vertices in this similarity, then we need an operation to switch counterclockwise to clockwise ordering as we go from $\triangle ABC$ to $\triangle DEF$. A reflection will do this. The complex conjugate of $z = z_1 + iz_2$ is denoted z^* or $\overline{z}$ and is defined by $\overline{z} = z_1 - iz_2$. The mapping $z \to \overline{z}$ is the reflection of the complex plane in the real axis. It reverses the clockwise orientation of the vertices of a triangle. Sending $\triangle ABC \to \triangle\overline{ABC}$ by conjugation and then $\triangle\overline{ABC} \to \triangle DEF$ by T, we get a similarity $S(z) = w\overline{z} + v$ that sends $\triangle ABC$ to $\triangle DEF$, as required.

triangle Average (5 and 6) concerns $\triangle GHI$ which is "midway" between $\triangle ABC \sim \triangle DEF$, where the similarity preserves the clockwise orientation of the triangles. Students will see that all these triangles are similar.

More generally, we could look at intermediate triangles λ of the way from $\triangle ABC$ to $\triangle DEF$. The vertices of such a $\triangle GHI$ are:

$$G = (1 - \lambda)A + \lambda D,$$
$$H = (1 - \lambda)B + \lambda E,$$
$$I = (1 - \lambda)C + \lambda F,$$

with $0 \leq \lambda \leq 1$ and with G, H, and I distinct. If $T(\triangle ABC) = \triangle DEF$, where $T(z) = wz + v$, then $D = wA + v$, $E = wB + v$, and $F = wC + v$. Using these expressions you can find G, H, and I by algebra. The formulas are all of the same form.

$$\text{Example: } G = (1 - \lambda + \lambda w)A + \lambda v.$$

The similarity $S(z) = [(1-\lambda)+\lambda w]z+[\lambda v]$ with $\lambda = 1/2$ sends $\triangle ABC$ to $\triangle GHI$, so these triangles are similar, which is what was to be shown.

This is what you might expect. The mapping $z \to 1z + 0v$, with a multiplier of 1 sends $\triangle ABC$ onto itself. The mapping $z \to wz + 1v$, with a multiplier of w rotates and shrinks $\triangle ABC$ so that it matches $\triangle DEF$ in size and orientation. To get λ of the way between these extremes, we would need a multiplier of $(1 - \lambda)1 + \lambda w$. The corresponding translation needed is λ of the way from 0 to v, or $(1 - \lambda)0 + \lambda z = \lambda z$, which is what we have.

There is one thing more to check. If the multiplier of z were zero, the points G, H, and I would have to be identical and G, H, and I would not be vertices of a triangle that is similar to $\triangle ABC$. Since G, H, and I are distinct, $[\lambda+(1-\lambda)w] \neq 0$, and S is a genuine similarity.

There are many relationships that students can discover by experiment, either with paper and pencil or using *Geometer's Sketchpad*™. There must be some way to prove that these relationships must hold. However, the proofs may require methods beyond the range of introductory geometry. This seems to be such a result. The difficulty in this case is that we need some way to describe all similarities of the plane, so that we can smoothly interpolate between the identity map that sends $\triangle ABC$ onto itself and the similarity between $\triangle ABC$ and $\triangle DEF$.

Matrices could be used in place of complex numbers, but this would require an understanding of rotation matrices, a matter no less technical than the use of complex numbers. We could work with rotations and dilations about the origin, but this is equivalent to using complex numbers in polar form, as above. If you or your students should find a royal road to this result, I would be delighted to see it.

Seeing and Calculating. The first thing students must do in **Set I**, Newton's Figure (7 and 8), Nine Triangles (9–11), and Chess Puzzle (12–18) is to see what is going on in the figures. Then they go on to description, logical analysis, and calculation. Chessboard Puzzle is unmasked as a trick by simple calculations. Some students will want to see exactly where the fraud is. Don't give the show away and spoil their fun, but one approach uses similar triangles to calculate how far the point B lies above the segment AE. The answer is 15/8 units, just short of the 2 units needed to make the picture correct, and the method of the fraud becomes clear.

Art, Electricity, and Algebra. **Set II**. Exercises 19–24 are tied to the Italian painter Piero della Francesca's work on proportions and perspective. In Electrician's Formula, (25–30), we see that when two resistors of values R_1 and R_2 work in parallel, they are equivalent to a resistance of $\frac{R_1R_2}{R_1+R_2}$. This expression gives the harmonic mean of R_1 and R_2. The harmonic mean also appears in the formula for the length of time it takes two workers to complete a job if one can complete the job in time R_1 and the other can complete it in time R_2. We see similar algebra in Deja Vu (43–47) used in a geometric context.

Original Proofs (37–42) and Mascheroni Constructions (48–50), which are constructions done with compass alone, are *pure geometry.*

The **Set III** puzzle on areas has a solution within reach of students who follow the line suggested here.

Lesson 5. Proportions and Dilations. The Side-Splitter theorem and the AA Similarity Theorem establish much of what we need for applications of similarity. In this lesson we see how what we know can be used to show the proportionality of corresponding segments in similar figures. See Theorem 46 on the altitudes of similar triangles. The figure on page 408 correctly suggests that we should think of the similarity $\triangle ABC \sim \triangle DEF$ as establishing a correspondence not just between the vertices of these triangles but also between the feet of their altitudes and between all corresponding pairs of points on or within the triangle as well. The *lengths of the segments between corresponding pairs of points under a similarity transformation are always proportional with the same proportionality ratio as the sides of the triangles.*

Set I. Fish Story (1–18) drives home more general ideas about similar figures visually, and shows how these ideas are used in computations. Grid Exercises (25–30) relate similarity transformations to coordinates.

Set II. Congruence and Similarity (35–61) links these ideas. Geometrically similar shapes are congruent, *if* they are the same size. We would expect that the conditions that force congruence of triangles would also force similarity of triangles *in a weakened form.* This is true. For instance, ASA conditions on two triangles force congruence, while the weaker AA condition forces the triangles to be similar. Likewise, the equality of the corresponding sides of two triangles, SSS, forces congruence. Requiring only that that the sides of the one triangle be *proportional* to the corresponding sides of the other triangle forces the triangles to be *similar* (SSS similarity). No student would doubt that a triangle with sides 6, 8, and 12 is similar to one with sides 3, 4, and 6, and we hope that students will freely use such facts, whether or not they are asked to work through the proofs of the SAS and SSS Similarity Theorems (Exercises 40–61).

A practical understanding of similarity can be based on Exercises 35–39 and class discussion. At this stage, interested students could pursue the proofs outlined in Exercises 40–61, if they wish. These proofs are routine applications of methods we have already seen.

Set III lets students explore the camera theme of this lesson. This is good *project.*

Lesson 6. Perimeters and Areas of Similar Figures. If AB and A′B′ are corresponding segments of similar figures and if $A'B'/AB = r$, then the figures' perimeters, P and P′, have ratio $P'/P = r$, and their areas, A and A′, have ratio $A'/A = r^2$. These are rules of scaling. More generally, the 3-dimensional volumes of similar figures scale as r^3, where r is the ratio of the lengths of corresponding segments in these figures. If a quantity is n-dimensional it will scale as r^n in similar figures, where lengths of corresponding segments in these figures scale as r^1. See Lesson 8, Similar Solids in Chapter 15.

The exercises show scaling ideas at work in sports, in reading maps, in quilting designs, and in comparing the sizes of circles in a cranberry harvest. Notice that the areas and perimeters of circles scale the same way as the areas and perimeters of polygonal figures. Scaling of perimeters and areas of similar figures has many applications. **Set III** drives home this point with a surprising version of the Pythagorean Theorem. (Compare Set III with Euclid VI.31.)

Summary and Review. The key idea in this chapter is that similar figures have the same shape, so that the lengths of corresponding segments are in the same ratio, r, which determines the proportionality of the similarity, and corresponding angles in similar figures are equal. The number r that gives the proportionality between the two figures lets us work out the numbers in all of the applications in the Summary and Review exercises. The **Set I** exercises cover all of the ideas in the chapter. The Trick Card and other examples in **Set II** are appealing and should be considered.

Chapter 11. The Right Triangle.

To the Student. As exercises in the Chapter 10 show, you can draw a reasonable map using a plane table without *any* calculations. But to map a township, state, or nation, you need measurements giving precise numbers. Similar right triangles and trigonometry are the tools for this job. They let you calculate lengths that can't be measured directly, for example, the height of Mount Everest.

The Pythagorean Theorem's $a^2 + b^2 = c^2$ lets us find the length of the third side of a right triangle given the length of two of the sides. Trigonometry extends these ideas to other triangles by using the Laws of Sines and Cosines (Lesson 7), allowing us to "solve" triangles, working out all sides and angles.

To apply these ideas, you need to be able to see right triangles in problems and to understand slopes. Many problems can be solved using *special triangles*: isosceles right triangles or 30°, 60°, 90° triangles. Learn to look for these special triangles, too.

The altitude to the hypotenuse divides a right triangle into similar triangles, both similar to the original. This has many applications, explored in Lesson 1. In Lesson 2, we use this observation to prove the Pythagorean Theorem. We apply similarity and the Pythagorean Theorem in this lesson's exercises. Lesson 3 looks at special triangles and their applications. We come to the trigonometric ratios in Lessons 4 and 5. In Lesson 6,

we study slope and its uses, returning to trigonometry in Lesson 7 on the Laws of Sines and Cosines.

This chapter shows how to solve triangles using trigonometry. Key skills are: *Being able to identify similar right triangles* in the figure of an application. *Seeing how to apply the Pythagorean Theorem* in these situations. *Recognizing and using special triangles. Knowing the definitions and uses of sines, cosines, and tangents. Knowing that more general triangles can be solved using the Laws of Sines and Cosines.* (Not all programs will hold students responsible for these laws.)

Right Triangles, Maps, and Coordinates. Numbers rule in our age. Every point on the surface of the earth has a latitude and longitude (measured relative to the geodetic, geocentric, or astronomical systems). A GPS unit can tell you where you are within a few meters, and advanced units have an accuracy to within a few centimeters. US topographical maps show a Universal Transverse Mercator (UTM) grid, and a GPS will give you your coordinates relative to this.

Maps are flat and the earth is curved, even somewhat rough. To make a flat map of a curved thing, we must relate points in the landscape to points on an ideal surface (the Clark spheroid or variants are used). This is the surface of an ideal sea that can be adjusted to better match local terrain to reflect the bulge at the equator and flattening at the poles. Such a surface is called a *datum*. By looking at the grid ticks showing overlapping UTM grid patches on a map, you can see the distortions that are inevitable when we try to impose a square grid on a curved surface.

The refinements in mapping are many. From its beginning, geometry gave us tools to map farmer's plots and to think about the shape of the earth. Recall Lesson 2 of Chapter 1, with Aristarchus' estimate of the earth's circumference and Lesson 5, with the Egyptian assessor's formula. Geometry is about numbers as well as shapes, and its trigonometrical side goes back to the Greek astronomer Hipparchus (about 161–126 BC).

Lesson 1. Proportions in a Right Triangle. The nautilus shell has a self-similar spiral. The shell in the photograph is mapped onto itself by a counterclockwise rotation of 180° combined with a dilation by a factor of 1.75 about its center. Self-similar spirals are also called *logarithmic spirals*, and each cuts every ray from its center at a constant angle characteristic of that spiral. The closer this angle is to 90°, the closer the successive turns of the spiral are. (What if the angle is 90°?)

The figures at the top of page 429 show that the altitude to the hypotenuse of a right triangle splits the triangle into two triangles, both similar to each other and to the original triangle. This is easy to see. The three proportions that follow from this *need* to be *studied* and *applied* in different situations until using them becomes automatic. In the figures on this page, the segment that is the geometric mean is always shown in red. These figures deserve careful study.

Set I. Chameleon Tail (1–7) builds on a second self-similar spiral. In Shadows (8–14) the idea of a *projection* is related to the projection of shadow by a light source; this helps give meaning to projections. These exercises use *perpendicular projection*. If the sun is not overhead, we get *parallel projection*, also a useful idea but not needed here. Find the Lengths (15–19), these provide practice with these proportions. Turn about is fair play, so teachers might ask their students to make up three problems, one using each of these proportionalities, and then collect these and use some in a quiz. Exercises 20–30 combine numerical calculation with geometric insight. The final two questions in this series can be answered using the numerical results in hand. Is there a way to see that $AD/DB = (AC/CB)^2$ in a flash of geometric intuition? Yes: $AD/DB = [\text{Area}(\triangle ACD)]/[\text{Area}(\triangle CBD)] = (AC/CB)^2$, because these triangles are similar ($\triangle ACD \sim \triangle CBD$). The ratios of their areas is equal to the square of the ratios of any pair of corresponding sides. The sides used here are the hypotenuses.

Rectangle and Square (31). When we reach Chapter 12, Circles, we will see that this observation gives us a way to construct the geometric mean of lengths a and b and thus to "Square a Rectangle" with sides a and b, getting a square of side $\sqrt{ab}$ having the same area (Euclid VI.13). This uses the result that a triangle inscribed on a semicircle with the diameter as its long side is always a right triangle. See also the rectangle inscribed in a circle in Exercises 32–37. This fact about angles inscribed in semicircles is Corollary 2 of Theorem 63 in Chapter 12.

Set II. Strongest Beam (32–34). In this sequence both *exact* and *approximate* calculations are used.

Zig Zag (38–45). What we have here is a series of triangles and segments that form various geometric series. Following the steps a student will be able to work all of this out easily, but there is another way to think about this.

Zig Zag, Similarity, & Geometric Series. Consider the similarity T consisting of reflection in the bisector of $\angle AOB$ followed by a dilation of magnitude r about O sends $T(\triangle AOB) = \triangle BOC$, $T(\triangle BOC) = \triangle COD$, and so on. When T is repeated to get T^2 we have a dilation of magnitude r^2 about O. Therefore, $T^2(AC) = CE$ and $T^2(CE) = EG$.

Draw the bisector of $\angle AOB$ and let it intersect AB at O_1, BC at O_2, and so on until it intersects FG at O_6. The dilation of magnitude r about O sends O_k to O_{k+1}. The similarities we are considering all leave O fixed and send this bisector onto itself. Looking only at points on this bisector, our mapping behaves as if it were a

dilation of magnitude r about the fixed point O. Using similarity, we can find S, the lengths of the segments:

$$\begin{aligned}(*)\quad S &= O_1O_2 + O_2O_3 \\ &\quad + O_3O_4 + O_4O_5 + O_5O_6 \\ &= 1(O_1O_2) + r(O_1O_2) + r^2(O_1O_2) \\ &\quad + r^3(O_1O_2) + r^4(O_1O_2) \\ &= (1 + r^1 + r^2 + r^3 + r^4)(O_1O_2).\end{aligned}$$

Looking at S another way, we get

$$\begin{aligned}S &= O_1O_6 = OO_6 - OO_1 \\ &= r^5(OO_1) - 1(OO_1) = (r^5 - 1)(OO_1).\end{aligned}$$

Now use $(O_1O_2) = (OO_2) - (OO_1) = r(OO_1) - 1(OO_1) = (r-1)(OO_1)$ to replace OO_1 by $\frac{1}{(r-1)}(O_1O_2)$ and get

$$(**)\qquad S = \frac{(r^5 - 1)}{(r - 1)}(O_1O_2).$$

Equating the final expressions for S in (*) and (**), cancelling the (O_1O_2) factors on both sides, and generalizing from 5 to n gives

$$(1 + r^1 + r^2 + r^3 + \cdots + r^n) = \frac{(r^{n+1} - 1)}{(r - 1)}.$$

This is a geometric derivation of the formula for the sum for a finite geometric series. This sum is calculated in Proposition 35 of Book IX of Euclid.

Using algebra we can get this result more easily than by Euclid's geometric methods. For nice geometric proofs of this result, see *Proof Without Words* by Roger B. Nelson, Mathematical Association of America, washington, DC, 1993, pages 119 and 120. More generally, you will find many wonderful geometric derivations of algebraic results in the Nelson book and in *The Book of Numbers* by John H. Conway and Richard K. Guy, Springer-Verlag New York, 1996, especially in Chapter 2, "Figures from Figures: Doing Arithmetic and Algebra by Geometry."

Set III. The Golden Rectangle exploration combines construction and calculation. The Golden Number or Golden Section and the Golden Rectangle are sources of endless study and speculation. *Project*: Look up material on the Golden Section in the library or on the Web. As a ratio, it goes by the names of Phi (ϕ) and Tau (τ) and is the positive root of the equation $x^2 - x - 1 = 0$. It satisfies $x = 1 + \frac{1}{x}$, which suggests how it can be written as a continued fraction. See Chapter 14, Lesson 2, Set III notes. It is the limit of the quotients of successive Fibonacci numbers. See also Euclid, Proposition 30, Book VI.

Lesson 2. The Pythagorean Theorem Revisited. Geometry is a coherent whole, and argument by which we derive the Pythagorean Theorem by using proportions shows us one of the links binding the subject together.

In this lesson, we use proportions and $c^2 = a^2 + b^2$ to work out lengths in problems. Note that many of the exercises in **Sets I** and **II** build familiarity with standard or easily computed right triangles, including Pythagorean Triples, Exercises 18 –25. There are several ways to get all Pythagorean Triples. One is to consider all triples of the form $2k\ell$, $k^2 + \ell^2$, $k^2 - \ell^2$, where k and ℓ are distinct integers.

There is a wealth of material on Pythagorean Triples. Researching and reporting on them would be a good *project*. Another interesting *project* would be to research some of the ancient units referred to here and elsewhere in this book. In *Geometry Civilized*, Oxford University Press, Oxford, United Kingdom, 1998, John Heilbron wrote on page 153 that the "sacred cubit" is 25.025 inches long. By e-mail, Heilbron comments that this three-place precision was a joke. We really do not know exactly how long many of these units were, but it is a good guess that a cubit was about the length of a forearm. But whose forearm and how was it measured? One ch'ih is about a foot. An interesting resource on measurements is *Measure for Measure* by Richard A. Young and Thomas J. Glover, Sequoia Publishing, Littleton, Colorado, 1996, which gives conversion factors for all manner of measurements.

Distance Formula (28–30) uses coordinate methods as does Grid Exercise (37–41). The **Set II** Oil Well problem (47–50) has a surprisingly nice solution. The numbers involved will be simplified if students work in thousands of feet, but a calculator will do the job without this simplification.

Applications in the exercises includes figuring the sag in a suspension bridge, working out the turning radius of a car, and relativistic length contractions.

Set III is a gentle exploration of Fermat's Last Theorem (FLT). Students who want to learn more can go to the Web. Try Eric Weisstein's site mathworld.wolfram.-com/FermatsLastTheorem.html. Weisstein notes that Marylin vos Savant's book on FLT states that the solution of the problem by Andrew Wiles and Richard Taylor is faulty. Vos Savant faults Wiles and Taylor for using the theory of elliptic curves in their work. She identified elliptic curves with a type of non-Euclidean geometry (specifically, elliptic geometry, the Lobachevskian geometry we study in Chapter 16). Since vos Savant questions the validity of non-Euclidean geometry, she questioned Wiles and Taylor's proof. But the elliptic curves of their work have nothing to do with elliptic geometry, and a proof based on non-Euclidean geometry is as correct as a proof using other.

Lesson 3. Isosceles and 30°-60° Right Triangles. Pythagorean triples and the special triangles of this lesson give us all of the examples that students can work out in exact algebraic form. These are the basis for all examples that can be handled easily without a calculator. More importantly for the development of geometry and trigonometry, special triangles kicked off an era of trigonometrical calculation that began over 3700 years ago, as the lead example of this lesson shows. I give a brief sketch of the development of such calculations in "To learn how to calculate better" found at the end of the notes for this chapter. These efforts gave us today's scientific calculator — a boon for us and for students.

Even today it is important that students be familiar with these special triangles. This familiarity is the concrete basis for the trigonometric ideas that come in Lessons 4, 5, and 7. *These special triangles often show up on standardized tests.*

Set I gives students practice with special triangles and their applications. **Set II**. Not Quite Right (34–41) asks students to compare approximate and exact answers involving special triangles. More generally, there are many places in these exercises where students may get answers either in exact form or as decimal approximations. Class discussion of which form is most appropriate should help. The key is understanding what is being asked for or what is needed to proceed practically. If you want a diagonal brace for a square framework made of two-by-fours, you don't need the exact answer of $\sqrt{2}$ meters, or even a decimal answer to four places. The best approach would be to lay the frame of two-by-fours on a pair of sawhorses, square it up, and cut the brace to fit. No calculation would be needed.

Trigonometric Angles (42–47) shows how ideas about angles are generalized to angles larger that 90°. Calculators are able to deal with a wide range of angles, expressed in degrees or radians. These exercises open the door to exploration of these features on the calculator and give access to material beyond the trigonometry of acute angle triangles that is the focus here.

Construction Exercise (48–52) involves construction, measuring, and exact and approximate calculations. Area Problem (53–56) involves problem solving and exact and approximate calculation.

Moscow Challenge, Exercise 57, requires thinking outside the box. If a student sees how to do it it, it is easy; otherwise, it isn't.

Set III. A Big Mistake, invites students to think critically and nail a glaring error. This exercise is within reach of most students.

Lesson 4. The Tangent Ratio. Indirect measurement is a major use of trigonometry, and finding the height of Mount Everest is a striking example of this use. *The Great Arc* by John Keay, footnoted in this lesson, outlines the work of the Great Trigonometrical Survey. Everest was just one height determined by this survey. See page 472 of *Geometry*, Triangulation, for more or look up the Great Trigonometrical Survey on the Web.

The calculator is essential here. In the exercises it works along with exact calculations and with what we learned earlier about special triangles. See **Set I** Exercises 17–28. Note the first steps toward solving triangles in Exercises 29–34.

Set II. Fly Ball (35–38) shows that a baseball appears to rise steadily as it comes to a fielder. This seems surprising. If you check sites on the physics of baseball, you should find a Java applet that gives a more realistic and detailed picture. If you use the usual simple model for the flight of the ball, ignoring air resistance, you can find the ball's trajectory. If te trajectory is projected into a plane perpendicular to the line from the batter to the fielder and passing through the batter, it appears as uniform linear upward motion. The calculation of this motion is the same as the calculation of the slope of the tangent as the limit of a secants passing through a point on a parabola. This latter calculation is usually done in the first weeks of a calculus class.

The lighthouse of Exercises 39–43 is the Pharos. See the map on page 8 of *Geometry*. Grid Triangle (48–52) compares exact and approximate results. Students will see that exact calculations are needed to prove you have a right triangle, although sufficiently precise approximate ones suffice to show when an angle is not a right angle.

Set III. What Time Is It? This is an exercise in photographic interpretation. We are given that the camel's shadows are three times their heights, and this gives the tangent of the sun's elevation. Given information on sunrise and where the sun will be at noon, we can find the time the photograph was taken. But how could we find out that the shadows are three times as long as the camels are high? To do this we must know something about camels, namely that a full-grown camel is almost exactly as long, head-to-tail, as it is tall, hoof-to-hump. Checking side views of camels in other books shows that this is about right.

Picking a camel in the lower left and one in the upper right that are fairly separate from the others and measuring their lengths and the lengths of their shadows in printer's points, I got:

shadow length = 36 points
camel length = 14 points
tan = 0.3888
angle = 21.25°
time after sunrise =(angle/90) x 6 hours + 6:00 am
= 7:25 am

and

shadow length = 38 points

camel length = 16 points
tan = 0.42
angle = 22.83°
time after sunrise =(angle/90) x 6 hours + 6:00 am
= 7:31 am

These measurements and calculations are in close agreement with the assumptions given in the exercise. Students might be asked how they could check whether the camel's shadows were three times the camel's height. If we know that this is morning, we can also figure out the compass bearing the camels are walking along. See page 661 for another photographic interpretation exercise.

Lesson 5. The Sine and Cosine Ratios. The glide path of a flying squirrel introduces this lesson on sines and cosines. These are introduced for acute triangles, but Sines from a Circle (23–35) puts sine and cosine in the context of points on a quarter circle. These exercises are based on work of Bashkara, in 7th century India. Extremely accurate trigonometric methods were developed by astronomers in the southern Indian region of Kerala in the 15th century. See the outline of the trigonometric calculations following the notes on this chapter. If a coordinate system is set up with point P in the figure on page 457 as the origin and using centimeters for the scales on both axes, the points on the circle shown have coordinates $(10\cos\theta,\ 10\sin\theta)$, where the angle θ is measured counterclockwise from the positive x-axis.

The **Set I** exercises for this lesson focus on using the sine and cosine and working from sines and cosines back to angles. Finding Lengths and Angles (23–28) takes us further toward solving triangles. The calculator is a key tool, and it is used to get approximate values for the angles in Pythagorean Triangles (13–22).

Set II turns to applications. Note the vector ideas in Force Components, (41 and 42). SAS Area (54 and 55) also has wide use: $\alpha\triangle\text{ABC} = \frac{1}{2}ab\sin\text{C} = \frac{1}{2}\|\overrightarrow{\text{CA}}\times\overrightarrow{\text{CB}}\|$, where the final expression is half the length of the cross product of the vectors $\overrightarrow{\text{CA}}$ and $\overrightarrow{\text{CB}}$. See, for example, Eric Weisstein's World of Mathematics at mathworld.wolfram.com on vector operations. See also "Dot Products, Rotations, Matrices" in the supplementary material for Chapter 11 in these notes.

Set III. Star Distance. If a student use a calculators to find the answer to this question, they may give the distance as 25,240,298.6593 million miles. This is computational accuracy beyond the accuracy of the data. If the calculations are done in scientific notation, as Jacobs has them, you get an answer that is correct, more easily understood, and gives the right sense of accuracy. Assuming thata parallax of 0.76 seconds implies the figure is correct to ± 0.005 seconds, then the distance lies between 25.075 and 25.407 trillion miles or 4.27 and 4.32 light-years.

An interesting source on parallax measurements is *Parallax: The Race to Measure the Cosmos* by Alan W. Hirshfeld, W. H. Freeman and Company, New York, 2001. The Hipparcos satellite (High Precision Parallax Collecting Satellite) launched in 1989 measured parallax to an average accuracy of a thousandth of an arc second. A satellite planned for launch in 2009, GAIA, is set to measure parallaxes down to 1/5,000,000ths of an arc second.

Lesson 6. Slope. The basic definition of slope and the relationships between slopes of parallel and perpendicular lines are the key *mathematical* ideas of this lesson. An intuitive feeling for meaning of slopes in physical situations and applications is the key *nonmathematical* subject.

The lesson begins with the slope of a line ℓ

$$\text{Slope of } \ell = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\text{A},$$

where the line ℓ passes through points $\text{P} = (x_1, y_1)$ and $\text{Q} = (x_2, y_2)$. Angle A is *the angle of inclination of* ℓ. On a calculator $\angle\text{A}$ is the arctangent of the slope of ℓ, which may be found using the "atan" or "$\tan^{-1}$" buttons on the calculator. Often this requires using a second function button.

Geometrically, $\angle\text{A}$ is the angle that ℓ makes with the x-axis. Angle A is measured by starting at the intersection X of ℓ with the x-axis, and finding the counterclockwise rotation about X needed to bring the ray running from X in the increasing direction on the x-axis into line with ℓ. For example, the line ℓ with equation $y = -x$ has slope -1. This line intersects the x-axis at X = the origin $= (0, 0)$. My calculator gives $\text{atan}(-1) = -45°$, so $\angle\text{A} = -45°$. This means that the positive x-axis must be turned through a counterclockwise angle of $-45°$ (that is, backwards 45°) about X the to bring it into line with ℓ. A *clockwise* rotation of 135° leaves the line at the same inclination as a *counterclockwise* rotation by $-45°$, because two rotations about X that differ by 180° result in the same inclination. Depending on the conventions we adopt, we may think of the inclination of this line as either $-45°$ *counterclockwise* or 135° *counterclockwise*. (But keep in mind that for rotations $-45°$ *counterclockwise* is equal to 45° *clockwise*.) Calculators give us angles of inclination that range from $-90°$ to $+90°$, and for ease of use, we should stick to these. Students should see how slopes and angles of inclination are related by experimenting with their calculators as well as by looking at different lines in a coordinate system.

Theorem 52 tells us that nonvertical lines are parallel precisely if they have the same slope. Intuitively, the slope is the same if and only if the angle of inclination is the same, so either both lines are parallel to the x-axis or they both make the same angle with the x-axis. In

either case, they are parallel if and only if they have the same slope. Exercises 52–59 prove the forward implication based on similarity. To prove the converse, show that if $m_1 = m_2$, then $\triangle ABC \sim \triangle DEF$, so $\angle 1 = \angle 2$, and so $\ell_1 \parallel \ell_2$.

Theorem 53 tells us that two nonvertical lines are perpendicular if and only if the product of their slopes is -1. The coordinate example on page 463 can be generalized to prove this. A conventional proof of the forward implication is given in Theorem 53 (60–65). The converse is handled like that of Theorem 52.

The Point-Slope Equation of a Line (42–46), is another useful result found in this lesson. See the Algebra Review "Graphing Linear Equations" on pages 481 and 482 for slope-intercept form of a line and related material.

It will seem intuitive to students that *lines* have *linear* equations. This fact is so "blazingly obvious" that students are impatient if you take the time to prove it. This phenomenon was noted by Keith Henderson, and the phrase is his. The fact that lines have linear equations is easy to establish by arguments using similar triangles.

Set I exercises give practice with the basis ideas, **Set II** cover more applications and proofs, and **Set III** tells us something interesting about temperature scales. Set III is within reach of most students.

Lesson 7. The Laws of Sines and Cosines. Label $\triangle ABC$ so that side a is opposite vertex A, side b is opposite vertex B, and side c is opposite vertex C. Then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad c^2 = a^2 + b^2 - ab\cos C.$$

The first set of equalities is the Law of Sines, and the second is the Law of Cosines. As you see in this lesson, the Law of Sines boils down to the observation that you get the same area for a triangle, no matter which pair of sides and included angle you use to find that area. See the SAS Area exercises on page 459.

The Law of Cosines is an extension of the Pythagorean Theorem that holds for arbitrary triangles. It is proved here for the acute case, but with such identities involving trigonometrical functions, if they hold over one range of variables, then they are universally valid. This is a general principle that students should find reasonable. We have looked at sines, cosines, and tangents for acute angles, but a calculator will give us the values of these functions for arbitrary angles. Experiment suggests that the formulas we use work for angles in general, and so they do.

