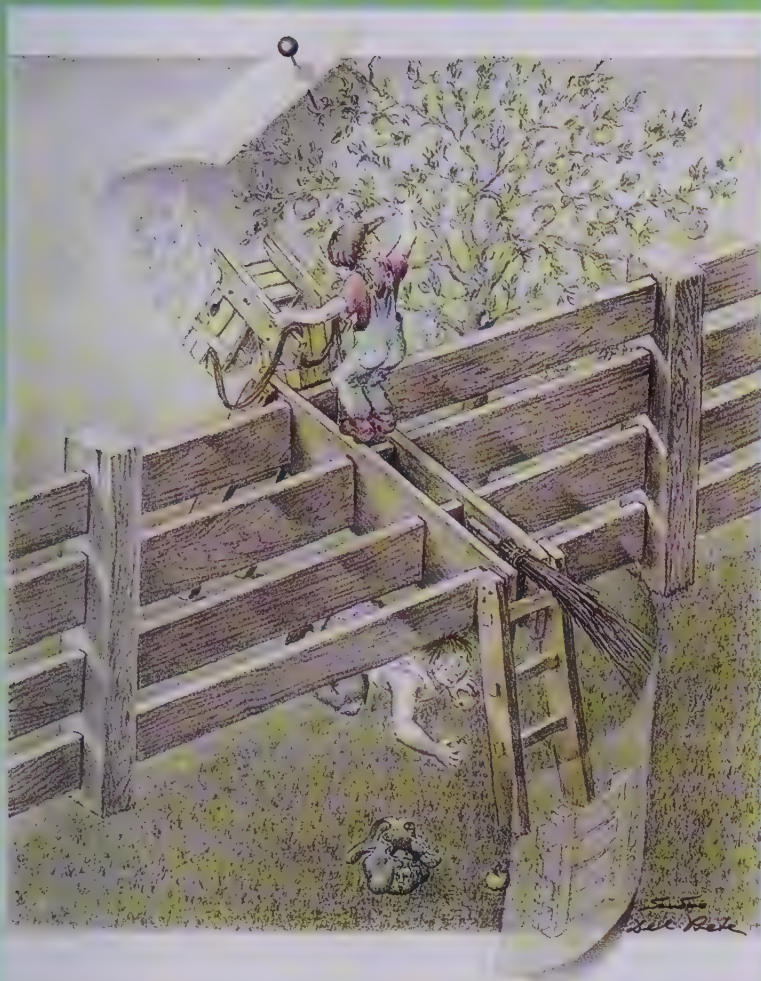


Chapter 6

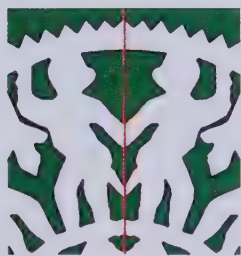
Parallel Lines



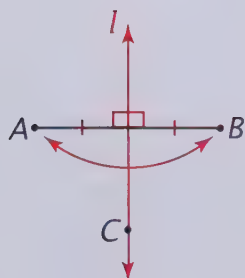
The word “parallel,” from the Greek word *parallelos*, meaning “side by side,” suggests the idea of lines that have the same direction, something that we take for granted. In this chapter, we consider several ways to prove that lines are parallel and encounter the Parallel Postulate, an assumption that has been the subject of more controversy than everything else in Euclid’s *Elements* combined! From the Parallel Postulate follow several more useful theorems about triangles.

LESSON 1

Line Symmetry



fold



This symmetric design was cut from paper. Although it contains eight pairs of monkeys (16 monkeys in all), the artist cut out only one of them.* A trick that children sometimes use in cutting paper made it possible: folding the paper in advance.

One pair of the monkeys and the fold that produced them are shown in the figure at the left. The monkeys are *symmetric* with respect to the line of the fold. The second figure at the left suggests how to define this symmetry in terms of points: A and B are symmetric with respect to l if they coincide when the paper is folded along l ; they will coincide if l is the *perpendicular bisector* of segment AB; a point such as C on line l coincides with itself.

Definition

Two points are *symmetric with respect to a line* iff the line is the perpendicular bisector of the line segment connecting the two points.

Although the perpendicular bisector can be found by folding, it can also be constructed with straightedge and compass. You already know how.† The construction is based on our first postulate that *two points determine a line*—more specifically, on the fact that *two points each equidistant from the endpoints of a line segment determine a line* that is the perpendicular bisector of the line segment.

Theorem 16

In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment.

**The Art of Chinese Papercuts*, by Zhang Daoyi (Foreign Language Press, 1989).

†Construction 1. To bisect a line segment (Chapter 1, Lesson 4, page 25).

In the first figure at the right, the two points are on opposite sides of the line segment. In the second figure, they are on the same side. Either way, the proof is the same.

Given: $CA = CB$ and $DA = DB$.

Prove: CD is the perpendicular bisector of AB .

Proof

Because $CD = CD$, $\triangle ACD \cong \triangle BCD$ (SSS); so $\angle 1 = \angle 2$. Also, $CE = CE$; so $\triangle ACE \cong \triangle BCE$ (SAS), and so $\angle 3 = \angle 4$. From the figure, $\angle 3$ and $\angle 4$ are also a linear pair; so their sides are perpendicular and $CD \perp AB$. Finally, $AE = BE$ because they also are corresponding parts of $\triangle ACE$ and $\triangle BCE$; so CD bisects AB .

This proof provides an easy way to construct a line perpendicular to a given line and passing through any given point.

Construction 6

To construct a line perpendicular to a given line through a given point.

The following steps show that the method is the same whether or not the point is on the line.

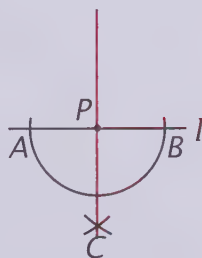
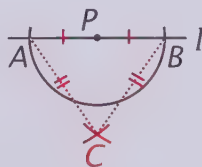
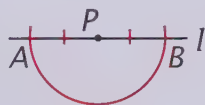
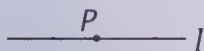
1. Let l be the given line and P be the given point.

2. With P as center, draw an arc that intersects l in two points (A and B). Note that P is equidistant from A and B .

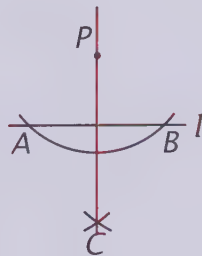
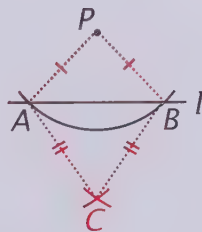
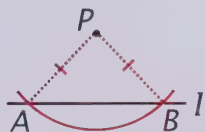
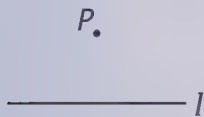
3. Use the compass to find another point C that is equidistant from A and B .

4. Draw PC .

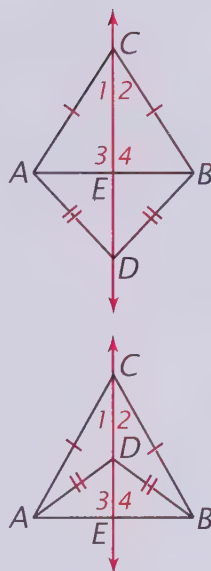
Point on line



Point not on line



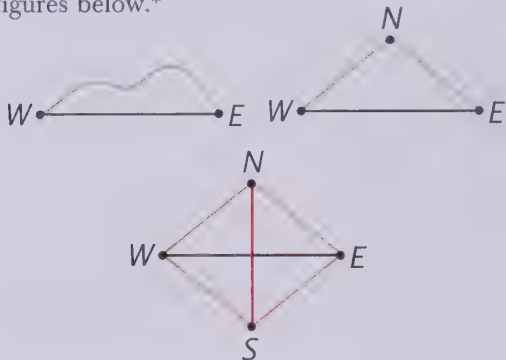
Line PC is the perpendicular bisector of segment AB according to Theorem 16, because P and C are equidistant from the endpoints A and B of the segment by construction.



Exercises

Set I

Rope Trick. An old way to make perpendicular lines on the ground is shown in the figures below.*

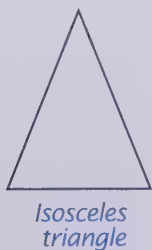


The ends of a long rope are tied to posts at W and E. Someone holding the rope at its center walks north until the rope is tight on both sides and places a post at the point N. Point S is located in a similar way by walking to the south.

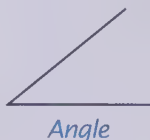
1. Draw the third figure and mark the lengths on it that must be equal from this procedure.
2. Why is the line NS perpendicular to WE?
3. Does NS bisect WE?
4. Does WE necessarily bisect NS? Explain.

Lines of Symmetry. Copy each of the following figures and sketch its line(s) of symmetry.

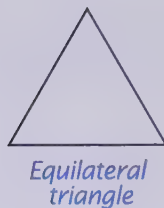
5.



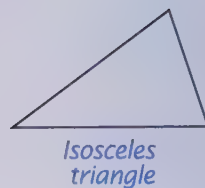
6.



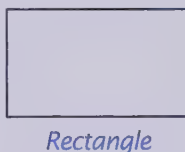
7.



8.



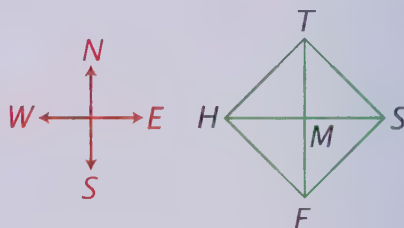
9.



10.



Baseball Diamond. According to major league rules, a baseball diamond should be laid out so that the batter is facing east.



11. Use your ruler to draw line segment HS 8.5 cm long. Construct a line as its perpendicular bisector. Label the midpoint M and mark off points F and T so that $MH = MS = MT = MF$. Connect points H, F, S, and T to form a diamond.
12. Measure HF in centimeters.

In your drawing, 1 cm represents 15 feet. How many feet is it from home plate to

13. first base?
14. second base?
15. In what direction is the sun in the late afternoon?
16. Why are baseball diamonds laid out in the directions that they are?
17. Why is a pitcher who is left-handed called a "southpaw"?

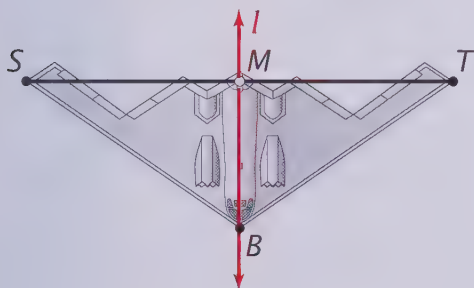
**Geometry Civilized*, by J. L. Heilbron (Clarendon Press, 1998).

Paper Chain. The figure below shows a paper chain. The chain was produced by cutting out just one penguin.*



18. How many times do you think the paper was folded before it was cut?
19. How many lines of symmetry between pairs of neighboring penguins are there in all?
20. How many times would a paper have to be folded to produce a chain of 16 penguins in this way?

Stealth Bomber. In this view of the B-2 Stealth Bomber, points S and T are symmetric with respect to line l .



21. What can you conclude about line l ?

Give a reason for each of the following statements.

22. $SM = MT$.
23. $\angle BMS$ and $\angle BMT$ are right angles.
24. $\angle BMS = \angle BMT$.
25. $MB = MB$.
26. $\triangle BMS \cong \triangle BMT$.
27. $BS = BT$.

*Wild Animal Paper Chains, by Stewart Walton and Sally Walton (Tupelo Books, 1993).

Set II

Race to the Fence. Acute Alice and Obtuse Ollie were standing on the opposite sides of a long straight fence when they decided to race to some point along it. Because Ollie was closer to the fence, Alice said it was only fair that they run to the point on the fence equidistant from both of them.†

Alice



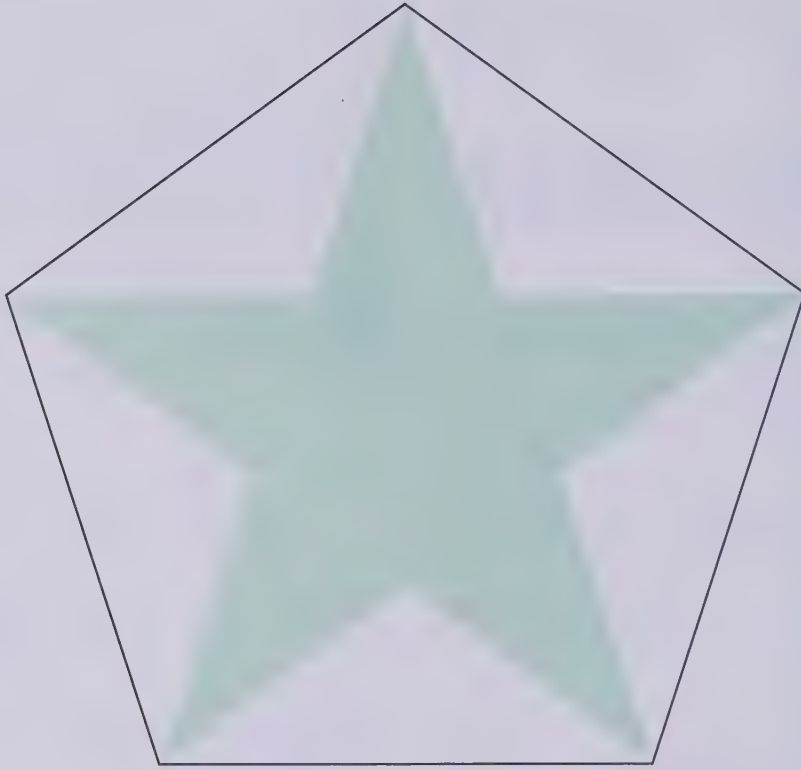
28. Draw a figure similar to but larger than the one shown above. Use your straightedge and compass to find the point to which they ran. Also draw their paths.

Alice



29. If Alice and Ollie were initially at equal distances from the fence, as shown in this figure, where would the point have been? Describe its location in words and draw a figure to illustrate it.

