

THE MANGA GUIDE™ TO

COMICS
INSIDE!

LINEAR ALGEBRA

SHIN TAKAHASHI
IROHA INOUE
TREND-PRO CO., LTD.



PRAISE FOR THE MANGA GUIDE SERIES

"Highly recommended."

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"Stimulus for the next generation of scientists."

—SCIENTIFIC COMPUTING ON *THE MANGA GUIDE TO MOLECULAR BIOLOGY*

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—THE SAN FRANCISCO EXAMINER

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—HACK A DAY ON *THE MANGA GUIDE TO ELECTRICITY*



"*The Manga Guide to Databases* was the most enjoyable tech book I've ever read."

—RIKKI KITE, LINUX PRO MAGAZINE

"The *Manga Guides* definitely have a place on my bookshelf."

—SMITHSONIAN'S "SURPRISING SCIENCE"

"For parents trying to give their kids an edge or just for kids with a curiosity about their electronics, *The Manga Guide to Electricity* should definitely be on their bookshelves."

—SACRAMENTO BOOK REVIEW

"This is a solid book and I wish there were more like it in the IT world."

—SLASHDOT ON *THE MANGA GUIDE TO DATABASES*

"*The Manga Guide to Electricity* makes accessible a very intimidating subject, letting the reader have fun while still delivering the goods."

—GEEKDAD BLOG, WIRED.COM

"If you want to introduce a subject that kids wouldn't normally be very interested in, give it an amusing storyline and wrap it in cartoons."

—MAKE ON *THE MANGA GUIDE TO STATISTICS*

"A clever blend that makes relativity easier to think about—even if you're no Einstein."

—STARDATE, UNIVERSITY OF TEXAS, ON *THE MANGA GUIDE TO RELATIVITY*

"This book does exactly what it is supposed to: offer a fun, interesting way to learn calculus concepts that would otherwise be extremely bland to memorize."

—DAILY TECH ON *THE MANGA GUIDE TO CALCULUS*

"The art is fantastic, and the teaching method is both fun and educational."

—ACTIVE ANIME ON *THE MANGA GUIDE TO PHYSICS*

"An awfully fun, highly educational read."

—FRAZZLEDDAD ON *THE MANGA GUIDE TO PHYSICS*

"Makes it possible for a 10-year-old to develop a decent working knowledge of a subject that sends most college students running for the hills."

—SKEPTICBLOG ON *THE MANGA GUIDE TO MOLECULAR BIOLOGY*

"This book is by far the best book I have read on the subject. I think this book absolutely rocks and recommend it to anyone working with or just interested in databases."

—GEEK AT LARGE ON *THE MANGA GUIDE TO DATABASES*

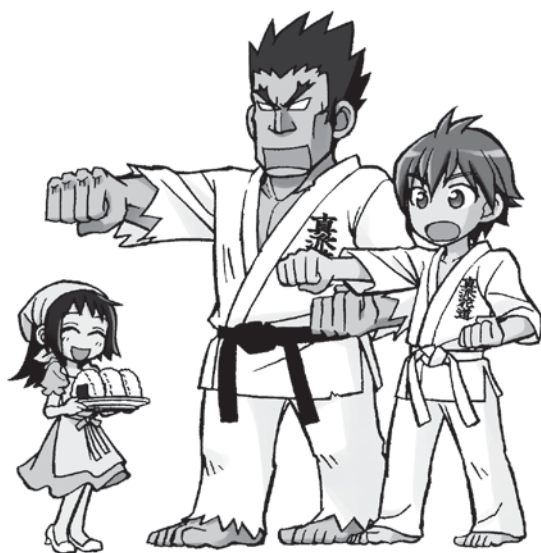
"The book purposefully departs from a traditional physics textbook and it does it very well."

—DR. MARINA MILNER-BOLOTIN, RYERSON UNIVERSITY ON *THE MANGA GUIDE TO PHYSICS*

"Kids would be, I think, much more likely to actually pick this up and find out if they are interested in statistics as opposed to a regular textbook."

—GEEK BOOK ON *THE MANGA GUIDE TO STATISTICS*

THE MANGA GUIDE™ TO LINEAR ALGEBRA



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LINEAR ALGEBRA

SHIN TAKAHASHI,
IROHA INOUE, AND
TREND-PRO CO., LTD.



THE MANGA GUIDE TO LINEAR ALGEBRA.

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PREFACE

This book is for anyone who would like to get a good overview of linear algebra in a relatively short amount of time.