This lesson gives us the tools needed to solve triangles in general. The **Set I** exercises cover the key ideas. Calculators are essential in working out answers.

Set II exercises include applications such as land surveying and triangulation (25–30) and triangulation extending to the moon (31–33). Note that although we know that side BM in the Distance to the Moon exercises is longer than side AM because BM is opposite the larger angle, this does not show up numerically, at least not on my calculator. Likewise, the **Set III** exercises are something of a challenge in precision calculation. Jacobs has a nice discussion of these in his answer section.

Chapter 11. Summary and Review. In addition to the results listed here, teachers may want to include some of the results on graphing lines from the Algebra Review on pages 473 and 474.

The cuneiform tablet in Number Mystery (65–66) is known as Plimpton 322. It has been extensively studied and analyzed. Down the years authors have concluded that it shows that Pythagorean Triples were known long before Pythagoras. But recent articles suggest other interpretations. Search the Web for the latest under Plimpton 322 and Eleanor Robson, of Oxford University, whose article "Words and Pictures: New light on Plimpton 322" in the February 2002 *American Mathematical Monthly* argues that this tablet was used as an aid for Babylonian mathematics teachers to create exercises that came out with simple answers.

▶ *Similarity and Non-Euclidean Geometry.* Similarities are unique to Euclidean geometry. In Euclidean geometry we can pick a center O and transform every shape in the plane by a dilation T of magnitude λ about O. This dilation about O has a simple expression if we use a Cartesian coordinate system with O as its origin. Expressed in such coordinates, T sends $P = (x, y, z)$ to the point $T(P) = P' = (\lambda x, \lambda y, \lambda z)$. Similarly, we write Q' for $T(Q)$. Using the coordinate from of this dilation or geometry, you can see that the distance between P' and Q' will be λ times the distance between P and Q.

Because the surface of a sphere is curved, geometry on the surface of a sphere is not Euclidean. The closest we can come to a straight line on a sphere is a great circle, the intersection of a plane passing through the center of the sphere with the surface of the sphere. The path of a hockey puck sliding without friction on a spherical planet will be a great circle. A spherical triangle is formed by arcs from three great circles. The Ruler Postulate must be modified in order to fit the geometry of the sphere because the lines of this geometry are great circles and have finite length and curve back on themselves. As you will see in Chapter 16, Non-Euclidean Geometries, spherical triangles have the ASA, SAS, and SSS congruence properties of Euclidean triangles. But there is a difference in behavior with respect to the sum of the angles in a triangle.

Spherical triangles with the same angle sum have the same area. Two spherical triangles whose corresponding angles are equal must be congruent; this is the AAA

congruence theorem for triangles on a given sphere. Similar triangles at different scales can't exist on a given sphere. The *spherical excess* of a triangle is defined as the sum of its angles minus a straight angle. Spherical excess is proportional to area for triangles on a given sphere. For general information on spherical triangles, see Eric Weisstein's *World of Mathematics* at mathworld.wolfram.com. For areas of spherical triangles, read on in these notes.

Besides the geometry of the sphere, there is a second sort of non-Euclidean geometry, the geometry explored by Saccheri, Lobachevsky, Bolyai, and Gauss. See Lessons 2–4 of Chapter 16. This *Lobachevskian* geometry is like the geometry on a sphere — a sphere whose radius is an imaginary number. Johann Heinrich Lambert made this observation in his work on the parallel postulate published in 1786. See Morris Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, 1972, page 868 and pages 875–877, where you will see how using imaginary values for the radius gives a trigonometry that fits with Lobachevskian geometry.

The sum of the measures of the angles of a triangle in Lobachevskian geometry is always *less* than a straight angle. The area of a triangle is proportional to its *Lobachevskian deficit*, defined as 180° minus the angle sum. As in spherical geometry, we have the AAA Congruence Theorem and the congruence theorems of Euclidean geometry. In Lobachevskian geometry as in spherical geometry, two triangles cannot be similar without being congruent.

It may seem odd that the areas of triangles should be related to angular excesses or deficits in non-Euclidean geometry. But here is a way to see why this must be the case. If you draw a triangle and divide it into subtriangles in any way, you will see that these deficits (or excesses) add up so that the deficit (or excess) for a large triangle is always the sum of those for smaller triangles into which it can be decomposed, no matter how this decomposition is done. This is exactly the property we expect for areas, so it is not surprising that the angular deficit (or excess) of a triangle is a measure of its area in non-Euclidean geometry. The reason is that the standard non-Euclidean geometries are spaces of constant curvature. The spherical excess of a spherical triangle measures the total curvature over the area the triangle encloses. The deficit in a Lobachevskian triangle should be thought of as the total negative curvature of the interior of that triangle. Lobachevskian geometry is just the geometry of a surface with constant negative curvature.

Areas of Spherical Triangles. The areas of spherical triangles can be worked out by methods independent of calculus. To do this we need to think of a spherical triangle T as the intersection of three hemispheres. We also need the Principle of Inclusion and Exclusion explained below. This is a way to count objects in a union of sets or find the area of the union of these sets.

Let the notation #X stand for the number of things in the set X. The number of things in the *union* of a family of sets is given by the alternating sum: (sum of the number of things in the sets taken one at a time) − (sum of the number of things in the pairwise intersections of the sets) + (sum of the numbers of things in the intersections of the sets taken three at a time) $-\cdots$, and so on. For example, for three sets S, U, and V, this gives us $\#(S \cup U \cup V) = [\#S + \#U + \#V] - [(\#(S \cap U) + \#(U \cap V) + \#(V \cap S)] + \#(S \cap U \cap V)$. This is the Principle of Inclusion and Exclusion, and it works for figuring areas as well as for counting elements, provided that all the numbers involved are finite.

Using the Principle of Inclusion and Exclusion, the area covered by the three hemispheres whose intersection is T is given by the sum of the areas of these three hemispheres, minus the areas of their pairwise intersections, plus the area of T, which is their intersection taken three at a time. If two hemispheres overlap forming angle A radians on a sphere of radius R, their area is $2AR^2$. The sphere has area $4\pi R^2$. The union of the three hemispheres leaves the triangle T′ antipodal to T uncovered. Note that T and T′ are congruent. We denote their common area by Δ. The area of the sphere is the area covered by the three hemispheres plus the area of T′. Algebraically, $4\pi R^2$ = (Area Covered by the Three Hemispheres) + Area of $T' = (2\pi R^2 + 2\pi R^2 + 2\pi R^2 - 2AR^2 - 2BR^2 - 2CR^2 + \Delta) + \Delta$, where the angles of T are A, B, and C in radians. Solving for the area Δ of T gives

$$\text{Area of T} = \Delta = R^2[(A + B + C) - \pi].$$

This result is known as Girard's Theorem. You can find this proof in John C. Polking's *Geometry on the Sphere*, available on the Web with pictures at mathforum.org in 2003. A nice proof is also found in both editions of Jeffrey Weeks's book *The Shape of Space*, Marcel Dekker, New York, 1985, with a Second Edition in 2001.

Flatness. Euclidean geometry is the geometry of a flat surface, one whose curvature is zero. Hence the sum of the angles in a triangle is always 180° and the total curvature of the area inside the triangle is always 180° − 180° = 0 (or π in radians). This conclusion is equivalent to the Angle Sum Theorem for triangles.

Equilateral Triangles. Let's see how equilateral triangles behave in the spherical, Lobachevskian, and Euclidean geometries. If you look at larger and larger equilateral (and hence equiangular) triangles centered on the north pole, you will see that the small ones have an angle sum close to 180°, while those which are close to the equator have an angle sum close to 3 × 180°. There is a limit to the angle sum for spherical triangles and for the area of such triangles. None has an

area greater than that of a hemisphere. Similarly, in Lobachevskian geometry, all triangles have a positive angle sum, although this sum is always less than 180°. The limiting area for a triangle in Lobachevskian geometry is the area that would be associated with an asymptotic triangle having angle sum 0°. Only in Euclidean geometry can we have equilateral triangles of arbitrarily large area, and in this case the sum of the angles is always 180°.

Equilateral Triangles and Uniformity of Curvature. The geometries we consider all have the usual congruence properties for triangles: SAS, ASA, SSS. Let us fix our unit length so that we can speak of segments of length s. Because of the congruence properties we assume, an equilateral triangle of side s has angles and an area that are determined by s and are independent of where the triangle is or how it is oriented. The angular deficit or excess depends only on s. The curvature of the surface within such a triangle is the same wherever the triangle is, and hence the curvature must be constant throughout the plane. This is geometrically clear for a sphere, whose curvature is the reciprocal of the square of the radius of the sphere. (For an n-sphere the curvature is r^{-n}, where r is the sphere's radius.) The Lobachevskian case of negative curvature cannot be visualized in such a simple way, but the same uniformity of curvature holds for it.

Angle Sums: Subtriangles and Supertriangles. The congruence postulates imply that, whether it is positive or negative, the curvature must be same everywhere in the plane. Suppose the Angle Sum Theorem (AST) holds for *some* triangle T. Divide T into *subtriangles.* If you draw a diagram, you will see that the angular excess or deficit for T is the sum of the excesses or deficits for its subtriangles. If any of these excesses were positive (or negative), then they all are positive (or negative) and the excess for T would be positive (negative) contrary to AST for T. So if T satisfies AST, then so do all subtriangles of T. Putting four copies of T together, you can create new triangles with sides twice the lengths of the original, and this *supertriangle* will also have angular deficit or excess equal to 0. If a triangle T′ is congruent to some triangle contained in T, then T′ satisfies the Angle Sum Theorem. Otherwise, T′ will be congruent to some triangle inside some similar enlarged version of T created by successive doublings of T. By this we see that if any T satisfies the AST, then so does any other triangle T′.

Similarity and the Angle Sum Theorem. Suppose we have two triangles $\triangle ABC \sim \triangle A'B'C'$, where the unprimed triangle has the larger sides. We may assume that the smaller triangle has been copied so that A and A′ coincide and B′ lies on AB and C′ lies on AC. Then the quadrilateral B′BCC′ must have angle sum 360°. Drawing the diagram and using the fact that the angle sum must be consistently greater or equal to 180° or less than or equal to 180°, you will see that $\triangle BB'C$ and $\triangle CC'B'$ must both satisfy the Angle Sum Theorem. So if we have any pair of similar triangles that are not congruent, then every triangle satisfies the Angle Sum Theorem.

John Wallis (1616–1703) was a British mathematician who deciphered coded messages for the Cromwell government and for Charles II's government after the Restoration. He proposed replacing the Euclidean parallel postulate with the postulate that given any $\triangle ABC$ and any segment EF, there is $\triangle EFG$ with $\triangle ABC \sim \triangle EFG$. Wallis favored his postulate over Euclid's or Playfair's because his does not involve lines being prolonged arbitrarily. Wallis's assumption is stronger than the existence of just one pair of noncongruent but similar triangles. We have shown above how the existence of just one such pair of triangles implies that the Angle Sum Theorem holds. As we will see below, the Angle Sum Theorem implies the usual Parallel Postulate. So Wallis's postulate is stronger than is needed to derive the Parallel Postulate uding our reasoning. Wallis needed his stronger assumptions because he used simpler arguments, ones not based on ideas of curvature of space.

Parallels, Pythagoras, and Similarity. The existence of similar figures is a distinguishing feature of Euclidean geometry. It is equivalent to the Parallel Postulate. With the Parallel Postulate, we were able to prove the Pythagorean Theorem. This reasoning can be reversed. If the Pythagorean Theorem is true, then we can construct an isosceles right triangle $\triangle ABC$ with hypotenuse BC. Say its legs have length a, so its hypotenuse has length $a\sqrt{2}$. The right angle is at A, and we let the hypotenuse be bisected by point A′. The segment AA′ divides $\triangle ABC$ into triangles $\triangle AA'C \cong \triangle AA'B$, by SAS. These right triangles have hypotenuses of length a. Because one leg of each has length $a/\sqrt{2}$, the Pythagorean Theorem ensures that the other leg also has length $a/\sqrt{2}$, hence they are isosceles.

The base angles of these triangles must then be 45°, because one pair of these four equal base angles adds up to give the right angle of the original right triangle. Hence the original triangle is a 45°–45°–90° triangle, and thus satisfies the Angle Sum Theorem. Now that we have one triangle $T = \triangle ABC$ that satisfies the Angle Sum Theorem, we know that all triangles must satisfy this theorem.

Angle Sum Theorem and Parallel Postulate. Now we show that if the Angle Sum Theorem is true, then the Parallel Postulate must hold. Given a line ℓ and a point P not on ℓ, we drop the perpendicular k from P to ℓ, meeting ℓ at some point we call O. Then the line m through P and perpendicular to the ℓ must be parallel to ℓ. (See Construction 7, page 225.) We must show that any line n other than n and passing through P must cross ℓ.

Draw the diagram of ℓ, m, O, and P so that ℓ has the same directions as the bottom edge of the paper. Marking the points at distance OP from O on ℓ and from P on m, you get two squares that share OP as an edge. It is the Angle Sum Theorem that lets you prove that these quadrilaterals are equilateral rectangles. If n is different from m, then it must slant downward toward ℓ as it enters one of these squares. Having entered that square, the line n must leave it. If n does this by crossing ℓ, then n is not parallel to ℓ and we are done. Otherwise n must cross the far side of this square at some distance δ below line m. We continue our row of squares between lines ℓ and m and follow the line n in the same direction. Each time it crosses the far boundary of one of these squares, line n must be closer to line ℓ by distance δ. Because δ is positive, eventually within one of these squares, n must cross from the upper side of ℓ to the lower side of ℓ, and hence n and ℓ cannot be parallel.

On Pythagoras, Angle Sums, and Parallels. The Pythagorean Theorem implies the Angle Sum Theorem, which implies the Parallel Postulate, and the existence of similar but noncongruent triangles implies the Angle Sum Theorem. We already know that the Parallel Postulate implies the Pythagorean Theorem, so each of these properties for a geometry implies the others. *The possibility of finding similar but noncongruent figures is the distinguishing feature of Euclidean geometry*, as contrasted to the non-Euclidean geometries discussed in Chapter 16.

To Learn How to Calculate Better. In a conversation with Andrew M. Gleason about his suggestions on the draft of this edition of *Geometry*, we asked ourselves what we wanted students to understand. We were asking, "What is the purpose of mathematics?" One answer Andy gave was "to learn how to calculate better." This idea is a shock when courses emphasize "understanding" and try to avoid getting bogged down with standard algorithms and fussy details. The importance of calculation seems particularly clear in trigonometry.

The definitions of trigonometric functions based on simple geometry, right triangles, and the Pythagorean relationship, $\sin^2 + \cos^2 = 1$, will let students work out the sides of various special triangles and the sines, cosines, and tangents of special angles. Knowing these things and having a calculator is enough for most practical purposes. More complicated triangle problems can be solved using the Laws of Sines and Cosines, if this material is covered.

The addition formulas are important for understanding how rotations are represented by matrices and for interpreting de Moivre's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. These addition formulas are used to work out the derivatives of the trigonometric functions in calculus. But when trigonometry was young, the addition formulas and the half-angle formulas were used for *calculation.*

Hipparchus of Nicæa (about 161–126 BC) was the first to work out an accurate table of trigonometric ratios and is considered the father of trigonometry. (See for instance *Isaac Asimov's Biographical Encyclopedia of Science and Technology*, more details can be found in *The Dictionary of Scientific Biography*, Charles Coulston Gillispe, editor in chief, New York, Scribner, 1970–1980. The Supplement has a long article on Hipparchus.)

Hipparchus must have used geometric versions of the addition formulas and half angle formulas to work out his tables, much as Ptolemy did in his *Almagest* (around 150 AD). For a beautiful description of these methods, see "How Ptolemy Constructed Trigonometry Tables" by Brother T. Brendan in *The Mathematics Teacher*, February 1965, pages 141–149, reprinted in *From Five Fingers to Infinity*, edited by Frank J. Swetz, Open Court, Chicago and La Salle, Illinois, 1994. For material on trigonometry that is easy to find, see Eli Maor's *Trigonometrical Delights*, Princeton University Press, 1989 (try http://pup.princeton.edu/books/-maor/chapter_2).

Construction of trigonometric tables was driven in part by the desire to measure distances. For instance, Hipparchus determined that the moon was 30 earth diameters from the earth, which is about right (30×8000 miles $= 240,000$ miles). One method he might have used is outlined below. It depends only on observations that can be made from a single location plus knowing that the area covered by totality during a solar eclipse is quite small. For the geometry of eclipses see pages 516 and 518 of *Geometry.*

Experimenting with shadows cast by disks or spheres, the Greeks found that the length of the dark conical shadow region (the umbra) cast by the moon and by the earth is about 108 times the diameter of the body casting the shadow. One way of checking this is to hold a disk so that it just eclipses the full moon. Measure the distance from the disk to your eye as a multiple of the diameter of the disk. This is one way to get the multiple 108 used here.

The sun lies at a great and relatively constant distance from the earth-moon system, so this multiple of 108 is approximately correct for the earth's and moon's umbrae because these umbrae are approximately similar figures. Because the umbra of the moon during a solar eclipse is at most a very small spot, the Greeks knew that the moon was in an orbit about 108 moon diameters from the earth. In addition, observations during lunar eclipses showed the Greeks that the diameter of the earth's umbra where the moon passes through it is about 2.5 lunar diameters. Lunar eclipses are fairly common and can be viewed over a large region, so this number is not too difficult to come by.

Now for the geometry. If x is the distance from the earth to the moon, then Lunar Diameter $= \frac{x}{108}$. Let D be the earth's diameter. If we are at distance x from

the earth to the apex of its umbra, we are at distance $108\text{D} - x$ from that apex. By similarity, the width of the earth's umbra at distance x from the earth is given by

$$\text{Earth Umbra Diameter} = \text{D}\frac{108\text{D}}{108\text{D} - x}.$$

Setting Earth Umbra Diameter = 2.5 × Moon Diameter gives us

$$\text{D}\left(\frac{108\text{D}}{108\text{D} - x}\right) = 2.5\left(\frac{x}{108}\right). \quad \text{Solving, } x = \frac{108}{3.5} \approx 31.$$

My numbers come from the online notes for Michael Fowler's University of Virginia course, Galileo and Einstein, galileoandeinstein.physics.virginia.edu. Checking with a current astronomy text and computing the numbers, you will find the following bounds for the distance to the moon, x, in terms of the earth's diameter: $28.5 \leq x \leq 31.8$. The length of the umbrae vary between 105 and 109 times the diameter of the sphere casting the shadow depending on the varying distance from the sun to the earth. See Earth Shadow, page 521 of *Geometry*, where the data give 108.6 earth diameters for the length of the earth's shadow.

The Greeks knew that the distances in the earth-moon-sun geometry varied as well as did the angles of the umbrae. One piece of evidence bearing on this is that some solar eclipses were total while at other times the moon was centered on the sun but left the rim of the sun uncovered (giving an annular eclipse). The angular size of the moon has been measured by various instruments over the millennia. Sosigines, in the 2nd century AD, noted that at some times the moon spans 11 fingerbreadths while at other times it spans 12 fingerbreadths, so the distance from earth to moon varies.

The Greeks knew that the distance they were trying to compute was not fixed, and they knew that their data were subject to errors. They would have taken any of their own numbers "with a grain of salt," and we should do so too. Hipparchus used observations of the solar eclipse of 14 March 190 BC to estimate the distance to the moon by a second method. See NASA's Goddard Space Flight Center website. Try www-istp.gsfc.nasa.gov/stargaze/Shipparc.htm or search on Hipparchos or Hipparchus and Distance to the Moon.

The Epilogue to the Introduction of Otto Neugebauer's *A History of Ancient Mathematical Astronomy*, Springer-Verlag, New York, Heidelberg, Berlin, 1975 contrasts ancient and modern astronomy as follows:

"Both Babylonian and Greek astronomy are based on a set of relative few data, like period relations, orbital inclinations, nodes and apogees, etc. The selection of these data undoubtedly required a great number of observations and much experience to know what to look for. Nevertheless, a mathematical system constructed at the earliest possible stage of the game was generally no longer systematically tested under modified conditions. ... Ancient observers were aware of the many sources of inaccuracies which made individual data very insecure. ... On the other hand period relations, e.g. times between planetary oppositions and sidereal periods, can be established within a few decades with comparatively high accuracy .... If a theory was capable of guaranteeing correct periodic recurrence of the characteristic phenomena the intermediate deviations would matter little, in particular when one had no means to distinguish clearly between theoretical and observational errors. Thus we notice an outspoken tendency in the development of astronomy to take refuge behind mathematical schemes, rather than to embark on systematic observational programs, which are so characteristic for astronomy since the invention of the telescope. ... Rightly, Tycho Brahe called his observational program the foundation of an *Astronomia Nova*. *Ancient Astronomy* is overwhelmingly *mathematical* astronomy."

The ancient goal was to develop and explore models for the motion of the stars, sun, and planets. Much of this work was mathematical, and developing methods of calculation was an important part of it. The system developed by Claudius Ptolemy some 200 years after Hipparchus held sway for over 1300 years until the emergence of the ideas of Copernicus, Kepler, and Newton.

David Bressoud's article "Was Calculus Invented in India?" in the January 2002 issue of *The College Mathematics Journal* shows why Hipparchus needed to go beyond similarity and develop a table of chords in order to work out the geometry of his model of the motions of the sun and earth. Hipparchus observed that spring and summer were longer than fall and winter, so he could not match these observations by placing the earth at the center of a circle and having the sun run around that circle at a uniform speed. In order to work out the details of Hipparchus' improved model we need a table of chords (or of sines). See Digression on Ancient Astronomy in these notes, Chapter 12, Lesson 5. Bressoud finds that the Greek estimate for $\sin 1°$ was accurate to about 1 part in 10,000.

Pushing this work forward, Indian astronomers in the state of Kerala developed the power series expansions for sine, cosine, and the arctangent before 1500 AD. Bressoud estimated that using these expansions would have allowed the Keralans to reduce the error to 1 part in 10^{12}. The methods are themselves capable of accuracy to any finite degree of precision, but Bressoud considered that practical limits of computation would have prevented the Keralans from going beyond 1 part in 10^{12}. (Source, e-mail exchange with Bressoud) We think of such expansions as part of calculus, but they were used in Kerala 200 years before Newton and Leibnitz. The need for more precise calculation often gives us new mathematics.

Power Series. You can work the other way — going from calculus and the idea of power series back to trigonometry. Take a leap and consider $e^x = \sum x^k/k!$, summed for $0 \le k$. Formal multiplication of power series and the Binomial Theorem shows that, term-by-term, $e^x e^y = e^{x+y}$, so this series behaves as an exponential function should. Differentiating term-by-term, we find formally that $\frac{d}{dx}e^x = e^x$, as it should. Using de Moivre's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, and recognizing that cosine is an even function and sine is an odd function, we might hope that the sum of the even terms in the power series expansion of $e^{i\theta}$ would give a power series for $\cos\theta$, and it does. The sum of the terms with odd exponents gives us $i\sin\theta$. Term-by-term differentiation of these series gives formulas agreeing with the expected

$$(*) \qquad \sin' = \cos \text{ and } \cos' = -\sin.$$

Note that this tells us that *the variable is in radians, not degrees.*

Assuming the validity of these term-by-term operations with these trigonometric series, we note that $\sin^2(0) + \cos^2(0) = 1$. Why? Let $S(\theta) = \sin^2(\theta) + \cos^2(\theta)$. Note that $\frac{d}{d\theta}S(\theta) = \frac{d}{d\theta}[\sin^2(\theta) + \cos^2(\theta)] \equiv 0$, by the usual rules of differentiation and (*) above, obtained by term-by-term differentiation of the series. So $S(\theta)$ is constant. Because $S(0) = 1$, $S(\theta) \equiv 1$, and these series satisfy geometric condition implied by the Pythagorean Theorem. Our series for sine and cosine give points $P(\theta) = (\sin\theta, \cos\theta)$ on the unit circle.

The arc $P(\theta)$ covers as θ goes from 0 to x has length

$$\int_0^x ((dx/d\theta)^2 + (dy/d\theta)^2)^{\frac{1}{2}} d\theta =$$
$$\int_0^x (\sin^2\theta + \cos^2\theta)^{\frac{1}{2}} d\theta = \int_0^x 1\, d\theta = x,$$

so these series give the sine and cosine in radians. This *must* be the case because the arc length traversed along the unit circle by $P(\theta)$ as θ goes from 0 to x is just x, which is the radian measure of the angle from 0 to θ on the unit circle.

Next, using $e^{i\theta}e^{i\phi} = e^{\theta+\phi}$ and de Moivre's formula we can work out the angle sum formulas for the trigonometric functions as follows:

$\cos(\theta+\phi) + i\sin(\theta+\phi) = e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
$= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi).$

Equating the real parts of the initial and final expressions above, we get $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$. Similarly, the angle sum formula for sine is obtained by equating the imaginary parts. These formulas agree with those you get from the geometric definitions of the sine and cosine functions.

To construct a set of trigonometric tables, we could follow Ptolemy. See the article by Brother Brendan, cited above, for details. What follows is a sketch of the method.

First, find accurate values for special angles found in regular polygons and differences of such angles, for example for angles 0°, 90°, 72°, 45°, 30°, 60°, 15°, 12°. Next use half angle formulas to get to smaller angles. We continue working in degrees. For $0 < \phi < \theta < 90°$, the the area of the sector between angles 0 and θ on a unit circle is $\pi\theta/360°$. Call this area Sector(θ). The area of the triangle spanned by the radii that bound this sector is $\frac{1}{2}1\sin\theta$. Call this area Triangle(θ). Drawing the figure, you can see that the ratio Sector(θ)/Triangle(θ) increases with the angle in this range. Ptolemy gives a proof of this inequality by Euclidean methods. So,

$$\frac{\pi\phi/360°}{\frac{1}{2}\times 1\,\sin\phi} = \frac{\text{Sector}(\phi)}{\text{Triangle}(\phi)}$$
$$< \frac{\text{Sector}(\theta)}{\text{Triangle}(\theta)} = \frac{\pi\theta/360°}{\frac{1}{2}\times 1\,\sin\theta}.$$

Taking the left hand side and the right hand side of this inequality and looking at what these tell us about the sine we get

$$\frac{\sin\theta}{\sin\phi} < \frac{\theta}{\phi}.$$

We know Triangle(θ) < Sector(θ) as well, which gives us $\sin\theta < \frac{2\pi\theta}{360°}$. Using values for the sine of $\frac{3}{4}°$ and $\frac{3}{8}°$, Ptolemy concludes $\sin(\frac{1}{2}°) = 0.008726851$, where I used a calculator to convert from Ptolemy's sexagesimal system to our decimal one. The same HP 48G+ gives $\sin(0.5°) = 0.0087265355$. These values agree to better than six places.

Ptolemy used linear interpolation to fill in the chords for chords whose angles ran from 0° to 1°, going by intervals of one minute. These correspond to sines running from 0° to $\frac{1}{2}°$ tabulated for half minute increments. The rest of his table can be filled out using the values already computed, linear interpolation, and the addition formulas.

The astronomers of Kerala improved on the idea of using linear interpolation and developed a polynomial approach that anticipates the power series presented above. The Kerala series were developed for a circle of radius R, the variable was the arc length, denoted by α in Bressoud's article, so $\alpha/\mathrm{R} = \theta$, where θ is the corresponding angle expressed in radians. The numbers computed by the Kerala series were the "half bowstring" (*ardha-jyā*, later just *jyā*), corresponding to our sines rather than to Ptolemy's chords. Explicitly, $\text{jyā}\ \alpha = \mathrm{R}\sin(\alpha/\mathrm{R}) = \mathrm{R}\sin\theta$.

Using series is one means toward more accurate and easier calculation. Napier's logarithms offered another means — and were also developed with trigonometric calculations in mind. Napier's logarithms reduced the problems of multiplication and division to the simpler ones of addition and subtraction — once you had his tables. Briggs's base 10 logarithms were much easier to use than Napier's.

To see how bold Napier's plan was and how different his logarithms were from ours look at "Napierian Logarithms and Natural Logarithms" by Howard Eves. This was published in *The Mathematics Teacher*, May 1960, and reprinted in *From Five Fingers to Infinity*, edited by Frank J. Swetz, Open Court, Lasalle, 1994. For Napier the logarithm of $10,000,000$ was 0, and for him the logarithm of a product was not the sum of the logarithms of its factors.

Section 1.2.2 of Volume 1, *Fundamental Algorithms* of Donald E. Knuth's series *The Art of Computer Programming*, Addison-Wesley, Reading, Massachusetts, First edition 1968 and Third Edition 1997 is devoted to "Numbers, Powers, and Logarithms." It shows how logarithms and powers can be defined and computed by simple algorithms that do not depend on power series or integrals, not yet invented in the time of Napier and Briggs. In the answers given in the back of Knuth's book, there are references to similar methods for calculating trigonometric functions.

My calculator whips out answers to 12 places, the 1 part in 10^{12} accuracy that David Bressoud felt was within reach using the methods of the Kerala astronomers around 1500 AD. These calculations are effortless for me, the mere pushing of a few buttons. The problem of carrying out trigonometric calculations is solved at this level by the microchip, but students still need to learn *what* to calculate and *what* the answer means.

How does your calculator or computer calculate sines, cosines, exponentials, and logarithms? Probably by completely different approach from those mentioned above. A common method is the CORDIC algorithm, and there are many web sites that explain it.

Learning how to calculate better is not simply a matter of being able to grind something out to a few more places accuracy. The world of high precision computing has reached previously inconceivable levels. In October of 1995, Takahashi and Kanada computed π to 6,422,450,938 decimal places. Let's put this in context. In the 15th century Kerala astronomers did work Bressoud felt capable of 1 part in 10^{12} accuracy. The 1995 work calculated π to an accuracy of 1 part in $10^{6,422,450,938}$. For late news on π, check the Web. Search on Jonathan and Peter Borwein (at Simon Frazer University), Simon Plouffe, David Bailey, Victor Adamchik, Stan Wagon, Richard Crandall, Yasumasa Kanada, and PiHex.

Surprisingly, it turns out that there are ways to find far off digits in the base sixteen (hexadecimal) expansion of *pi* without calculation of the earlier hexadecimal digits. This follows from work of David Bailey, Peter Borwein, and Simon Plouffe, who discovered the following formula:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

Using this and related formulas, the quadrillionth binary digit of *pi* has been calculated. It's 0. A quadrillion is 10^{12}, a large number indeed. See PiHex on the Web or the paper "Pi: A 2000-year Search Changes Direction" by Victor Adamchik and Stan Wagon, on the Web as of 2003. The suspicion is that no such "spigot algorithms" exist that will calculate any given decimal digit of π without calculating those before your target digit, but no one is sure about this.