†Based on a problem in *You Are a Mathematician*, by David Wells (Wiley, 1995).

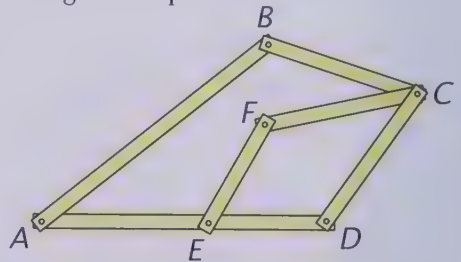


Clairvoyance Test. The following problem appeared in a “clairvoyance” test.*

30. Make an accurate copy of this figure by placing your paper over it and marking the five corners. Draw the figure and then choose a point anywhere inside it. Use your straightedge and compass to construct perpendicular lines from your point to each of the pentagon’s five sides. (Extend the sides with your ruler if necessary.)
31. Add the lengths of the five perpendicular line segments by marking them along the edge of a sheet of paper. Measure the total length with your ruler to the nearest half inch.

32. Why do you suppose this problem might be used on a test of “supernatural power”?

Linkage Problem. Linkages are used to control motion in machines. The linkage below is made of six rods that are equal as follows: $AB = AD$, $CB = CF = CD$, and $EF = ED$. The rods can pivot about their ends; so the figure can change its shape.†



**The Incredible Dr. Matrix*, by Martin Gardner (Scribner’s, 1976).

†*Mathematics Meets Technology*, by Brian Bolt (Cambridge University Press, 1991).

Regardless of the shape of the linkage, a line through AC is always perpendicular to a line through BD.

33. Why?
34. What other pairs of points determine two lines that are always perpendicular?
35. Draw the figure by using line segments to represent the rods. Also draw AC and EC.
36. Regardless of the shape of the linkage, $\angle B$ is always equal to $\angle D$. Why?
37. Is $\angle B$ always equal to $\angle F$? Explain why or why not.

One Problem, Two Solutions. The figure below by Scott Kim has line symmetry, but the line is not shown.

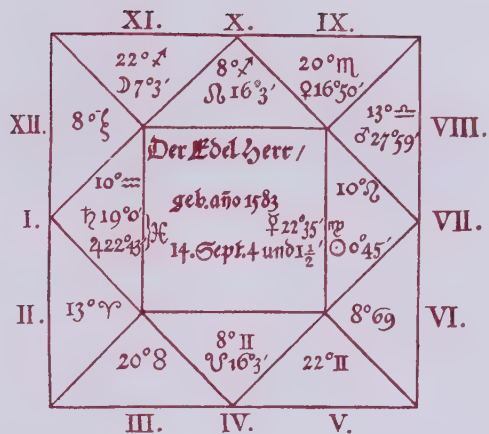
NO. 1502
PROBLEM

PROBLEM
SOLUTION

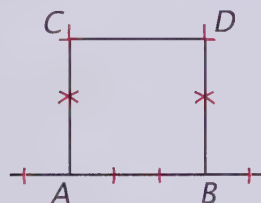
38. If the figure were folded on the symmetry line, where would the fold be?
39. To what letter on the other side of the line does the letter S correspond?
40. To what letter does U correspond?
41. What is strange about the figure?

Kepler's Diagram. The figure at the top of the next column was drawn by the astronomer Johannes Kepler in 1608.*

**Age of Kings*, by Charles Blitzer (Time-Life Books, 1967).



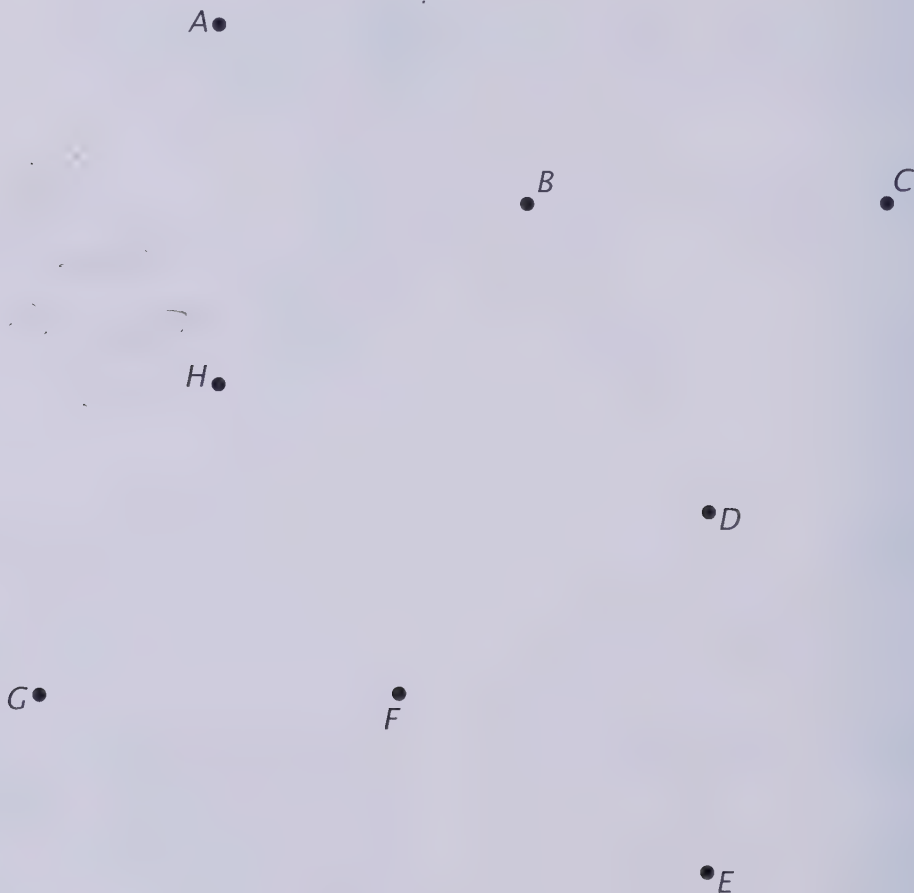
42. Draw the figure using the following procedure. Use your ruler to draw a horizontal line across your paper and mark a line segment 3 inches long on it for the base of the figure. Construct perpendicular segments at each endpoint of the base as illustrated here.



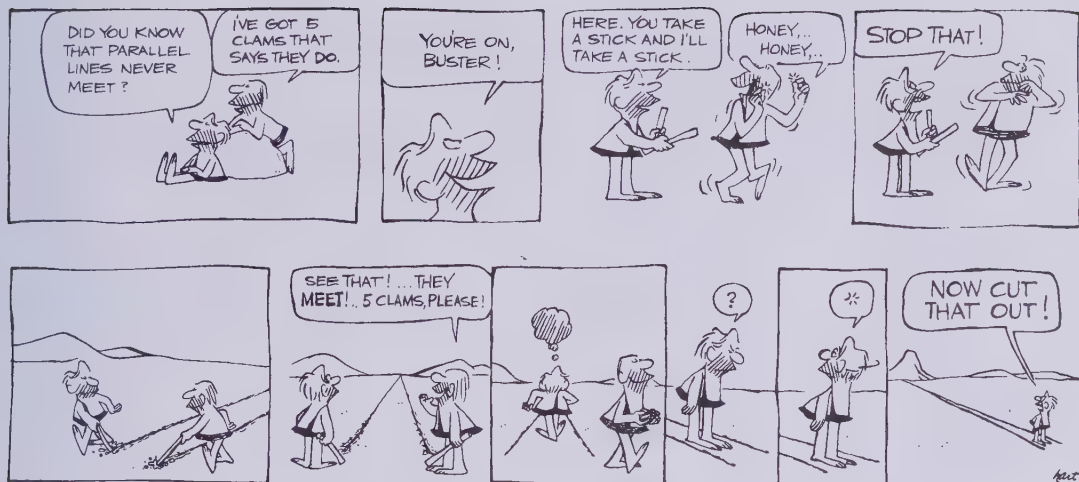
- Copy AB on each perpendicular to determine C and D. Use only your straightedge and compass and what you notice about the symmetry of Kepler's figure to complete your drawing.
43. How many squares does Kepler's figure appear to contain?
44. What kind of triangle do all of the triangles in the figure appear to be?
45. How many triangles do there seem to be in all?

Set III

*An Unusual Set of Points.** Make an accurate tracing of the figure below. Choose any two points and connect them with a line segment. Construct the perpendicular bisector of the line segment. What do you notice? Do it again with several other pairs of points. What seems to always be true?



* *Wheels, Life, and Other Mathematical Amusements*, by Martin Gardner (W. H. Freeman and Company, 1983).



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LESSON 2

Proving Lines Parallel

So far in our study of geometry, we haven't paid much attention to parallel lines. The idea that they never meet is the basis for our definition of them, as you may recall.

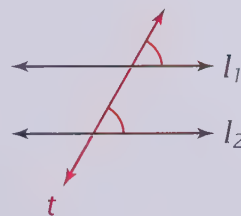
Definition

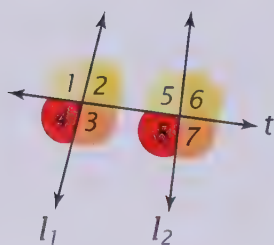
Two lines are **parallel** iff they lie in the same plane and do not intersect.

B. C. and Peter's approach to trying to determine whether two lines meet illustrates the fact that, in practice, our definition of parallel lines is difficult to use. Lines are infinite in extent. We can't look everywhere along them to find out if they intersect.

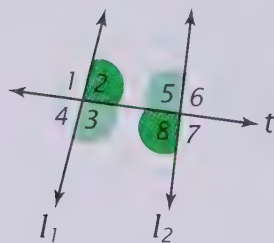
The solution to this problem is to look at a third line that cuts across the two lines that we are wondering about. For example, lines l_1 and l_2 in the figure at the right appear to be parallel. The third line, t , intersects them to form a number of angles. If we know that certain of these angles are equal, such as the pair indicated, then it is easy to prove that the lines must be parallel.

The third line is called a **transversal**. The word "transversal" comes from the Latin word *transversus*, meaning "turned across." A **transversal** is a line that "goes across"—that is, "intersects" two or more lines in different points.

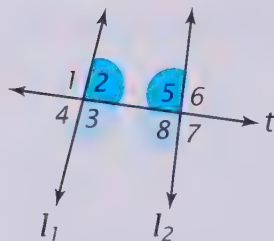




Corresponding angles



Alternate interior angles



Interior angles on the same side of the transversal

When a transversal intersects two lines that lie in the same plane, it forms pairs of angles that are given special names. In the figure at the left, $\angle 1$ and $\angle 5$ are called **corresponding angles**. The other pairs of corresponding angles are $\angle 2$ and $\angle 6$, $\angle 4$ and $\angle 8$, and $\angle 3$ and $\angle 7$.

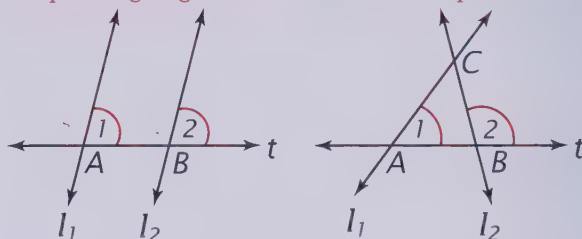
The transversal also forms two pairs of **alternate interior angles**: $\angle 2$ and $\angle 8$ are alternate interior angles, as are $\angle 3$ and $\angle 5$.

Finally, angles such as $\angle 2$ and $\angle 5$ are called **interior angles on the same side of the transversal**. So are $\angle 3$ and $\angle 8$.

All we need to know that two lines are parallel is to know that the angles in one of these pairs are related in a special way. We begin by proving that, if a pair of corresponding angles are equal, then the lines are parallel.

Theorem 17

Equal corresponding angles mean that lines are parallel.



Given: Lines l_1 and l_2 with transversal t , $\angle 1 = \angle 2$.

Prove: $l_1 \parallel l_2$.

Our proof, from Euclid's *Elements*, is indirect. Look at the figure at the right above, which illustrates the possibility that the lines might intersect, and you may see the contradiction immediately.

Proof

Suppose l_1 and l_2 are *not* parallel. Then they must intersect (because they lie in the same plane). If l_1 and l_2 intersect in some point C , then they form a triangle, $\triangle ABC$. But $\angle 2$ is an exterior angle of this triangle, and $\angle 1$ is a remote interior angle; so $\angle 2 > \angle 1$, which contradicts the hypothesis that $\angle 1 = \angle 2$. So the assumption that the lines are not parallel is false, and $l_1 \parallel l_2$.

Knowing that equal corresponding angles mean that lines are parallel makes it easy to establish some other ways to prove lines parallel as well.

Corollary 1. Equal alternate interior angles mean that lines are parallel.

Corollary 2. Supplementary interior angles on the same side of a transversal mean that lines are parallel.

Corollary 3. In a plane, two lines perpendicular to a third line are parallel.

Exercises

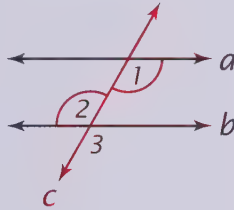
Set I

Complete the following proofs of the corollaries of this lesson by giving the missing statements.

1. **Corollary 1.** Equal alternate interior angles mean that lines are parallel.

Given: $\angle 1 = \angle 2$.

Prove: $a \parallel b$.



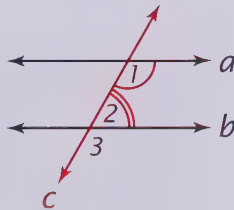
Proof

Statements	Reasons
1. $\angle 1 = \angle 2$	Given.
2. $\angle 2 = \angle 3$	Vertical angles are equal.
3. $\angle 1 = \angle 3$	Substitution.
4. $a \parallel b$	Equal corresponding angles mean that lines are parallel.

2. **Corollary 2.** Supplementary interior angles on the same side of a transversal mean that lines are parallel.

Given: $\angle 1$ and $\angle 2$ are supplementary.

Prove: $a \parallel b$.