Those who will get the most out of *The Manga Guide to Linear Algebra* are:

- University students about to take linear algebra, or those who are already taking the course and need a helping hand
- Students who have taken linear algebra in the past but still don't really understand what it's all about
- High school students who are aiming to enter a technical university
- Anyone else with a sense of humor and an interest in mathematics!

The book contains the following parts:

Chapter 1: What Is Linear Algebra?

Chapter 2: The Fundamentals

Chapters 3 and 4: Matrices

Chapters 5 and 6: Vectors

Chapter 7: Linear Transformations

Chapter 8: Eigenvalues and Eigenvectors

Most chapters are made up of a manga section and a text section. While skipping the text parts and reading only the manga will give you a quick overview of each subject, I recommend that you read both parts and then review each subject in more detail for maximal effect. This book is meant as a complement to other, more comprehensive literature, not as a substitute.

I would like to thank my publisher, Ohmsha, for giving me the opportunity to write this book, as well as Iroha Inoue, the book's illustrator. I would also like to express my gratitude towards re_akino, who created the scenario, and everyone at Trend Pro who made it possible for me to convert my manuscript into this manga. I also received plenty of good advice from Kazuyuki Hiraoka and Shizuka Hori. I thank you all.

SHIN TAKAHASHI
NOVEMBER 2008

PROLOGUE

LET THE TRAINING BEGIN!





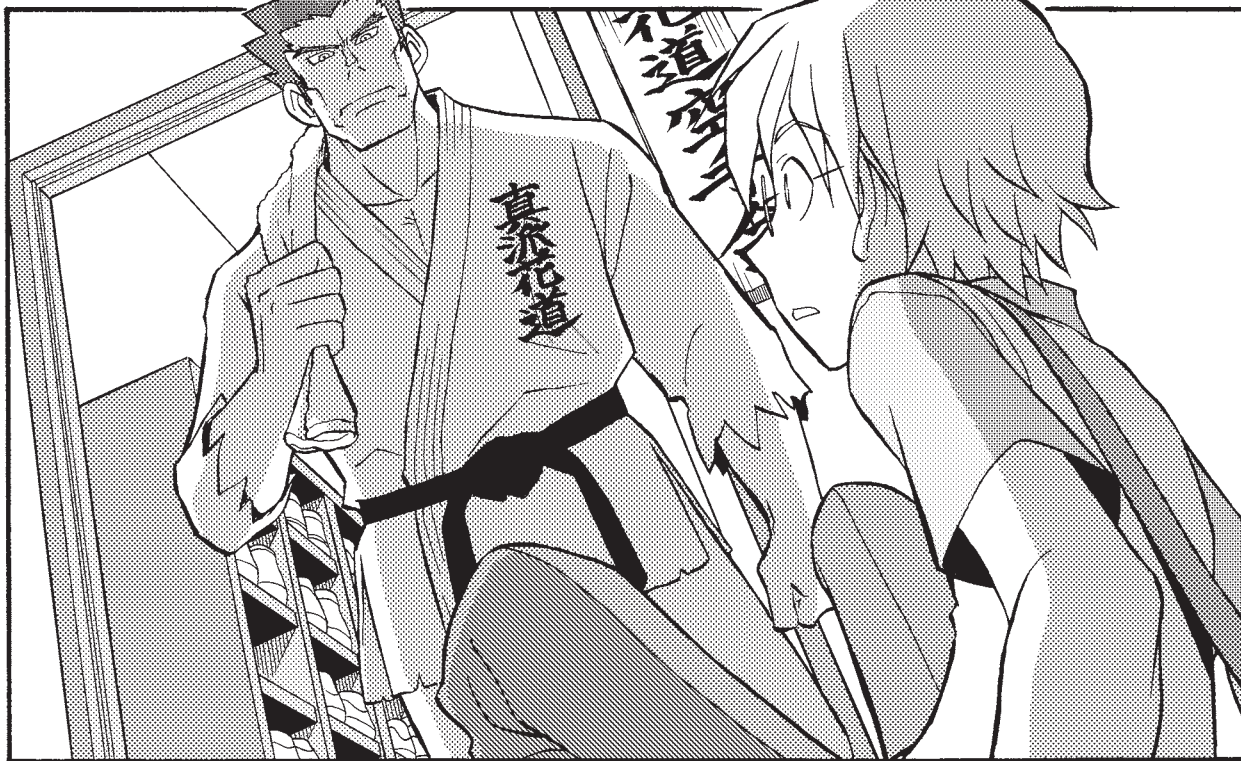
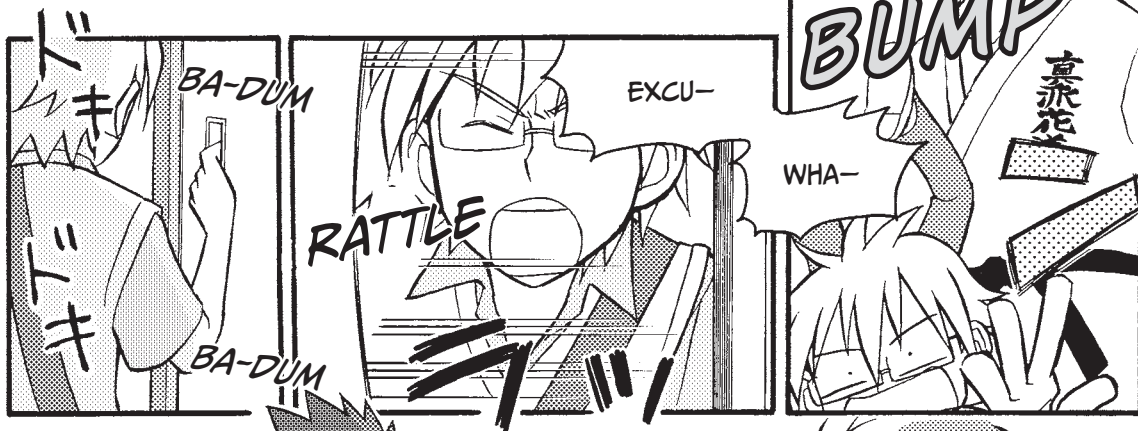
NOTHING TO BE
AFRAID OF...