A calculator gives $\sqrt{2} \approx 1.14142135624 = \mathrm{A}$. But paper and pencil shows:

$$\mathrm{A}^2 = 1.14142135624^2 = 2.000\,000\,000\,074\,098\,693\,76\,.$$

So this A is a bit too large to be $\sqrt{2}$. Even without any arithmetic we knew that the twentieth and last decimal digit of A^2 would be a 6. Thinking along this line, we can see why the square of a finite decimal expression with a proper fractional part can never be an integer.

What else can we find out about roots? If $\left(\frac{m}{n}\right)^2 = 2$, algebra gives $m^2 = 2n^2$. Could both m and n be integers? No, because if they were, then the integer on both sides of the equation $m^2 = 2n^2$ would be an exception to the theorem that *every integer has a unique factorization into primes.* On the right side, the isolated factor of 2 together with whatever factors of 2 there might be in n^2 tells us that there are an odd number of factors of 2 in $2n^2$. On the left hand side, there must be an even number of factors of 2 in m^2 — perhaps 0. An integer cannot have both an even and an odd number of factors of 2. So $\sqrt{2}$ is not the ratio of two integers; it is irrational.

Proving irrationality requires a different sort of calculation, a calculation that tells us what must be true if the prime factors on both sides of an equation are to balance. Working this out for each prime factor of ℓ, we can see that that an integer ℓ has a rational kth root only if ℓ is a perfect kth power of an integer, otherwise it is impossible to balance the prime factors in an equation of the form $m^k = \ell n^k$, where m and n are integers. So $\ell^{1/n}$ is either an integer (when ℓ is a perfect nth power of an integer) or it is irrational.

The argument that shows $\sqrt{2}$ is irrational is based on the fact that a positive integer can be written as products of primes in only one way, disregarding order. After working with a few examples, we should be willing

to believe that positive integers have unique prime factorizations. The proof of this fact goes back to Euclid (Book VII.1 – 3 and Book IX.14). The key to the proof is the fact that if a prime p divides the product of two integers m and n, then p divides at least one of m and n. The proof of this, in turn, is based on a computational method that writes the Greatest Common Divisor (G.C.D.), d, of integers k and ℓ as the least positive integer that can be written as $mk + n\ell$, where m and n are integers. This method is called the Euclidean algorithm. Generalized, this algorithm is the basis for breaking more elaborate algebraic structures into prime components in group theory and linear algebra.

Suppose you wanted to find the G.C.D. of 4293 and 2816. Rather than factoring them, you might observe that the common divisors 4293 and 2816 are exactly the same as the common divisors of 2816 and $1477 = 4293 - 1 \times 2816$. The largest number in this new pair of numbers is the smaller of the two numbers in our initial pair. By subtracting multiples of the smaller of our two numbers from the larger we can get a new pair with the same G.C.D., and we can do this so that the largest number in this new pair will always be the smallest number in our previous pair. This decreasing chain of integer pairs must terminate with a pair d, 0. The d in this pair is the G.C.D. Keeping track of the final result in terms of the original integers gives the G.C.D as an integral combination of those original numbers. This is the Euclidean algorithm. The only way to really understand it is to either do examples by hand or write a program to carry it out yourself. Applying this algorithm to 4293 and 2816 gives us $1151 \times 2816 - 755 \times 4293 = 1$, so these two numbers have a G.C.D. of 1. Their prime factorizations also show that they are relatively prime: $4293 = 3^4 \times 53$ and $2816 = 2^8 \times 11$.

Suppose prime p divides the product km. We want to show that p divides one of the factors, k and m. If p divides k evenly, then we are done for p divides the first of the two factors. If not then the G.C.D. of p and k is 1 because the only positive divisors of a prime are 1 and itself. So we can find integers s and t such that $sp + tk = 1$. Multiplying this equation on both sides by m we see that $spm + tkm = m$. But p divides spm and tkm, whose sum is m, so p divides m, the second of the two factors. In either case a prime that divides a product of integers must divide one of the factors. Knowing this, you can prove that positive integers have unique prime factorizations by using cancellation to show that no smallest counterexample to unique factorization could exist. From unique factorization it follows that $\sqrt{2}$ is irrational.

The argument that $\sqrt{2}$ is irrational is based on prime factorization of integers, a result established by essentially computational methods. But we use computing in this analysis for something quite different than finding the decimal expansion of $\sqrt{2}$.

"The purpose of computing is insight, not numbers." This is the maxim at the front of Richard Hamming's classic *Numerical Methods for Scientists and Engineers*, Second Edition, McGraw Hill, New York, 1973. No matter how closely we measure the diagonal of a unit square, we will never be able to conclude from measurements that its length is irrational. In practical measurement, there is only finite precision; infinite decimals play no role. No matter how many decimal places we compute of $\sqrt{2}$, these numbers will never give evidence that $\sqrt{2}$ is irrational. More subtle reasoning, using algebra and number theory, is needed to show that quantities are irrational multiples of the unit.

Mathematics *is* a practical tool, and computation is essential. But computation also plays a role in mathematical theory. Reasons beyond the practical drive the pursuit of accuracy to 1 part in 10^{12}, and beyond — even to the insistence on *exact* algebraic answers. Without exact calculations, we would never understand that some numbers are irrational.

Without exact calculations we could never show that 12 congruent regular pentagons fit together *exactly* to form a regular dodecahedron. This result is given in Euclid, Book VIII, Proposition 17. The construction is found in *Geometry* in Dodecahedron Construction, page 681. All that is needed to match Euclid's proof is to carry out the computations algebraically and to see that the faces fit as the figure suggests.

To understand how precise 1 part in 10^{12} is, consider that the meter was intended to be 1/10,000,000th of the length of a quadrant on the earth's surface — measured along meridian starting at the North Pole, passing through Paris, and ending on the equator. To measure this quadrant with an accuracy of 1 part in 10^{12}, we would have to get the distance right to within $\pm 1/100$th of a millimeter. Such accuracy is unattainable.

We don't need the 1 part in 10^{12} accuracy unless we are following some intricate chain of calculations through many steps. Even in chains of triangulations, uncertainties in the initial data introduce errors larger than those likely to arise from errors in the trigonometric functions. But to decide whether the predictions of the Ptolemaic system match the movements of the sun, moon, and planets, we need to know that deviations between observations and predictions are real, and not due to errors in calculations. A theoretical model like this is supposed to work *exactly*, and so it is appropriate to ask for extreme, even absolute, precision in testing it. This is one reason that astronomers played an important part in trigonometrical advances.

A Colorful Contributor to Trigonometry. Some of those who worked on trigonometry led exciting lives. Consider Naṣīr ad-Din aṭ-Ṭūsī, who is described by Otto Neugebauer in Part One of his previously cited *History of Ancient Mathematical Astronomy* as follows:

"In 1255 the Mongol prince Hulāgū Kahn, grandson of Genghis Kahn, succeeded in taking the mountain stronghold of the Assassins, al-Alamut, which had been a major obstacle for a Mongol domination of Persia and Iraq. Among the prisoners taken by Hulāgū in al-Alamut was Naṣīr ad-Din aṭ-Ṭūsī, who had been held as an honored prisoner and astrological advisor by the 'Old man of the Mountain,' the Grand Master of the Assassins. Āt-Ṭūsī once more became the trusted advisor of his master, now Hulāgū, and seems to have played in that capacity a somewhat sinister role in the end of the last Abbasid Caliph when Baghdad was taken by the Mongols in 1258."

Neugebauer is more concerned with astronomy, so we turn to the Revised and Enlarged Edition from 1919 of Florian Cajori's *A History of Mathematics* for an appraisal of aṭ-Ṭūsī's mathematics:

"During the supremacy of Hulagu, lived Nasir-Eddin (1201–1274), a man of broad culture and an able astronomer. He persuaded Hulagu to build him and his associates a large observatory at Magara. Treatises on algebra, geometry, arithmetic, and a translation of Euclid's *Elements* were prepared by him. He for the first time elaborated trigonometry independently from astronomy and to such great perfection that, had his work been known, Europeans of the fifteenth century might have been spared their labors."

Nasir Eddin's work on non-Euclidean geometry is mentioned on page 669 of *Geometry.*

Lines, Vectors, Transformations, and Matrices. The notes below outline some other ways of looking at lines and their equations, in the plane, and in higher dimensions. For more on these matters, see standard texts or Web resources such as Eric Weisstein's MathWorld.

Equations of Lines, Distances. The general form for the equation of a line ℓ in the coordinate plane is

$$(*) \qquad Ax + By = C, \text{ where } A^2 + B^2 \neq 0.$$

The slope of ℓ is $m = -\frac{A}{B}$ and is independent of C. Using this we can find the point on ℓ closest to the origin. The line $k \perp \ell$ through the origin has slope $\frac{-1}{m} = \frac{B}{A}$, so its equation is $-Bx + Ay = 0$. Finding the intersection of k and ℓ by algebra gives us the following coordinates for the point P closest to the origin on ℓ:

$$x = \frac{AC}{(A^2 + B^2)} \text{ and } y = \frac{BC}{(A^2 + B^2)}.$$

The distance from ℓ to the origin O is the distance from P to O. By the distance formula this is

$$(**) \quad \text{Distance from } \ell \text{ to the origin} = \sqrt{\frac{C^2}{(A^2 + B^2)}}.$$

When $C = 0$ we get the line that is parallel to the general line ℓ and that passes through the origin. Call this line n. Other values of C give lines parallel to n and at varying distances from the origin. Using C in place of $\sqrt{C^2}$ in (**), we get a *signed distance from ℓ to the origin* O.

$$\text{Signed Distance from } \ell \text{ to the O} = \frac{C}{\sqrt{(A^2 + B^2)}}.$$

This distance is positive when ℓ is on the same side of n as the point with coordinates (A, B), zero when $\ell = n$, and negative when ℓ is on the opposite side of n from the point (A, B). This analytic geometry ties in with the vectors as studied in linear algebra.

Vectors and Lines. In Cartesian coordinates we can think of points in the coordinate plane as vectors, say v and w. These we conceptualize as arrows extending from the origin to the points defining the tips of the arrows. Suppose our v and w are specified by coordinate points as follows: $v = (A, B)$ and $w = (x, y)$. The length of a vector is written $\|v\|$ and defined in Cartesian coordinates by

$$\text{Length of vector } w = \|w\| = \sqrt{x^2 + y^2},$$

which makes this length the distance from the origin to the point of the arrow, by the Pythagorean Theorem.

The *dot* or *scalar product* of these two vectors is written $v \cdot w$ and defined by

$$\text{Dot Product of } v \text{ and } w = v \cdot w = \|v\| \|w\| \cos\theta,$$

where θ is the angle between v and w. In Cartesian coordinates the dot product is the sum of the products of the coordinates. For example, $v \cdot w = Ax + By$.

If $v \neq 0$, the multiples of v, $\{ \lambda v \mid \lambda \text{ is a real number} \}$, form a line k through the origin, O. The vectors $\{ w \mid v \cdot w = d\|v\| \}$ form a line ℓ perpendicular to the line k and at a directed distance of d from the origin. If d is positive, ℓ is displaced in the direction that v points. If d is negative, ℓ is displaced in the opposite direction from the origin. This agrees with the directed distance as developed above.

Dot Products and Cosines. As above, think of v and w as two vectors in the plane represented by arrows from the origin. Let θ be the angle between these arrows. Then $v \cdot w = \|v\| \, \|w\| \cos\theta = Ax + By$, so

$$\left\{ \begin{array}{c} \text{Cosine of the Angle} \\ \text{between } v \text{ and } w \end{array} \right\} = \cos\theta$$
$$= \frac{v \cdot w}{\|v\| \, \|w\|} = \frac{Ax + By}{\|v\| \, \|w\|},$$

which gives a simple way to find $\cos\theta$, if we know the Cartesian coordinates of the points of these vectors.

The dot product is built in as a vector operation on calculators such as the TI-89 (under 2nd function, Math, Matrix, Vectr ops) or the HP-48G+ (under Math, Vectr). In Cartesian coordinates in n-dimensions the dot product of two vectors $v = (v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1w_1 + v_2w_2 + \cdots + v_nw_n = \|v\|\,\|w\| \cos\theta$, where $\|v\|$ is the length of the vector v and this length is defined by $\|v\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$ and $\|w\|$ is the similarly defined length of w. So this method for finding the angle between two vectors generalizes to higher dimensional Euclidean spaces. In order to have this work in our coordinates, we must use a rectangular system of coordinates with equal units along each axis. In this case, the generalization of the Pythagorean formula for lengths between points is valid. These are called *orthonormal coordinates.*

Dot Products, Rotations, Matrices. Suppose u is a unit vector whose tail is at the origin, O. Then its tip lies somewhere on the unit circle about O. Find the smallest positive counterclockwise angle θ so that a rotation about O through angle θ carries $(1,0)$ (on the x-axis) onto the tip of u. Then the tip of u has coordinates $(\cos\theta, \sin\theta)$. The same counterclockwise rotation carries the point $(0,1)$ on the y-axis to $(\cos(\theta + 90°), \sin(\theta + 90°)) = (-\sin\theta, \cos\theta)$, because (1,0) is 90° counterclockwise from $(1,0)$ on the unit circle about O. This gives us a second vector v whose tail is at O and whose tip is at $(-\sin\theta, \cos\theta)$. I always draw a diagram to work these relations out. The words are only good if you back them with your own pictures.

We know $u \cdot u = v \cdot v = 1 = \|u\| = \|v\|$ because these vectors run from O to points on the unit circle, and computation gives $\cos^2\theta + \sin^2\theta = 1$ for $u \cdot u$ and $v \cdot v$, confirming this. We expect $u \cdot v = 0$ because these vectors are at right angles to each other and $u \cdot v = \|u\|\,\|v\| \cos\phi$, where ϕ = the angle between u and v, which is +90°. And computation confirms this, too.

Additivity. If T is a rotation through the counterclockwise angle of θ, what is $\mathrm{T}\big((x,y)\big)$, where these points are given in terms of our original coordinate system? The vector sum of vectors w and z is defined by the Parallelogram Law to be the point $w + z$ such that O, w, $w + z$, w form a parallelogram. Because rotations are isometries, they preserve distances and send parallel lines into parallel lines, hence $\mathrm{T}(w + z) = \mathrm{T}(w) + \mathrm{T}(z)$, because these vector operations are defined in terms of parallelism. A mapping that distributes over addition the way that T does is said to be *additive.*

Scalar Multiplication. Multiplication of a vector, w, by a number, *lambda*, is defined by requiring that the point of the vector λw lie on the ray from O through the head of the arrow representing w at a distance $\lambda\|w\|$ from O. This operation is *scalar multiplication* of vectors and is defined geometrically in terms of distances, given the origin, O. Hence, scalar multiplication is respected by isometries that leave O fixed. That is, $\mathrm{T}(\lambda w) = \lambda\mathrm{T}(w)$. This paragraph and the previous one show us that

$$\text{Linearity} \quad \mathrm{T}(\alpha u + \beta v) = \alpha\mathrm{T}(u) + \beta\mathrm{T}(v),$$

for all scalars α and β and all vectors u and v. Mappings with this property are *linear.*

Using the linearity of T, we can express the rotation T using the coordinates x and y of our point to find the coordinates x' and y' of its image.

$$\begin{aligned}\mathrm{T}\big((x,y)\big) &= \mathrm{T}\big(x(1,0) + y(0,1)\big)\\ &= x\mathrm{T}(1,0) + y\mathrm{T}(0,1)\\ &= x\,(\cos\theta, \sin\theta) + y\,(-\sin\theta, \cos\theta)\\ &= (x\,\cos\theta - y\,\sin\theta, x\,\sin\theta + y\,\cos\theta)\\ &= (x', y').\end{aligned}$$

Matrix Notation. This gives a compact way of writing T. We write our vectors as columns. The transformation T is represented by a 2×2 array of numbers. The matrix operates on the coordinate vectors following the Rows × Columns pattern seen in the right-hand equality below.

$$\begin{aligned}\begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\\ &= x\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + y\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.\end{aligned}$$

The left-hand equality above shows how x' and y' arise as the sum of column vectors representing the images of $(1,0)$ and $(0,1)$ in our coordinate system.

Using this pattern, we can define transformations using coordinates and matrix notation. For example, we may define $\mathrm{S}\big((x,y)\big) = (x', y')$ by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}.$$

Here you can put in any numbers you like for the a_{ij} and you will get some sort of transformation, S. Whatever you choose for these numbers, $\mathrm{S}\big((0,0)\big) = (0,0)$ so the origin is mapped onto itself. Furthermore, lines are mapped onto lines or collapsed to single points. Triangles may be distorted or flattened, but lines never become curved.

Area and the Determinant. A key geometric experiment is to see what S does to the square whose vertices are $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$, called *the unit square.* S will always send this square onto a parallelogram, although that parallelogram may be flattened onto a line. The signed area of that parallelogram is given by the determinant of the matrix A, written $\det|\mathrm{A}|$ and defined by the following computation

$$\det|\mathrm{A}| = \det\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{21}.$$

This area is counted as positive if a clockwise path around the unit square gives a clockwise tour around its image, and the signed area is negative if the orientation is reversed.

The determinant used to be studied in Algebra 2, in connection with Cramer's Rule, where it was a formal device used in solving systems of linear equations. Now it has been recognized that Gauss-Jordan elimination is an easier, more efficient, and more comprehensible way to solve such systems. The determinant has moved to the linear algebra course, a new addition to the curriculum that you can read about at such sites as www.cde.ca.gov/ standards/math/html in 2003.

Why does $\det|\mathrm{A}|$ give the area of S(UnitSquare)? In order to be able to see the geometry and work out the algebra, assume that $w = \mathrm{S}\big((1,0)\big)$ lies in the first quadrant and $z = \mathrm{S}\big((0,1)\big)$ lies in the second quadrant. The other possibilities can be handled similarly, with careful attention to the signs of all quantities. The UnitSquare has vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ in the counterclockwise order. The corresponding vertices of S(UnitSquare) are: 0 (the vector zero), w, z, and $w+z$, and they are also in counterclockwise order, so the orientation is preserved, and the signed area of S(UnitSquare) will be positive.

If the matrix representing S is $\mathrm{A} = (a_{ij})$, we have the following, where we shift from row to column vector representation for w and z:

$$w + z = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}. \end{pmatrix}$$

Because w is in the first quadrant, a_{11} and a_{21} are both positive. Because z is in the second quadrant, a_{12} is negative while a_{22} is positive. S(UnitSquare) is a parallelogram with vertices 0, w, $w+z$, and z. Drawing the figure, you will see that these lie on the sides of the rectangle R with vertices $(a_{12},0)$, $(a_{11},0)$, $(a_{11},a_{21}+a_{22})$, $(a_{12},a_{21}+a_{22})$. The width and height of R are $-a_{12}+a_{11}$ and $a_{21}+a_{22}$, where the $-a_{12}$ is needed because a_{12} is negative while the other a's are positive. From the diagram you see that

$$\begin{aligned} &\text{Area of R} = (-a_{12}+a_{11})(a_{21}+a_{22}) \\ &= \alpha\big(\mathrm{S}(\text{UnitSquare})\big) + 2\times\frac{1}{2}a_{11}a_{21} + 2\times\frac{1}{2}a_{22}(-a_{21}), \end{aligned}$$

where α denotes the area and the last two summands account for the areas of the four right triangles that surround S(UnitSquare) in R. Solve this for αS(UnitSquare) and you will find $\alpha\big(\mathrm{S}(\text{UnitSquare})\big) = a_{11}a_{22} - a_{21}a_{21}$, which is how $\det|\mathrm{A}|$ was defined.

Note that the area found above can also be expressed by $\|w\|\,\|z\|\sin\theta$, where θ is the counterclockwise angle between vector w and vector z. This tells us that

$$\sin\theta = \frac{1}{\|w\|\,\|z\|}\det\begin{vmatrix} w_1 & z_1 \\ w_2 & z_2 \end{vmatrix} = \frac{w_1z_2 - z_1w_2}{\|w\|\,\|z\|}.$$

The numerator on the left gives the magnitude of the *cross product* of the vectors w and z, denoted, $w\times v$, an operation that can be found in standard sources such as Eric Weisstein's MathWorld on the Web.

▶ *Matrices and Computer Graphics.* Software that displays material or sends it to a printer usually has the ability to resize, rotate, and shift the image. Matrix operations are built into this software. Postscript™ is a example of such software. It manipulates images by sending what was at the point (x,y) to a new point with coordinates (x',y') which are determined by a matrix as follows.

$$(x',\ y',\ 1) = (x,\ y,\ 1)\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ t_x & t_y & 1 \end{pmatrix}.$$

We multiply using Rows×Columns, but Postscript™ uses row vectors, so these must be placed left of the matrix that operates on them. The matrix shown above is the Current Transformation Matrix (CTM) in Postscript™. Multiplying we get

$$\begin{aligned} x' &= ax + cy + t_x \\ y' &= bx + dy + t_y. \end{aligned}$$

The third component is always 1, and the t_x and t_y are used to perform translations of the coordinates. The Postscript™ language is built into many graphics devices. It allows you to create many interesting geometric patterns by downloading Postscript™ programs directly to your printer. There is a wealth of material on Postscript™ and creation of designs both in books and on the Web. Search on Postscript™, designs, and, if you wish, fractals.

The matrices representing two sorts of transformations in this augmented vector pattern are given below.

Rotation about O through angle θ:
$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Dilation of magnitude λ about (x_0,y_0):
$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ (1-\lambda)x_0 & (1-\lambda)y_0 & 1 \end{pmatrix}.$$

Chapter 12. Circles.

To the Student. The sun or moon appear as disks. A drop of oil on water spreads out in a circle. Wheels are round. A circle is defined by its center and its radius. A circle has perfect symmetry about its center and is sent onto itself by reflection in any line through its center and by rotations about its center. Your blood pushes outward in all directions equally, so arterial cross-sections are circular. In order for the bearing on an axel to allow the wheel to turn freely, it must have rotational symmetry, so it must be circular.

Here we study this simplest curved shape, its uses, and its properties.

Lesson 1. Circles, Radii, and Chords. Here we formally meet the circle and its radii, diameters, and chords. We learn that if a line through the center of a circle is perpendicular to a chord, it bisects that chord (Theorem 56) and that if a line through the center bisects a chord that is not a diameter, it is perpendicular to the chord (Theorem 57). These results are combined in Proposition 3 of Book III of Euclid.

Set I exercises cover the definitions and key ideas. Exercises 16–26 give direct proofs of Theorems 56 and 57 and are good as spiral review and for learning more about circles.

Exercises 27–29. The indirect proof Theorem 58, which states that the perpendicular bisector of a chord goes through the center of the circle, can be replaced by the following direct proof. If AB is the chord of circle O, then the center O is equidistant from A and B, as is C, the point that bisects AB. Hence, by Theorem 16, page 212, O and C determine the perpendicular bisector of AB. Hence O lies on the perpendicular bisector of the chord AB, as was to be shown.

Set II has many applications, including the argument Ole Rømer (1644–1710) used to make the first determination of the speed of light, Exercises 37–39. The Three Circles figure is quite surprising when completed, Exercises 43–47.

Set III. Island Puzzle yields to a nice "Aha!"

Lesson 2. Tangents. We open this lesson with the hammer throw. The proof that a tangent is perpendicular to the radius to the point of tangency (Theorem 59) is direct and geometric. A proof that a line ℓ perpendicular to a radius OA at A (Theorem 60), can be given using the Pythagorean Theorem. If B $\neq$ A is on ℓ then $\mathrm{OB} = \sqrt{\mathrm{OA}^2 + \mathrm{AB}^2} > \mathrm{OA}$. So B lies strictly outside of circle O, and ℓ touches the circle O only at A. Thus ℓ is tangent to O at A.

Algebraic methods based on established results are as valid as geometric ones. The choice of method of proof depends on personal preference or on which sort of reasoning seems easiest to follow for a given result. What is clearest for me may not be clearest for you.

Set I. This covers the core material for the lesson. Tangents allow us to smoothly piece together portions of circles and segments. See Band Saw (1–3) and Railroad Curve (19–23). In practice, railroad curves are designed to be even smoother than those described in (19–23). They keep the "jerk derivative" within bounds. This is the derivative of the acceleration. When a railroad car rounds a curve, the radius of curvature, r, of the track and velocity, v, of the train determine the "centrifugal force," which has magnitude v^2/r. This is the force that makes you lean toward the outside of a curve on a train. If this force shifts suddenly, the passengers will lurch from side to side in the car. That would not be good. So engineers need to make the lead-in to a curve more gentle than our considerations suggest. You can look up jerk derivative on the Web if you want more information about this.

Set III asks about a point from which the earth and moon have the same apparent diameter. You might ask your students if they think there are other points with this property.

There is a second such point beyond the moon, where the moon just eclipses the earth. The locus of all such points is a *circle of Apollonius*, a circle on which the ratio of distances to the centers of the moon and of the earth remains constant. If d_e is the distance from our point P to the center of the earth and d_m the distance to center of the the moon, while r_e and r_m are the radii of the earth and moon, the condition $d_e/d_m = r_m/r_e$ will insure that the two have the same apparent diameter. Inversion in this circle of Apollonius interchanges the points that are the centers of the earth and moon. Search on circles of Apollonius or inversive geometry at websites such as Eric Weisstein's MathWorld. See also, *Geometry*, Hollow Earth, page 668 for the inversion in a sphere that interchanges points P and P′.

Lesson 3. Central Angles and Arcs. Arcs and arc measure are properly introduced here along with *reflex angles.* Intuition and common practice let us deal with clockwise or counterclockwise rotations through any number of degrees. For technical reasons, Euclid stuck to angles no larger than a straight angle. We need angles larger than 180° to deal with major arcs. Postulate 10, The Arc Addition Postulate, can be proven from the Protractor Postulate by analyzing the different possibilities for arrangements of minor and major arcs determined by points A, B, and C.

Curvature and the Measure of Arcs. The local curvature at each point of a circular arc is $1/r$, where r is its radius. The product of the length, S, of the arc by its curvature gives the total curvature of the arc, S/r. This total curvature is the radian measure of angle through which the arc's tangent turns as the

arc is traversed. Using calculus, we can define the *radius of curvature, r,* for a smooth curve at each of its points and a corresponding *curvature, 1/r,* at each point of the curve, and the radian measure of the angle through which the tangent of the curve turns as we move along any stretch of the curve is the integral of ds/r, where ds is the element of length for the curve.

The tangent may turn counterclockwise or clockwise depending on the direction of travel along the arc and on which way the arc is bowed. For a closed curve, we can take this into account by taking the curvature to be $= 1/r$ when the arc is bowed outward and $-1/r$ when the arc is bowed inward. This will make our accounting come out right no matter which way we go around the curve.

For a smooth simple closed curve, the integral of ds/r around a single circuit must be 2π radians. This is because, taking clockwise and counterclockwise turnings with the proper accounting of signs, the tangent must turn through a total of one full rotation. To convert to degrees, multiply by $360°/2\pi$ in all calculations and the integral becomes 360°.

If we are dealing with arcs of circles smoothly put together, this integral becomes a sum over the arcs of the arc measures, taking the direction of turning (clockwise or counterclockwise) into account. This is why the sum of the arc measures in The Square That Isn't (30–36) must be 360°. The same argument holds for the arc measures in the Set III exercise, so Obtuse Ollie gets this one right.

Set I. Nice coverage of key ideas. Arc Illusion (19–22) is provocative.

Set II. Note the original proofs, Exercises 43–47. The Night Sky (40–42) asks intriguing questions.

Lesson 4. Inscribed Angles. "In a circle the angle at the center is double the angle at the circumference, if the angles intercept the same arc" as we read in Euclid, Book III, Proposition 20. This becomes Theorem 63 when stated in terms of inscribed angles and arc measure. Corollary 1 to Theorem 63 tells us that inscribed angles that intercept the same arc are equal (Euclid Book III, Proposition 21). These are useful in applications and lead to further by allowing us to show that various triangles connected with circles are similar. See Exercises 55–58 of this lesson and Exercises 6–10 of Lesson 6, page 512.

The proof of Theorem 63 (6–19) uses algebra applied to angle measures.

Basic ideas are covered in **Set I** exercises. **Set II**, intersecting Triangles and Circles (55–58) is beautiful, and **Set III**'s Finding the Center is a practical problem that gives a nice "Aha!"

Lesson 5. Secant Angles. Given two lines that both cut a circle, we can find the angle between them by knowing the arcs they cut off between them. The proofs of formulas are given in Exercises 1–12, and they offer practice with the ideas of the previous lesson.

Here is a more dynamic way to look at these theorems. Look at Figure 1 on page 511. Ask yourself: How much angular rotation is needed to turn line DC counterclockwise so as to coincide with line EA? The process is done in two stages. Keep D fixed as a pivot and rotate C around so the line runs through A. You have a counterclockwise rotation of $80°/2 = 40°$, by Theorem 63. Now your line runs through D and A. Pivoting it about A, rotate the lower part counterclockwise of it until it passes through E. This is an additional counterclockwise rotation of $40°/2 = 20°$. So the total counterclockwise rotation is $40°+20° = 60°$. This is the rotation required to bring one side of $\angle$CBA into line with the other; it is also half the sum of the intercepted arcs. This works whenever B is inside the circle. When B is outside the circle, the rotation associated with the arc closest to B will have the opposite direction, and we get a rotation of magnitude half the measure of the larger arc take away the measure of the smaller arc. For general angles use an algebraic version of the examples done above.

Set I. Note the heavier arcs on the circle in the Danger Angle (17–21). These suggest the right approach. **Set II**, The Seasons (45–48) are intended to be worked out informally, but they can be dealt with quantitatively as well. See the digression on ancient astronomy below. Angles Problem (56–59). This figure holds the secret to working out the angles in the Tantalus problem, the subject of the Set III exercise on page 97. The person who devised the Tantalus problem removed the clues that link it to this regular 18-sided polygon.

The **Set III** exercise will become clear if students look at and think about the figure at the bottom right of the page.

A Digression on Ancient Astronomy. In The Seasons exercises students have a chance to think about early efforts to model the movement of the sun in the sky from season to season. This is the "solar theory" of ancient astronomy. Astronomy was a practical matter because the regular annual changes in the sky were a guide to when crops needed to be planted or gathered. Astrology was a strong motivation as well, because the stars and planets were thought to influence human actions.