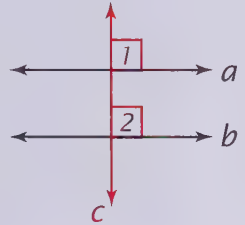
Proof

Statements	Reasons
1. $\angle 1$ and $\angle 2$ are supplementary	Given.
2. $\angle 2 + \angle 3 = 180^\circ$	The angles in a linear pair are supplementary.
3. $\angle 1 = \angle 3$	Supplements of the same angle are equal.
4. $a \parallel b$	Equal corresponding angles mean that lines are parallel.

3. **Corollary 3.** In a plane, two lines perpendicular to a third line are parallel.

Given: $a \perp c$ and $b \perp c$.

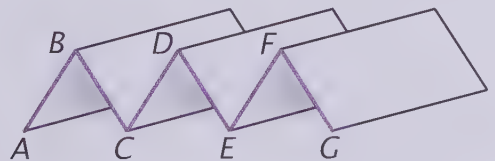
Prove: $a \parallel b$.



Proof

Statements	Reasons
1. $a \perp c, b \perp c$	Given.
2. $\angle 1 = \angle 2$	Perpendicular lines form right angles.
3. $\angle 1 = \angle 2$	All right angles are equal.
4. $a \parallel b$	Equal corresponding angles mean that lines are parallel.

Folded Paper. The figure below shows a sheet of paper folded into an accordion shape. The pleated edge outlined in color lies in a vertical plane.

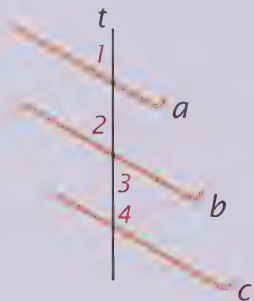


- What are $\angle B$ and $\angle C$ called with respect to lines AB, BC, and CD?
- If $\angle B = \angle C$, why is $AB \parallel CD$?
- If $CD \perp DE$ and $DE \perp EF$, why is $CD \parallel EF$?

Snake Track. As a sidewinding snake moves across the desert, its track appears as a series of lines, each at an angle of about 60° to the snake's line of travel.*

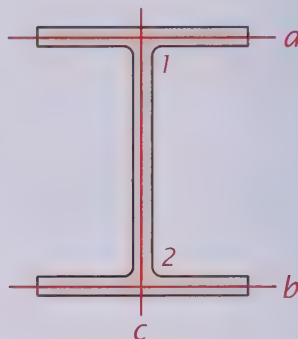


In the figure below, t is the snake's line of travel and a , b , and c are parts of its track.



7. What are $\angle 1$ and $\angle 2$ called with respect to the transversal?
8. If $\angle 1$ and $\angle 2$ each equal 60° , what can you conclude about the lines of a and b ?
9. Why?
10. What are $\angle 3$ and $\angle 4$ called?
11. If $\angle 3 = 60^\circ$ and $\angle 4 = 120^\circ$, how are $\angle 3$ and $\angle 4$ related?
12. What can you conclude about the lines of b and c ?
13. Why?

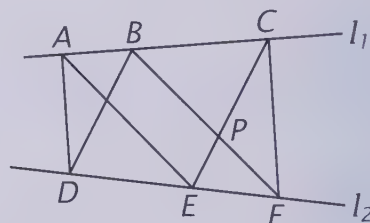
I-Beam. A shape commonly used in beams of structural steel is shown below.



Can you conclude that $a \parallel b$ if you know that

14. $\angle 1 = \angle 2$?
15. $\angle 1$ and $\angle 2$ are supplementary?
16. $a \perp c$ and $b \perp c$?

Parallel Lines. Some of the line segments in the figure below appear to lie in parallel lines.

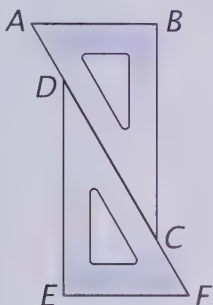


17. Copy the figure and mark its angles as needed to answer the following questions.
18. Which lines are parallel if $\angle AED = \angle BFD$?
19. Why?
20. Which lines are parallel if $\angle DBF = \angle BPC$?
21. Why?
22. Which lines are parallel if $\angle DAC$ and $\angle ACF$ are supplementary?
23. Why?

*"How Snakes Move," by Carl Gans (*Scientific American*, June 1970).

Set II

Drafting Triangles. Two identical drafting triangles are sometimes placed together by a navigator in plotting a chart.

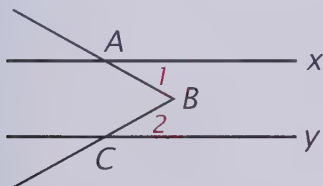


24. As long as points A, D, C, and F are lined up, AB must be parallel to EF. Why?
25. Why must BC be parallel to DE?

Optical Illusion. In the optical illusion below, two lines cross a V-shaped pattern.*



26. Draw the figure as shown below, in which all but one of the Vs have been removed.

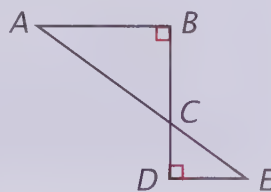


27. If $\angle 1 = \angle 2$, can you conclude that $x \parallel y$? Explain.

Suppose line AB is drawn to intersect y at D.

26. (continued) Extend line AB so that it intersects line y . Label the intersection D.
28. If $\angle 1 = \angle 2$ and $BD = BC$, can you conclude that $x \parallel y$? Explain.

Different Proofs. Here are two different proofs for the same problem. Tell the missing reasons in each proof.



Given: $\angle ABD$ and $\angle BDE$ are right angles.

Prove: $AB \parallel DE$.

- | | | |
|-----|--|----------------|
| 29. | <i>Proof</i> | |
| | <i>Statements</i> | <i>Reasons</i> |
| | 1. $\angle ABD$ and $\angle BDE$ are right angles. | Given. |
| | 2. $AB \perp BD$ and $BD \perp DE$. | Why? |
| | 3. $AB \parallel DE$. | Why? |
-
- | | | |
|-----|--|----------------|
| 30. | <i>Proof</i> | |
| | <i>Statements</i> | <i>Reasons</i> |
| | 1. $\angle ABD$ and $\angle BDE$ are right angles. | Given. |
| | 2. $\angle ABD = \angle BDE$. | Why? |
| | 3. $AB \parallel DE$. | Why? |

Construction Exercise. Use your construction tools to do the following construction as accurately as you can.

31. Construct $\triangle ABC$ with base $AB = 11$ cm, $AC = 9$ cm, and $BC = 7$ cm. Construct the line that bisects $\angle C$ and let D be the point in which the line intersects AB. Construct

*Can You Believe Your Eyes? by J. R. Block and Harold E. Yuker (Brunner/Mazel, 1992).

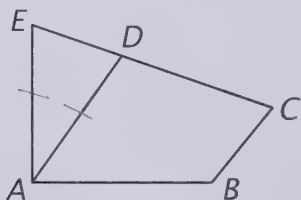
the perpendicular bisector of CD ; let E be the point in which the bisector intersects AC , let F be the point in which it intersects CD , and let G be the point in which it intersects BC . Draw DE and DG .

Mark your figure as needed to answer the following questions.

32. How many right angles are in the figure?
33. Why is $\triangle CFE \cong \triangle CFG$?
34. Why is $EF = FG$?
35. Why are all four triangles surrounding F congruent?
36. Why is $CG = GD = DE = EC$?
37. Why is $CG \parallel DE$ and $GD \parallel EC$?
38. Is $EG \parallel AB$?

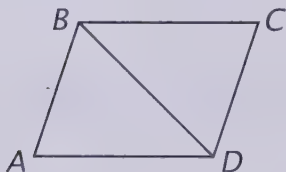
Write complete proofs for each of the following.

39.



Given: $AE = AD$ and $\angle E = \angle BCE$.
Prove: $AD \parallel BC$.

40.



Given: $AB = CD$ and $AD = BC$.
Prove: $AB \parallel CD$.

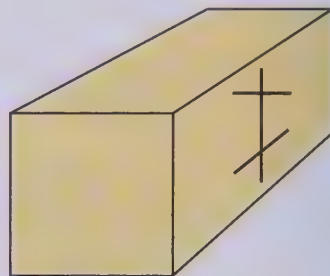
Set III

What Do You See? Sometimes what you *think* you see determines how you see it.*

1. Which cross bar in the figure below is perpendicular to the vertical bar?



2. Which cross bar in the figure below is perpendicular to the vertical bar?



3. If you didn't give the same answer to both questions, why not? Aren't they the same figure?
4. Is it possible that *both* cross bars are perpendicular to the vertical line? Explain.
5. If they are, does it follow that they are parallel? Explain.

**Human Information Processing: An Introduction to Psychology*, by Peter H. Lindsay and Donald A. Norman (Harcourt Brace Jovanovich, 1977).



LESSON 3

The Parallel Postulate

The most famous and enduring riddle of geometry arose from a construction.

Construction 7

To construct a line parallel to a given line through a given point.

The idea behind this construction is simple: equal corresponding angles mean that lines are parallel. To construct the parallel line, we draw a transversal and copy an angle.

1. Let l be the given line and P be the given point.
2. Choose any point O on line l and draw line PO . Name one of the angles formed at O $\angle 1$ as shown in the second figure at the right.
3. Copy $\angle 1$ at P as shown in the third figure. Name it $\angle 2$.

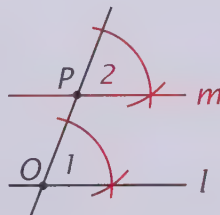
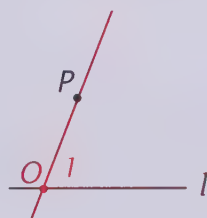
Because $\angle 1$ and $\angle 2$ are equal corresponding angles formed by lines m and l and transversal PO , $m \parallel l$.

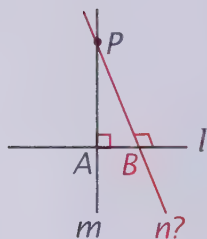
The *problem* is:

Is line m the *only* line that can be drawn parallel to l through point P ?

P .

_____ l

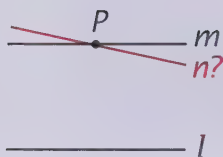




The answer surprised mathematicians when it was discovered in the 1830s, and it is not what you might expect even today. To see why, we need to take another look at what we know about perpendicular lines.

We know how to construct a line m perpendicular to a given line l through a given point P . Suppose we do so, with l and m intersecting at A as shown at the left. Could there be *another* line n passing through P and perpendicular to l with intersection point B as added in color? No, because, if there were, then the remote angle PAB of $\triangle PAB$ would be *equal* to the triangle's exterior angle at B . This equality would contradict the fact that an exterior angle of a triangle is *greater* than either remote interior angle and proves that

Through a point not on a line, there is exactly one line perpendicular to the given line.



Can we use a similar proof by contradiction to show that there can't be two different lines m and n through P and *parallel* to a given line l ? This riddle took more than 2,000 years to work out. The surprising answer came with the discovery of non-Euclidean geometries, which we will learn about in Chapter 16. There are consistent geometries that satisfy all of the postulates and theorems that we have learned so far in which *more* than one line can be parallel to a given line through a point not on it! This fact astonished people when it was discovered in the 1830s.

Euclid knew that he needed some further assumption to go forward with his development of geometry (his Fifth Postulate), and we need such a postulate, too. We call ours the Parallel Postulate.

Postulate 7. The Parallel Postulate

Through a point not on a line, there is exactly one line parallel to the given line.

With this postulate, it is easy to prove the following theorem.

Theorem 18

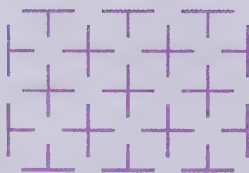
In a plane, two lines parallel to a third line are parallel to each other.

Exercises

Set I

Optical Illusion.

Every pair of lines in this optical illusion appears to be either perpendicular or parallel.



1. How many lines does the figure appear to have?

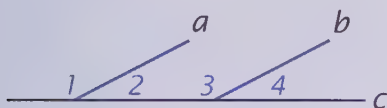
What determines whether or not two lines are

2. perpendicular?
3. parallel?
4. What do you see in the figure that "isn't there"?



Windshield Wipers. One type of windshield wipers has two blades that remain parallel as they move back and forth.

In the figure below, the lines of the wipers, a and b , form angles with the transversal c .



5. Which angle corresponds to $\angle 1$?
6. If these two angles are always equal, why do the wipers always stay parallel?
7. What are $\angle 2$ and $\angle 3$ called with respect to lines a , b , and c ?
8. If these two angles are always supplementary, why are the wipers always parallel?
9. Name another pair of angles that, if they were equal, would ensure that the wipers are parallel.
10. If $\angle 1$ and $\angle 4$ are supplementary, does it follow that the wipers are parallel? Why or why not?

Exactly One. The words “exactly one” have a twofold meaning: that there is *at least one* and that there is *no more than one*.

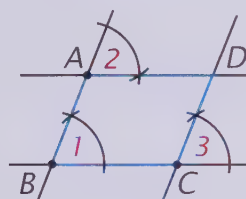
11. Which would be easier to prove: that a cat has at least one flea or that it has no more than one flea?

Tell whether one or more of the following expressions could correctly replace the question mark in each of the following statements: “at least,” “no more than,” “exactly.”

12. A line segment has ? one midpoint.

13. An angle has ? one ray that bisects it.
14. A triangle contains ? one right angle.
15. Two lines intersect in ? one point.
16. Lines are perpendicular if they form ? one right angle.
17. Through a point on a line, there is ? one line perpendicular to the line.
18. Through a point not on a line, there is ? one line perpendicular to the line.

Parallelogram. The figure below shows a method for constructing a *parallelogram*, a quadrilateral with two pairs of parallel sides.