The Babylonian mathematical astronomy was based on records dating back to the 8th century BC. Its methods were primarily arithmetic. See, for example, Otto Neugebauer's *The Exact Sciences in Antiquity*, available from Dover Publications in 2003, or search for Babylonian astronomy on the Web.

Greek mathematical astronomy used *geometric models.* These exercises are based on Hipparchus' model for the movement of the sun along its path (the ecliptic)

through the zodiac. The zodiac is made up of constellations of fixed stars. These are so far away that they represent directions that are unchanged by shifts on the scale of the diagram seen here.

Given the earth's position in this figure, point A lies in the direction of the sun relative to earth at the vernal equinox. Viewed from the sun, the equinoxes are the points when the earth's equatorial plane cuts through the center of the sun. At such a point, the sun is directly overhead at the point on the earth's equator where it is noon. At this moment, the dividing line between night and day on the earth runs through both poles. Consequently, days and nights are of almost equal duration at this time of year, hence the name *equinox* from *equal night.*

Viewed from the earth, the equinoxes are the times when the sun's path (the ecliptic) intersects the celestial equator. The *celestial equator* is the projection of the earth's equator from the earth's center onto the celestial sphere (which is simply the set of directions represented by points in the sky, fixed in relationship to the fixed stars). The celestial equator divides the northern and southern hemispheres of the heavens. When the sun stands at the celestial equator as viewed from the earth, it is directly overhead of some point on the earth's equator (where it is high noon) and the sun illuminates a hemisphere that passes through both poles. This is exactly the configuration described in the previous paragraph, except that we are viewing it from the earth rather than from the sun.

The *summer solstice* is the instant when the earth's northern hemisphere is most turned toward the sun. Looking at this from the earth's point of view, it is when the sun is farthest north on the ecliptic. This time of year gives the longest days in the northern hemisphere. Similarly, the *winter solstice* is when the sun reaches its furthest south point on the ecliptic and the days are shortest in the northern hemisphere.

In the diagram for Exercises 45–48, the number of degrees from the vernal equinox at A to the summer solstice at B is given as $93.14° \approx \frac{94.25}{365.25} \times 360°$, where Hipparchus used 94.5 days as the number of days in spring and 365.25 days as the number of days in the year. (Neugebauer in his *History of Ancient Mathematical Astronomy*, cited earlier, gives Hipparchus's value of the length of the year more precisely as $365\frac{1}{4} - \frac{1}{300}$ days. His source is Ptolemy in the *Almagest.*) By a similar calculation, the 92.5 days from the summer solstice at B to the autumnal equinox at C corresponds to the 91.17° of the length of the arc corresponding to summer.

The standard way for the Greeks to deal with circular movements over time was for them to assume that the circular movements were uniform, proceeding at a constant rate. But that does not fit this picture, since $\angle AEB = \angle BEC = \angle CED = \angle DEA = 90°$. So how could Hipparchus make a sun moving uniformly around a circle fit into the nonuniform pattern he found for the lengths of the seasons? This is the question students answer here. His solution is simple. He moves the earth off center *but keeps the perpendicular directions to the equinoxes and solstices the same.* This is OK because these directions are tied to points along the zodiac. These points are so far distant that the tiny shift in the earth-sun system leaves them essentially unchanged.

Spring and summer should account for $93.14° + 91.17° = 180° + 2 \times 2.155°$. So we need to move points A and C to new positions A′ and B′, each closer to D by an arc of measure 2.155°. Now if we relocate E to E′ on the segment A′B′, then spring and summer together will have the right duration if the sun moves along the circle at a uniform rate. Next we choose B′ on the major arc from A′ to C′ so that the arc from A′ to A′ has measure 93.14°. This ensures that spring and summer each have the right duration. Finally the foot of the perpendicular from B′ to segment A′B′ gives us the location of M′. This is the construction Neugebauer gives for Hipparchus' model of the motion of the sun.

If we assume a nominal radius of 60 for the circle here, we find that E′ is displaced 2.2562 units below E and 1.0314 units to the right. The total displacement determined by the Pythagorean Theorem is 2.3891 units, rounded. This displacement makes an angle of $-65.4360°$ with the ray EC in the original picture. If you compute the lengths of the seasons given by this model in degrees, you will find they match those given in the original figure.

These models were simply devices for making predictions that matched observations, particularly the observations made at solstices, equinoxes, conjunctions, eclipses, and other notable events. Just as we may find it easier to think of the sun as going around the earth or of the earth as turning on its axis beneath the sun depending on the problem at hand, the Greeks used first one model, then another. The models are mathematically equivalent, but quite different physically. On page 57 of his *History of Ancient Mathematical astronomy*, cited earlier, Neugebauer says "The frequent shift from one cinematic model to another is the best indication of the fact that none of these implied that there existed in nature a corresponding mechanical structure."

Not until Newton do we see a theory of the solar system that attempts to explain the motions of the solar system by a dynamic physical theory. What ancient mathematical astronomy gives is simply a mathematical description of what you will see. It says nothing about the mechanism involved.

Hipparchus knew that to properly track the motions of the sun, you needed to allow for a backward movement of the equinoxes on the ecliptic by 1° per century. This value is close to our modern value for the precession

of the equinoxes. Over the years this adds up. The vernal equinox is known as "the first point of Aries" because it originally took place as the sun entered the constellation of Aries. The vernal equinox now takes place when the sun is in Pisces. In another 600 years it will take place when the sun enters Aquarius (hence, the phrase "the dawning of the Age of Aquarius"). We understand the physical mechanism behind these slow changes. The earth precesses; its axis of rotation turns slowly about an axis perpendicular to the plane of the solar ecliptic like a giant top. Long ago, people *knew* of these effects, although they had no physical theory of them.

Lesson 6. Tangent Segments and Intersecting Chords. The geometry of eclipses of the sun and of the moon, Exercises 11–15, page 518, is important for the development of geometry and astronomy, and it is discussed at length earlier in these notes, Chapter 11 supplement, "To learn how to calculate better."

Each of the results in this lesson comes down to noticing that certain products of segments connected with segments related to a given circle must be constant. These products are constant because they are crossed products of terms in a proportion that arises from similar triangles. Given the general idea, you can fiddle around and get whichever result you want.

In the figure for Theorem 67, if you extend AC and DB until they meet at some point Q, then $\triangle QBA \sim \triangle QCD$, so $QA \cdot QC = QD \cdot QB = QX^2 = OQ^2 - r^2$, where QX is tangent to the circle O which runs through A, B, C, and D, and r is the radius of O. The limiting case of the tangent gives us a simple geometric interpretation of the quantity that is constant.

Doing calculations is the best way to drive home these results. **Set I** has the Lunar Eclipse geometry (11–15). Note that the distances AS, SE, and EB in the lunar eclipse figure are given in Exercise 53. **Set II**'s Ridiculous Question is a delight. The answer is obvious, if the question has a definite answer. When the reason this works sinks in, you see why the question is ridiculous. Chord Division (44–52) gives the geometry behind a proposed scheme to measure the curvature of the earth. Points C and D are at the ends of a long canal, whose midpoint is at A. Sighting from C to D the bulge (PA) the water makes in following the earth's surface at A is measured. We have $AP \cdot PB = CP \cdot PD = CP^2$. The relative error made by assuming the diameter AB is the same length as PB is small, so we may estimate the earth's radius as $\text{Diameter}/2 \approx CP^2/2AP$. This problem can also be solved using the *small angle approximation*, treating $\theta \approx \sin\theta \approx \tan\theta$ as if they are equal for small values of θ. The angle must be expressed in radians to have this work out. This approximation gives the same result as the approximation used above.

Set III will be within the reach of most students.

Summary and Review. To the key points listed here, I would add the symmetries of the circle, reflections in lines through its center or rotations about its center. Often these symmetries give us the keys to answer our questions or solve our problems. Exercises 9–16 are nice because the figure poses the complete exercise. The Golden Ratio (55–62) brings us back to the number $\phi = (1 + \sqrt{5})/2$ again. Algebra shows $\phi = 1 + 1/\phi$, from which you can get the continued fraction expansion of ϕ. (See, Chapter 14, Lesson 2, Set III in these notes for this fraction.)

Algebra Review. Solving Systems of Equations. This is a nice review of systems of linear equations. The method of using linear combinations of equations generalizes to any number of linear equations in any number of unknowns. If there are more unknowns than there are equations, the set of solutions will form a plane or higher dimensional *affine subspace* of the space of all possible values for the variables.

We take the equations we are given and combine them in various ways to get conclusions of the sort seen in Example 4, page 528: $y = 4$ and $x = -1$. These equations are implied by our original equations and they tell us what x and y must be.

Chapter 13. The Concurrence Theorems.

To the Student. "Once is chance, twice is coincidence, three times is enemy action," Auric Goldfinger warns James Bond, as Bond crosses him again. If two lines in a configuration cross, that is not surprising. We expect it, but when three lines are *concurrent at the same point*, then we suspect something is going on. This chapter is about such surprises and why they happen.

General Comments on Concurrences. When constructions lead to many lines all running through the same point, or many points that all fall on the same line, we should expect that there is some property that point has which makes all these lines run through it, or some special property of the line that forces all of our lines to run through it. This the pattern of most of the examples here.

There are other sorts of concurrence theorems. The Theorem of Pappus says that given rays r and s from point O and points A, B, and C on r and a, b, and c on s, if you connect each of these points to the two points labeled by different letters on the other ray, the three intersections of these segments will be collinear. It is very easy to convince yourself by experiment that this is true just using paper and pencil. I did so in high school, but I was never able to discover a geometric proof of this fact.

In *The Art of the Infinite*, Oxford University Press, New York, 2003, Robert and Ellen Kaplan combine geometry and algebra to give a nice proof of the Theorem

of Pappus. Their proof suggests how to prove Pascal's Theorem: If a hexagon is inscribed in a conic, then its opposite sides intersect in points that are all collinear. This is not exactly how they state Pascal's result, but the results are equivalent. See their Chapter 8, Back of Beyond.

These are theorems of projective geometry, so lines ordinarily considered to be parallel are regarded as meeting at points of an ideal *line at infinity.* There is a more geometric proof of the results of Pappus, Pascal, and Brianchon in *Geometry and the Imagination* by David Hilbert and Stephan Cohen-Vossen, in 2003 available as an American Mathematical Society Chelsea reprint, pages 104–106 and 117, and based on the geometry of hyperboloids, page 11. Pascal used algebra, not geometry, in his proof. The reason these results are true is because of the web of relationships that hold between the points and lines in the configuration. We should not expect some simple explanation, such as the fact that all of the perpendicular bisectors of chords of the same circle must be coincident *because* they all run through the center of the circle.

One projective result that has a direct geometric proof is Desargues's Theorem. It says that if the lines through corresponding vertices of $\triangle ABC$ and $\triangle A'B'C'$ are concurrent at some point O, then the intersections of the corresponding sides of these triangles are collinear on some line ℓ. See the exercise sequence Triangles in Perspective and its figures on page 622 of *Geometry.*

This is a projective theorem, so the point O or the line ℓ can be at infinity. To understand why this result must be true, we draw the diagram and think of realizing it as the projection into a plane of a pyramidal arrangement, with O as the apex of the pyramid. Then each of the triangles lies in a planar slice through this pyramid, and the line ℓ is the intersection of the planes making the slices that give triangles ABC and $\triangle A'B'C'$. The intersections of the corresponding sides of these triangles lie in both these planes, and hence on ℓ, as was to be shown.

In this chapter we begin with concurrences that have simple explanations and come to ones that require a more subtle approach, such as Ceva's Theorem. Let the surprises begin!

Lesson 1. Triangles and Circles. The circle of standing stones at Stonehenge gets us going in this lesson. We see that every triangle has a unique *circumcircle* and hence is a *cyclic* polygon, one that can be *inscribed* in a circle. The perpendicular bisectors of the sides of a triangle are concurrent *because* they all run through the center of the triangle's unique circumcircle.

Quit while You Are Ahead? Inductive Proofs. The ideas used to prove tat regular polygons are cyclic can be generalized to polygonal paths A Polygonal path with n segments, $n > 3$ is *cyclic* (all of its vertices lie on a given circle O) if and only if the bisectors of any three of its segments (taken in order) are concurrent. The point of concurrence will be the center of the circle O. If a cyclic path includes all but one of the sides of a polygon, then the final side which joins the initial and final vertices of the path is also inscribed in O, so the polygon is cyclic.

This result is obvious from consideration of the result for triangles and its method of proof. Jacobs wisely kept to the case of triangles. If there were a situation where a student needed to make this generalization, the student could figure it out using the triangular case as a guide. An example often suffices to show us what is true, and why. It is clear that the method used for the triangle could deal with polygonal paths of 4 steps or more. Thus the general assertion above about polygonal paths being cyclic is clearly true *by induction.* But to give a formal proof by induction requires setting up the machinery. This induction argument is no more convincing than the example. We avoid the formalities, but some teachers may want to show their students induction at work in geometry. The above is one result that could be established by induction. A second is Theorem 74, page 573 of *geometry*: Every regular polygon is cyclic.

The Two Color Theorem. Here are two geometric results that can be proven by induction. Every map having a finite number of countries all bounded by doubly infinite lines (possibly intersecting) can be colored with two colors so that no two countries with a common border have the same color. The same is true if the boundaries are formed by complete (though possibly intersecting) circles. These results are easily proved by induction.

Induction is a powerful tool in geometry. The proof of the Four Color Theorem for planar maps was done by showing that any counterexample can be reduced to a counterexample with fewer countries (reducibility). This argument is equivalent to using *complete induction*, where you suppose the proposition is true for *all* $k < n$, and then prove to show that the proposition must be true for $k = n$ — after showing that the initial cases are true, of course. Complete induction is equivalent to ordinary mathematical induction.

Recursion/Induction and Geometry. Successive division of intervals into tenths, hundredths, and so on is an inductive or recursive process. Such cascade processes are essential for measurement — of lengths, areas, and volumes. Fractals are constructed by an inductive or recursive process: the Koch snowflake, the Sierpinski polygons (pages 144–145), flowsnakes, and dragon curves, to mention a few. You can find these things on the Web or in many books, including Benoit Mandelbrot's book *The Fractal Geometry of Nature*, W. H. Freeman and Company, New York, 1982, where the idea of fractals was launched.

An interesting application of induction is working out tilings of the plane by *rep-tiles*, tiles that can be assembled in groups to form shapes similar to the basic tiles. See Reptiles exercises on page 422. As another example, four squares can be assembled to form a (similar) square at twice the original linear scale. Not very interesting, but let's look farther. Take a square and mark it into four equal squares, then cut off the upper left-hand subsquare giving you an ℓ-shaped tile. Four of these can be assembled into a larger similar ℓ-shape, and so on, *ad infinitum*. The kite and dart tiles devised by Roger Penrose tile the plane, but do this in a way that has no translational symmetries (*aperiodic tilings*). Penrose tilings are usually created by using an inductive construction called *inflation*, in which the tiles are first enlarged and then the enlarged tiles are subdivided into tiles congruent to the original dart and kite tiles.

For rep-tiles, see Martin Gardner's column of that title in Mathematical Games, SCIENTIFIC AMERICAN, May 1963, reprinted in *The Unexpected Hanging* by Martin Gardner, Simon and Schuster, 1969, available in 2003 from the University of Chicago Press. Also search the Web under rep-tiles, self-similarity, aperiodic tilings, and Penrose tilings. Other sources include *Penrose Tiles to Trapdoor Ciphers* by Martin Gardner, W. H. Freeman and Company, New York, 1989, which also covers Mandelbrot's work and fractals. *Tilings and Patterns* by Branko Grünbaum and Geoffrey Shephard, W. H. Freeman and Company, San Francisco, 1987, covers tilings in general, including aperiodic tilings. A shorter, paperbacked version was released later and does not cover aperiodic tilings.

Here is a tantalizing problem, unsolved in 2003. *Is there a single tile that will tile the plane only aperiodically?* This tile might be a rep- of frac- tile. This seems a simple question, but many have attacked it without success. Penrose found two tiles that only tile the plane aperiodically. Can we get it down to one, or is two the best that can be done? There is a lot on the Web about tilings, especially about beautiful or unusual tilings. Look at David Epstein's *Geometry Junkyard* on the Web or search on names such as John Conway, Charles Radin, together with subjects such as tiling, aperiodic, rep-tile (or reptile), or fractile.

Set I begins with a light review of the ideas. RGB Color, Exercises 15–17 feature construction, discovery, and proof, while Exercises 18–21 feature construction and discovery. Four Lines and Four Circles, Exercises 22–25 gives us an astonishing concurrence of four circles arising from a seemingly random configuration of four lines and their six points of intersection. This is a consequence of Miquel's Theorem, which can be easily proven after we know the properties of cyclic quadrilaterals. A Web search for Miquel's Theorem will bring a rich harvest. A wonderful source for Miquel's Theorem and other concurrence results is Ross Honsberger's *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library from the Mathematical Association of America, 1995. Honsberger is an artist. He makes everything beautifully simple and he has an eye for "mathematical gems."

Because there is so much about Miquel on the Web, I will be brief here. In the figure at bottom left, page 533 of *Geometry*, the key is $\triangle$ADC. Points B, F, and E are chosen along the lines determined by the three sides of $\triangle$ADC. By Miquel's Theorem, the red, orange, and green circles, which are called Miquel circles for this system, are all concurrent. A *Miquel circle* for a triangle and the points chosen, one on each of the lines determined by the sides of the triangle, is a circle through one of the vertices of the triangle and through the two chosen points that lie on sides running through that vertex. I leave you with the mystery of why the circumcircle for $\triangle$ADC must be concurrent with the three Miquel circles. This configuration is shown with no hints in the delightfully puzzling *Penguin Dictionary of Curious and Interesting Geometry* by David Wells, Penguin Books, 1991.

Nearest School (30–32) yields nice construction and discovery exercises. This is called *Fermat's problem*, and the point that solves it is the *Fermat point*. In the book cited above, David Wells gives the following physical interpretation of the Fermat point. Take three equal weights tied to strings, one passing frictionlessly through a hole at each vertex of the triangle. Tie all three strings together in a knot and keep the triangle in a horizontal plane. The weights will pull the strings toward each vertex with equal force. The position of equilibrium for the knot will be the Fermat point, which leaves as little string as possible on the plane of the triangle and puts as much of the string length below the plane of the triangle as possible. This puts the center of mass of the weights as low as it can be. Three equal forces in a plane can only be in balance if they make 120° angles with each other. Students could build a model as a project to see how this works. This interpretation is also given in Hugo Steinhaus's *Mathematical Snapshots*, Third Edition, Oxford University Press, 1983. This book, in any edition (try Dover Publications), has many wonderful applications, mostly presented visually with minimal text. It is highly recommended.

Set II. Folding Experiment (33–40) is a nice exercise in construction, discovery, and measurement. Babylonian Problem (46–48) includes construction, measurement, and comparison to exact results. Diameters and Sines (49–53) asks students to draw conclusions, give their reasons, and do calculations.

Set III. Slipping Ladders is an old but good problem. The answer is intuitive at one level but surprising at another. I doubted my initial insight until I checked it.

Lesson 2. Cyclic Quadrilaterals. A *convex* quadrilateral is cyclic if and only if it has *a* pair of opposite angles that are supplementary. I have restated the theorem to emphasize the condition given in the text that the quadrilateral must be convex. The difficulty arises because our angles are ordinarily no larger than 180°, so if you take a cyclic quadrilateral and flip one of its vertices so that it lies an equal distance from its neighbors, but is *inside* the circle instead of being *on* it, that angle and its opposite one are still supplementary — according to our usual way of measuring things.

The proof of the converse part of this theorem on page 537 shows that the locus of points D such that $\angle ADC$ is a constant (say k) is the arc of a circle. Actually, we get two arcs for this locus, one on each side of the line AC. This description of circular arcs is useful. You can use our results on secant angles to prove this. If D moves off the given arc, the angle at D, as calculated using intercepted arcs changes from the value, k, of the angle when D is on the arc.

Set I. These exercises are straightforward and interesting. A Different Proof (13–17) is so nice Jacobs kept it even though one reviewer felt that one proof was enough. In Equilateral Triangles on the Sides (24–28) the point P happens to be the Fermat point for $\triangle ABC$. From arguments using angle and arc measure it is clear that $\angle APC = \angle CPB = \angle BPA = 120°$. It is this condition that forces all of these circular arcs to go through P. This depends on the *locus property* for circular arcs and angles noted above.

Set II. Cyclic or Not? (29–31) combines visual thinking and reasoning and therefore, proof. Bramagupta's Theorem (32–38) combines construction, discovery, and proof. Area Formula (39–41), also due to Bramagupta, would make a nice *project* for exploration using *Geometer's Sketchpad*™. The library or on the Web might shed light on the history of this result. Ptolemy's Theorem (42–55 and on to 57) and its corollaries were used by Ptolemy to work out his table of chords. Specializing it so that AB is a diameter and the quadrilateral is an isosceles trapezoid, you can get the angle addition formula for the sine. See John L. Heilbron's *Geometry Civilized*, Oxford University Press, Oxford, 1998, pages 162 and 163.

Set III. There are two other such sets of points. See Jacobs's answer section. This could lead to a nice class discussion. Why did some students find one set but not the other set of points?

Lesson 3. Incircles. Every triangle has an *incircle*, whose center is determined by the intersection of any two of its (consequently) concurrent angle bisectors. The results on incenters are exactly analogous to those on circumcenters in Lesson 1. These results generalize in the same way. If $n > 3$, a n-gon P has an incircle if and only if the bisectors of any three consecutive angles of P are concurrent.

If the center of the incircle for P is O, we start at some vertex A and find O as the intersection of the angle bisectors at neighboring angles A and B. Then our condition tells us that the angle bisector for the next point, C, around P also runs through O, and so on. If Z is the last vertex before we return to A, and all the other angle bisectors run through O, then the bisector of Z *must* run through O as well. The incircle we have from looking at earlier vertices is tangent to sides YZ and ZA, so the ray ZO bisects angle Z.

Set I. Exercises 1–9 cover the basic ideas. Properties of circles and tangents are used in Exercises 10–19. Construction Problem (27–30) shows that two noncongruent triangles can have the same incircle and circumcircle. Poncelet's porism says that if an n-gon can be simultaneously circumscribed about circle O and inscribed in circle Q, then no matter what point you start from on Q, you can simultaneously inscribe an n-gon in Q and have this same n-gon circumscribed about O. See Eric Weisstein's MathWorld entry on this topic on the Web. In 2003 there was an animation of the porism on this site.

Porisms. A related result that can be understood completely using circular inversions is Steiner's porism. Here is Steiner's construction: Given circle O inside circle Q, construct a chain of circles starting with C_0, tangent to O and Q. Each $C_{(i+1)}$ in the chain $C_0, C_1, C_2, \ldots$ is tangent to O, Q, and C_i. Steiner's porism is that this chain either closes up with some C_n just touching C_0 as well as O, Q, and $C_{(n-1)}$ or it never closes up. Which of these alternatives happens depends on O and Q but is independent of the position of C_0.

Steiner's porism can be proved by using a circular inversion to reduce the initial situation to a pair of concentric circles, for this configuration the assertion of the porism is obvious. Since inversions preserve circles and lines, this special symmetric case allows you to show that the general version of the porism holds. This is a comment of H. S. M. Coxeter's from his *Introduction to Geometry, Second Edition*, John Wiley and Sons, New York, 1969, page 87.

Porisms and Technology. These porisms cry out for animations, which have been created using various software, generally *Geometer's Sketchpad*™. Search on keywords such as Java applet, hexlet, bowl of integers, Steiner chains, seven circles theorem. Students will find many things that could be the basis for *projects* in this area.

Set II. Ship Location, Exercises 31–39. The data suggest the ship is located on each of these lines ℓ_1, ℓ_2, and ℓ_3, but they are not concurrent. The point P, which is equally close to ℓ_1, ℓ_2, andℓ_3 is a good candidate for our estimate of the ship's true position. The results found in Incircle Problems 1 (40–47) and 2 (48–58) are explained by Incircle Problem 3 (59–62). Trisector Challenge, 63, is easy enough if you don't get distracted by angle trisection.

Set III. A nice exercise. See Jacobs's comments in his teacher materials.

Lesson 4. The Centroid of a Triangle. For finding balance points we can treat a solid body as if its mass were concentrated at one point, its *center of mass*. The center of mass of a uniform triangle is its *centroid*. The medians of a triangle are concurrent at the triangle's centroid. Because a median bisects each infinitesimal strip of the triangle that crosses it parallel to the side that the median bisects, each of these strips will balance perfectly, if allowed to pivot independently about the median. So the triangle as a whole is perfectly balanced if allowed to pivot on a median.

Suppose we knew that a body has a center of mass about which it balances perfectly on any axis. Then we can conclude that that point must be at the intersection of any two medians of a uniform triangle. This is a physical argument for the concurrence of the medians of a triangle. See Medians as Bisectors, Exercises 38–46. The geometrical proof is found in Exercises 5–12.

It is somewhat surprising that the concurrence of the medians can be used to prove that the altitudes of a triangle are concurrent. The intersection of the altitudes is the *orthocenter*. See Exercises 13–19.

Set I. The proofs of Theorems 71 and 72 (5–19) are cleanly outlined and standard. Median Construction (20–26) involves construction and discovery strands. Altitude Construction (27–31) involves construction, visual thinking, and discovery.

Set II. Medians as Bisectors (38–46) tells us why triangles balance along their medians. Doing Without a Compass (47–54) is a beautiful application with an element of discovery. Engineering Challenge (55 and 56) is a nice challenge.

Set III. Balancing Point. The answer is within reach. Discovering why it is right, possibly using *Geometer's Sketchpad*™ or other aids would be an interesting *project*.

Centers of Mass, Medians, and Ceva's Theorem. A more flexible way to think of this is to suppose that masses m_A, m_B, and, m_C are at the vertices of $\triangle$ABC. If we are working with coordinates or thinking of these points as vectors, the center of mass (C.M.) of this arrangement will be

$$\text{C.M. for } \triangle\text{ABC} = \frac{1}{M}(m_\text{A}\text{A} + m_\text{B}\text{B} + m_\text{C}\text{C}),$$

where $M = m_\text{A} + m_\text{B} + m_\text{C} > 0$. Archimedes did not have algebra and vectors to aid him. He used his Law of the Lever, which states that the center of mass of the weights at B and C will be at a point Y on segment BC such that $m_\text{B}/m_\text{C} = CY/YB$. This comes down to saying that the masses at B and C act as a single mass $m_\text{BC} = m_\text{B} + m_\text{C}$ at the point Y, which turns out to be the point we would write in vector or coordinate form as

$$\begin{aligned}\text{Y} &= \text{C.M. for masses at B and C}\\ &= \frac{1}{m_\text{BC}}(m_\text{B}\text{B} + m_\text{C}\text{C}).\end{aligned}$$

Then Archimedes would find the center of mass for all three masses by locating the center of mass for the system consisting of the mass m_A at A and the mass m_BC at Y, again using his Law of the Lever.

This calculation is

$$\text{C.M. for } \triangle\text{ABC} = \frac{1}{m_\text{A} + m_\text{BC}}(m_\text{A}\text{A} + m_\text{BC}\text{Y}).$$

This tells us that the center of mass of a triangle $\triangle$ABC with masses at its vertices lies on the segment from A to the Y, the center of mass of the masses on side BC opposite from A.

Take points X, Y, and Z on the sides of $\triangle$ABC as follows: X is on side AB, Z is on side AC, and Y is on side BC, and choose these so that they satisfy the equation below, which is the hypothesis of Ceva's after clearing the fractions. See the next lesson in *Geometry*, Ceva's Theorem, page 555, top, for the formula and the figure.

$$(*) \qquad \text{AX} \cdot \text{BY} \cdot \text{CZ} = \text{AZ} \cdot \text{BX} \cdot \text{CY}.$$

Next define masses m_A, m_B, and m_C by

$$m_\text{A} = \text{BX} \cdot \text{CY}, m_\text{B} = \text{AX} \cdot \text{CY}, \text{ and } m_\text{C} = \text{AX} \cdot \text{BY}.$$

Using (*) and Archimedes' Law of the lever, algebra shows that

$$\begin{aligned}\text{Y} &= \text{center of mass of } m_\text{B} \text{ and } m_\text{C},\\ \text{X} &= \text{center of mass of } m_\text{B} \text{ and } m_\text{A}, \text{ and}\\ \text{Z} &= \text{center of mass of } m_\text{A} \text{ and } m_\text{C}.\end{aligned}$$

So the Cevians are the lines from each vertex through the center of mass of the masses on the other vertices. Thus they are concurrent because they all run through the center of mass of the three masses we assigned to the vertices.

The concurrence of the medians and of the Cevians can be looked at purely geometrically, as Jacobs does in *Geometry*. Alternatively, these results can be looked at physically using centers of mass, or algebraically, as I have suggested, using vectors. Each point of view has advantages and drawbacks. If you have vector machinery at hand, the vector approach is simplest, because it just uses *calculation* — which is easier than the *inspiration* geometric proofs require. Developing familiarity with vector methods would divert Jacobs from his main task. So he chose a geometrical approach.

Lesson 5. Ceva's Theorem. This is an extension of the result that the medians of a triangle are concurrent. The note above shows how it is related to centers

of mass. The approach in the text is direct and geometric.

The facts of this lesson are more important than the proof, and more memorable. If students go on and use this result or learn other approaches to it, the geometric proof will be less important than knowing that the theorem is true and knowing that there are several ways to look at it.

Andrew M. Gleason noted that the Ceva Condition is conveniently expressed and used in this alternative form

$$AZ \cdot CY \cdot BX = AX \cdot BY \cdot CZ.$$

Start at any vertex then take the product of every other segment going clockwise for one side of the equation, then do the same thing going counterclockwise from the starting vertex to get the other side of the equation. See Ollies Equation (25–27), for variants on the usual form of the Cevian condition.

In the forward direction of the proof of Ceva's Theorem, you show that a $(AX/BX)(BY/YC)(CZ/AZ) = 1$ if the Cevians are concurrent. Given X and Z, we know by algebra that there is only one Y on the side BC such that

$$(*) \qquad (BY/YC) = \frac{1}{[(AX/BX)(CZ/AZ)]}.$$

We know that we will find a Y′ that satisfies (*) by drawing the line BP, where $P = CX \cap BZ$. So we know $Y = Y'$. This is the proof of the converse, and it is a corollary of the fact that the algebraic condition (*) determines the position of Y when the positions of the other points are given.

The converse of the Pythagorean Theorem has this same character. It seems that when you prove the converse, you are repeating what you have already established. You are, but with a twist. This can puzzle students, who think they have already proven the converse, and with good reason.

Set I. Which is Which? (1–5) covers the basics: recognizing things and naming them. Using Ceva's Theorem (6–9) gives students practice in seeing and solving. A calculator will be useful here. Equilateral Triangle (10–14) requires construction, measuring, and conjecture.

Set II. The Theorem of Menelaus (28–33) gets us off to a nice start. Concurrent or Not? (34–36) uses Ceva's Theorem to establish a generalization of the theorem that the medians are concurrent. The result in What Kind of Triangle? (37–47) seems surprising. The Gergonne Point (48–51) is a nice place to use Gleason's form of the Ceva Condition. Another Look at Altitudes (52–59) gives us another proof that the altitudes of a triangle are concurrent. It is also a nice review of similarity arguments.

Set III. Area Puzzle. This is remarkable, and the more so for there being two ways to arrive at the right answer.

Lesson 6. Napoleon's Discovery and Other Surprises. There are an absolutely staggering number of results that involve concurrences, constructions of various figures on the sides of other figures, special points relating to a triangle, and so on. Ross Honsberger's *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, New Mathematical Library from the Mathematical Association of America, 1995 is a wonderful source of these things. Also search on the Web for such topics as The Nine Point Circle, Symmedian Point, Euler Line, and Clark Kimberling's Triangle Centers. Other sources are H. S. M. Coxeter's *Introduction to Geometry*, Second Edition, John Wiley & Sons, New York, 1961, and *Geometry Revisited* by Coxeter and S. L. Greitzer, The Mathematical Association of America, 1967, and Robert and Ellen Kaplan's *The Art of the Infinite*, Oxford University Press, New York, 2003.

The Euler line is covered in Exercises 6–34. This sequence is broken into parts that can be assigned separately. The lead-in for this lesson is covered in Exercises 61–63, Napoleon Triangles. **Set III**, Alternating Triangles appeared as the discovery exercise, Equilateral Triangle Mystery, on page 275. The rest of **Sets I and II** have their surprises and rewards.

Set III. The secret of proving that the natural conjecture here is true is to *simplify*. The sides of the resulting parallelogram are parallel because they are parallel to one or the other of the diagonals of the original quadrilateral. So if you draw *only* the parts of the diagram needed to check this, *and* if you draw *one* of the diagonals, then things are simpler. Once simplified, you can see why one of the sides is parallel to one diagonal. Then you have the solution.

▶ *Circles and Triangles.* The diagram, bottom left, on page 565 of *Geometry* shows the situation for Napoleon's Theorem. An alternative approach can be based on observing that the circumcircles of the equilateral triangles are concurrent. This observation allows you to dispense with segments EB, DA, and CF. The fewer lines the better. The circles through CBD and CAE intersect at a point P. The inscribed $\angle CPB$ and $\angle CPA$ are both 120°, so it follows that $\angle APB$ must also be 120°, and hence the circle through ABF also passes through P. This is because the locus of all points X such that X lies on the same side of the line AB as point C and $\angle AXB = 120°$ is the arc of a circle. The locus of points on the opposite side of line AB from C and such that $\angle AYB = 60°$ is the other arc of this same circle, because AXBY is a convex quadrilateral and so $\angle AXB + \angle AYB = 180°$, as we expect. So E is on the other arc of this circle as shown.

Now we need to show $\angle ZYX = \angle YXZ = \angle XZY = 60°$ so that $\triangle XYZ$ is equilateral. The argument is to

observe that XY bisects both the minor arcs from A to P, so $\angle AYG = \angle PYG$. Similarly, $\angle BYH = \angle PYH$. So $\angle AYG + \angle BYH = \angle PYG + \angle PYH$, because equals added to equals give equals. The sum of these four angles is the central angle $\angle AYB$. Because $\angle AYB$ is the central angle for the inscribed angle $\angle AFB$ ($= 60°$), we know $\angle AYB = 120°$. So $\angle ZYX = \angle HYG = \angle PYG + \angle PYH = 60°$. And the same argument shows that the other angles of $\triangle XYZ$ are 60° as well, so $\triangle XYZ$ is equilateral.

Summary and Review. There is a mix of review and discovery in these exercises. All can be done at the level asked, but exercises like Five Circles (28–31) may challenge some students to go beyond what is asked for and find proofs of some amazing patterns. All is fair in going after such problems. After all, Euler used algebra to find his line. The geometric proof came later. Explorations with *Geometer's Sketchpad*™, library research, or Web browsing may be needed to unravel some of these mysteries. The example below is a case in point.

Going at It Hammer and Tongs. The Five Circles exercises would be a baffler were it not for the comments Jacobs in the draft materials for the teacher. Looking at the diagram on page 568, we have the cyclic quadrilateral ABCD in a circle we will call O. Let X be the midpoint of $\overset{\frown}{AB}$, Y the midpoint of $\overset{\frown}{BC}$, Z the midpoint of $\overset{\frown}{CD}$, and W the midpoint of $\overset{\frown}{DA}$. Let Q be the intersection of the chords XZ and YW of O.

$$\angle XQY = \frac{1}{2}(\overset{\frown}{XB} + \overset{\frown}{BY} + \overset{\frown}{ZD} + \overset{\frown}{DW}) =$$

$$\frac{1}{4}(\overset{\frown}{XB} + \overset{\frown}{BY} + \overset{\frown}{YC} + \overset{\frown}{CZ} + \overset{\frown}{ZD} + \overset{\frown}{DW} + \overset{\frown}{WA} + \overset{\frown}{AX}) =$$

$$\frac{1}{4}360° = 90°, \text{because } \overset{\frown}{AX} = \overset{\frown}{XB},\ \overset{\frown}{BY} = \overset{\frown}{YC}, \text{ etc.}$$

So XZ and YW are perpendicular. Let S be the center of the incircle of $\triangle ABC$ and T be the center of the incircle for $\triangle ABD$. We show that reflection in line XZ interchanges S and T. We locate S by taking the intersection of the bisectors of $\angle ACB$ and $\angle BAC$. If you draw the diagram with A, X, B, Y, C, Z, D, and W, and mark the arcs that are equal, you will see that these bisectors are AY and XC.

We locate T, the incenter of $\triangle ABD$ in a similar way as the intersection of angle bisectors BW and XD. Reflection the line XZ interchanges segments AY and BW and it interchanges segments XC and XD, hence it interchanges points S and T. By the same sort of argument, reflection in line XY interchanges the incenter of $\triangle BCD$ with that of $\triangle ACD$.

Applying like reasoning to reflection in the line YW we find that these four incenters are permuted by reflection in this line as well. Hence the set of these four incenters is invariant under reflections in the two perpendicular lines XZ and YW. Hence these four points are the vertices of a rectangle that has XZ and YW as lines of symmetry.

Chapter 14. Regular Polygons and the Circle.

To the Student. The equilateral triangle is the simplest regular polygon; next comes the square, the pentagon, the hexagon, ... All are symmetric by rotations about their centers and ahve line symmetries by reflections in lines through their centers. The more sides a regular polygon has, the more symmetries it has and the closer it is a circle. We find the area of a circle, πr^2 as the limit of the areas of inscribed regular polygons. The limit of the perimeters of these polygons gives us the circle's circumference, $2\pi r$.

Limits and Bounds. Archimedes showed that the area of a circle is equal to that of a right triangle whose altitude is the radius r of the circle and whose base is equal to the circumference c of the circle. Algebraically: Area $= (1/2)rc$. This is Proposition 1 of his *Measurement of the Circle.* The ratio π is Circumference/Diameter $= c/d$, and $d = 2r$, so Archimedes result gives our Area $= \pi r^2$.

A good source on Archimedes' methods is Sherman Stein's *Archimedes: What Did He Do Besides Cry Eureka!*, Mathematical Association of America, Washngton DC, 1999. Continuing his work, Archimedes got the following bounds for the value of π: $3\frac{10}{71} < \pi < 3\frac{1}{7}$. *Archimedes used bounds*; he used inscribed and circumscribed regular polygons to find upper and lower bounds for π. In this chapter, *we use limits*, defining the area and circumference of a circle as the limit of the areas of inscribed polygons and the circumference as the limit of the perimeters of inscribed polygons.

Calculators give us the easy informal path to limits used in this chapter. The method of finding upper and lower bounds following in Archimedes footsteps is still of interest. See the supplementary notes on this chapter for details. The idea is sketched below.

If you inscribe a regular n-gon in the unit circle, it will have a perimeter $2n\sin(\frac{180°}{n})$, as given by Theorem 75, on page 581 of *Geometry.* If you *circumscribe* a regular n-gon about the unit circle so that it is tangent to the circle at the vertices of the inscribed n-gon, you can see that the perimeter of this circumscribed n-gon is $2n\tan(\frac{180°}{n})$. The circumference of the unit circle, 2π, must lie between the perimeters of these inscribed and circumscribed n-gons. So

$$(*) \qquad 2nr\sin(\frac{180°}{n}) < 2\pi r < 2nr\tan(\frac{180°}{n}).$$

This form relates directly to the perimeters and circumference, but dividing by $2r$ upper and lower bounds for π.

$$n\sin(\frac{180°}{n}) < \pi < n\tan(\frac{180°}{n}).$$

With a calculator we can work these out as numbers and see that they bracket π and are close to π when n is

large. With $n = 6000$ the average of these bounds gives π to six decimal places on my calculator. There is something self-referential in this confirmation because calculators, in effect, use a stored value of π when they find trigonometric functions. Our understanding of these calculations is based on the geometry of area and circumference developed in this chapter.

The circumference of a circle of radius r may be thought of as the single number lying between the upper and lower bounds established by (*) for every n. Alternatively, it may be thought of as $C = \lim_{n\to\infty} 2nr\sin\left(\frac{180^\circ}{n}\right)$. Either way, its formula is $C = 2\pi r$, and we evaluate it by calculator. For practical purposes, we need not concern ourselves with the fact that our machine only gives the answer to ten decimal places. In the physics and engineering, few things can be measured to even six decimal place accuracy.

Lesson 1. Regular Polygons. A polygon is *regular* if it is convex and if all of its edges are congruent to each other and all of its angles are congruent to each other. These are *local* conditions. If a convex polygon on a soccer field is convex, you can check its regularity even in a dense fog. Pace around the polygon. Check that the sides are the same length. Check that all the angles are equal.

If *local* measurements show the fog-bound convex polygon is regular, then Theorem 74 tells you that it has an incenter, and you can work out its shape and size from your measurements. The proof shows how the *local geometry* determines the *global structure. Local regularity* implies that the polygon has symmetries interchanging any two of its sides or angles.

This is the pattern for regular figures in general. We require that the various faces and edges of regular polyhedra be equivalent, a *local* condition, and we find that there are *global* symmetries that interchange these elements as well. The study of regular figures is tied up with the effort to find local characterizations that force the existence of global symmetries.

A regular polygon with n sides is symmetric by reflection in lines through its incenter and a vertex. It is also symmetric under rotations about its center by any multiple of the central angle subtended by one of its sides from its center.

Set I. These are exercises in seeing regular polygons and analyzing how they fit into other patterns. Set I covers the basic material in this lesson.

Set II. Mason's Triangle is a beautiful construction. The angles are not quite right, being off by less than a degree. This might be a good test of *technology*: Can students see what is up using *Geometer's Sketchpad*™?

In Folding an Octagon (33–43) the problem is to see that the octagon is regular. What follows is keyed to the figure on page 577 of *Geometry*. The figure is symmetric by reflections in the perpendicular lines GE and HF, which run through O, the center of the square EFGH, as well as by reflections in the diagonals AC and DB which also run through O. These symmetries show that

$$OH = OG = OF = OE,$$

because each of these is segments can be carried any other of them by one of these reflections. For the same reason,

$$OI = OJ = OK = OL.$$

We see GCFO is a square with diagonals GF and OC which intersect at a point we label M. Diagonal OC divides square GCFO into isosceles right triangles, one of which is $\triangle GCO$. The folding shows $\angle CGK = \angle KGM$, so both of these are 22.5° angles. It follows that $\angle KGO = 45^\circ + 22.5^\circ = 67.5^\circ$. Because $\angle GOK = 45^\circ$ it follows from the Angle Sum Theorem that $\angle GHO = 67.5^\circ$ as well, and $\triangle GOK$ is isosceles. Hence, OG = OK, and all of the segments in the displays above must be equal. The reflective symmetries can send any of the central angles the octagon GKFJEIHL makes at its center O into any other of these central angles, so they are all congruent, and as we have seen, all of these vertices are equidistant from O, so this is a regular polygon *in the global sense.*

A rotation by 45° about O sends octagon GKFJEIHL onto itself. So it is equiangular and equilateral, and it is clearly convex, so it meets our *local* definition of a regular polygon.

Hexagonal Fastener (44–50) looks at the geometry that allows wrenches to work. What is important is the relative difference between the radius of the incircle and that of the circumcircle of the head of the fastener. Square and nuts are common, fire hydrants are designed for pentagonal wrenches (to discourage frivolous uses), and we do not see septagonal or octagonal nuts. An interesting source on design and engineering of common things, including fasteners, is *The Evolution of Useful Things* by Henry Petrovski, Vintage reprint, 1994.

Inscribed Pentagon (51–61) develops key facts about the regular pentagon. Referring to the labels on page 578 of *Geometry*, this figure is symmetric by reflection in the line FE. The calculations here provide a good review of secant angles. An interesting property of the 36°, 72°, 72°, triangle $\triangle AFB$ is that there is a point X on side AB such that $\triangle AFB \sim \triangle FXB$ and AX = XF = FB. Let the ratio of similarity be r such that rBX = FB. Then BA = BX+XA = BX+BF = BX+rBX = r^2BX, because scaling BX up by r gives us BF and scaling BF up by r gives us BA. Geometrically: BX is to XA as XA is to BA. The ratio r is then the Golden Number, ϕ, the positive root of $r^2 - r - 1 = 0$, $\frac{1+\sqrt{2}}{2}$.

Given segment AB the point X such that BX is to XB as XB is to BA, is said to divide AB in *extreme and mean ratio.* The construction for dividing AB in extreme and mean ratio is given in Proposition 11, Book II of Euclid. Extreme and mean ratio reappears in at

the start of Book XIII leading to constructions of the regular pentagon, the dodecahedron, and icosahedron. The diagram in these exercises is found in Heath's Euclid for Proposition 8, Book XIII. *Once* X *has been constructed* on AB, the regular pentagon ABCDE is easily constructed. The construction of this triangle is found in Euclid, Proposition 10, Book IV. Euclid, Proposition 11, Book IV gives a construction for inscribing a regular pentagon in a given circle.

The fact that the 36°, 72°, 72° triangle can be split into a similar triangle plus an isosceles triangle leads to the possibility of recursive constructions, including tilings. The aperiodic Penrose tilings are related to pentagons and 36°, 72°, 72° triangles, and the Golden Number is intimately linked to these aperiodic tilings. See, for example, Figure 10.3.14 of *Tilings and Patterns* by Branko Grünbaum and G. C. Shephard, W. H. Freeman and Company, New York, 1987.

Set III. A Close Construction is a mixture of construction and analysis, with calculators playing a role.

Lesson 2. The Perimeter of a Regular Polygon. Here the concrete example of a regular pentagon shows us the situation for a regular n-gon. If you want an upper bound on the circumference as well, circumscribe a regular pentagon about the circle with its sides tangent to the vertices of the inscribed regular pentagon. Its sides have length $2r\tan\frac{180°}{5}$, so its perimeter is $5 \times 2r\tan\frac{180°}{5}$. For the general n-gon, replace 5 by n and get

$$2nr\sin(\frac{180°}{n}) < \text{Circumference} < 2nr\tan(\frac{180°}{n})$$

for the circumference of a circle of radius r. Here we use the principle Archimedes used: If one bounded convex region lies inside another, then the inner region has the smaller area and smaller perimeter.

Set I. Perimeter Equations (1–9) gives practice with the formula developed and shows how N tends to π as the number of sides increases so that these regular polygons approach a circle. The rest of Set I covers the basic ideas of the lesson. Hydrogen Fluoride (19–29) is a nice application. The Ångström is 10^{-10} meters; this is not a *Systeme International* (SI) unit but it lies between the SI nanometer (10^{-9} meter) and the picometer (10^{-12} meter). For a better understanding of the sizes of things in the universe, students should see *Powers of Ten*, either the movie by the Office of Charles and Ray Eames or the the SCIENTIFIC AMERICAN Library book, New York, 1983, by Philip and Phylis Morrison and the Office of Charles and Ray Eames. These are highly recommended, and there is a good deal of related material on the Web.

Set II. Applications to cell phones (33–40). Ten Pentagons (41–48) combines geometric thinking and use of a calculator. Euclid's Discovery (49–58) is Proposition 10 of Book VIII. This is a discovery exercise.

Set III. Golden-Heather. Here we see the 36°, 72°, 72° triangle mentioned above in connection with Inscribed Pentagon. If we look at the ratio of AE to CE, which is the ratio of the distance between nonconsecutive petals to that between consecutive ones, this is the same ratio as BA/BF because $\triangle$ABF (of page 578) $\sim$ $\triangle$CAE (of page 578). As the discussion of Inscribed Pentagon in the notes for the previous lesson shows, this ratio is ϕ, the Golden Number or Golden Ratio.

A broader pattern is

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sin(90° - \theta),$$

from which the first trigonometric surprise flows. Recognizing $2\sin 54°$ as ϕ, the positive root of $r^2 - r - 1 = 0$, we can see by algebra that $\phi = 1 + (1/\phi)$. Successive replacements of ϕ by $1 + (1/\phi)$ give us the continued fraction expansion of ϕ:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}.$$

Lesson 3. The Area of a Regular Polygon. The dissection by Józef Kürscák is very beautiful, and it is explored further in Kürscák Triangles, Set II, 41–51.

The area formula has the extra cosine factor and is not as simple as the result for perimeters. Archimedes did his approximations for the case of perimeters when he estimated π. This is because the situation for perimeters is simpler. But Archimedes had already shown for a circle of radius r: Area $= \pi r^2$ by a rigorous argument using exhaustion. The Greeks defined π as the ratio of the diameter of a circle to its circumference — a *geometric* definition, independent of estimating π by limits or bounds.

Set I covers all the basic ideas. In Area Equations (1–7) the calculator tempts us to plug in a very large value for n to see if our result looks like πr^2. Formulas should be *understood* as well as used, and Inscribed Polygons (8–11) and The Area of a Square (12–19) build an understanding of the formulas of this lesson. Close But Not Quite (20–28) gives a lesson in the ways calculation goes *beyond* what can be seen.

Set II begins with Honeycomb Geometry (29–40), which includes reviewing the K, which uses 30°, 60°, 90° triangles. This builds the trigonometric muscles.

Kürscák Triangles (41–51). The assumption that all of the yellow triangles are congruent is made in order to simplify these exercises. But the geometry is determined by insisting that ABCD be a square and the purple triangles be equilateral, with the layout as shown.

Here is why: $\triangle IBC \cong \triangle HBA$ by SSS. Then by congruence and properties of isosceles triangles, $\angle IBC = \angle ICB = \angle HAB = \angle HBA = 15°$. Where we use the fact that $\angle IBC + \angle IBH + \angle HBA = 90°$. Similar reasoning on angle sums shows that $\angle GCD = \angle EAD = 15°$, so yellow triangles $\triangle EAD \cong \triangle GCD \cong \triangle IBC \cong \triangle HBA$ by SAS. Now, by adding up the angles about points E and F we can conclude $\angle DEH = \angle DGI = 150°$, which is the size of the obtuse angles of all the yellow triangles we have so far proven congruent. The short sides of $\triangle DEH$ and $\triangle DGI$ are equal and equal to the short sides of all the yellow triangles so far proven congruent. The obtuse angles of all these triangles are 150°, so these two yellow triangles are congruent to the lot already analyzed. The last pair of yellow triangles is $\triangle FHD$ and $\triangle FID$. These are congruent by SSS, and congruent to the rest of the yellow triangles.

The students are told to assume all the yellow triangles are congruent in order to avoid this long argument.

Rats! (52–58) involves conjecture and computation. Calculators will be used here.

Set III. Congruent Pentagons. Marjory Rice has done a good deal of interesting work on tilings, particularly with pentagons. You can find information on her work by searching the Web.

Lesson 4. From Polygons to Pi. The circumference of a circle is the limit of the perimeters of the the polygons that can be inscribed in it. More generally this approach allows us to define the length of the boundary of any bounded, convex region on the plane. The limit is taken as the number of sides of the inscribed polygons is allowed to go to infinity. It is not necessary that the inscribed polygons be regular. For circles, regularity allows us to have a formula giving the perimeter of the inscribed polygon as a function of n. This makes it easy to apply the idea of a limit.

Archimedean Alternative. Archimedes used the principle that when one convex region is properly included in another, the smaller region has the smaller area and the smaller perimeter. This principle lets us see that the perimeter of a bounded, convex region R is the least upper bound of the perimeters of all convex polygons contained in R.

Given a point X and a convex region R not including X, there is always a unique $X' \epsilon R$ that is closest to X, and this X' is necessarily on the boundary of the smaller region. To see why this is true, think of blowing up a circular balloon centered on X until it just touches R at X'.

Archimedes' principle about the perimeters of bounded convex regions can be proven by thinking about what happens when points on the boundary of the larger region are mapped to the closest point in the smaller region. If X and Y are on the boundary of the larger region, let X' and Y' be the corresponding closest points in the smaller region. You can show that $XY \geq X'Y'$, and that some pairs of boundary points of the larger region must be mapped to pairs on the boundary of the smaller region that are actually *closer* together than the original two points. This means that the length of the boundary is shrunk by this mapping, and the smaller region must have that smaller perimeter. The boundary shrinks as it is drawn inward.

Set I covers all basic ideas. Viète's Calculation would carry us through 16 doublings of the number of sides, starting from a hexagon. The 5 doublings shown in my appendix to this chapter's notes, give 3 decimal places accuracy (4 if you average the upper and lower bounds), and each doubling reduces the error to 1/4th of what it was before, so the additional 11 doublings of Viète would give something like 10 place accuracy.

Set II has nice applications, including Videotape (52–56) and distances along the earth's orbit in Going in Circles. Both help students get a sense of sizes and distances and how things can be worked out using geometry.

Set III. Tire Change is practical and within the capabilities of average students. Nothing tricky here.

Lesson 5. The Area of a Circle. One way to get circles or spheres is to pump up a uniformly stretchy boundary. For spheres, a soap bubble is the nicest example. A drop of oil spreads out on smooth water to form a circle.

Here we see why a city built against a relatively straight boundary and defended by as short a wall as can be managed will be defended by a (roughly) circular wall. The problem of getting the most land for the least wall (or stretched ox hide) is called Dido's Problem and has been studied widely.

Here is a way of proving that for a given length L, the area bounded on one side by a curve C of length L and on the other by a line ℓ can be a maximum only if C is a semicircle whose diameter lies on ℓ. I refreshed my memory on this by looking at *The Enjoyment of Mathematics* by Hans Rademacher and Otto Torplitz, Princeton University Press, 1957, Sections 21 and 22. This book is highly recommended and available from Dover Publications in 2003.

You have to draw the diagram to see this. Let the curve C start at point A on line ℓ, which is parallel to the bottom of your sheet of paper. C curves upward and returns to ℓ at point B, to the left of A. Suppose that C encloses as large an area as possible for the given length. Let X be *any* point on C other than A or B. Then we have $\triangle AXB$, and the arcs of C above the chords AX and BX. We regard these arcs as rigidly fixed above their chords, but the chords are to be regarded as *hinged* at X. If $\angle AXB \neq 90°$ we can increase the area of the arrangement while keeping the length of

the upper bounding curve the same by opening or closing this angle to make it 90° while sliding its ends along ℓ. Thus, if C encloses the maximum area for the given arc length $\angle AXB = 90°$ for every X on C. This means that C is the locus of all points above line ℓ such that $\angle AXB = 90°$; in short, C is a semicircle whose diameter lies on ℓ, as was to be proven.

This is an *isoperimetric problem*, that is, a problem of maximizing or minimizing one quantity while holding some other parameter constant. One good source on such problems is Stefan Hildebrandt and Anthony Tromba's *Mathematics and Optimal Form*, SCIENTIFIC AMERICAN Books, New York, 1986, later revised as *The Parsimonious Universe*, Copernicus Books, New York, 1996.

For isoperimetric problems I also recommend *Mathematical Snapshots* by Hugo Steinhaus, available from Dover Publications and Oxford University Press in 2003. This book has so many wonderful things in it, most of them presented with figures, drawings, or pictures. (The First Edition, printed in Poland, had a pocket in the back containing 3-D glasses, a pop-up dodecahedron, and so on. This is a hands- and minds-on book. Now you have to make your own models, but that may be an advantage.)

When thinking of the area of a circle, you should allow yourself to think of a circle *as a triangle whose apex is at its center and whose base is its circumference.* This is something of a geometric distortion, but it gives you a correct understanding of the relationship between the circumference and the area. See Equal Areas, Exercise 17. It might be easier to think of the circle in this exercise as consisting of thin flexible concentric layers. If you sliced from C to A in this figure and let the layers uncurl on a line tangent to the lowest point on the circle, you would get a triangle with apex at A, symmetrical about the vertical line AC and having a base of length $2\pi r$, which is the same as the length AB seen on page 601 of *Geometry*.

Set I gives practice seeing circular areas in various situations and using the formulas we have to work problems. The photograph in Ripples shows the complex structure to the expanding ring of capillary waves from a stone dropped into still water. Careful observation of these spreading rings of waves will give food for thought. There are a few nice photographs of capillary waves on the Web, but I found no simple, definitive source on this.

Central-Pivot Irrigation (18–24) is a guestion that has appeared in abstract form on the MCAS mathematics test and in NCTM materials. The flash of insight is that we are looking at the same ratio for both these cases, the portion of a square area filled by the square's incircle — and this is a correct intuition. In these exercises, the student will see by a calculation that both arrangements water the same fraction of the surrounding square. A question for teachers is: Should you insist on a calculation here or would you give a student full credit for the insight alone?

Circle on a Grid (25–30). Observation, calculation, and judgment are all needed in these exercises. This matter is given serious attention by David Hilbert and Stephan Cohen-Vossen in *Geometry and the Imagination*, in 2003 an American Mathematical Society Chelsea reprint. See the beginning of Chapter II, Regular Systems of Points. These questions lie in an area known as the *geometry of numbers* and extensively studied by Hermann Minkowski. Hilbert and Cohen-Vossen get the following neat result using these ideas: On a square lattice, a parallelogram with lattice points as vertices that contains no lattice points in its interior has area 1 square unit.

Set II. Area Problems (31–36) are those wonderful exercised where you see and calculate the answer. Cable Disaster (37 and 38) is a lesson in the importance of understanding geometry. You cannot expect 2 cables of $\frac{1}{2}$ inch diameter to have the strength of one cable 1 inch in diameter. Even using 4 cables $\frac{1}{2}$ inch in diameter in place of 1-inch cable might not be prudent. If the load were to come unequally on one of these 4, then if the over-stressed cable failed, then the other 3 would be at risk.

Set III. Lull's Claim. It is rewarding for students to see that they can get something right that a sage so famous as Lull got wrong. Modern tools trump fuzzy thinking.

Lesson 6. Sectors and Arcs. It is reasonable to suppose that mathematical objects *exist* and we merely investigate and discover some of their properties. In a general way, this is the *Platonist* view. But it is also reasonable to take the view that we need to develop theories of the length of curves and areas of regions in order for these notions to have a definite meaning. In an introductory course, we are Platonists. We have neither the time nor tools to construct notions like area from scratch — and the students would not appreciate the effort. The areas and perimeters of bounded convex regions and of regions that can be broken down into a finite number of such regions (or their complements, intersections, and so on) all have perfectly definite meanings, and these can be worked out by methods of the this chapter.

This is the lesson where students get to use whatever methods they can bring to bear to calculate areas and lengths. *Proportionality* is the guide in working these things out for sectors of circles:

$$\frac{\text{Sector Angle}}{\text{Full Circle}} = \frac{\text{Sector Area}}{\pi r^2} = \frac{\text{Sector Arc}}{2\pi r}.$$

▶ *Fractal Troubles?* Paradoxes will arise if we assume that the boundary of any region can be assigned a finite length. Fractals have so seeped into the general consciousness that we know that the Koch Snowflake has a boundary of infinite length, and we know that trying to define the length of the coast of Britain is a tricky business. See *The Fractal Geometry of Nature* by Benoit Mandelbrot, W. H. Freeman and Company, New York, 1982, or other books. Plate 47 in Mandelbrot's book is an interesting example. Six congruent areas surround a seventh, also congruent, to form a similar figure enlarged by a linear scale of $\sqrt{7}$. But if you look at the boundary of the scaled up region, you will see that it consists of 6 curves each of which is congruent to half of the boundary of the original region. If the boundaries of these regions had finite length, we would expect the enlarged region to have a boundary of length $6 \times \frac{1}{2} \times$ (Boundary Length Of The Smaller Regions) on the one hand. On the other hand, we would expect the boundary length to scale up linearly with the diameter of the set, and hence, that it would be multiplied by a factor of $\sqrt{7}$. But $6 \times \frac{1}{2} = 3 \neq \sqrt{7}$. So our assumption that this region's boundary has a finite length must be wrong. ◀

Set I. Many applications here: slicing oranges, dolphin sonar, latitude, driveways, hearing and location of sounds, and Reuleaux triangles (these are used in Wankel engines). We meet The Radian (26–30). Some discussion with students of the various modes on calculators would be in order.

Set II. The range of examples here is also great. Four Crescents (49–56) is an extended see-and-compute series with a surprising finale.

Set III. Pizza Puzzle is a nice exploration. It would be ideal for a *Geometer's Sketchpad*™ project. After you discover what the answer is, there is still the challenge of finding a reasonable way to prove you are correct. Jacobs has a nice discussion in his answer section and his reference gives further details.

Summary and Review. One theorem here is that all regular polygons are cyclic. This implies that they are symmetric by easily found reflections and rotations. While we have focussed on the details needed to carry us through this material, there is a larger picture, which involves ideas about symmetry. This can be mentioned as well in summing up the chapter.

Set I. Molecule (1–3). The conventions in representing organic molecules include not showing the hydrogen atoms, which is one reason the line formula looks simpler that the space-filling model. There may be some flexibility in the molecule, and structures may be able to rotate about a single bond, so one molecule may appear in several different looking forms, either as a space model or as a diagram.