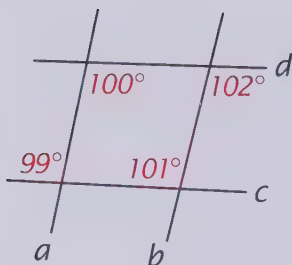
19. Draw two intersecting lines and mark points A, B, and C on them as shown in the figure. Use your straightedge and compass to construct parallelogram ABCD by copying $\angle 1$ as illustrated.
20. Why is $AD \parallel BC$ and $CD \parallel BA$?
21. What else seems to be true about segments AD and BC as well as about segments CD and BA?

What relation seems to exist between

22. $\angle 1$ and $\angle BCD$?
23. $\angle 1$ and $\angle ADC$?
24. Does parallelogram ABCD appear to have line symmetry? Explain why or why not.

Euclid's Assumption. Euclid made the following assumption:

If two lines form interior angles on the same side of a transversal whose sum is less than 180° , then the two lines meet on that side of the transversal.



The figure above shows four lines intersecting at four points.

25. How many of these lines can be considered to be transversals?
26. Copy the figure and write the measures of the other angles surrounding each of the four points.

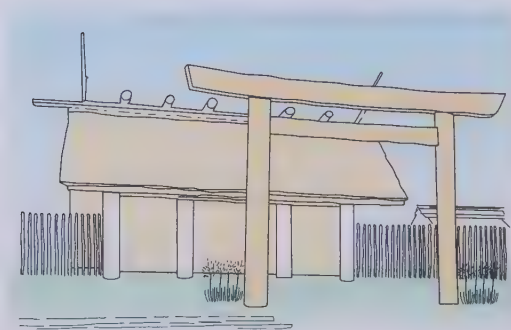
In what direction (left, right, up, or down) do lines

27. a and b intersect?
28. c and d intersect?

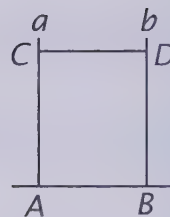
Set II

Gateway. Architect Francis Ching wrote:

Two points can denote a gateway signifying passage from one place to another. Extended vertically, the two points define both a plane of entry and an approach perpendicular to it.*



* *Architecture: Form, Space, and Order*, by Francis D. K. Ching (Wiley, 1996).



29. Draw the gateway described by doing the following.
 - (1) Mark two points 4 cm apart and label them A and B.
 - (2) Draw line AB and construct two lines, a and b , perpendicular to it at points A and B.
 - (3) Mark point C on line a and point D on line b so that $AC = BD = 5$ cm.
 - (4) Draw CD.

In the plane of the gateway (that is, the plane of your paper), how many lines can be drawn that are perpendicular to AB

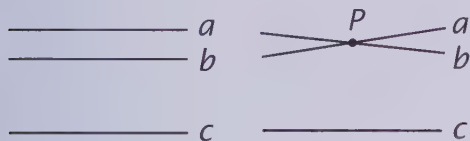
30. through point A?
31. through point D?
32. What relation do lines a and b have?
33. Why?
34. What relation do lines AB and CD seem to have?
35. How many lines can be drawn that are parallel to AB through point C?
36. Why?
29. (continued) Draw line segments AD and BC.

Tell whether you think we have enough information to be able to prove each of the following statements.

37. $\triangle ADB \cong \triangle BCA$.
38. $AD = BC$.
39. $\triangle ADC \cong \triangle BCD$.
40. $\angle ACD = \angle BDC$.
41. $\triangle ABC \cong \triangle CDA$.

Theorem 18. Complete the following indirect proof of Theorem 18.

In a plane, two lines parallel to a third line are parallel to each other.



Given: Lines a , b , and c lie in a plane;

$a \parallel c$ and $b \parallel c$.

Prove: $a \parallel b$.

Proof

42. Suppose ? .

If a and b are not parallel, then they intersect in some point P .

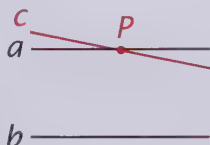
43. But $a \parallel c$ and $b \parallel c$. Why?

44. This contradicts the fact that ? .

45. Therefore, what we assumed is false and ? .

Proclus's Claim. The Greek geometer Proclus said, "In a plane, if a line intersects one of two parallels, it will intersect the other also."

In the figure at the right, $a \parallel b$ and line c intersects line a at P .



46. Does it follow that line c must also intersect line b ?

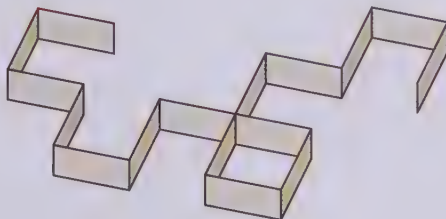
47. Explain why or why not.

Set III

A Dragon Curve. The figure at the right, a model of a "dragon curve," can be formed by folding paper.*

1. Cut a strip of paper $12\frac{1}{2}$ inches long and $\frac{1}{4}$ inch wide. Fold it in half by putting the narrow ends together. Fold the result in half in the same direction. Continue in this way two more times. Open the paper so that, when viewed from the edge, every fold forms a right angle.
2. On graph paper, draw a pair of axes extending 10 units in each direction from the origin. Plot the following points and connect them in alphabetical order with line segments:

A (-4, 8),	G (-2, 4),	M (2, 6),
B (-7, 9),	H (1, 3),	N (5, 5),
C (-8, 6),	I (0, 0),	O (6, 8),
D (-5, 5),	J (3, -1),	P (9, 7),
E (-6, 2),	K (4, 2),	Q (8, 4),
F (-3, 1),	L (1, 3),	
3. How is the figure that you have drawn related to your paper strip?
4. Which point corresponds to the position of the first fold?
5. Which points correspond to the position of the second fold?
6. In what way is the second half of the figure (from I to Q) related to the first half (from A to I)?
7. What do you notice about the coordinates of the corners of the two halves of the figure?



**Mathematical Magic Show*, by Martin Gardner (Knopf, 1977).

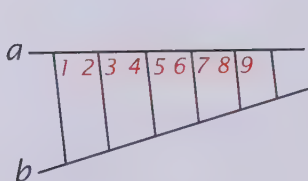


LESSON 4

Parallel Lines and Angles

Biologists see a lot of geometry when looking at a spider's web. Jean Henri Fabre, in his book *The Life of the Spider*, wrote:

In each sector, the various chords, the elements of the spiral windings, are parallel to one another. . . . With the two radiating lines that frame them they form obtuse angles on one side and acute angles on the other; and these angles remain constant in the same sector, because the chords are parallel.



In the figure at the left, lines a and b are the “radiating lines” and the segments between them are the parallel “chords.” Fabre said that angles such as $\angle 1$ and $\angle 3$ are equal because the chords are parallel. In other words, parallel lines form equal corresponding angles with a transversal.

This statement seems like a theorem that we have already proved, but it is not. Look at the two statements written in if-then form:

Theorem 17. If two lines form equal corresponding angles, then the lines are parallel.

Statement about spider web. If two lines are parallel, then they form equal corresponding angles.

It is tempting to assume that these two statements say the same thing, but they do not: one is the *converse* of the other. Remember that the converse of a true statement may be false. Here is another pair of statements about parallel lines that obviously do *not* say the same thing:

Statement. If two lines are parallel, then they lie in the same plane. (True.)

Converse. If two lines lie in the same plane, then they are parallel. (False.)

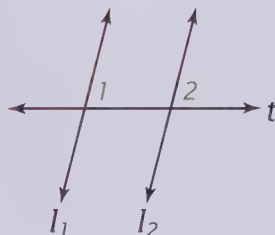
This is a reminder that, even though we may think that the converse of a theorem that we have already proved is true, to add it to our deductive system we must either assume it as a postulate or prove it as a new theorem. In regard to Fabre's statement about the spider web, we can easily prove it by using the Parallel Postulate.

Theorem 19

Parallel lines form equal corresponding angles.

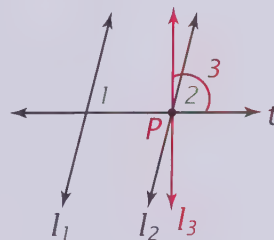
Given: Lines l_1 and l_2 with transversal t , $l_1 \parallel l_2$.

Prove: $\angle 1 = \angle 2$.



Proof

Suppose that $\angle 1 \neq \angle 2$. Draw a line l_3 through point P so that $\angle 3 = \angle 1$ (the Protractor Postulate). Equal corresponding angles mean that lines are parallel; so $l_1 \parallel l_3$. But $l_1 \parallel l_2$ (given); so, through point P, we have two lines parallel to l_1 . This contradicts the Parallel Postulate, which says that, through a point not on a line, there is exactly *one* line parallel to the line. So our assumption that $\angle 1 \neq \angle 2$ is false, and hence $\angle 1 = \angle 2$.



Now that we have proved the converse of Theorem 17, it is easy to prove the converses of its corollaries as well.

Corollary 1. Parallel lines form equal alternate interior angles.

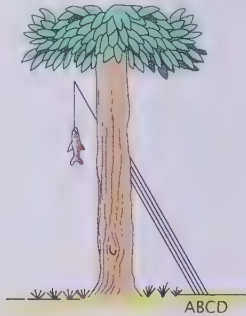
Corollary 2. Parallel lines form supplementary interior angles on the same side of a transversal.

Corollary 3. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other.

Exercises

Set I

Which Fishpole? The figure below appeared in a book of puzzles.* The four lines labeled A, B, C, and D are parallel.



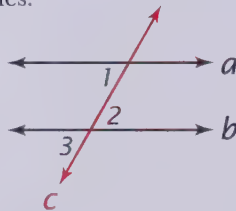
1. What can you conclude about the acute angles at A, B, C, and D?
2. Why?
3. Which fishpole caught the fish?
4. What is “puzzling” about this puzzle?

Corollary Proofs. Complete the following proofs of the corollaries of this lesson by giving the missing reasons.

5. **Corollary 1.** Parallel lines form equal alternate interior angles.

Given: $a \parallel b$.

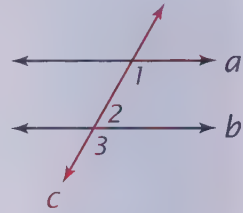
Prove: $\angle 1 = \angle 2$.



Proof

Statements	Reasons
1. $a \parallel b$.	?
2. $\angle 1 = \angle 3$.	?
3. $\angle 2 = \angle 3$.	?
4. $\angle 1 = \angle 2$.	?

6. **Corollary 2.** Parallel lines form supplementary interior angles on the same side of a transversal.



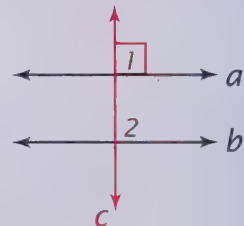
Given: $a \parallel b$.

Prove: $\angle 1$ and $\angle 2$ are supplementary.

Proof

Statements	Reasons
1. $a \parallel b$.	?
2. $\angle 1 = \angle 3$.	?
3. $\angle 2$ and $\angle 3$ are supplementary.	?
4. $\angle 2 + \angle 3 = 180^\circ$.	?
5. $\angle 2 + \angle 1 = 180^\circ$.	?
6. $\angle 1$ and $\angle 2$ are supplementary.	?

7. **Corollary 3.** In a plane, if a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.



Given: $c \perp a$ and $a \parallel b$.

Prove: $c \perp b$.

Proof

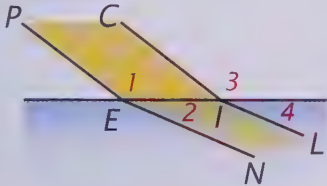
Statements	Reasons
1. $c \perp a$.	?
2. $\angle 1$ is a right angle.	?
3. $\angle 1 = 90^\circ$.	?
4. $a \parallel b$.	?
5. $\angle 1 = \angle 2$.	?
6. $\angle 2 = 90^\circ$.	?
7. $\angle 2$ is a right angle.	?
8. $c \perp b$.	?

*Challenge! by Charlie Rice (Hallmark Editions, 1968).

Bent Pencil. The pencil in this cup appears to be bent because light rays are bent when they go from air into water.

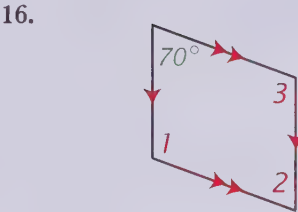
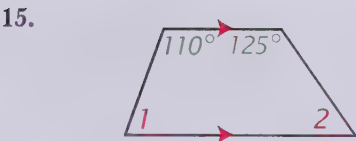
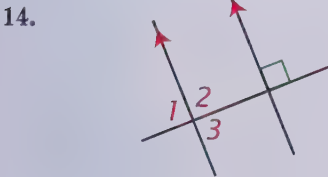
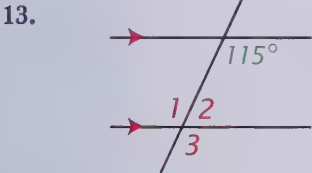


In the figure below, $PE \parallel CI$ and $\angle PEN = \angle CIL$.

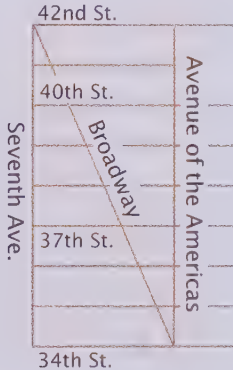


8. Why is $\angle 1 = \angle 3$?
9. Why is $\angle PEN = \angle 1 + \angle 2$ and $\angle CIL = \angle 3 + \angle 4$?
10. Why is $\angle 1 + \angle 2 = \angle 3 + \angle 4$?
11. Why is $\angle 2 = \angle 4$?
12. Why is $EN \parallel IL$?

Find the Angles. In each of the following figures, the arrowheads indicate parallel lines. Find the measures of the indicated angles.



City Streets. This map shows some of the streets in New York City. The streets running from left to right on the map are parallel.



Seventh Ave. and Avenue of the Americas are perpendicular to 34th St.

17. What relation must Seventh Ave. have to 42nd St.? Why?
18. What relation must Seventh Ave. and Avenue of the Americas have? Why?

One of the acute angles at the intersection of Broadway and 37th St. is 67° .