Regular 15-gon (4–14). This is the figure for Proposition 16 of Book IV in Thomas L. Heath's edition of Euclid. The construction is given there. This series reviews the formulas developed and involves calculation and calculators. A nice added question here is: How do you locate the other vertices of the regular 15-gon?

Penny Farthing Bicycle (15–20) uses calculators. Area Comparisons (23–26) has a nice Aha! discovery at the end. Gothic Arch involves both exact calculations and the use of a calculator. Mathematical constructions of ornamental features are found in every place and age. Moorish tilings come to mind immediately, but examples of arches and traceries like this one are also common. See, for example, John L. Heilbron's *Geometry Civilized*, Oxford University Press, 1998, Chapter 5, From Polygons to Pi.

Time Zones (39–43) helps students build a geometric understanding of the earth and its geography. There is 3-dimensional thinking is needed here.

From Decagon to Square (44–49) is a dissection with a lesson: In dissections areas are preserved, although perimeters may not be. In Exercise 45, I note that

$$2 \sin \frac{180°}{12} \cos \frac{180°}{12} = \sin \frac{180°}{6} = \sin 30° = \frac{1}{2}.$$

Students will not know this, but my calculator gives 0.5 for this answer to 10 decimal places.

Chord and Ring (50–54) is a classic. If you *assume* the question has a definite answer, then you can find that answer by considering a degenerate case — here when the inner circle has 0 as radius. The 3-dimensional version of this conundrum involves a ball with a cylindrical hole centered on the ball's center. The length of the hole is L. What is the remaining volume of the ball?

Biting region (63 and 64). Beware of the dog!

▶ *How Archimedes Measured the Circle.* This is Archimedes with improvements by Andrew M. Gleason. Archimedes begins by observing that if a bounded convex region B is properly contained in the bounded convex region C, then the perimeter of B is smaller than that of C, and likewise for their areas.

Here is our setup: A regular n-gon with side length $s = 2$ is circumscribed about a circle of radius a_n, here "a_n" stands for *apothem* (the word for the radius of the incircle of a regular polygon). We let b_n stand for the radius of the circumcircle for our regular n-gon. Gleason observes:

$$\text{Incircle} \subset \text{Regular n-gon} \subset \text{Circumcircle}$$

$$\text{giving, for perimeters } 2\pi a_n < 2n < 2\pi b_n.$$

By algebra we get the following bounds on π

$$(*) \qquad \frac{n}{a_n} > \pi > \frac{n}{b_n}.$$

When $n = 6$ we have a regular hexagon with side length $s = 2$, then the radius is $b_6 = 2$ and the apothem is $a_6 = \sqrt{3}$. This gives us the crude bounds

$$3.5 > \frac{6}{\sqrt{3}} > \pi > \frac{6}{2} = 3,$$

rounding up on the high side.

Archimedes was careful and made sure that at each stage where he made an approximation, he made it in the direction that preserved the inequality he was interested in. As we move to decimal numbers, I will treat the calculator answer as *exact*, although I should follow Archimedes in his careful rounding. We will not push this so far that rounding at the 10th decimal would have any noticeable effect. If we wished to go further, every rounding or truncation would have to be made with due care.

These estimates can be easily improved by a series of steps, each step doubling the value of n. To do this draw a diagram showing a vertical side of our inscribed regular n-gon, s, of length 2, with the apothem, a_n, going to the left horizontally from the midpoint of s to the center O of the incircle and the circumcircle. Then add radii, b_n, from O to the endpoints of s. To double the number of sides to $2n$, you extend a_n through O by a distance b_n to get our new center, O′, which is on the circumference of the circumcircle of our original n-gon. Then the angle subtended by s at O′ is half the angle subtended by s at O, and s will be a side of our new $2n$-gon with center O′. From the figure you can see that

$$\text{(Scheme)} \qquad a_{2n} = a_n + b_n \text{ and } b_{2n} = \sqrt{a_{2n}^2 + 1}.$$

This shows us how to compute a_{2n} and b_{2n} from a_n and b_n. Two additions, one squaring, and one extraction of a square root are all we need. These operations were routine even in ancient times. The Babylonians knew how to extract square roots. Guess-and-test is a good method, if you don't have to do too many square roots. Another method is to make a guess at the square root of a positive number d; call your guess g. You know that $\sqrt{d}$ lies between g and d/g, if these are equal, your first guess was on the mark. If not let your second guess be $\frac{1}{2}(g + \frac{d}{g})$. This method can be carried out using approximate decimal arithmetic or using exact fractional arithmetic. It converges rapidly and can be handled by machine or by paper and pencil.

Gleason's scheme is a way of calculating the value of π to high precision using methods known before Euclid. It is easy to program a spreadsheet or programmable calculator to compute a_n and b_n and establish successive limits on π using this method. It is based on Archimedes' approach of successive doublings of the number of sides, but it uses the trick of getting its upper bound for the number π by *inscribing* a circle in our n-gon, rather than *circumscribing* an n-gon outside our fixed circle. It allows the circles to approximately double in linear size at each step, while the sides remain at length 2. This simplifies the algebra. And we use *algebra*, which makes our work easier than that of Archimedes. Finally, we let the computer to do the messy stuff for us.

What *exactly* are these a_ns and b_ns? Drawing the diagram of one side s of our n-gon. The apothem a_n goes from the center O of the n-gon to the midpoint M of our side s. Let P be one end point of the side s and P′ be its other endpoint. The distance OP is b_n. Because we have a *regular* n-gon, $\angle\text{MOP} = \frac{360°}{2n} = \frac{180°}{n}$. So we have the right triangle $\triangle\text{OPM}$ with $a_n = \text{OM}$, $\text{MP} = 1$, and $\text{OP} = b_n$. Trigonometry gives us

$$b_n \sin\left(\frac{180°}{n}\right) = 1 \quad \text{and} \quad b_n \cos\left(\frac{180°}{n}\right) = a_n.$$

When (*) is rewritten in trigonometric terms it becomes

$$(\dagger) \qquad n \sin\left(\frac{180°}{n}\right) < \pi < n \tan\left(\frac{180°}{n}\right).$$

If you look at Theorem 75, page 581 of *Geometry*, you will see that the left-most quantity is Jacobs's N.

To understand (†) in direct geometric terms, we need another diagram. Let ray m bisect $\angle\text{POP}'$. This ray m follows the apothem of our regular n-gon to M, the midpoint of chord PP′, then cuts the circumference of the unit circle at a point we label N. The tangent t to the radius OP of our unit circle at P cuts the ray m at point we call T. Then $\angle\text{POT} = \frac{360°}{2n} = \frac{180°}{n}$. We have three convex regions:

$$\triangle\text{POP}' \subset \text{Sector POP}' \subset \text{Quadrilateral POP}'\text{T}.$$

Their perimeters must be unequal in the same order as the set inclusions, by Archimedes' assumptions. They all share OP and OP′ as sides, so we can ignore these and see that

$$\text{Chord PP}' < \text{Arc Length PP}' < \text{PT} + \text{TP}'.$$

But $\text{Chord PP}' = 2\sin\frac{180°}{n}$, $\text{Arc Length PP}' = \frac{2\pi}{n}$, and $\text{PT}+\text{TP}' = 2\tan\frac{180°}{n}$. So the above inequality is

$$2\sin\frac{180°}{n} < \frac{2\pi}{n} < 2\tan\frac{180°}{n}.$$

Multiplying by $n/2$, we get (†).

If we *shift from degrees to radians*, then $\frac{180°}{n} = \frac{\pi}{n}$ radians, and (†) gives us the inequality

$$\sin\frac{\pi}{n} < \frac{\pi}{n} < \tan\frac{\pi}{n}, \text{ or, } \sin\theta < \theta < \tan\theta,$$

which holds where θ is in radians and lies in the range $0 < \theta < \pi/2$. To go from submultiples of π to arbitrary angles θ requires observing that the length of the arc

bounding sector with central angle θ, in radians, is just θr. This is intuitive and can be proven once you have the formal tools for defining the lengths of curves.

Archimedes started with a hexagon, our $a_6 = \sqrt{3}$ and $b_6 = 2$, which comes from 6 congruent pie-shaped pieces, each of which is an equilateral triangle of side length 2. Our table of Archimedes/Gleason Bounds gives a_n and b_n and bounds π for $n =$ 6, 12, 24, 48, 96, and 192. Our best value for π based on these computations would be the average of the upper and lower bounds on π for $n = 192$. This estimate is 3.1416627612, to my HP 48G+'s 10 places. To 10 places $\pi = 3.1415926536$, so the average of our upper bound and lower bounds exceeds π by 0.0000701076 or roughly 1 part in 45,000.

Let's get an upper bound on the gap between the upper bound, U_n, and the lower bound, L_n that we get for π. For a given $n \geq 4$, we have

$$L_n = n \sin\frac{\pi}{n} < \pi < n \tan\frac{\pi}{n} < U_n.$$

Now we put an upper bound on $U_n - L_n$ that holds for $n \geq 6$. In this calculation we use the fact that $n\sin(\pi/n) < 4$, which is true because the left side of this inequality is a lower bound for π and hence less than 4. We also use that fact that $\cos(\pi/n) > \cos(\pi/4) = 1/\sqrt{2}$, which holds because $0 < \pi/n, \pi/4$. Finally, we use the fact that $1 - \cos(\pi/n) < \frac{1}{2}(\pi/n)^2$, which more familiar in calculus, but is part of geometry as well.

$$\begin{aligned} U_n - L_n &= \left(n\sin\frac{\pi}{n}\right)\left(\frac{1}{\cos(\pi/n)}\right) \\ &< 4\left(\frac{1-\cos(\pi/2)}{\cos(\pi/2)}\right) \quad \left\{\text{Use } n\sin\frac{\pi}{n} < 4.\right\} \\ &< 4\sqrt{2}(1-\cos(\pi/n)) \left\{\text{Use } \cos\left(\frac{\pi}{n}\right) > \frac{1}{\sqrt{2}}.\right\} \\ &< 4\sqrt{2}\left(\frac{1}{2}(\pi/2)^2\right) < 2\sqrt{2}\left(\frac{4}{n}\right)^2 = \frac{32\sqrt{2}}{n^2}. \end{aligned}$$

Putting this on one line, we have

$$(\ddagger) \qquad U_n - L_n < \frac{32\sqrt{2}}{n^2}.$$

This shows that $\lim_{n\to\infty} U_n - L_n = 0$. We know $L_n < L_{2n} < \pi < U_{2n} < U_n$, so if we proceed by doubling the number of sides our lower bounds increase from one step to the next and our upper bounds decrease. The common limit along such a sequence of doubling of n's must be π. Indeed, π could be defined *analytically* as this common limit. We have chosen to define π *geometrically*, so these calculations are simply a way of working out the value of π numerically — a reworking of the calculations Archimedes did so long ago.

This inequality ($\ddagger$) implies that $|L_n - \pi| < 32\sqrt{2}/n^2$ and $|U_n - \pi| < 32\sqrt{2}/n^2$, so the upper and lower bounds both converge to π as $n \to \infty$. In fact, the lower bounds increase as n increases while the upper bounds decrease as n increases. This is not as obvious as it is when we double the number of sides, but it is true.

Archimedes/Gleason Bounds for π

n	a_n	b_n	Bounds on π
6	$\sqrt{3}$	2	3.465 Upper_6
6			3 Lower_6
12	$2+\sqrt{3}$	$2\sqrt{2+\sqrt{3}}$	3.2153903092
12			3.1058285412
24	7.5957541127	7.6612975756	3.1589113607
24			3.1326286138
48	15.2570516883	15.2897882987	3.1460862151
48			3.1393502030
96	30.5468399870	30.5632039091	3.1427145996
96			3.1410319509
192	61.11004389661	61.1182253095	3.1418730500
192			3.1414524723

Chapter 15. Geometric Solids.

To the Student. We live in a three-dimensional world and can deal with three-dimensional shapes with ease. Here we learn how mathematical ideas from plane geometry allow us to work out the properties of three-dimensional shapes and calculate surface areas and volumes. Euclid's Elements build toward solid geometry (Book IX) and the regular polyhedra (Book XIII), which we cover in this chapter.

Postulates 11 and 12. Postulate 11 says that if two points on the line ℓ lie in the plane Π, then $\ell \subset \Pi$. We need this in order to know that lines do not somehow stick out of the planes they should lie in. Postulate 12 says that if two planes intersect, they intersect in a line. This tells us that ours is a three-dimensional space.

In four dimensions, the two-dimensional planes

$$\Pi = \{(u, v, 0, 0) \mid u, v \text{ real numbers}\} \text{ and}$$
$$\Sigma = \{(0, 0, w, x) \mid w, x \text{ real numbers}\}$$

intersect in a single point, $(0, 0, 0, 0)$, not a line. This cannot happen in three dimensions.

These are the only postulates we need to add for the study of solid geometry. Given these, Euclid proved the results in the supplemental notes for this chapter and outlined in Solid Geometry as a Deductive System, pages 626 and 627 of *Geometry*.

Lesson 1. Lines and Planes in Space. This lesson and the summary on pages 626 and 627 Of *Geometry* are the basis for our study of solid geometry. Our

approach is based on intuition, with all assumptions clearly stated. Students will need to use the tools of plane geometry freely as they study solid geometry.

We know that lines that are parallel *have the same direction.* When thought of this way, *it is immediate that two lines parallel to a third line must be parallel to each other.* However, this transitivity of parallelism is not easy to prove from the formal definition. Teaching the spirit of parallelism is more important than getting hung up on the letter of parallelism. We want students to trust their intuitions, but also to be able to check them, as needed.

Likewise, planes are parallel if and only if they are *perpendicular* to the same line or to lines in the same family of parallel lines.

This lesson introduces lines and planes in space and their relationships. The goal is to link intuition and common understanding to geometric terminology. This lesson sharpens vision and builds vocabulary.

Set I. These are exercises in seeing what is portrayed and interpreting it as solid geometry: visual thinking, three-dimensional relationships, passing from a two-dimensional picture to the three-dimensional geometry, and reasoning about shapes in space.

I like all of these exercises, and I was tickled to see Kilroy, of "Kilroy was here," put to good use in Kilroy in Space (24–29).

Set II. Triangles in Perspective (31–37). This is Desargues's Theorem, discussed at the beginning of Chapter 13, The Concurrence Theorems. In its projective geometry form this theorem says: Two triangles are perspective from a point if and only if they are perspective from a line.

Matsumassa Anno's Impossible Slice (38 and 39) is a delight and gives us a chance to think clearly about what is clearly impossible. A calculator is needed in Parallel Bars (40–43). Lopsided Box is excellent as an exercise in visual thinking and deductive skills. You cannot just go with intuition here.

Cube and Hypercube (54–57) shows how we can reason from the net for a cube to get properties of the cube and extends this reasoning to draw conclusions about hypercubes.

Set III. Escher's *Belvedere* lets students find and describe geometric impossibilities.

Geometry as a Deductive System. This brief outline of the basis for solid geometry sets out the essentials.

Lesson 2. Rectangular Solids. The subject of this lesson is as solid and common as a brick, and as crucial to building our understanding of geometry in three dimensions as building stones and bricks are to architecture.

Vocabulary introduced or reviewed here includes: polyhedron, faces, edges, rectangular solid, diagonal, and the common uses of length, width, and height in describing rectangular solids.

Theorem 79, which gives the formula for the diagonal of a rectangular solid, also tells you that the distance between points $P = (x, y, z)$ and $P' = (x\prime, y\prime, z\prime)$ is given by

$$\|P - P'\| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

where we use a set of Cartesian Coordinates in space. Such axes are mutually perpendicular and have equal units. This distance, written $\|P - P'\|$, is just the diagonal of a rectangular solid whose faces are parallel to the coordinate planes and whose opposite vertices are P and P′.

Set I. Inside or Outside? (1–8) is an exercise in visual thinking. Deck of Cards (9–13) will need a calculator. Integer Lengths (14–17) acquaints students with an open research problem. Slicing a Cube (20–28) shows how to figure out what plane slices of a solid look like. For more complex and less familiar three-dimensional shapes the problem of determining what various slices can look like is quite difficult. For instance, what is the maximum number of pieces into which a doughnut can be sliced by three planes? This is Macalester College's Problem of the Week for Week 866. In 2003 it was on the Math Forum website. Search on Stan Wagon and see *Which Way Did the Bicycle Go?* by Joseph D. E. Konhauser, Dan Velleman, and Stan Wagon, Mathematical Association of America, 1996 for more problems from the Macalester group.

Set II. An Inside Job (29–31) is problem-solving with an Aha! at the end. Diagonals (32–37) involves visual thinking, three-dimensional relationships, a review of angles and trigonometric ideas, and use of a calculator. Hole through a Cube (38–42) has a conclusion that is quite surprising, until you understand what is going on. Bond Angles (43–47) involves visual thinking, spatial sense, and use of a calculator. Soma Puzzles (48–54) involves visual thinking, spatial sense, logical reasoning, and problem-solving. This sequence shows how local analysis of the pieces of a puzzle allows you to figure out what is happening. This sort of reasoning is the key to solving "hidden block puzzles."

Set III. Flattened Cube. This is an fresh sort of problem based on nets for a cube. It is within the reach of anyone who will draw the nets on paper and work at it.

Lesson 3. Prisms. These are the next step in building our tool kit of solid shapes and their properties.

Set I gets off to an easy start thinking about 3-D Lettering (1–3). Simple, but it forces students to think about what they see and what gives things their 3-D

look. A Hexagonal Surprise (19–28) is surprising. See Avoiding Trouble.

Avoiding Trouble. This Hexagonal Surprise is a fact whose proof depends on tracking relations between quantities that are interdependent. There must be a purely geometric approach that will prove that this hexagonal configuration always gives two congruent triangles, associated with the two sets of non-neighboring vertices of the hexagon, and that these two triangles are translates of each other. I gave up looking for such a proof.

The geometry suggests more complexity than is there. It looks as if we need to consider *any* six points in the plane, but this is not so because our construction and our conclusions can be stated only using parallel lines. Working with parallel lines alone, we cannot pin down congruence, but we can determine whether a transformation is a translation or not, and that is what we need here.

Simplify. Parallelism can be checked by seeing that slopes are the same. Slopes are the same in *one* coordinate system iff they are the same in all coordinate systems (where we allow $m = \infty$ for the moment). Suppose our hexagon is ABCDEF. We simplify the problem by setting up coordinates so that $\mathrm{F} = (0,1)$, $\mathrm{A} = (0,0)$, $\mathrm{B} = (1,0)$, $\mathrm{C} = (c_1, c_2)$, $\mathrm{D} = (d_1, d_2)$, $\mathrm{E} = (e_1, e_2)$. The axes here will not necessarily be at right angles nor will they necessarily have scales that correspond to the distance in the plane, but that is of no matter, because all we are interested in is parallelism, something that is determined by the same algebraic condition in all such sets of coordinates.

Now we find the coordinates of the points A', $\mathrm{B}', \ldots \mathrm{F}'$ of the Hexagonal Surprise. Please refer to Hexagonal Surpriise for what follows. FABA' is a parallelogram, and the diagonals of a parallelogram bisect each other. The bisector of FB is the point whose coordinates are determined by $\frac{1}{2}(\mathrm{F} + \mathrm{B})$. Likewise, the bisector of AA' is $\frac{1}{2}(\mathrm{A} + \mathrm{A}')$. Setting these equal, and solving for A' we get

$$\frac{1}{2}(\mathrm{F} + \mathrm{B}) = \frac{1}{2}(\mathrm{A} + \mathrm{A}'), \quad \text{so} \quad \mathrm{A}' = \mathrm{F} + \mathrm{B} - \mathrm{A}.$$

And the analogous algebra gives us all of the other primed points. Then calculation gives

$$\begin{aligned}
&\mathrm{A}' = (1,1),\ \mathrm{B}' = (c_1 - 1, c_2),\\
&\mathrm{C}' = (1 - d_1 - c_1, d_2 - c_2),\\
&\mathrm{D}' = (e_1 + c_1 - d_1, e_2 + c_2 - d_2),\\
&\mathrm{E}' = (d_1 - e_1, 1 + d_2 - e_2), \text{ and } \mathrm{F}' = (e_1, e_2 - 1).
\end{aligned}$$

Checking coordinates we find

$$\begin{aligned}
\mathrm{A}' - \mathrm{D}' &= \mathrm{C}' - \mathrm{F}' = \mathrm{E}' - \mathrm{B}'\\
&= (1 - c_1 + d_1 - e_1, 1 - c_2 + d_2 - e_2).
\end{aligned}$$

So $\triangle \mathrm{D}'\mathrm{F}'\mathrm{B}'$ is a translate of $\triangle \mathrm{A}'\mathrm{C}'\mathrm{E}'$, and we have a proof of the property observed in this series.

This would be an interesting problem to explore using technology. I worked out an approach using vectors, but I was unable to reconstruct it. It is interesting that there is no generalization to octagons, let alone $2n$-gons for $n > 3$.

This result can be proven by algebra, and a computer algebra system might help get the result without simplifying things by choosing a convenient coordinate system. Transformations could be used to bring the original hexagon into a standard position, but this is equivalent to the simplification of choosing the coordinates as is done above.

If there is a neat and purely geometric way to do this, I would like to see it. For problems of this nature, the simplest approach is often to grind out the algebra — after setting things up so as to simplify the work.

Set II. Magic Box (42–49). Andrew M. Gleason mentioned that he had built such a box and that it worked well.

Euler's Discovery (50–53). For connected planar maps, if we count the number of Countries, Edges, and Vertices we find:

(∗) $\qquad$ Countries − Edges + Vertices = 1.

There are simple inductive proofs of this result. There are so many good sources that I hesitate to single out one. Euler's formula, (∗), for planar graphs can be related to the Euler characteristic for polyhedra by means of the Schlegel diagrams introduced on page 688 of *Geometry*. In this interpretation, the Faces of P translate into the Countries of our map. See also the sequence More on the Euler Characteristic, page 651 of *Geometry*.

Set III. This is a simple start at understanding how the observable geometry crystals can tell us something about the arrangement of atoms. When this work was done devices such as the atomic force microscope had not been developed and our understanding of crystal structure came from indirect observation.

Lesson 4. The Volume of a Prism. This is a concrete and simple introduction to volume. All geometric measurement comes down to counting how many congruent copies of the basic unit it takes to fill out the shape being measured. We must allow fractional parts of the unit to be used: tenths, thousandths, and so on, *ad infinitum*. This is where the decimal or other notations come in.

This is how we know that the volume of a rectangular solid of length l, width w, and height h is

(∗) Volume = Length × Width × Height, or $V = lwh$.

The beauty is that this is obvious when the dimensions are integral multiples of the unit length. Because of the way our notation works, this ensures that the same formula holds when we add in decimal submultiples of the unit length. It is important that we *use the same units for the dimensions being multiplied* if we want our answer in standard cubic units. There are other sorts of units, such as acre-feet, but they are uncommon.

Postulate 13, Cavalieri's Principle, is intuitive and gives us the volumes of any solid shape we are likely to deal with. This principle underlies the slicing methods used to find volumes in calculus. See Keng-chih's Principle, Exercises 26–28. Postulate 13 follows from Cavalieri's Principle by the argument given in the last paragraph of page 641 of *Geometry*.

We need Postulate 13 because we rely on intuitive ideas of volume, namely the properties suggested by the top two figures in the margin of page 641 and referred to in the starred paragraph. To make Postulate 13 into a theorem, we would need to develop a *theory* of volume, following the idea that volumes can be *defined* by measuring how many unit cubes or thousandths of a unit cube, and so on can be packed into a given solid. Using this idea we could prove the results in (*), and have Postulate 14 as a consequence of Cavalieri's Principle. In calculus or analysis, a theory of volume is developed, and these results are theorems.

Set I. These are lovely exercises: concrete, interesting, and thought provoking. You learn something from each of them. The Weathering of Rock sequence (16–25) describes a recursive fragmentation process. This is a fractal idea, and there is considerable interest in fractal geometric processes in geology, especially in subsurface flows through fractal fissures in connection with recovery of oil from oil fields and in connection with the spread of pollutants in ground water.

Set II. Ream of Paper (38–40) gives students an idea of how thick a sheet of paper is. Emilio Segrè told me that when he worked with Enrico Fermi's group in Rome, they made it a practice to think about common things that could be used to get a quantitative grip on things in everyday life and in the laboratory. Segrè mentioned the thickness of sheets of paper, the volume of a drop of water, and so on. But note that some drops are larger than others and some sheets are thinner than others. *Geometry* by Jacobs is almost 800 pages (400 sheets) and it is less than an inch thick, not counting the covers, but the 500 sheet ream of paper in these exercises is two inches thick. See also Set III, below.

Binomial Cube (41–46) brings algebra and geometry together with visual thinking. Advertising Claim (57–60) brings critical thinking into play, part of the numeracy or quantitative literacy theme discussed following Set III in these notes.

Set III. Molecule Experiment. The picture here is from the WGBH series *The Ring of Truth,* discussed below. The pond is in a park in central Boston and the oil slick is of the finest olive oil. The idea of this experiment has been around for ages, but it is difficult to get consistent experimental results, and the results are hard to interpret. In the book by Frederick P. Greenleaf, mentioned below, this was done as an experiment in a laboratory section. Greenleaf's lab manual describes some of the difficulties in getting consistent results.

Irving Langmuir and Katherine B. Blodgett did work on thin films at GE. Out of this work came the idea of antireflective coatings for optics, which has wide uses today. Accounts of their work can be found in the library or on the Web.

Quantitative Literacy. This is an emerging theme in the curriculum, and *Geometry* teaches it throughout the book. In this lesson, we are working out volumes of real things: candy bars, suitcases, swimming pools, reams of paper, and the volume of individual molecules.

It is important to understand that *quantitative literacy* means having a general idea of what numbers *make sense* for various quantities, not simply in the ability to figure out volumes once someone else has given you numbers to work with. For example, in *Innumeracy* John Allen Paulos mentions a question on page 12, said to be used by a consultant from MIT to weed out candidates for jobs. (In 2003, *Innumeracy* was available from Hill & Wang in a 2001 paperbacked edition.) Here is the question: Suppose trucks come every fifteen minutes, twenty four hours a day without interruption, are instantly filled with earth and rock, and carry their loads away without getting in each other's way. How long would it take to cart away an isolated mountain, say Mount Fuji, to ground level? The only information Paulos gives is that Mount Fuji is 12,000 feet high, so this must be used to get our answer.

A few pages later, and without explanation, Paulos gives his answer: 5000 years. This seemed too short to me, so I decided to work it out myself. There are three things we need to know in order to work this out, starting from the reasonable assumption that Mount Fuji can be considered to be a cone. 1. What is the altitude of this cone? This includes deciding what we consider "ground level" for this problem. 2. What is the area of the base of this cone? Given the answer to 1, this comes down to deciding how steep the sides of Mount Fuji are. The answer we get will depends on our assumptions here. 3. How much rock can a truck carry in one load?

Looking at a contour map of Mount Fuji suggests that clearing it from its present height of 3776 meters down to the 1000 meter level would make the remnant blend in with the surrounding country side. The map also shows that at the 1000 meter contour we can consider Mount Fuji's base to be a circle with a radius of 12 kilometers. A call to a local importer of heavy Japanese earth-moving trucks got the information that the largest of these could carry 18.5 cubic yards. I rounded up to

20 cubic meters. Do the calculation and you will find that it would take 600,000 years to truck away the top 2776 meters of Mount Fuji.

Paulos did not tell me all of his assumptions, but he did say that he assumed Mount Fuji was a cone as wide at its base as it was high. This would make its sides slope upward at a 60° angle. Very impressive. Here is where his quantitative literacy should have made him *check it out*, always a good thing to do if you think a calculation is really important.

The above analysis is from my article, "Thoughts on *Innumeracy:* Mathematics Versus the World," published in *The American Mathematical Monthly*, October 1993. A reply from John Allen Paulos followed my article.

Paulos did not name his consultant from MIT. But I know of a similar question asked in a job interview. David Root was teaching at Purdue University around 1970, when he decided to leave academia and work on the problem of energy supplies.

A reigning power then was M. King Hubbert, who correctly predicted in 1956 that US oil output would hit a maximum in the early 1970s and then decline (this was "Hubbert's Peak"). Having studied energy-related matters for a year at Purdue, Root flew to interview with Hubbert. When they met, the first question Hubbert asked Root was: How many population doublings did it take to go from Adam and Eve to the current population? (then about 3.5 billion) Root answered "31," not missing a beat. Hubbert said something like, "Closer to 32." This is practical numeracy. David Root made a career with the US Geological Survey.

We should teach numeracy. I believe that Jacobs has done as much as anyone to help students see how they can use mathematical ideas to make sense of the world around them. He began with his students, for whom he wrote *Mathematics: A Human Endeavor,* now in its third edition. Many other teachers have taught from this book. Jacobs has made realistic and interesting examples a hallmark of his texts. Working such examples shows the value numeracy and teaches it at the same time.

Numeracy, quantitative reasoning, and critical thinking are terms you can search on the Web to see what is available in this area. For a broad appreciation of measurement and calculation in science, I recommend *The Ring of Truth* by Philip and Phylis Morrison, both a book (Random House, 1987) and a television series (WGBH, available in VHS format in 2003). Also of interest is *Quantitative Reasoning: Understanding the Mathematical Patterns of Nature,* the text developed for a core course at New York University by Frederick P. Greenleaf and available from McGraw Hill, Primus in 2003.

Numeracy is important in evaluating economic, social, educational, and medical ideas, but data on things such as student achievement are difficult to gather and subject to varying interpretation. This takes us away from geometry, where things are easier. Here is what Lynn Steen wrote in the Epilogue to *Mathematics and Democracy: The Case for Quantitative Literacy,* which he edited for the Woodrow Wilson Foundation and the National Council on Education and the Disciplines.

"... [N]umeracy is not so much about understanding abstract concepts as about applying elementary tools in sophisticated settings. ... [T]his is no simple feat. Numeracy takes years of study and experience to achieve."

A Web search on various key words and names mentioned above will turn up current material on numeracy and quantitative literacy. The best approach to these skills is to use real numbers and find interesting examples that can be dealt with using numbers in subjects across the curriculum. *Geometry* has such examples, so Jacobs is on the right track.

Lesson 5. Pyramids. Pyramids are a perennial favorite topic and a Web search on the names of any of them (Bent Pyramid, Pyramid of Cheops) will turn up interesting material. Imhotep is the legendary architect of pyramids.