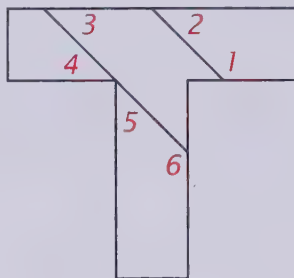
19. How large are the other three angles at that intersection?
20. How large are the angles at the intersection of Broadway and 40th St.?
21. How do you know?

T Puzzle. This four-piece puzzle in the shape of the letter T was once used to advertise sausage! Its edges are parallel and perpendicular, and the two slanted cuts are parallel.



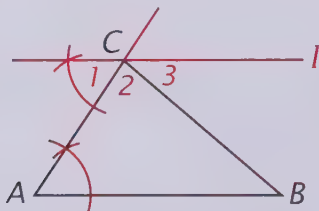
Tell in what way each of the following pairs of angles is related and *why*.

22. $\angle 1$ and $\angle 2$.
23. $\angle 2$ and $\angle 3$.
24. $\angle 3$ and $\angle 4$.
25. $\angle 2$ and $\angle 4$.
26. $\angle 4$ and $\angle 5$.
27. $\angle 5$ and $\angle 6$.



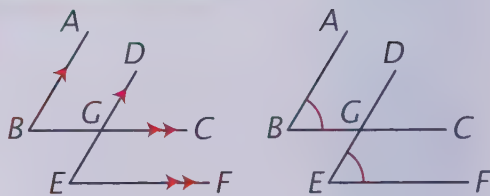
Set II

Parallel Construction. In the figure below, line l has been constructed through point C by copying $\angle A$ of $\triangle ABC$ as $\angle 1$.



28. Draw a figure similar to this one and do the construction.
29. Why is $l \parallel AB$?
30. Why is line l the *only* line that can be constructed parallel to AB through point C?
31. Why is $\angle 3 = \angle B$?
32. What is $\angle 1 + \angle 2 + \angle 3$?
33. What is $\angle A + \angle 2 + \angle B$?

Overlapping Angles.



In the figure at the left above, $BA \parallel ED$ and $BC \parallel EF$. It looks as if $\angle B = \angle E$.

34. Is this necessarily true? Explain why or why not.

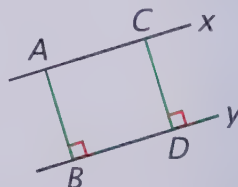
In the figure at the right above, $\angle B = \angle E$. It looks as if $BA \parallel ED$ and $BC \parallel EF$.

35. Is this necessarily true? Explain why or why not.

Sand Dune Lines. When desert sand dunes are seen from a high altitude, they often seem to form parallel lines.*



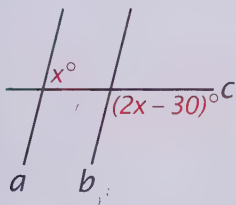
One way to describe parallel lines is to say that they are “everywhere equidistant.” This description suggests that, if $x \parallel y$ in the figure below and AB and CD are both perpendicular to y , then $AB = CD$.



**The Flying Circus of Physics*, by Jearl Walker (Wiley, 1977).

36. Copy the figure and add to it as needed to answer each of the following questions.
37. We can add line AD to the figure. Why?
38. Why is $\angle CAD = \angle BDA$?
39. Why is $AB \parallel CD$?
40. Why is $\angle BAD = \angle CDA$?
41. Why is $AD = AD$?
42. Why is $\triangle ABD \cong \triangle DCA$?
43. Why is $AB = CD$?
44. In general, what would you measure to find the distance between two parallel lines?

SAT Problem. In the figure below, $a \parallel b$.

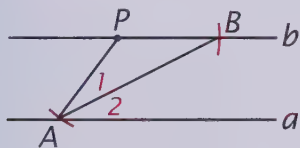


45. Write an equation relating the measures of the two indicated angles.

Find the measure of

46. the acute angle.
47. the obtuse angle.

Construction Problem. In the figure below, line b has been drawn parallel to line a through P. Also $PA = PB$.

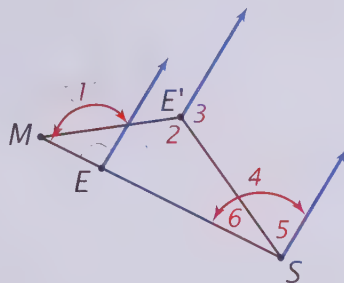


48. Draw line a and point P. Use your straightedge and compass to construct $b \parallel a$ and $PA = PB$.
49. What seems to be true about $\angle 1$ and $\angle 2$?
50. Use what you know about angles and parallel lines to prove it.



Set III

Astronomer Johannes Kepler was the first to figure out the orbits of the planets.



He used diagrams like the one above to calculate the shape of Earth's orbit.* The points M and S represent Mars and the sun and E and E' represent two positions of Earth. The three lines in blue are parallel and everything lies in the same plane. The angles numbered 1, 2, and 3 can be measured from Earth. From these angles, Kepler was able to figure out the measures of the angles numbered 4, 5, and 6 and the angle at M.

How did he figure out the measure of

1. $\angle 4$?
2. $\angle 5$?
3. $\angle 6$?
4. $\angle M$?

Starting with SM as one side, Kepler was then able to draw $\triangle MSE'$ and hence determine the position of Earth at E'.

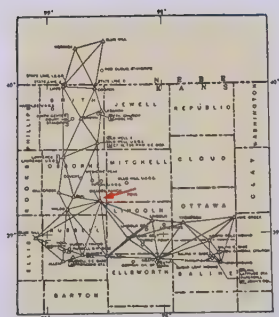
5. Suppose that $\angle 1 = 84^\circ$, $\angle 2 = 124^\circ$, and $\angle 3 = 122^\circ$. Find $\angle 4$, $\angle 5$, $\angle 6$, and $\angle M$.
6. Show how Kepler drew $\triangle MSE'$ by using your ruler and protractor to draw it. Let $MS = 10$ cm.

**Constructing the Universe*, by David Layzer (Scientific American Library, 1984).



LESSON 5

The Angles of a Triangle



When Lewis and Clark set out on their historic expedition across America, most of the land west of the Mississippi was unknown territory. The map that they used, shown above, had been drawn in 1802. It would be many years before the entire country was surveyed.

To map the United States, surveyors used a method called *triangulation*—so named because it locates points by means of a network of triangles. The entire country is now “criss crossed with an elaborate network of many thousands of triangles, measured by surveyors’ transits, and the corners of all of them are marked by permanent stones called ‘bench marks.’”^{*} The initial point from which the triangulation began was in a pasture in Meades Ranch, Kansas. Part of the network of triangles starting from this point is shown in the map at the left. Meades Ranch is marked with an arrow.

Surveyors check their accuracy over short distances by measuring the angles of the triangles in their network. That the sum of the angles of a triangle is 180° is a famous theorem of plane geometry.[†] Now that we have the Parallel Postulate, we can prove it.

^{*}*Triangles*, by Henry M. Neely (Crowell, 1962).

[†]Chapter 2, Lesson 6.

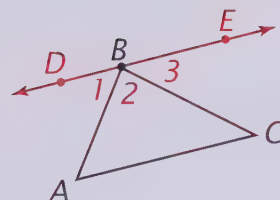
Theorem 20. The Angle Sum Theorem

The sum of the angles of a triangle is 180° .

Given: $\triangle ABC$.

Prove: $\angle A + \angle B + \angle C = 180^\circ$.

Proof



Statements

1. $\triangle ABC$.
2. Through point B,
draw line $DE \parallel AC$.
3. $\angle 1 = \angle A$ and $\angle 3 = \angle C$.
4. $\angle 1 + \angle 2 = \angle DBC$.
5. $\angle DBC$ and $\angle 3$ are
supplementary.
6. $\angle DBC + \angle 3 = 180^\circ$.
7. $\angle 1 + \angle 2 + \angle 3 = 180^\circ$.
8. $\angle A + \angle B + \angle C = 180^\circ$.

Reasons

Given.
Through a point not on a line,
there is exactly one line
parallel to the line.
Parallel lines form equal
alternate interior angles.
Betweenness of Rays
Theorem (BD-BA-BC).
The angles in a linear pair
are supplementary.
The sum of two supple-
mentary angles is 180° .
Substitution (steps 4 and 6).
Substitution (steps 3 and 7).

This theorem has some useful corollaries.

Corollary 1. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.

Corollary 2. The acute angles of a right triangle are complementary.

Corollary 3. Each angle of an equilateral triangle is 60° .

A fourth immediate consequence of Theorem 20 concerns the measure of an exterior angle of a triangle. We already know that an exterior angle of a triangle is greater than either remote interior angle. Now we can prove that it is equal to their sum.

Theorem 21

An exterior angle of a triangle is equal to the sum of the remote interior angles.

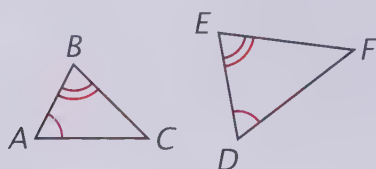
Several proofs of this theorem have been discovered. We will consider one of them in the exercises.

Exercises

Set I

Corollary Proofs. Complete the following proofs of the corollaries of this lesson by giving the reasons. The proofs are written in paragraph form.

Corollary 1. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal.



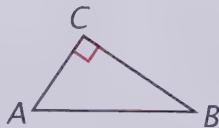
Given: In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$ and $\angle B = \angle E$.

Prove: $\angle C = \angle F$.

Proof

1. $\angle A + \angle B + \angle C = 180^\circ$ and $\angle D + \angle E + \angle F = 180^\circ$. Why?
2. It follows that $\angle A + \angle B + \angle C = \angle D + \angle E + \angle F$. Why?
3. Because $\angle A = \angle D$ and $\angle B = \angle E$, it follows that $\angle D + \angle E + \angle C = \angle D + \angle E + \angle F$. Why?
4. Therefore, $\angle C = \angle F$. Why?

Corollary 2. The acute angles of a right triangle are complementary.



Given: $\triangle ABC$ is a right triangle with right $\angle C$.

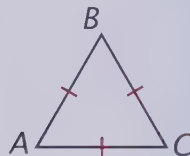
Prove: $\angle A$ and $\angle B$ are complementary.

Proof

5. $\triangle ABC$ is a right triangle with right $\angle C$; so $\angle C = 90^\circ$. Why?

6. Because $\angle A + \angle B + \angle C = 180^\circ$, it follows that $\angle A + \angle B + 90^\circ = 180^\circ$. Why?
7. So $\angle A + \angle B = 90^\circ$. Why?
8. Therefore, $\angle A$ and $\angle B$ are complementary. Why?

Corollary 3. Each angle of an equilateral triangle is 60° .



Given: $\triangle ABC$ is equilateral.

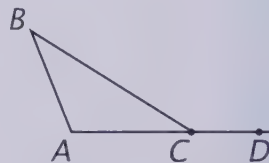
Prove: $\angle A = 60^\circ$, $\angle B = 60^\circ$, $\angle C = 60^\circ$.

Proof

9. $\triangle ABC$ is equilateral; so it is also equiangular. Why?
10. Because $\triangle ABC$ is equiangular, $\angle A = \angle B = \angle C$. Why?
11. Because $\angle A + \angle B + \angle C = 180^\circ$, it follows that $\angle A + \angle A + \angle A = 3\angle A = 180^\circ$. Why?
12. So $\angle A = 60^\circ$. Why?
13. It follows that $\angle B = 60^\circ$ and $\angle C = 60^\circ$. Why?

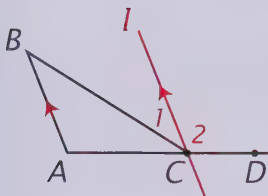
Exterior Angle Theorem. One way to prove Theorem 21 is given here. Complete the proof by giving the reasons.

Theorem 21. An exterior angle of a triangle is equal to the sum of the remote interior angles.



Given: $\angle BCD$ is an exterior angle of $\triangle ABC$.

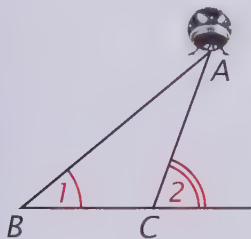
Prove: $\angle BCD = \angle A + \angle B$.



Proof

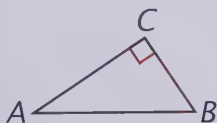
14. Through point C, draw line l parallel to AB. Why is this possible?
15. $\angle 1 = \angle B$. Why?
16. $\angle 2 = \angle A$. Why?
17. $\angle BCD = \angle 1 + \angle 2$. Why?
18. Therefore, $\angle BCD = \angle A + \angle B$. Why?

UFO Angles. The angles of elevation of a UFO at A are measured from two positions, B and C.



19. Write the two inequalities that follow from the Exterior Angle Theorem.
20. Write the equation relating the measures of $\angle 1$, $\angle 2$, and $\angle A$.
21. Find $\angle A$ if $\angle 1 = 42^\circ$ and $\angle 2 = 70^\circ$.

Right and Wrong. Obtuse Ollie told Acute Alice that, because $\angle C$ is the right angle of the triangle below, $\angle A$ and $\angle B$ must be the wrong angles.*



*From an idea in *The Heart of Mathematics*, by Edward B. Burger and Michael Starbird (Key College Publishing, 2000).

22. What kind of angles must they actually be?

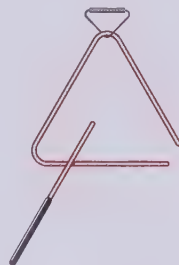
Alice told Ollie that, if he is correct, then “two wrongs make a right.”

23. What does she mean?

Another Kind of Triangle.

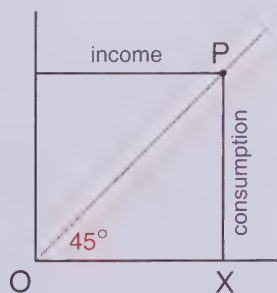
The triangle in this picture is made of steel.

24. What kind of geometric triangle has a similar shape?
25. What is the triangle in the picture used for?
26. How large are its “angles”?



Economics Graph. The following words from an economics book describe the graph of the “consumption” function.[†]

At any point on the 45° line, the distance up from the horizontal axis (which is consumption) exactly equals the distance across from the vertical axis (which is income).



27. From the fact that $XP \perp OX$, what kind of triangle is $\triangle OPX$?
28. What is the relation of $\angle OPX$ to $\angle POX$?
29. What is the measure of $\angle OPX$?
30. Why is $XP = OX$?

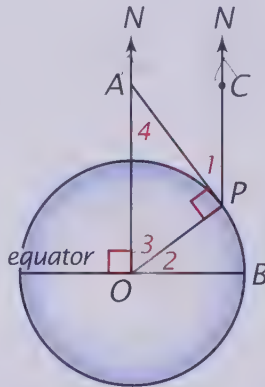
[†]*Economics*, by Paul A. Samuelson and William D. Nordhaus (McGraw-Hill, 1989).

Set II

North Star and Latitude. In Peculiar, Missouri, the North Star is always 38° above the horizon. The angle between Peculiar and the equator also is 38° , which isn't really peculiar, because we can prove it.*

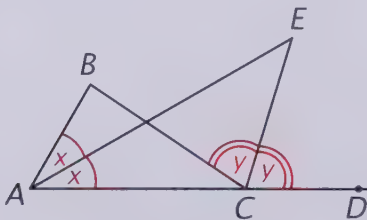
In the figure at the right, the angle of elevation of the North Star at P is $\angle 1$; the latitude of P is $\angle 2$. Also, $OA \parallel PC$, $OA \perp OB$, and $OP \perp PA$.

Explain why $\angle 1 = \angle 2$ by answering each of the following questions.



31. Why is $\angle 1 = \angle 4$?
32. What relation does $\angle 4$ have to $\angle 3$?
33. Why?
34. What relation does $\angle 2$ have to $\angle 3$?
35. What relation does $\angle 4$ have to $\angle 2$?
36. Why?
37. Why is $\angle 1 = \angle 2$?

Angle Bisectors. In the figure below, the bisectors of an interior and an exterior angle of $\triangle ABC$ meet at E.



*Peculiar, Missouri, is on highway 71 in Cass County.

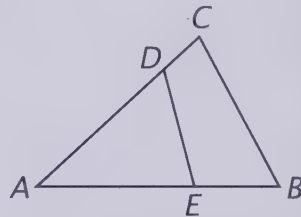
38. Draw a larger $\triangle ABC$ with exterior $\angle BCD$. Use your straightedge and compass to bisect $\angle BAC$ and $\angle BCD$. Label the point in which the bisectors meet E.
39. Use your protractor to measure $\angle B$ and $\angle E$. How do they seem to compare in size?

Tell why each of the following equations is true.

40. $\angle BCD = \angle BAC + \angle B$ and $\angle ECD = \angle EAC + \angle E$.
41. $2y = 2x + \angle B$ and $y = x + \angle E$.
42. $2y = 2x + 2\angle E$.
43. $2x + \angle B = 2x + 2\angle E$.
44. $\angle B = 2\angle E$.

Write proofs for the following.

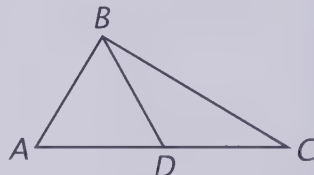
45.



Given: In $\triangle ABC$ and $\triangle ADE$,
 $\angle ADE = \angle B$.

Prove: $\angle AED = \angle C$.

46.

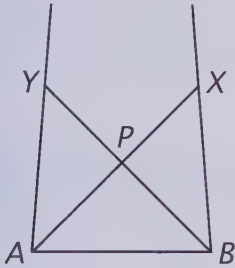


Given: $\triangle ABC$ is a right triangle with right $\angle ABC$; $\angle ABD$ and $\angle C$ are complementary.

Prove: $\triangle ABD$ is isosceles.

Set III

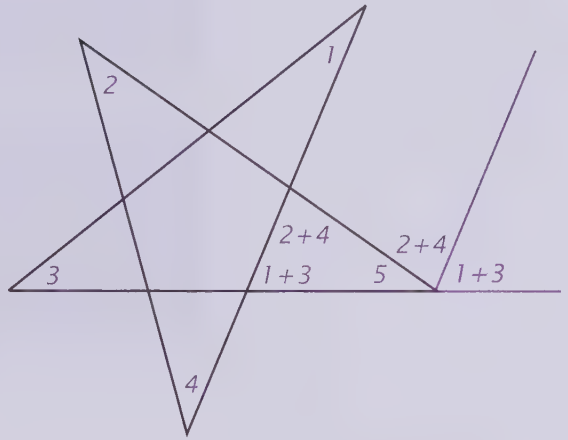
Catch Question. This problem appeared in a collection of “catch” questions.*



Suppose lines AX and BY bisect two of the angles of a triangle whose third vertex is out of view. If $AX \perp BY$ and the base AB of the triangle is 10 inches long, how tall is the triangle?

47. Draw the figure and mark this information on it.
48. Reasoning from your figure, explain why this is a “catch” question.

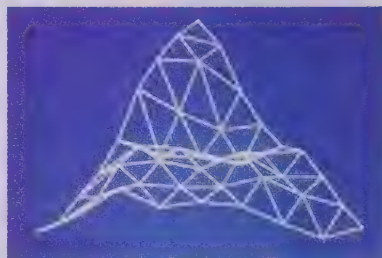
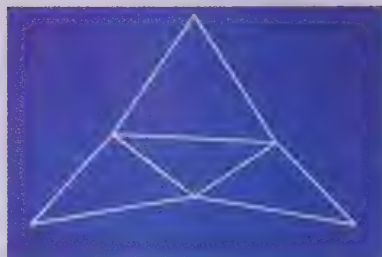
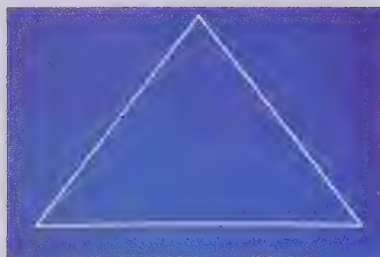
This proof appeared in a book of proofs without words.†



1. What do you think is being proved?
2. Briefly explain in words how the proof works.

**The Unexpected Hanging and Other Mathematical Diversions*, by Martin Gardner (Simon & Schuster, 1969).

†Proof by Fouad Nakhli in *Proofs Without Words: Exercises in Visual Thinking*, by Roger B. Nelsen (Mathematical Association of America, 1993).

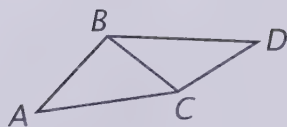


LESSON 6

AAS and HL Congruence

Surveyors have been using triangulations—networks of triangles sometimes numbering in the thousands—for centuries. Triangulations of *millions* of triangles are now used in computer programming.

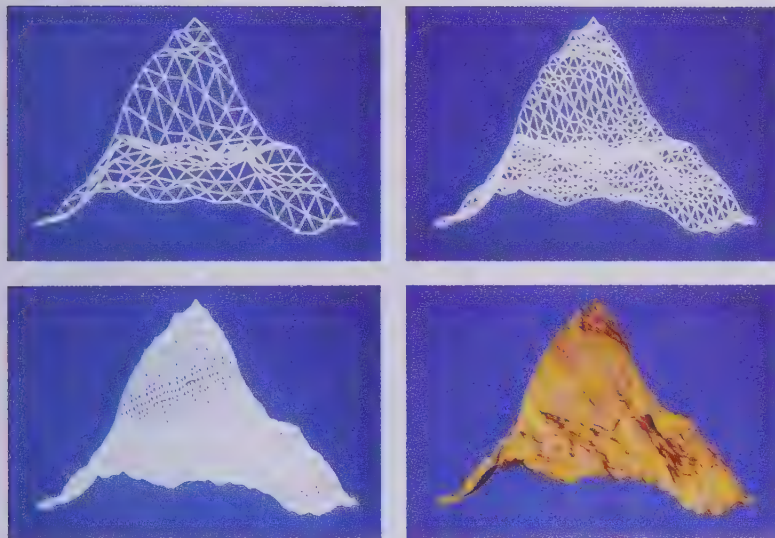
The images above show how the image of a mountain can be generated from repeated divisions of triangles.* The sequence, *Fractal Demo*, by Alvy Ray Smith, begins with a single triangle in “three-dimensional space.” The triangle is divided into 4 triangles by connecting the midpoints of its sides. The figure is then “bent” at the midpoints to give it the impression of being three-dimensional. Each of the 4 triangles is then transformed in the same way, resulting in 16 triangles. This process is repeated as shown, and the triangles are eventually filled in to create the picture of the mountain.



Consider a pair of triangles formed by this process, as shown at the left. Are they congruent? Although it is hard to tell from their appearance, we know several sets of conditions that are sufficient to prove triangles congruent. In fact, we have given them names: ASA, SAS, and SSS.

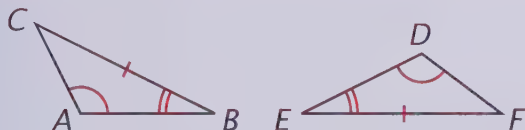
Are there any other sets of parts that can be used to prove triangles congruent? Yes; we will consider two of them in this lesson.

**Visualization: The Second Computer Revolution*, by Richard Mark Friedhoff and William Benzon (Abrams, 1989).



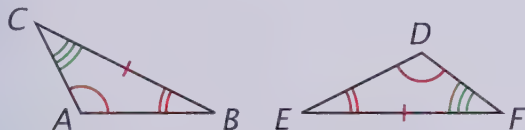
Theorem 22. The AAS Theorem

If two angles and the side opposite one of them in one triangle are equal to the corresponding parts of another triangle, the triangles are congruent.



Given: $\triangle ABC$ and $\triangle DEF$ with $\angle A = \angle D$, $\angle B = \angle E$, and $BC = EF$.

Prove: $\triangle ABC \cong \triangle DEF$.



Take a good look at the figures above and you may see the proof. If two angles of one triangle are equal to two angles of another triangle, the third pair of angles also are equal. Consequently, AAS follows from ASA.

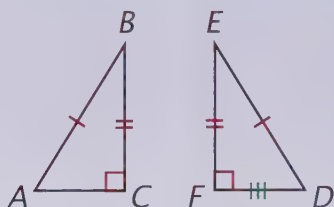
Proof

In $\triangle ABC$ and $\triangle DEF$, $\angle A = \angle D$ and $\angle B = \angle E$. Therefore, $\angle C = \angle F$, and so, with $BC = EF$, $\triangle ABC \cong \triangle DEF$ by ASA.

As the name of the next theorem, HL for “hypotenuse–leg,” implies, the theorem applies only to right triangles.

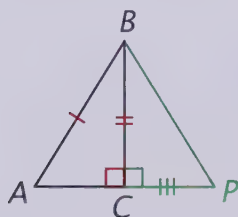
Theorem 23. The HL Theorem

If the hypotenuse and a leg of one right triangle are equal to the corresponding parts of another right triangle, the triangles are congruent.

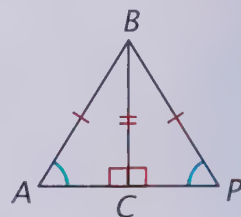


Given: $\triangle ABC$ and $\triangle DEF$ are right triangles (with right angles C and F); $AB = DE$ and $BC = EF$.

Prove: $\triangle ABC \cong \triangle DEF$.



$\triangle DEF \cong \triangle PBC$ (SAS)



$\triangle ABC \cong \triangle PBC$ (AAS)
so $\triangle ABC \cong \triangle DEF$

Again, a careful look at the figures above may reveal the proof.

Proof

Extend line AC (two points determine a line) and choose point P on it so that $PC = BC$ (the Ruler Postulate); also, draw BP.

$\triangle PBC \cong \triangle DEF$ (SAS); so $PB = DE$ (corresponding parts of congruent triangles are equal). Because $AB = DE$ (given), $AB = PB$ (substitution).

AB and PB are equal sides of $\triangle ABP$; so $\angle A = \angle BPC$ (if two sides of a triangle are equal, the angles opposite them are equal).

$\triangle ABC \cong \triangle PBC$ (AAS); so $\triangle ABC \cong \triangle DEF$ (two triangles congruent to a third triangle are congruent to each other).

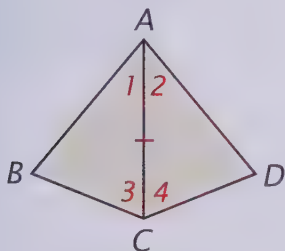
Exercises

Set I

Hang Glider. A popular type of hang glider is the Rogallo.



The design of its sail consists of two triangles: $\triangle ABC$ and $\triangle ADC$. They share a common side; so $AC = AC$.*



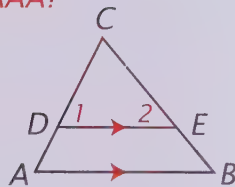
Name the way of knowing that $\triangle ABC \cong \triangle ADC$ if

1. $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.
2. $AB = AD$ and $BC = CD$.
3. $\angle 1 = \angle 2$ and $AB = AD$.
4. $\angle B = \angle D$ and $\angle 3 = \angle 4$.
5. $\angle B = \angle D = 90^\circ$ and $BC = CD$.

Congruent or Not? SSS names a way to prove triangles congruent. Does AAA name another way?

**Hang Gliding and Soaring*, by James E. Mrazek (St. Martin's Press, 1976).

AAA?

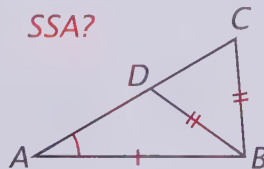


In $\triangle ABC$, $DE \parallel AB$.

6. Why is $\angle 1 = \angle A$ and $\angle 2 = \angle B$?
7. How many pairs of angles in $\triangle DEC$ and $\triangle ABC$ are equal?
8. Is $\triangle DEC \cong \triangle ABC$?

AAS names a way to prove triangles congruent. Does SSA name another way?