Thinking about the middle line of figures on page 648 of *Geometry* the right way will let you see why these three pyramids all have the same volume. Using Cavalier's Principle, all you need to notice is that each pair of these pyramids can be seen as having bases that are congruent and apexes that are identical. The trick is that you must think of the bases as being on different rectangular sides of the prism when you switch to a different pair of pyramids. This insight will reward the industrious.

Set I. These exercises cover the the lesson. Leonardo's Claim (18–20) involves visual thinking, number sense, and the ability to decide whether something looks like a reasonable picture of what is described.

Set II. Divided Cube (27–32) is a geometric discovery that we would think of as requiring visualization. It is notable that it was made by a blind mathematician. The physicist Plateau, who studied the geometry of soap films (Web-search on Plateau's Problem) was also blind, and there are a number of successful blind mathematicians today, among them Lawrence Wos, who works on computer-aided reasoning at Argonne National Laboratories, and Lawrence Baggett, who served as Chairman of the Mathematics Department at the University of Colorado, Boulder. See Allyn Jackson's article from the November 2002 *Notices of the American Mathematical Society,* titled "The World of Blind Mathematicians" for more about blind mathematicians present and past. This was available from the AMS website over in 2003.

Pyramid Numbers (33–37). Although the cited reference does not say where this is to be found in Herodotus,

digging reveals that it is in Book II (Euterpe), paragraph 124, where you will find "Each face is eight plethra and the height is the same." Looking under "Weights and Measures" in *The Oxford Companion to Classical Literature,* Oxford University Press, 1989 for "Plethron" sheds light on this matter. A plethron is *both* a measure of length and of area. As a measure of length, it is 100 Greek feet, a Greek foot being 16 Greek fingers. As a unit of area, it is the Greek acre. That is, a plethron is the nominal area for a field that a man can plow in one day, and this source gives it as 100 feet square, that is, 10,000 Greek square feet. This area seems low compared to the modern acre, which is 1 chain by 10 chains or 66 by 660 feet and is the nominal amount of land that a man can plow in a day. But these "feet" may not be the same, and we had better plows than the ancient Greeks, even before we got tractors. If we think of both plethera in the quotation as being areas, then that second must be gotten somehow from the height, and squaring is the only reasonable way — and the numbers work out reasonably closely, so this might be what was intended, or it might not.

Richard Gillings, in his book *Mathematics in the Time of the Pharaohs*, available in 2003 from Dover Publications, says that he sees no evidence that the pyramids were constructed on the area principle given above. It is interesting that he also suggests that Egyptians may have used the method of the Divided Cube above to figure the volume of pyramids.

See Gillings also relating to The Frustum of a Pyramid (48–57), which was something the Egyptians seem to have known about.

Set III. How Many Faces? Student versus SAT. Student wins. Inspiring.

Lesson 6. Cylinders and Cones. The general notion of *a cone* C *over a set* S *with apex* P is that C is the union of all segments sP, such that s$\in$ S. The results in this lesson are true for volumes of cylinders and cones, whether or not their bases are circular, as can be seen by Cavalieri's Principle.

Considering Cones. Take a pyramid P whose base has the same area, A, and lies in the same plane as the base of the cone C. Let the apex of P be at the same height above the common base plane of P and C as the apex of C is. Taking a slice parallel to the base plane at distance λ below the common plane of the apexes of P and C, we find that the areas are λ^2A for both the cross-section of P and that of C. This is true because of the general scaling principle for areas under similarity (Lesson 6, Chapter 10). Because slices at equal heights have equal areas, the volumes are equal. Both volumes are $\frac{1}{3}Bh$, the formula for pyramids.

Set I. Primitive Perspective (1 and 2), visual thinking and three-dimensional geometry. Italian into English (3–9) involves general problem-solving. Solids of Revolution (10–20) involves visual thinking and three-dimensional geometry. Sliced Solids (23–26) brings in key ideas relating to contour maps. These sorts of slicings are used to build three-dimensional models by sticking together slabs cut to the contours. There is a lot out there. I found a nice contour map of Mount Fuji on the Web. Sausages in a Can (27–31) is a practical sort of calculation.

Set II. This set is loaded with quantitative comparisons of geometrical quantities arising in everyday life. An understanding of Rolling Cones plays a key role in Michael Gilbert's classic 1950 murder mystery, *Smallbone Deceased.*

Set III. Iron Ore, photographic interpretation again. See also What time is it? on page 453 and my notes on it. In 2003 the Smithsonian Institution had some coverage on Photo Interpretation and there were many college-level and commercial websites devoted to this topic. Books such as *Below from Above* by George Gerster, Abbeyville Press, 1986, cited on page 453 of*Geometry* and *Earth from Above* Yann Arthus-Bertrand, Harry N. Abrams, 2002 (photographs also available in a large-sized postcard book of the same title; see www.yannarthusberthold.com) offer many striking images that could be the basis for photo-interpretation exercises.

Lesson 7. Spheres. The material of this lesson and the arguments supporting it are at just the right level of detail and completeness. Cavalieri's Principle is the tool to find the volumes that Archimedes determined by his method of exhaustion and that we find today by integration. The volume of a sphere is a third of its surface area times its radius, for reasons given on page 664. Without making any argument, one can jump to this conclusion by the analogy outlined below. This may be easy for your students to grasp and remember *because* it is outrageous.

Volume of the Sphere. We may view the circle as a triangle whose base is its circumference and having one vertex at its center, with its altitude being r, so

$$\text{Area} = \frac{1}{2}[\text{Base} \times \text{Altitude}] = \frac{1}{2}[\,(2\pi r)\,r] = \pi r^2.$$

In the same spirit, we may think of a spherical ball as a pyramid whose base is its surface and whose apex lies at its center, then its volume is given by

$$\text{Volume} = \frac{1}{3}[\text{BaseArea} \times \text{Altitude}] = \frac{1}{3}[\text{SphereArea} \times r].$$

Area of the Sphere. Consider a sphere like our earth, with a given axis and equator. Archimedes knew that if you project the points of a sphere outward from its axis onto a cylinder tangent to the sphere at its equator, then this mapping preserves areas. In map making this is Lambert's Equal Area Projection. If r is the radius of the sphere, the image of the sphere lies on a

cylinder of diameter $2r$ and height $2r$. The area of this cylinder is $\pi \times 2r \times 2r = 4\pi r^2$, and if this map projection preserves area, then the area of a sphere of radius r must be $4\pi r^2$, as given by Theorem 84 of *Geometry*.

The argument that Lambert's Projection is area-preserving involves looking at an narrow symmetric strip with a north-south width of w about a parallel of latitude. To fix the geometry so a diagram can be drawn, assume this parallel is at a north latitude of θ on our sphere of radius r. This parallel is a circle of radius $r\cos\theta$ about the axis and lying distance $r\sin\theta$ north of the equatorial plane. For small w, its area on the sphere can be thought of as that of a rectangle of length $\ell = 2\pi r\cos\theta$ and width w, so we may take its area to be Area $= \ell \times w = 2\pi r\cos\theta \times w = 2\pi rw\cos\theta$.

Our strip projects to a strip around the cylinder tangent to the equator. This cylinder has radius r, and so the image is a strip of length $\ell' = 2\pi r$. The north-south distance w lies in the surface of the sphere and makes an angle of θ with the vertical generators of the cylinder we are projecting to, so width w on the sphere projects to a width $w' = w\cos\theta$ on the cylinder. The area of the strip as it is projected onto the cylinder is then Area$' = \ell' \times w' = 2\pi r \times w\cos\theta = 2\pi rw\cos\theta$, which is the same as the area before it was projected, and the Lambert Projection is area-preserving.

This is a calculus argument. Archimedes used his method of exhaustion, which is a geometric form of integration. See his *On the Sphere and Cylinder* or, better, Sherman Stein's book *Archimedes: What Did He Do Besides Shout Eureka?*, cited earlier. When we worked out the area of our strip on the sphere, we approximated the curved area on the sphere, call it SphereStrip, by the area of a flat strip of the same length and width, $2\pi r\cos\theta \times w$. This works out because

$$\lim_{w\to 0} \frac{\text{SphereStrip}}{2\pi r\cos\theta \times w} = 1.$$

As $w \to 0$, the approximation becomes better and better and the ratio of the original and Lambert-projected areas is 1.

Carlos A. Furuti had a nice website covering map projections in 2003; search for map projections on the Web or at Eric Weisstein's MathWorld.

Set I. Archimedes' Discoveries (7–10). A good source on this material is Stein's book on Archimedes, cited earlier. Flatland (11–15). Abbott's book is available in an inexpensive Dover edition as well as in an elaborate, annotated version by Ian Stewart. These exercises show how three-dimensional geometry can be worked out using what we know about two-dimensional geometry. Sound Wave (16–18). The *intensity* of the wave at radius r from the source varies as $1/r^2$. The intensity integrated over a spherical shell gives the outward flow of sound power through the sphere of radius r about the source. This flow will be independent of the radius r under steady-state conditions.

Volume and Density (19–23). This sequence helps students understand the difference between density and mass — commonly a confusing matter. Chocolate Packing (24–27). See Center-Pivot Irrigation, Chapter 14, Lesson 5, page 601 of *Geometry* and my notes on it.

Hollow Earth (32–37). Points P and P′ are interchanged by *inversion* with respect to the sphere S with center O and radius OA. Inversion in S is a *conformal mapping* that interchanges the outside of S with the inside of S. The origin is interchanged with the point at infinity. Conformal mappings preserve angles but may distort lines and distances. The symmetries of the Poincaré model of the Lobachevskian non-Euclidean plane are inversions in circles that meet bounding circle of this model at right angles. See the Lobachevsky Meets Escher sequence on page 716 of *Geometry*. Search the Web for *inversive geometry* for more. You will find Java applets generated by *Cabri Geometry* or *Geometer's Sketchpad*™.

Melting Snowball (38–43). This is a related-rate problem. The assumption is that the volume is decreasing at a constant rate, which means the flow of calories into the snowball is constant over time until the snowball melts away. There is the sublimating mothball problem: the rate of sublimation (in grams per second) is assumed to be proportional to the surface area. If it loses half its mass in two months, how long before it evaporates completely?

Earth Under Water (44 and 45). The assumptions are clear and the fact that there is so much surface water on the earth is surprising. A calculator shows us how these two estimates compare. They agree to about 4 parts in 10,000. We treat the earth as if it were not rotating, so it has no equatorial bulge.

Set III. *Raiders of the Lost Ark* should give your student's brains some exercise, and it is fun to recall this action movie.

Lesson 8. Similar Solids. Scaling was one the "two new sciences" Galileo wrote about in his dialogues. There are many sources on scaling, but a favorite for me is *On Size and Life* by Thomas McMahon and John Tyler Bonner, Scientific American Library, W. H. Freeman and Company, New York, 1985.

The bottom line for this lesson is that if $S_1 \sim S_2$ in three-dimensional space with ratio of similarity r, so that $rS_1 \cong S_2$, and if the areas of these shapes are A_1 and A_2 and their volumes are V_1 and V_2, then,

$$\frac{A_2}{A_1} = r^2 \text{ and } \frac{V_2}{V_1} = r^3.$$

In words, areas scale as the square and volumes scale as the cube, as found in Theorems 83 and 84.

Scaling and Strength. The tensile strength of a cable scales in proportion to its cross-sectional area. The resistance to bending for a beam is proportional to the (second) moment of inertia of its cross-section, and hence, for similar beams, the resistance to bending scales as the fourth power of their linear sizes. See J. E. Gordon's *Structures; Or Why Things Don't Fall Down*, Pelican Books, 1978.

If a cable is made sufficiently long, it will break under its own weight — independent of its width. It is weight per unit area that determines the failure point in tensile stress, and that is the same for thick and thin cables of the same length hanging under their own weight. Likewise, a pile of bricks sufficiently high will crush the bricks at the bottom of the stack, regardless of the width of the pile. The thinner pile is more subject to toppling, however. Scaling arguments let us understand why there are no giant humans, geometrically similar to us but three times our size.

Twenty six of the exercises in this lesson deal with questions of what happens when you change the scale of a creature or device. This sort of thinking is very useful, but some caution is needed. If you take a bridge design and linearly scale it up to cross a wider span, we know that the weight of the bridge will go up as the cube of the linear scale, while the bearing area for this weight, the footing area, will only go up as the square of the linear scale. We conclude that geometrically similar designs will fail at some larger scale.

In the same way, you know that $0.0001r^3$ will eventually become larger than $10,000r^2$ if r is made large enough, but you have to go a long way to see this effect. If the bridge was designed to carry large loads compared to its weight, then scaling it up will initially result in *increased* carrying capacity. It is a mistake to think that larger things of a similar shape must be weaker.

See the note on Scaling and Fractals at the end of the notes on this lesson.

Set I. Similar or Not? (1–5) involves visual thinking and drawing three-dimensional shapes. Scaling Up (6–11) involves understanding how strength of support and weight scale up from a model to a full-scale structure. Giant Windmill (12–15) continues this exploration of scaling, this time for renewable energy windmills. Gulliver in Lilliput (16–18) is a classic example of scaling. Note that Johnathan Swift knew his maths!

Set II. Doubling the Cube (29–36) covers basic scaling calculations. Plato's Method (37–46) is somewhat more theoretical, with a review of some ideas from right triangle similarity. Rubber Bands (44–46) is a wonderful practical example, and utterly original. Height and Weight (51–54) is very practical. Heated Ring (55–56) is a classic. Ask your physics colleagues about this. Teddy Bear Trouble (57–60) is about how things of different dimensionality scale: Buttons are like points, 0-dimensional, and when you scale by a factor of r you need $r^0 = 1$ times as many as you had. For cloth, which is 2-dimensional, you need to multiply by a factor of r^2. In general the factor is r^d, where the thing you are concerned with is d-dimensional. See Scaling and Strength above and Scaling and Fractals below.

Set III. Biological scaling, and an example within the reach of every student. It would be natural for students to look up material about giant clams to see how reasonable their estimates were.

Scaling and Fractals. Some things do not follow scaling laws with simple or expected exponents. The lungs of mammals of various sizes are similar in function and may look like geometric scale models of each other. But their development allows the surface area for the exchange of gasses to increase by a ratio greater than the r^2 that geometric scaling would suggest. The reason is that as the lung grows to a scale appropriate to a larger mammal, it develops more branches in its air passages and more tiny breathing chambers (alveoli) to *fill out* the larger volume of a larger lung. See, for example, Plates 164 and 165, of Benoit Mandelbrot's *The Fractal Geometry of Nature*, W. H. Freeman and Company, New York, 1982 for a two-dimensional analogy of the bronchial tree. See also Mandelbrot's discussion of lungs on pages 157–160.

For lungs of mammals, the area available for exchange of gasses between blood and air scales more as r^3 than as r^2. This is contrary to geometric scaling laws. But *fractal geometry* comes to the rescue. Benoit Mandelbrot showed that such anomalous scaling was a sign that we should think of the exchange surface of the lung as a *surface* whose dimension is closer to 3 than to 2. Here he used the idea of *fractal dimension*. Initially, he used Haußdorf-Besicovitch (HB) dimension, a way of defining the dimension of any set lying in a space that allows you to find distances between any pair of its points (a metric space). The tricky HB dimension is often replaced by the easier to understand and compute *self-similarity dimension*, the most commonly encountered fractal dimension.

The boundaries of fractal shapes typically have peculiar dimensions. The Koch snowflake is a familiar example. Starting with an equilateral triangle as the 0th stage, you erect outward equilateral triangles on the middle third of each of its faces to get the 1st stage, and continue in this manner to get the 2nd stage, and so on, to get the snowflake as the limit. See Mandelbrot or other sources. The boundary curve of this snowflake has (similarity) dimension $\frac{\log 4}{\log 3} \approx 1.2618$ versus the dimension of 1 we would expect for a smooth curve.

Similarity dimension is defined for shapes that have a strict sort of fractal symmetry, namely they can be broken into a finite number of pieces so that when these pieces are reduced in scale by a factor of r and you take N sets of the reduced pieces, then you can reassemble

these pieces to get the original shape. In short, the thing is the sum of N copies of itself reduced in scale by r. An example is the square which is the sum of $N = 4$ copies of itself at scale $r = \frac{1}{2}$. A shape allowing such a decomposition has

$$\text{Similarity Dimension} = \frac{\log N}{\log(1/r)}.$$

A quick check will show that this gives the right dimension for familiar shapes such as an interval, a square, and a cube.

The limit for the Sierpinski polygon found on pages 144 and 145 of *Geometry* is a continuous image of a circle. Imagine the circumcircle of the first stage being deformed to match this first polygon. At each stage the deformation becomes more complicated. In the limit, you get a map of the circle onto a curve that covers every point of a square. You can see the outline of this square in the fourth stage shown in *Geometry*. This limiting "curve" has dimension 2, versus 1, which is the dimension of any smooth image of the circle.

Fractals are intriguing and their shapes are beautiful. They are practical as well, because they give us a *geometric* way of understanding things that were mysterious before the fractal approach was developed.

Kleiber's Law is an empirical observation that the metabolic capacity of an organism scales as $m^{3/4}$, where m is the organism's mass. Many efforts have been made to derive this law from first principles. The latest direction is work by Geoffrey West, James Brown, and Brian Enquist in 1997 that sees this law as arising from the fractal dimension of the capillary beds in the body. See the McMahon-Bonner book cited above for Kleiber's Law or look on the Web, where you will find current material on the fractal aspects of this law. In April 2003, *New Scientist* ran an article by Phillip Hunter, "The Power of Power Laws" on allometry, scaling, and their fractal interpretations. See also "Elephants, Mice, and Red Herrings" by Gary P. Burness, *Science*, 17 May 2002. This pot will be bubbling for years.

Fractals are extensively used in geosciences, in meteorology, and in practical matters such as designing better battery plates. My introduction to these shapes came more than a decade before Mandelbrot showed that they were useful. It was in a real variables class taught by A. S. Besicovitch (the B of HB dimension). Besicovitch gave us the definition of fractional dimension and showed us how it applied to various shapes. He also cautioned us that this was merely a mathematical curiosity and had no physical meaning or practical use. You can know all about something without being able to see what it might be good for.

Lesson 9. The Regular Polyhedra. The tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron: these are the five convex regular polyhedra. The geometry of the first three can be understood by starting with the cube, whose properties are easy to grasp. Notice that if you take the two skew diagonals on opposite faces of a cube, these are opposite skew edges of a tetrahedron, all of whose edges are diagonals of the same cube. The octahedron is related to the cube as shown in Octahedron and Cube (37–42) in Set II. as shown in Octahedron and Cube (37–42) in Set II.

Euclid relates the dodecahedron to the cube by the Dodecahedron Construction (43–46), in Set II. (See Proposition 17, Book XIII, especially the Porism and its figure in the Heath edition.) It is a surprise that the cube can be used in constructing a dodecahedron because their symmetries seem so different. Once we have the dodecahedron, we can get the icosahedron by taking the centers of the faces of the dodecahedron as the vertices of our icosahedron. The dodecahedron and icosahedron are *duals*. If you take the centers of the faces of one, these points will serve as the vertices of the other. See the exercise sequence Octahedron and Cube at the start of Set II.

The regular polyhedra can be constructed starting from a cube, which has an insphere and a circumsphere. With some care we can see that all the vertices of the constructed regular polyhedra are equidistant from the center of these spheres. With a bit more care, you can figure out just what symmetries each regular polyhedron has. All of these will be rotations about axes going through the center O of the circumsphere, or reflections in planes passing through that center, or compositions of these (e.g., the mapping that sends every point to its antipodal point with respect to O).

We defined regularity for these convex polyhedra by the *local conditions* that all the faces should be congruent regular polygons and that all the vertices should be surrounded by an equal number of these polygons. We find that this makes these regular polyhedra so symmetric that there are isometries that send the polyhedron onto itself and interchange any given pair of faces or any given pair of vertices. This is *global* regularity.

Set I. Numbers and Names (1–5) the essential vocabulary. Two Tetrahedra (10–11). Figuring out that the ratio of similarity for these two tetrahedra is be $\frac{1}{3}$ requires looking at the configuration and seeing how to use what we know about plane geometry in several intersecting planes to get the result. You can start with a plane containing an edge E of the larger tetrahedron and one e of the smaller tetrahedron, with $E \parallel e$. This gives us a triangle with e as a segment to which the Side-Splitter Theorem applies. Because the centroids of the faces lie $\frac{2}{3}$rds of the way from vertex to the base, you can conclude that e is $\frac{1}{3}$rd the length of E. Da Vinci Polyhedron (12–19) involves visual thinking, counting, calculating, and seeing how these numbers give the number of edges and vertices of the dodecahedron. Escher Box (20–27) does the same thing for the icosahedron. Note that the

earlier set has a hint after Exercise 17 that is not found in the latter set. Assign these together?

Euclid's Claim (30–36) is the argument that shows no more than five regular polyhedra can be made. The problem is that we cannot even get *one* vertex to work out with an angle sum less than 360° except for the arrangements that we know give us the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. If the angle sum exceeds 360° then the polyhedron cannot be convex at that vertex. This is Proposition 21, Book XI of Euclid. Euclid proves this by cutting off the vertex from the interior of the convex polyhedron. This gives us a pyramid with the vertex as its apex. Projecting the sides of the pyramid perpendicularly onto its base projects each angle at the vertex into an angle larger than it in the plane of the base, and the sum of these projected angles is 360°, so the sum of the angles at the apex must be less than 360°. The reason that the angles at the vertex are smaller than the angles projected perpendicularly into the base is because their corresponding sides are longer, as can be seen by using the Pythagorean Theorem.

Set II. Octahedron and Cube (37–42) another pair of dual polyhedra. The geometry here is easier to work out than in the case of the tetrahedron and its 1/3rd scale dual tetrahedron. Dodecahedron Construction (43–46). Here we work from the diagram and the assumption that this construction works to get the ratio of the diagonal of a regular pentagon to its side. Euclid worked the other way around. He knew that the ratio of the diagonal of a regular pentagon to its side was the Golden Number, ϕ. We know this as well. (For us, ϕ is the positive root of $x^2 - x - 1 = 0$.) See Set III, Golden Heather, for Lesson 2 of Chapter 14, page 584, and my notes.

Using this he showed that the arrangement of roofed structures pictured here fits together *exactly* as shown. Here we take the side-lengths AB, and so on, to be 1, and the side-lengths of the cube to be ϕ. Our task is to show that the pentagons of the figure are really planar. One way to check this is to compute the slopes of the planes of the two portions of these faces that join on an edge. This computation shows that each of the faces made up of an isosceles trapezoid and an isosceles triangle in this figure lies in one plane. For me this came down to seeing that $1/\phi = \phi - 1$. Algebra proved that this rather surprising construction works. To verify this, you must work algebraically, without approximations.

Round About (47–60). The fact that regular polyhedra have inspheres and circumspheres is essential to our understanding of their symmetry. The results found in this series are counterintuitive, which is an attraction, but it takes many steps to get to the punch line.

Set III. Tetrahedron Puzzle is just baffling enough, and it involves a model, which is good.

Models. To understand geometry, you must put your hands to the subject, not just think about it. Constructions and models are essential. How far to go? There are many good sources on the Web, and in 2003 George W. Hart's *Virtual Polyhedra: The Encyclopedia of Polyhedra* was a good place to start. Hart and Henry Picciotto have a 2001 Key Curriculum black-line masters book, *Zome Geometry,* of projects for people using the Zome geometric model-building system.

Older technology is featured in *Polyhedral Models* by Magnus J. Wenninger, Cambridge University Press, 1971, and in *Mathematical Models,* Revised edition, by H. Martyn Cundy and A. P. Rollett, Oxford University Press, 1960. Cundy and Rollett cover all sorts of models, including mechanical ones.

Sometimes looking at pictures and thinking work — if the pictures and words are good enough. They are in *Shapes, Space, and Symmetry* by Alan Holden (since 1991 a Dover Publications paperback).

Summary and Review. The areas of right circular cones and cylinders were covered in exercises. See Cylinder and Cone Areas, page 660 in *Geometry.* These might be added to the list of formulas, although these areas are easy to work out when the occasion arises. It is important to distinguish between the *lateral* and *total* areas for these shapes. Confusion on this could give trouble on facing standardized tests.

If you have given more time to coordinate methods, then you might mention the equation of a sphere S of radius r about point $O = (x_0, y_0, z_0)$ namely

Sphere Equation $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

Set I. Flipping Cubes (1–6) and Seeing Things (7 and 8) are about visual thinking and three-dimensional representations. Box Pattern (11 and 12) is about a polyhedral net. Octagonal Building (13–16) is about volumes. Screen Illumination (17–19) touches on geometry, scaling, and the intuition behind the inverse square law for intensity of light. Stars (20 – 24) is about scaling applied to astronomy. Divided Cube (25–31) is a nice study of the regular tetrahedron as inscribed in a cube.

Set II. According to Archimedes (32–42) is a nice review of volume and area results for the sphere, right circular cone, and right circular cylinder. Twisted Ring (43–45) and SAT Problem (46–48) involve visual thinking and problem-solving. Wine barrel (49–56) involves thinking about why various volume formulas work and how they relate to other formulas. Euler characteristic (57–61) introduces Schlegel diagrams.

Solid Geometry as a Deductive System. In Books XI–XIII of his *Elements*, Euclid developed solid geometry up to the construction of the regular polyhedra found in Lesson 9 of this chapter. At the start, his reach exceeds his grasp, as Heath notes in his edition of Euclid. Euclid gives some shaky arguments attempting to prove our Postulates 11 and 12. From that point on

he is solid in his arguments. I am struck by how neatly he got all of this down. The following is largely a distillation of his work. We have advantages over Euclid because once we have established a few key facts, we can use coordinates and algebraic insights and tools to simplify things.

The first step is to understand how planes can be determined. I set these results out as a series of exercises, with the solutions in brief following the statements. These exercises could be used as honors assignments.

Determining Planes.

Three noncollinear points determine a unique plane $\mathcal{P}$. This is Postulate 2. Show that each of the following configurations determines a unique plane $\mathcal{P}$ containing it.

1. A line m and a point O not on m. Points A and B on line m are not collinear with O. Why? The plane $\mathcal{P}$ determined by these points contains two points of m, and hence, by Postulate 11, all of m, and $\mathcal{P}$ contains O. Using Postulate 12, show that any plane $\mathcal{Q}$ that contains O and m must be identical with $\mathcal{P}$. So $\mathcal{P}$ is unique.

Proof. O does not lie on m by hypothesis, so O, A, and B are noncollinear. Remark, $\mathcal{P}$ contains A and B, and hence it contains line AB, which is identical with m. Reasons: two points determine a line and Postulate 11.

If plane $\mathcal{Q}$ were not identical to $\mathcal{P}$, then $\mathcal{P}$ intersects $\mathcal{Q}$ in a line, by Postulate 12. This contradicts the fact that both $\mathcal{P}$ and $\mathcal{Q}$ contain m and the point O, not on m. Hence $\mathcal{P}$ and $\mathcal{Q}$ must be identical, and the plane $\mathcal{P}$ found by using A, B, and O is the unique plane determined by point O and line m which misses O.

2. Intersecting lines m and n. Take A on m and B on n, different from the intersection, O. Show that $\mathcal{P}$, determined by O, A, and B, is the unique plane containing lines m and n.

Proof. This is Euclid XI.2. Postulate 12 is Euclid XI.3 First, $\mathcal{P}$ contains the line OA $= m$, because the two points O and A on m determine m. Second, because O and A are points in $\mathcal{P}$, the line OA lies in $\mathcal{P}$, by Postulate 11. The same reasoning shows that $\mathcal{P}$ contains line OB $= n$. If $\mathcal{Q}$ were another plane containing lines m and n, then $\mathcal{P}$ would intersect $\mathcal{Q}$ in a line (Postulate 12), which contradicts the fact that both $\mathcal{P}$ and $\mathcal{Q}$ are assumed to contain distinct lines m and n. Hence, no such second plane, $\mathcal{Q}$, exists, and the plane $\mathcal{P}$ we have found is the unique plane determined by intersecting lines m and n.

3. Parallel lines m and n. To be parallel, these lines must lie in a common plane, which we will call $\mathcal{P}$. Show no other plane, $\mathcal{Q}$, can contain m and n.

Proof. This is Euclid XI.7. If lines m and n also lie in the distinct plane $\mathcal{Q}$, then, by Postulate 12, $\mathcal{P}$ intersects $\mathcal{Q}$ in a line, contrary to the fact that planes $\mathcal{P}$ and $\mathcal{Q}$ both contain lines m and n. So no such distinct plane $\mathcal{Q}$ containing m and n can exist, and $\mathcal{P}$ is the only plane containing both m and n.

Determining a Perpendicular Plane.

The plane perpendicular to a line ℓ at O is determined by any pair of lines perpendicular to ℓ at O. *Proof.* Take points A and A′ equidistant from O on ℓ, as in your diagram. Let the blue lines m and n be perpendicular to line ℓ at O. These lines will be perpendicular bisectors of segment AA′ in two different planes. The blue lines determine a plane $\mathcal{P}$ passing through O. To show that $\mathcal{P}$ is perpendicular to ℓ, we must show that any line OX, where X is in $\mathcal{P}$, is perpendicular to ℓ. We draw a green line in the plane $\mathcal{P}$, passing through X, and crossing m and n at points we name W and Y.

This is Euclid XI.4. Your figure should show ℓ looking vertical and blue lines m and n shown stretching to the right with X on a green line running front to back at an angle but seeming in the plane $\mathcal{P}$. The green line runs through X and crosses n at a point labeled Y and n at a point labeled W.

4. AW = A′W and AY = A′Y. Why?

Proof. Because n and m are perpendicular bisectors of AA′, so points W on n and Y on m are equidistant from A and A′.

5. $\triangle$AWY $\cong$ $\triangle$A′WY, so $\angle$AWY = $\angle$A′WY. Why?

Proof. Since WY = WY, this congruence follows from 4 by SSS. These angles are equal by CPCTE.

6. $\triangle$AWX $\cong$ $\triangle$A′WX, hence, AX = A′X. Why?

Proof. The triangles are congruent by SAS. The segments are then equal by CPCTE.

7. In the plane determined by points A, A′, and X, OX $\perp$ AA′. Why? How does this show that the plane $\mathcal{P}$, that is determined by lines m and n, both perpendicular to ℓ, is itself perpendicular to ℓ?