SSA?

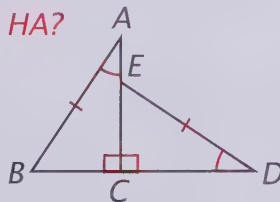


In $\triangle ABC$, $DB = CB$.

9. How many pairs of parts in $\triangle ABC$ and $\triangle ABD$ are equal?
10. Is $\triangle ABC \cong \triangle ABD$?

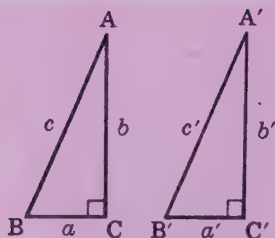
HL names a way to prove triangles congruent. Does HA name another way?

HA?



In this figure, $AC \perp BD$, $AB = DE$, and $\angle A = \angle D$.

11. How many pairs of parts in $\triangle ABC$ and $\triangle DEC$ are equal?
12. Is $\triangle ABC \cong \triangle DEC$?
13. Explain why or why not.



[가정]

$\triangle ABC$ 와 $\triangle A'B'C'$ 에서

$\angle C = \angle C' = \angle R$, $AB = A'B'$,
 $AC = A'C'$

[결론]

$\triangle ABC \equiv \triangle A'B'C'$

정리 7

빗변과 다른 한 변이 각각 같은 두 직각삼각형은 합동이다.
[RHS 합동]

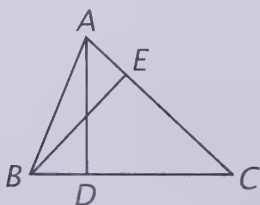
Korean Theorem. A theorem from a Korean geometry book is shown above.

14. What do you think $\angle C = \angle C' = \angle R$ means?
15. What do you think $\triangle ABC \equiv \triangle A'B'C'$ means?
16. What theorem do you think is illustrated? State it as a complete sentence.

The Korean name for the theorem, RHS, is remarkable in that it is an abbreviation of three words in *English* rather than Korean.

17. What do you think the three words identifying this theorem are?

Triangle Problem. In $\triangle ABC$, $AD \perp BC$, $BE \perp AC$, and $AD = BE$.



18. Copy the figure above and mark it as needed to answer the following questions.
19. Why is $\triangle BAE \equiv \triangle ABD$?
20. Why is $\angle BAE = \angle ABD$?
21. Why is $BC = AC$?
22. What kind of triangle is $\triangle ABC$?

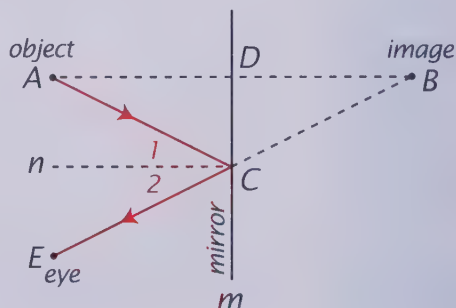
Set II

Mirror Distances.

A wall mirror is sometimes used to create the illusion that the space of a room is twice as large as it actually is.



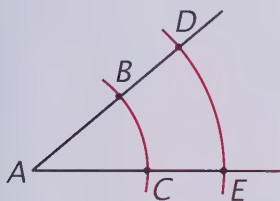
This illusion follows from the fact that, when we see light from an object reflected in a mirror, the object and its reflection seem to be at the same distance from the mirror.



In the figure above, line m represents the mirror. Both AB and n are perpendicular to m , and the angle of incidence, $\angle 1$, is equal to the angle of reflection, $\angle 2$.

23. Copy the figure and mark it as needed to answer each of the following questions.
24. Because $AB \perp m$ and $n \perp m$, $AB \parallel n$. Why?
25. $\angle 1 = \angle A$. Why?
26. $\angle 2 = \angle B$. Why?
27. Because $\angle 1 = \angle 2$, $\angle A = \angle B$. Why?
28. $DC = DC$. Why?
29. $\triangle BDC \cong \triangle ADC$. Why?
30. Therefore, $BD = AD$. Why?

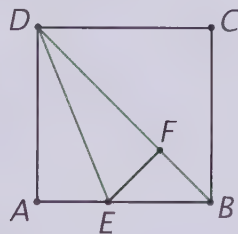
Angle Bisection. The figure below illustrates the first steps in a different way to bisect an angle.



Two arcs have been drawn intersecting the sides of $\angle A$ so that $AB = AC$ and $AD = AE$.

31. Use your straightedge and compass to make a large copy of the figure. Draw BE and CD .
32. Which parts of $\triangle ADC$ and $\triangle AEB$ can be used to prove them congruent?
33. Why is $\triangle ADC \cong \triangle AEB$?
34. Why is $\angle ADC = \angle AEB$?
31. (continued) Label the point in which BE and CD intersect P . Draw AP .
35. Which parts of $\triangle BDP$ and $\triangle CEP$ can be used to prove them congruent?
36. Why is $\triangle BDP \cong \triangle CEP$?
37. Why is $DP = PE$?
38. Which parts of $\triangle ADP$ and $\triangle AEP$ can be used to prove them congruent?
39. Why is $\triangle ADP \cong \triangle AEP$?
40. Why does AP bisect $\angle DAE$?

Origami Frog. The origami pattern for the Japanese jumping frog begins with a square sheet of paper and the two folds shown below.*



First, corner C is folded onto corner A . Then, side DA is folded onto the fold DB .

41. Copy the figure and mark the following information on it: $AB = BC = CD = DA = DF$. $DA \perp AB$, $EF \perp DB$, and $AB \perp BC$.
 42. Why is $\triangle ABD \cong \triangle CBD$?
 43. Why is $\angle ABD = \angle CBD$?
- What is the measure of
44. $\angle ABD$?
 45. $\angle FEB$?
 46. Why is $EF = FB$?
 47. What kind of triangles are $\triangle ADE$ and $\triangle FDE$?
 48. Why is $\triangle ADE \cong \triangle FDE$?
 49. Why is $AE = EF$?
 50. Why is $AE = FB$?

More on Equal Parts. Each of our methods for proving triangles congruent requires showing that three pairs of parts are equal. Can two triangles have more than three pairs of equal parts but not be congruent?

Consider, for example, one pair of equal angles and three pairs of equal sides.

51. Do you think it is possible that two triangles could have these equal parts and yet not be congruent? Explain why or why not.

**Folding the Universe*, by Peter Engel (Vintage, 1989).

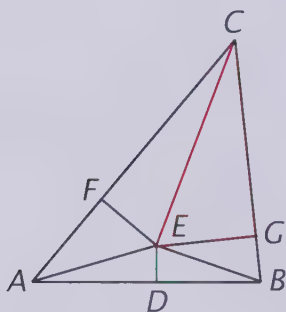
Consider two pairs of equal sides and three pairs of equal angles.

52. Use your ruler and protractor to draw, as accurately as you can, $\triangle ABC$ with $AB = 7.2$ cm, $\angle A = 21^\circ$, $AC = 10.8$ cm, $\angle B = 127^\circ$, and $\angle C = 32^\circ$ and $\triangle DEF$ with $DE = 10.8$ cm, $\angle D = 21^\circ$, $\angle E = 127^\circ$, $EF = 7.2$ cm, and $\angle F = 32^\circ$.
53. Do you think two triangles can have five pairs of equal parts and not be congruent?
54. Do you think two triangles can have six pairs of equal parts and not be congruent? Explain.

Set III

Is Every Triangle Isosceles? Here is a proof originally published in 1892 that is so strange that it has since become famous.*

The proof supposedly shows that *every* triangle is isosceles and is based on the figure below.

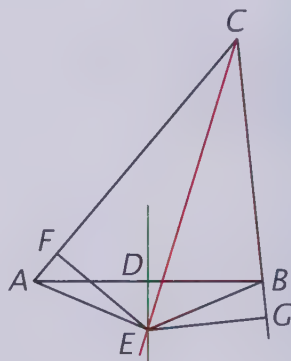


$\triangle ABC$ represents any triangle. The bisector of $\angle ACB$, CE , and the perpendicular bisector of AB , DE , intersect at E . Lines EF and EG are drawn from E so that $EF \perp AC$ and $EG \perp BC$. Finally, lines EA and EB are drawn.

1. Copy the figure and mark it as necessary to explain each of the following statements.

2. $\triangle CEF \cong \triangle CEG$. Why?
3. $EF = EG$. Why?
4. $\triangle EAD \cong \triangle EBD$. Why?
5. $EA = EB$. Why?
6. $\triangle AFE \cong \triangle BGE$. Why?
7. $AF = BG$. Why?
8. $AF + FC = BG + GC$. Why?
9. $AF + FC = AC$ and $BG + GC = BC$. Why?
10. $AC = BC$. Why?

Although it seems as if our reasoning is correct, something is obviously very wrong. All triangles are *not* isosceles. The problem with the proof is that we have assumed something from the figure that isn't really true. Here is the same triangle with the lines accurately drawn. Compare the steps of the proof with it.



11. Which statements in the proof are actually false?

**Mathematical Recreations and Essays*, by W. W. Rouse Ball (Macmillan, 1892).

CHAPTER 6 Summary and Review

Basic Ideas

Angles formed by a transversal 220
Line symmetry 212
Parallel lines 219
Transversal 219

Postulate

7. *The Parallel Postulate.* Through a point not on a line, there is exactly one line parallel to the line. 226

Theorems

16. In a plane, two points each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment. 212

17. Equal corresponding angles mean that lines are parallel. 220

Corollary 1. Equal alternate interior angles mean that lines are parallel. 220

Corollary 2. Supplementary interior angles on the same side of a transversal mean that lines are parallel. 220

Corollary 3. In a plane, two lines perpendicular to a third line are parallel. 220

18. In a plane, two lines parallel to a third line are parallel to each other. 226

19. Parallel lines form equal corresponding angles. 231

Corollary 1. Parallel lines form equal alternate interior angles. 231

Corollary 2. Parallel lines form supplementary interior angles on the same side of a transversal. 231

Corollary 3. In a plane, a line perpendicular to one of two parallel lines is also perpendicular to the other. 231

20. *The Angle Sum Theorem.* The sum of the angles of a triangle is 180° . 237

Corollary 1. If two angles of one triangle are equal to two angles of another triangle, the third angles are equal. 237

Corollary 2. The acute angles of a right triangle are complementary. 237

Corollary 3. Each angle of an equilateral triangle is 60° . 237

21. An exterior angle of a triangle is equal to the sum of the remote interior angles. 237

22. *The AAS Theorem.* If two angles and the side opposite one of them in one triangle are equal to the corresponding parts of another triangle, the triangles are congruent. 243

23. *The HL Theorem.* If the hypotenuse and a leg of one right triangle are equal to the corresponding parts of another right triangle, the triangles are congruent. 244

Constructions

6. To construct a line perpendicular to a given line through a given point. 213

7. To construct a line parallel to a given line through a given point. 225

Exercises

Set I

Flag Symmetries. At first glance, the flags below all appear to have line symmetry, but one does not.



Algeria



Barbados



Cuba



Jamaica

1. What is a simple test to determine whether such figures have line symmetry?
2. Which flag does not have line symmetry? Explain.
3. Describe the line (or lines) of symmetry of the other three flags.

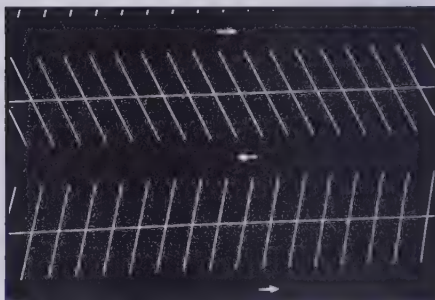
Draw two points A and B on your paper as shown in the figure below.

A.

B

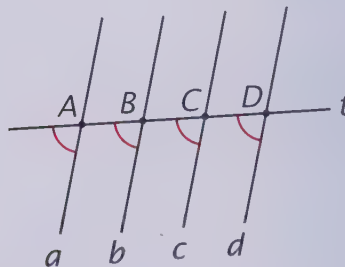
4. Use your straightedge and compass to construct the line such that A and B are symmetric with respect to it.
5. What relation does the line have to line segment AB?

Lot of Lines. A common occurrence of parallel lines and transversals is shown in this photograph.*



6. What does the photograph show?

The figure below illustrates one method by which the lines might have been made parallel.



7. What is it?

Construct the figure by doing the following.

8. Draw a line to show the transversal in the figure and use your compass to mark points A, B, C, and D equally spaced along it.
Without using your protractor, construct a line through A that makes a 60° angle with the transversal.
Finally, construct lines b , c , and d parallel to line a .

**Across the American Landscape*, by James Corner and Alex S. MacLean (Yale University Press, 1996).

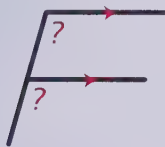
Alphabet. The shapes of some of the letters of the alphabet illustrate some of the definitions and theorems of geometry. Complete the definition or theorem suggested by each of the following figures.

9.



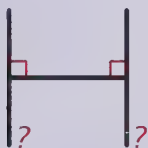
An exterior angle of a triangle is equal . . .

10.



Parallel lines . . .

11.



In a plane, two lines . . .

12.



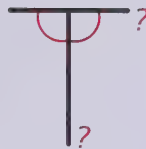
If two lines form a right angle . . .

13.



Equal alternate . . .

14.



If the angles in a linear pair . . .

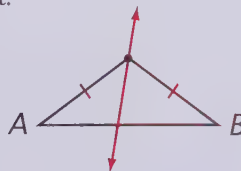
15.



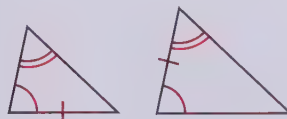
Vertical angles . . .

Ollie's Mistakes. In their big review for the geometry test, Obtuse Ollie misstated four theorems. Acute Alice drew a figure for each one to show him that something was wrong. Rewrite each theorem so that it is stated correctly.

16. "In a plane, a point equidistant from the endpoints of a line segment determines the perpendicular bisector of the line segment."



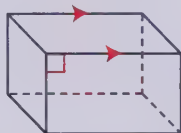
17. "If two angles and a side of one triangle are equal to two angles and a side of another triangle, the triangles are congruent."



18. "If two sides of one triangle are equal to two sides of another triangle, the third sides are equal."

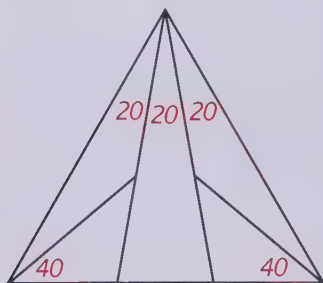


19. "A line perpendicular to one of two parallel lines is also perpendicular to the other."

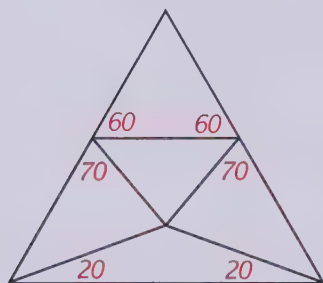


Triangle Division. Each of the equilateral triangles below is divided into five triangular pieces.* Make a large copy of each figure and write in the measures of the other angles.

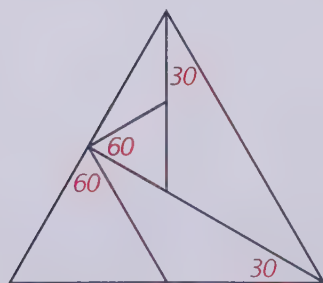
20.



21.



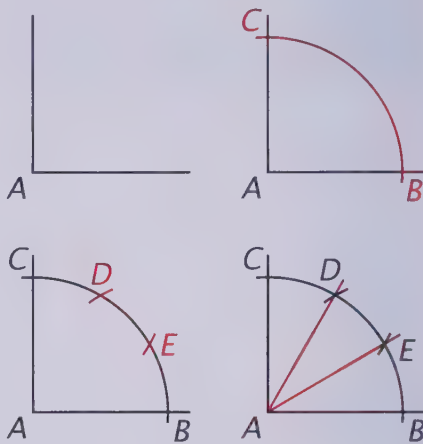
22.



* *Which Way Did the Bicycle Go?* by Joseph D. E. Konhauser, Dan Yelleman, and Stan Wagon (Mathematical Association of America, 1996).

23. What do all of the triangular pieces in all three figures have in common?
24. How many pieces in each figure are equilateral triangles?

Angle Trisection. Although it is impossible to trisect most angles with only a straightedge and compass, it is possible to trisect a right angle.



25. Use your ruler and protractor to draw right $\angle A$ as shown in the first figure above.

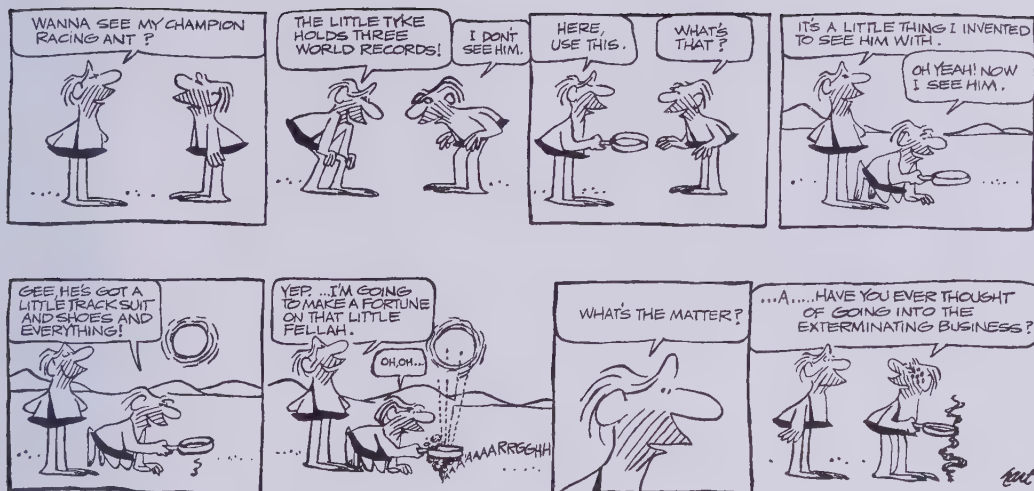
With A as center draw an arc intersecting the sides of the angle at B and C as shown in the second figure.

With B as center, draw an arc with the same radius intersecting arc BC at D as shown in the third figure. With C as center, draw an arc with the same radius intersecting arc BC at E.

Draw AD and AE as shown in the fourth figure.

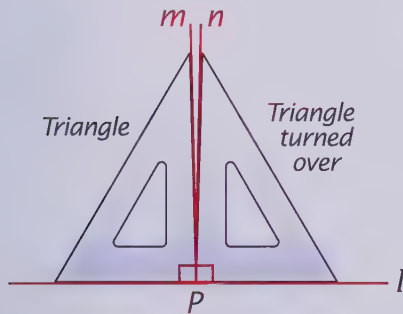
Finally, use your protractor to check the accuracy of your construction.

26. Draw DB and EC. What kind of triangles are $\triangle DAB$ and $\triangle EAC$?
27. What can you conclude about $\angle DAB$ and $\angle EAC$? Why?
28. What can you conclude about $\angle EAB$? Why?



Used by permission of Johnny Hart and Creators Syndicate, Inc.

Surveyor's Triangle. Surveyors use a right triangle in drawing perpendicular lines. If the triangle is defective, it can seem to produce two lines perpendicular to line l through point P as shown in the figure below.*



29. In a plane, through a given point, how many lines can be drawn perpendicular to a given line?
30. State the theorem that would be contradicted if lines m and n are both perpendicular to l .
31. What relation would the line that is actually perpendicular to line l at P have to the angle shown in red?

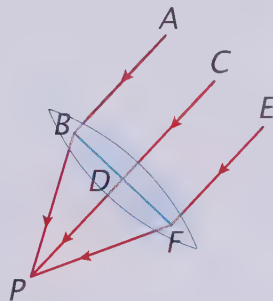
**Elementary Surveying*, by Paul R. Wolf and Russell C. Brinker (Harper Collins, 1989).

32. Draw line l , mark point P on it, and use your straightedge and compass to construct the line through P that is perpendicular to l .

Set II

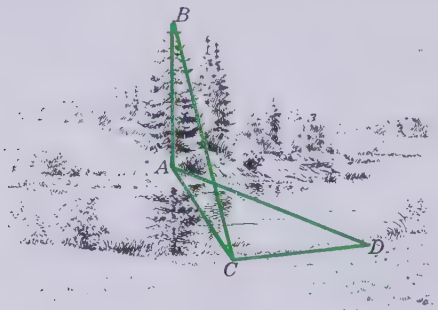
Magnifying Glass.

A magnifying glass can be used to focus the light rays of the sun on a small spot. In this figure, $AB \parallel CP \parallel EF$, and CP is the perpendicular bisector of BF .



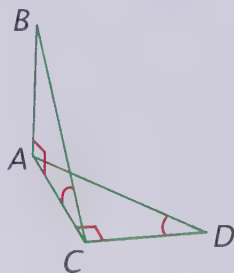
33. If all of the lines lie in the same plane, why must it be that $AB \perp BF$ and $EF \perp BF$?
34. Why does it follow that $\angle ABF = \angle EFB$?
35. What word describes lines such as BP , CP , and FP that meet in a single point?
36. Why is $\triangle BDP \cong \triangle FDP$?
37. Why does it follow that $\angle PBF = \angle PFB$?
38. What conclusion follows about $\angle ABP$ and $\angle EFP$?

Measuring a Tree. The figure below appeared in an old geometry book.*



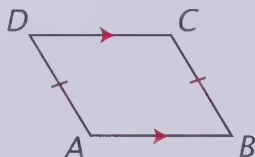
It illustrated a method for finding the height of a tree on the other side of a river. The directions said:

Find the point C directly opposite the tree where the angle of elevation ACB of the tree is 45° . Lay off from C a line CD perpendicular to AC and of such length that the angle ADC = 45° .



39. If this is done, what can you conclude about $\angle B$ and $\angle CAD$?
40. What can you conclude about BA and AC and about AC and CD?
41. Why?
42. Why would CD have to be equal to the height of the tree?

Quadrilateral Problem. In quadrilateral ABCD, $AD = BC$ and $AB \parallel DC$.



43. Name a pair of angles in the figure that must be supplementary.

**First Steps in Geometry*, by G. A. Wentworth and G. A. Hill (Ginn, 1901).

44. How do you know?

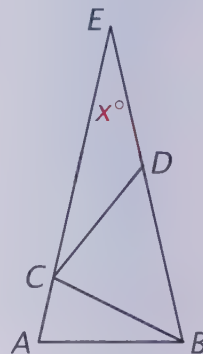
45. Do you know that any angles in the figure are equal? If so, which ones?

Isosceles Triangle. In isosceles $\triangle ABE$, $AB = BC = CD = DE$.

46. Draw the figure and mark it as needed to answer each of the following.

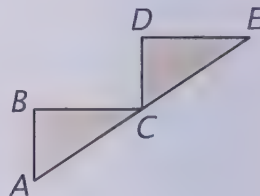
Find the measures of the following angles in terms of x .

47. $\angle ECD$.
48. $\angle CDB$.
49. $\angle DBC$.
50. $\angle ACB$.
51. $\angle A$.
52. $\angle ABE$.



53. Write an equation and solve for x to find the measure of $\angle E$.

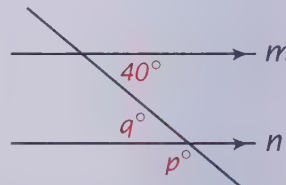
Stair Steps. In the figure at the right illustrating two stair steps, $\triangle ABC$ and $\triangle CDE$ are right triangles, $AB = CD$, and C is the midpoint of AE.



54. Copy the figure and mark it as needed to explain why $BC \parallel DE$.

SAT Problem.

The figure at the right appeared on an SAT exam.



Three angles formed by parallel lines m and n and the transversal have been marked.

55. The problem is to find $p - q$. Explain how this can be done.

Operations with Fractions

Addition and Subtraction

Recall that division is defined in terms of multiplication:

$$\frac{a}{b} = a \cdot \frac{1}{b}.$$

For fractions having the same denominator, it follows that

$$\frac{a}{b} + \frac{c}{b} = a \cdot \frac{1}{b} + c \cdot \frac{1}{b} = (a + c) \cdot \frac{1}{b} = \frac{a + c}{b}$$

and that

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}.$$

Before fractions having different denominators can be added or subtracted, they must be expressed as fractions having a common denominator.

In general,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

and

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}.$$

Example 1: $\frac{x}{3} + \frac{2}{y}$

Solution: $\frac{xy}{3y} + \frac{6}{3y} = \frac{xy + 6}{3y}$

Example 2: $\frac{x}{x-1} - \frac{x}{x+1}$

Solution:
$$\frac{x(x+1)}{(x-1)(x+1)} - \frac{x(x-1)}{(x+1)(x-1)} =$$

$$\frac{x^2 + x - x^2 + x}{(x+1)(x-1)} = \frac{2x}{x^2 - 1}$$

I Diese figur ist vii bedeyt ain fiertel von ainez
 III gangen/also mag man auch ain fünfftail/ain
 sechstail/ain sybentail oder zwat sechstail 2c. vnd alle
 ander bruch beschreiben/Als $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ | $\frac{1}{6}$ 2c.
 VI Diß sein Sechs achtail/das sein sechstail der
 VIII acht ain gangz machen.
 IX Diß figur bezaigt ain newen ayllfftail das seyn
 XI IX tail/der XI. ain gangz machen.
 XX Diß figur bezaiget/zwenzigt ainundreys
 XXXI figt tail /das sein zwenzigt tail .der aines
 undreißigt ain gangz machen.

This excerpt from a popular arithmetic book published in Germany in 1514 shows Roman numerals being used to write fractions. The

first fraction is $\frac{1}{4}$. Do you recognize the others?

Multiplication

Multiplying fractions is easy. In general,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Division

The definition of division expresses division as multiplication by the reciprocal; so, in general,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example 3: $5x \cdot \frac{x}{4}$

Solution: $\frac{5x}{1} \cdot \frac{x}{4} = \frac{5x^2}{4}$

Example 4: $\frac{x}{x+6} \div \frac{x}{x+3}$

Solution: $\frac{x}{x+6} \cdot \frac{x+3}{x} = \frac{x(x+3)}{x(x+6)} = \frac{x+3}{x+6}$

Exercises

Do the indicated operations.
Express your answers in
lowest terms.

1. $\frac{1}{2} + \frac{2}{3}$

2. $\frac{x}{4} - \frac{x}{5}$

3. $\frac{x+5}{2x} + \frac{5}{2x}$

4. $\frac{4}{x^2} - \frac{3}{x^3}$

5. $\frac{3}{4} \cdot \frac{8}{9}$

6. $\frac{x^2}{2} \cdot \frac{x^4}{4}$

7. $\frac{1}{2} \div \frac{7}{10}$

8. $\frac{x+2}{x} \div \frac{2}{x}$

9. $\frac{5x}{16} + \frac{7x}{16}$

10. $\frac{x+4}{10} - \frac{x+2}{10}$

11. $\frac{x+y}{12} \cdot \frac{2}{x+y}$

12. $\frac{10}{x} \div 5$

13. $\frac{x}{2} + \frac{y}{4} - \frac{z}{8}$

14. $\frac{\pi r}{2} \cdot \frac{8r^2}{3}$

15. $\frac{x}{y} - 1$

16. $\frac{6}{2x-3} \cdot \frac{x-3}{3}$

17. $\frac{3x}{x^2-1} - \frac{2}{x+1}$

18. $\frac{y}{xy-y^2} + \frac{x}{x-y}$

19. $\frac{x^2+8x+15}{x^2+x-6} \div \frac{x+5}{x-2}$

20. $x^2 + x + 1 + \frac{1}{x-1}$