Proof. We showed that a point X in the plane $\mathcal{P}$ is equidistant from A and A′, hence it lies on the perpendicular bisector of AA′ in the plane determined by ℓ and X, so OX $\perp$ ℓ. But X was any point in the plane $\mathcal{P}$ determined by m and n, so any line in the plane $\mathcal{P}$ passing through O must be perpendicular to ℓ. By definition, this means that plane $\mathcal{P}$ is perpendicular to line ℓ.

Perpendicular Lines are Coplanar.

If three lines are all perpendicular to a line ℓ at point O, the three lines are coplanar.

Let lines k, m, and n be perpendicular to line ℓ at point O. We show that line k lies in the plane $\mathcal{P}$ determined by m and n. Let $\mathcal{Q}$ be the plane determined by intersecting lines k and ℓ. $\mathcal{P}$ intersects $\mathcal{Q}$ at O. Hence, by Postulate 12, the intersection of $\mathcal{P}$ and $\mathcal{Q}$ is a line, which we call j.

8. Because j lies in $\mathcal{P}$, $j \perp \ell$ at O. Why?

Proof. This is Euclid XI.5. Plane $\mathcal{P}$ is determined by two lines m and n perpendicular to ℓ at O. Hence $\mathcal{P} \perp \ell$. Because j lies in $\mathcal{P}$ and intersects ℓ at O, $j \perp \ell$ at O.

9. Line j is the intersection of planes $\mathcal{P}$ and $\mathcal{Q}$, so it lies in plane $\mathcal{Q}$ with line k. Because both j and k are perpendicular to ℓ at O, we can conclude $j = k$. Why?

Proof. Because in the plane $\mathcal{Q}$ there is a unique perpendicular to ℓ at O.

10. So line k, m, and n lie in $\mathcal{P}$. Why?

Proof. Because $k = j$, and j is the intersection of $\mathcal{Q}$ and $\mathcal{P}$, and $\mathcal{P}$ is the plane determined by m and n. So $k(= j)$ lies in the plane $\mathcal{P}$ of m and n.

Planes Perpendicular to a Line.

Two planes perpendicular to the same line are parallel to each other.

11. Let plane $\mathcal{P}$ be perpendicular to line ℓ at point O, and plane $\mathcal{P}'$ be perpendicular to ℓ at O′. If $\mathcal{P}$ and $\mathcal{P}'$ intersect at some point X, lines XO and XO′ would be distinct perpendiculars from X to line OO′. Is this possible? Why does this show that $\mathcal{P}$ and $\mathcal{P}'$ must be parallel?

Proof. This is Euclid XI.14. No. Three points determine a plane, so O, O′, and X would lie in some plane, and in any plane there is just one line perpendicular to a given line (OO′) through a given point (X). Hence no such point of intersection X of planes $\mathcal{P}$ and $\mathcal{P}'$ can exist, so these planes must be parallel, by the definition of parallel planes.

Perpendicular Lines.

Lines perpendicular to the same plane are parallel. Let ℓ and ℓ' be lines perpendicular to plane $\mathcal{P}$, intersecting $\mathcal{P}$ at points O and O′, respectively. We will use the distance OO′ as our unit, and take point Y on line ℓ so that OY = 1 unit. In plane $\mathcal{P}$, take point X so that XO′ $\perp$ OO′ and O′X = 1 unit. Then by the Pythagorean Theorem, OX = $\sqrt{2}$ units = YO′, as you will see when you draw the figure. Because $\ell \perp \mathcal{P}$ and OX is in $\mathcal{P}$, OX $\perp$ OY, and by the Pythagorean Theorem, we find that XY = $\sqrt{3}$ units.

This is Euclid XI.6. This needs a figure. Lines ℓ and ℓ' are perpendicular to a seemingly horizontal plane $\mathcal{P}$, and both should look vertical. The points O and O′ are horizontal across from each other on the page. The configuration is such that Y might be the upper-back-left corner of a cube, O the lower-back-left corner of the same cube, O′ the lower-back-right corner of the cube, and X the lower-front-right corner of the cube. The lengths mentioned in the description should be on the diagram and the right angles marked. These would include the two angles made by segment YO where it hits segments OX and OO′, the right angle between ℓ' and OO′, in the plane of ℓ and ℓ', and the right angle in the plane $\mathcal{P}$ between OO′ and OX. Angle XO′Y is shown to be a right angle in 12, below, but this is not given in the setup. Color could help make your figure easier to read.

12. Why must ∠YO′X be a right angle?

Proof. Because $(\text{YO}')^2 + (\text{O}'\text{X})^2 = 2 + 1 = 3 = (\text{YX})^2$, so, by the converse of the Pythagorean Theorem △YO'X is a right triangle, and ∠YO'X is a right angle.

13. Lines ℓ', O′O, and O′Y must in a single plane. Why? We call this plane $\mathcal{Q}$.

Proof. Because these three lines are perpendicular to line O′X at O′. If three lines are all perpendicular to a line ℓ at point O, the three lines are coplanar.

14. Why must ℓ and ℓ' be parallel in $\mathcal{Q}$?

Proof. Because they are perpendicular to line OO′ in $\mathcal{Q}$. Lines in the same plane perpendicular to the same line are parallel.

Perpendicular Planes & Parallel Lines.

A plane perpendicular to one of two parallel lines is perpendicular to both of them.

This is proven using a figure with ℓ and ℓ' as the parallel lines, with $\mathcal{P}$ the plane perpendicular to ℓ at point O, and with X, O, O′, and Y arranged as for the previous result. The proof uses the Pythagorean Theorem and its converse.

This is Euclid XI.8. The next result proves the transitivity of parallelism for lines in space, something not entirely obvious from the definition.

Parallel Lines in Space.

Given lines $\ell \parallel m$, $m \parallel n$, $n \parallel \ldots$ there is a plane $\mathcal{P}$ perpendicular to all of these lines. Any two lines in such a collection must be parallel.

Let $\mathcal{P}$ be a plane perpendicular to ℓ. Because $P \perp \ell$, we know $P \perp m$, and hence then $P \perp n$, and so on, by the previous result. So $\mathcal{P}$ is perpendicular to all these lines. Then any two lines in this family are parallel, because they are perpendicular to the same plane.

Perpendicular Bisection in Space. The plane perpendicular to segment AA′ at its midpoint consists of the points equidistant from the A and A′.

Extensions of Plane Geometry to Space.

Like perpendicular bisection, other notions of plane geometry extend to space. Reflection in a plane can be defined analogously to reflection in a line. Reflections in planes are isometries, as are translations (which move all points an equal distance in the same direction or are given by reflections in two parallel planes), rotations about an axis, and so on. The Cartesian coordinates extend to space, using three axes, each at right angles to the other two.

To use the methods of plane geometry in three dimensions, we look at relations in the various planes determined by the points and lines of our problem in space. Within each of these planes we use results of plane geometry, then we put these results together to solve our three-dimensional problems.

Euclid XI.11 to XI.13 concern the construction and uniqueness of perpendiculars to planes in space. These results can be followed easily in Heath's edition of Euclid or as outlined under subheads below.

Euclid's XI.10 and XI.15 tell us that if intersecting pairs of lines m and n, and m' and n' are such that the corresponding lines are parallel, then the planes these intersecting pairs determine are parallel and the angles these make are equal. We have different definitions of prism, so these results are not as important for us as they were for Euclid. However, we need to know that the figures traced out in the two base planes of a prism by lines parallel to a fixed line are congruent. This fact follows from arguments based on translations in three-dimensional space, and the details can be established by reasoning outlined here.

The basic insight is that if two planes $\mathcal{P}$ and $\mathcal{P}'$ are parallel, any plane $\mathcal{Q}$ containing line ℓ that intersects $\mathcal{P}$ in a line m must intersect $\mathcal{P}'$ in a line m' that is parallel to m and hence makes the same angles with ℓ in $\mathcal{Q}$ as m does. By Postulate 12 we know that if two planes intersect, their intersection gives a line.

To see why the above basic insight must be true, we begin by assuming that $\mathcal{Q}$ is determined by ℓ and some point O′ in $\mathcal{P}$ not on ℓ. $\mathcal{Q}$ intersects both $\mathcal{P}$ and $\mathcal{P}'$, so it intersects these planes in our m and m'. Because m and m' lie in parallel planes $\mathcal{P}$ and $\mathcal{P}'$, these lines do not intersect, and because both lines also lie in the plane $\mathcal{Q}$, they must be parallel. Since ℓ is transversal to m, it is also transversal to m' in plane $\mathcal{Q}$, making equal angles with both lines, as was to be shown.

This argument shows that ℓ intersects $\mathcal{P}'$. Hence, any plane containing ℓ intersects $\mathcal{P}'$ in a line, and hence every such plane is determined by ℓ and some point O′ in $\mathcal{P}'$, and our argument covers all planes that contain ℓ. This result is Euclid XI.16.

If $\ell \perp \mathcal{P}$, then the above shows that $\ell \perp \mathcal{Q}$, because plane will $\mathcal{Q}$ contains many lines perpendicular to ℓ where they intersect, and thus $\ell \perp \mathcal{Q}$.

Constructing Perpendiculars.

From O not on plane $\mathcal{P}$, drop a perpendicular to some line m in $\mathcal{P}$, and let point O′ be the foot of this perpendicular. If OO′ is perpendicular to $\mathcal{P}$, we have our perpendicular from O to $\mathcal{P}$. Otherwise, find the perpendicular n to m at O′ in the plane determined by O and m. Now, in the plane of O and n drop the perpendicular to n from O′ to locate O″. I say, $m \perp$ OO′and $m \perp$ O′O″, so $m \perp \mathcal{Q}$, which we use to denote the plane determined by OO′O″. Now $m' \parallel m$, so $m' \perp \mathcal{Q}$, and hence $m' \perp$ OO″. But $n \perp$ O′O″ as well. So O′O″ $\perp \mathcal{P}$, which is the plane determined by m' and n. Thus we have constructed the perpendicular from O to $\mathcal{P}$, as required. This is Euclid XI.11.

Constructing a plane parallel to a given plane through a given point. The plane parallel to $\mathcal{P}$ through a point O not in $\mathcal{P}$ can be found by dropping the perpendicular from O to $\mathcal{P}$ (with its foot at O′) and considering the unique plane determined by any two lines perpendicular to OO′ at O′.

Coordinates in Space.

From the above results we see that given any plane $\mathcal{P}$ in space and a point O in $\mathcal{P}$, we may establish a Cartesian coordinate system with origin at O in $\mathcal{P}$ and any pair of perpendicular axes through O in $\mathcal{P}$ as our x- and y-axes. Taking our z- axis to be a line perpendicular to $\mathcal{P}$ at O, we know that every point in space lies on a plane parallel to $\mathcal{P}$ and hence perpendicular to our z-axis.

Using a ruler with the same units as on our x- and y-axes along our z-axis and with its 0 at the point O, we define the z-coordinate of a point P in space as the signed distance from O at which the plane parallel to plane $\mathcal{P}$ through point P cuts the z-axis. The x- and y- coordinates of P are defined to be the coordinates of the foot of the perpendicular (whose foot we will denote by P_{xy}) from point P to plane $\mathcal{P}$.

By the Pythagorean Theorem applied in the plane $\mathcal{P}$,

$$(OP_{xy})^2 = x^2 + y^2,$$

where x and y are the x and y coordinates of P_{xy} in $\mathcal{P}$. By the Pythagorean Theorem applied in the plane of the z-axis and point P_{xy}, we see that

$$(OP)^2 = (OP_{xy})^2 + z^2,$$

where z is the z-coordinate. So the distance from O to any P in space can be found by using:

$$(OP)^2 = x^2 + y^2 + z^2.$$

In a like way, the distance between any two points P = (x, y, z) and P′ = (x', y', z') is given by:

$$PP' = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2},$$

where these Cartesian coordinates are given in the usual way. The equation of a sphere was given at the discussion of the notes on the Summary and Review for Chapter 15. It is a consequence of the distance formula established above.

Analytic Geometry in Space. This is an extension of analytic geometry in the plane. The coordinates of the points lying in a given plane $\mathcal{Q}$ all satisfy:

$$Ax + By + Cz = D$$

for some constants A, B, C, and D. The vector point of view gives the nice interpretation that the vector from the origin O to the point (A, B, C) is *perpendicular* to the plane $\mathcal{Q}$. Furthermore, if $A^2 + B^2 + C^2 = 1$, then D is the signed distance from O to the plane $\mathcal{Q}$.

There are many results of this type. These are usually covered as part of calculus or linear algebra. The links with geometry rest on the introduction of coordinates as outlined above.

Chapter 16. Non-Euclidean Geometries.

To the Student. Geometry is the science of space and shapes. We know that our surroundings are not uniform. Up and down are different. To our right we see one thing; to our left we see another. But these differences are due to the particulars of our immediate situation. The earth below us pulls us down in a consistent way, even though as the earth rotates our "down" is a direction that whirls around against the sphere of the fixed stars.

We believe that aside from such differences due to our immediate surroundings, the space that we live in is uniform — in all directions and throughout the universe. The geometry that we believe underlies what we see *should be completely symmetric, giving no special role to any place or orientation.* If we are concerned solely with geometry, *a triangle behaves the same way wherever it is constructed and however its sides are oriented.*

The method of *superposition* by which a figure is directly compared with another and the congruence postulates both express our confidence in the uniformity of space, in the idea that *geometric* properties of a configuration are independent of its position.

It seems equally clear that the geometric properties do not depend in any essential way on the *size* of a configuration. As a practical observation, we know we can make scale models and that the geometric properties of things seem unaffected by changes in size. This fact can be *proven* once we have the Parallel Postulate. But because it seems so obviously a necessary consequence of the symmetry built into our ideas, and our world, there was a long and ultimately unsuccessful effort to prove the Parallel Postulate from the congruence postulates and other simple and *irrefutable* assumptions of Euclidean geometry.

Even at the beginnings of geometry, people realized that the geometry of triangles on the celestial sphere and on the surface of the spherical earth was different from the geometry of triangles in a plane. But although we now see *spherical geometry* as one form of non-Euclidean geometry, it was not thought of that way. Spherical geometry fitted into the Euclidean picture because the two-dimensional surfaces we are thinking of lie in a *Euclidean* space of three dimensions.

Even before Hipparchus (about 150 BC), the earth was believed spherical by astronomers based on the observation of the circular shadow the earth cast on the moon during eclipses. It was also believed to be spherical because this explained why the sun at noon on the summer solstice would be lower in the sky the farther north you go. This evidence was convincing for those who examined it. But for others, it was "flat earth all the way."

Geometry on a Sphere (Lesson 1) is the oldest non-Euclidean geometry, but it was not recognized as such. As a geometry, it has flaws that can only be eliminated by *identifying pairs of antipodal points.* This dizzying operation gives us a space in which each pair of points determines precisely one line containing both points and each pair of lines determines exactly one point where both lines concur. We have all the usual congruence axioms plus AAA congruence, but there can be no pair of similar but noncongruent triangles. Lines have finite length, so we need to make an appropriate modification of our Ruler Postulate and, because lines curve back on themselves, we need to make adjustments to the order axioms. These adjustments are somewhat fussy, but their general nature is clear.

This perfected version of spherical geometry is not easy to grasp intuitively. (It is called *elliptic geometry*, or *single elliptic geometry* to distinguish it from sphere geometry.) Here we only deal with sphere geometry or, later, *Riemannian geometry.* This is because we wish to stick to things that are easily understood by thinking of triangles on a sphere (a regular sphere, without identifying antipodal points).

The best resource for those who want to understand the possibilities for the structure of space is *The Shape of Space* by Jeffrey R. Weeks, Marcell Dekker, New York 1985 and second edition 2001. Also see books and other materials Weeks has developed. (Go to the Key Curriculum Press website, for instance.) Weeks has written for general readers, but ones who want to understand and develop intuition about the geometry of curved spaces.

Weeks is applying his ideas not just to geometry, but to the problem of what geometry the universe might have,

given the data we have from astronomy. Go to the Web for this breaking story.

The big break for non-Euclidean geometry came with the exploration of *hyperbolic* or *Lobachevskian* models of the plane. These models have infinitely long lines and our usual ruler and order postulates hold. They correspond to surfaces of constant *negative* curvature, as contrasted to spherical geometry, which is the geometry of surface with constant *positive* curvature.

Lesson 1. Geometry on the Sphere. A good source on the Web is John C. Polking's *Geometry on the Sphere,* available at mathforum.org in 2003. I outlined a proof of Girard's Theorem, which is one of Polking's main results in Areas of Spherical Triangles which is in the supplemental notes for Chapter 11, above. The geometry of the sphere, as a surface in Euclidean three-space is easily understood. People have thought about it since they began to study the sky and stars. But the geometry of the sphere with its antipodal points identified is not intuitive and is a relatively modern creation. You can explore it using the *disk model* for elliptic geometry, which has been realized using *Geometer's Sketchpad*™. I see references and samples on the Web. The tricky thing here is that points opposite each other on the boundary of the disk of the model are identified, so things are not as they seem. Go to the book *The Shape of Space* by Weeks for examples to help develop your intuition.

The uniform positive curvature of a sphere causes a spherical triangle's *spherical excess* (defined as the angle sum minus a half rotation) to be proportionate to the triangle's area. Additionally, it alters the way the circumference and area of a circle vary with the circle's radius r. Consider a sphere of radius R. Measure the radius r of a circle *on* this sphere *along the surface of the sphere* so that the arc this radius gives has radian measure r/R, then the circumference is $2\pi \sin(r/R)$ and its area on the surface of the sphere is $2\pi r^2(1 - \cos(r/R))$. Johann Heinrich Lambert (1728–1727) in his researches on the parallel postulate noticed that formulas related to the sphere (positive curvature $= R^{-2}$) go over to analogous formulas for hyperbolic space when the radius R is replaced by iR, where i is a square root of -1.

The circular functions with the variable in radians are expressed in terms of exponentials thus

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Replacing θ by $i\theta$ we get the corresponding *hyperbolic functions* sinh (pronounced "sinch") and cosh,

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \quad \text{and} \quad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}.$$

There is a wealth of material on hyperbolic trigonometry on the Web, including applets that show you the areas of hyperbolic triangles in the Poincaré disk model of the hyperbolic plane. See pages 116 and 117 of *Geometry* for this model.

Set I. Our Spherical Earth (1–4) and Basic Differences (5–11) familiarize students with sphere geometry and how it differs from plane geometry. We have *not* identified antipodal points here. Shortest Routes (12–15). In going from A to B on the surface of the earth, the shortest path will be the one that minimizes its curvature at all points along the way. Since there is no way you can avoid curving *with* the earth's surface, you must accept that curvature, but avoid other curvature by following a great circle. This is what airplanes do, although they will deviate from the shortest path to take advantage of favorable winds. Beach Ball (16–24) is an exploration of angle sums and areas in sphere geometry.

Set II. Spherical Triangles (25–28) relates to AAA congruence. Seemingly Parallel (29 and 30) involves visual and critical thinking. Euclidean and Sphere Geometries (31–40) involves discovery, conjecture, and critical thinking.

Set III. Antipodal Points is a marvelous map.

Lesson 2. The Saccheri Quadrilateral. Andrew M. Gleason points out that Euclid's parallel postulate, given here, has advantages over the Playfair postulate used in *Geometry.* Euclid tells you that you will *observe something*, the intersection of two lines. He also tells you *where* to look for this intersection. In contrast, the Playfair postulate is neither constructive nor explicit. Current practice favors Playfair's version and it works, so that is what Jacobs used.

Saccheri developed the conclusions of assuming that the summit angles of an isosceles Saccheri quadrilateral are acute — in modern terms, the conclusions of of working in a Lobachevskian or hyperbolic plane. But the significance of his 1733 work, featured in the chapter opening image, was not understood for perhaps a hundred years, as Jacobs notes. See Morris Kline's *Mathematical Thought from Ancient to Modern Times,* Oxford University Press, New York, 1972, page 879 *et seq.*

Set I. These exercises show how Saccheri developed the properties of this geometry using the careful reasoning we have used — proofs. This allowed Saccheri to reach consistent but counterintuitive conclusions. **Set II** brings in Nasir Eddin, mentioned in A Colorful Contributor to Trigonometry in the supplemental material for Chapter 11 in these notes. **Set III**, Offsets, is a stunning and inevitable consequence of trying to live by the square on a spherical planet. Quadrangle maps that show Universal Transverse Mercator girds will also illustrate this phenomenon, and these are available for most areas. GPS units give your location with respect

to the local UTM grid. Mapping and working out coordinates are geometric problems involving technical details once you get beyond the basics. There are good *projects* to be done in this area.

Lesson 3. The Geometries of Lobachevsky and Riemann. Bernard Riemann was assigned the topic "the foundations of geometry" by Gauss for his *Habilitationsvortrag* as *Privatdozent* at Göttingen. Riemann delivered this address in 1854 and in it he outlined what has become our modern approach to geometry. Building on Gauss's use of calculus to understand the geometry of surfaces (differential geometry), Riemann outlined a theory allowing n-dimensional geometric spaces whose curvature was defined by a local idea of distance (a *Riemannian metric*) whose nature could vary from one part of the space to another. Riemann's ideas allowed his spaces (now called *Riemannian manifolds*) to have positive curvature in some regions and negative curvature in others. Using the same side-lengths we might notice spherical excess for our triangle in one region, but if we assembled the same length sides in another region, we might find a spherical deficit. See the note at the end of this lesson on Curved Space and Spacetime.

Riemann turned to the study of spaces of uniform positive curvature. He found that these had the geometry of the sphere, so spherical geometries are called *Riemannian*, and also *elliptic*, following a later classification set up by Felix Klein. Morris Kline's oft-cited *History of Mathematics from Ancient to Modern Times* tells the story.

We can classify these geometries in terms of their uniform curvatures as follows: Lobachevskian or Elliptic — negative curvature (no two lines are parallel), Euclidean — zero curvature (unique parallels), Riemannian or Elliptic — positive curvature (several parallels to a given line through the same point off that line).

Set I. Theorems 1 and 2 of Lobachevskian geometry (1–18). Here we can use our methods of proof, except we replace the Parallel Postulate by the Lobachevskian Postulate.

Set II. Lobachevsky and His Geometry (19–23) is a good review of the ideas and easy. Lobachevskian Quadrilateral (24–30), another proof, as above. Riemannian Quadrilateral (31–34), only requires a general and visual understanding of the spherical geometry model. Good review and practice. This also holds for Riemannian Triangle (35 and 36). Triangular Pyramid (37–40) and Open Box (41–46). Both involve informal reasoning about Lobachevskian geometry.

Set III. Pseudosphere. This introduces one model of the Lobachevskian plane. Note that this model is not smooth and it runs off to infinity along an axis. It is impossible to find a model for the Lobachevskian plane that is a smooth surface in space and with the usual notion of distance along curves in the surface. But see Lobachevsky Meets Escher on page 716 of *geometry* for the Poincaré disk model.

Curved Space and Spacetime. This ties into the material of the next lesson. Our best understanding in 2003 is that the paths of light through space are as close as we can come to straight lines. Light paths through space do not behave like straight lines in Euclid's geometry. The strongest evidence is that we can see the same distant galaxy by looking in two (or even four) slightly different directions. For examples of this you can search the Web for gravitational lensing or Einstein cross. In 2003, Goddard Space Flight Center had some good material (gsfc.nasa.gov), including the picture of the day for 2000 April 23. There was also good material at astron.berkeley.edu, and an excellent photograph of the Einstein cross of gravitational lens Q2237 + 030 at www.astr.ua.edu/keel/agn/qso2237. Being able to see the same thing by looking in different directions violates the postulate that two points determine a line — if we take light paths to represent lines. This departure from Euclid is explained by Einstein's Theory of General Relativity, which states that mass and energy determine the curvature of spacetime. Even the less mind-bending idea of spacetime is a departure from Euclidean ideas.

According to Einstein's special relativity, we should think of time as an essential coordinate in describing events. The *Interval*, between events is defined by the quadratic expression

$$(*) \qquad \text{Interval}^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2,$$

where Δx is the observed difference of the x-coordinates for the two events, and so on for y and z, while Δt is the difference in the observed times of the two events and c is the speed of light. Observers whose relative velocity is unchanged between the occurrence of the two events will compute the same Interval relating these two events.

This is true even though they may observe differing time intervals or space intervals between these events. If the first event is the flash of a laser on earth aimed at the moon and the second event is the arrival at the moon of that flash, then $\text{Interval}^2 = 0$, because the space component is exactly matched by the negative time term, $-(c\Delta t)^2$. In Einstein's special relativity, light travels along *null-paths* between events whose *interval* of separation in space and time will always give us Interval = 0.

The spacetime geometry of special relativity is different from Euclidean geometry. For example, we have seen that two *different events* can be separated by an Interval of length 0. We have a *difference* in (*), so the Interval^2 can be positive, negative, or zero. When it is *positive*, the Interval is called *spacelike* and its length is $\sqrt{\text{Interval}^2} > 0$. When (*) gives us a negative value, we call the interval, *timelike*, and take the length to be $\sqrt{-\text{Interval}^2} > 0$. If two events are separated by a timelike interval, then one can influence the other by

signals traveling at or below the speed of light. If the interval between two events is spacelike, then neither can influence the other. If $\text{Interval}^2 = 0$, then a *null-interval* separates the two events. One can influence the other only by a signal that travels at the speed of light. All of this is very different from the Pythagorean distances of Euclidean geometry.

The counterintuitive surprises of relativity arise from the very different natures the geometry underlying special relativity and Euclidean geometry. To feel at ease with this geometry you must become good at and trust the calculations or retrain your geometric intuition. Both take considerable doing and would distract from mastery of Euclidean geometry, especially at the introductory level.

Good and relatively elementary treatments of special relativity are available. A proven resource is *Spacetime Physics* by Edwin F. Taylor and John Archibald Wheeler, W. H. Freeman and Company, First Edition, 1963, Second Edition, New York, 2000. Both editions are highly recommended, and Taylor has software packages that let you see how things behave in special relativity; (*Spacetime and Rellab* are two of these packages.

In general relativity, the four dimensional spacetime picture of special relativity holds *locally*, but the Riemannian geometry of spacetime is determined by the effects mass and energy exert on space itself. As John Archibald Wheeler puts it on the first page of *A Journey into Gravity and Spacetime*, SCIENTIFIC AMERICAN Library, HPHLP, New York 1990,

"Spacetime grips mass, telling it how to move; and mass grips spacetime, telling it how to curve."

In general relativity, particles or planets move along the straightest paths available in a spacetime manifold whose geometry is determined by Einstein's field equations. Such paths on curved surfaces or in curved space are *geodesics*. The *geodesics* on a sphere are great circles. In the Euclidean plane, they are ordinary straight lines. On the surface of a right circular cylinder, the geodesics are paths that make a constant angle with the vertical generating lines.

The density of massenergy (the words are mashed together as in spacetime) that determines the curvature of spacetime. The ideas are subtle and the formalism is daunting. But Wheeler's *Journey* book will give the essential ideas to a persistent general reader. Julian Schwinger wrote *Einstein's Legacy,* for the same SCIENTIFIC AMERICAN Library series, published in 1986. Schwinger kept the outline of how Einstein equations were derived, despite resistance by his editor. See his Preface, where he hopes that this material may tempt some readers to work through the mathematics itself, and see his pages 184–186 for the material.

The benefit of the relativistic point of view is that it reveals many things that were unimagined: for instance, the equivalence of mass and energy via the equation $E = mc^2$. It allows us to understand the orbital motion of the planets as resulting from these bodies moving along geodesics in our spacetime manifold at a constant four-dimensional velocity — a velocity that has a time component as well as its three space components.

Free float describes the trajectory of particles moving under the influence of gravity and inertia alone. These trajectories are geodesics in our curved spacetime. Each moving particle is considered to carry with it an ideal clock that records what is called *proper time.* Even though these clocks keep perfect time, two actors, each carrying a clock from event A to event B, may show different elapsed times for the journey. The elapsed time will be longer for the path that is closest to a geodesic from A to B. This gives us the principle of maximal ageing — the longest possible time interval measured in going from A to B is the time measured along a geodesic path. In their book, *Exploring Black Holes: An Introduction to General Relativity*, Benjamin Cummings, 2000, Edwin F. Taylor and John Archibald Wheeler use the principle of maximal ageing and calculus to work out relativistic effects near black holes. This book begins where their *Spacetime Physics* leaves off.

To work with relativistic ideas one must master a certain amount of mathematical machinery. Because there are practical applications of finding geodesics that include tent design and manufacturing of clothing, commercial software is available that automatically solves some of these geometrical problems. In 2003 the Elements Research website had material on geodesic computation (elesoft.com/geodcomp). The site had nice examples, although it also mentions the daunting machinery of Christoffle symbols.

The challenge in working with curved spaces is not in grasping the idea of curvature of space. That idea is clear from the analogy with curved surfaces. Rather, the challenge lies in having machinery that allows you to define specific curved spaces and work out their properties in detail. Software now allows us to explore and display properties of curved and higher-dimensional spaces.

Lesson 4. The Triangle Angle Sum Theorem Revisited. In supplemental notes at the end of my comments on Chapter 11's lessons, I cover Similarity and the Angle Sum Theorem, Similarity and Non-Euclidean Geometry, and related topics. From the note above on Curved Space and Spacetime, you can see that the geometry of our universe seems to be curved, with a curvature that varies. The way we understand these more intricate models is by smoothly patching together pieces of spaces of the sort we have already studied. We have seen the basic structures, and we understand the idea

of curvature of space and how it makes lines (geodesics) converge or diverge depending on whether the local curvature is positive or negative.

Our space is non-Euclidean, but it is by our study Euclidean geometry that we came to understand this fact. The models of non-Euclidean geometries are constructed within Euclidean geometry, so our understanding of the non-Euclidean is dependent on our understanding of the Euclidean. This chapter on non-Euclidean geometry is both a fitting climax of this book and a hopeful beginning for further geometric adventures.

Set I. These exercises are proofs related to angle sums of triangles.

Set II exercises are more informal that Set I and offer students a wider range of opportunities to look, see, reason, and draw conclusions. Even if you usually assign exercises from Set I, you should consider these for assignment.

Set III. Astronomical Triangles. You can never prove that the curvature of space is *exactly* 0 by measurements because of their finite precision.

Summary and Review. This gives a clear overview of the material of the chapter and introduces the beautiful Poincaré disk model for the Lobachevskian plane. If your students are using the Web to explore and research things, they will be able to find good models of non-Euclidean geometry there. Software packages such as *Geometer's Sketchpad*™ and *Cabri Geometry* can help, too.

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