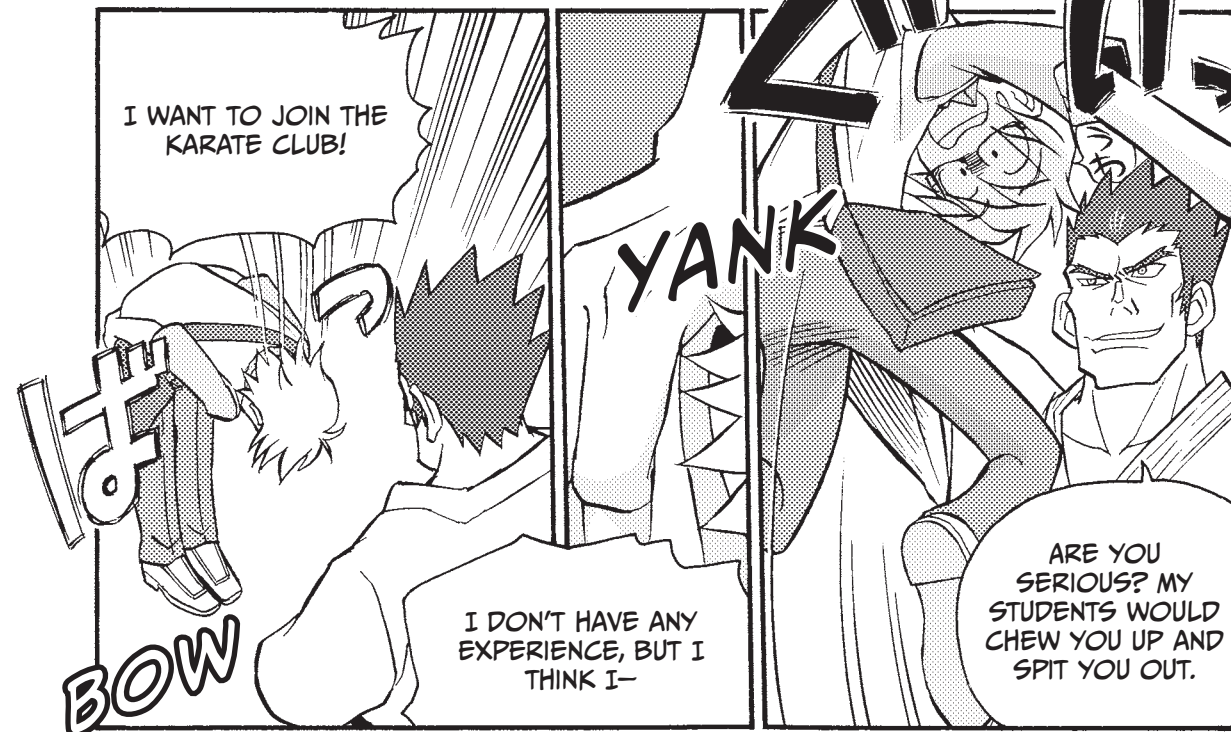
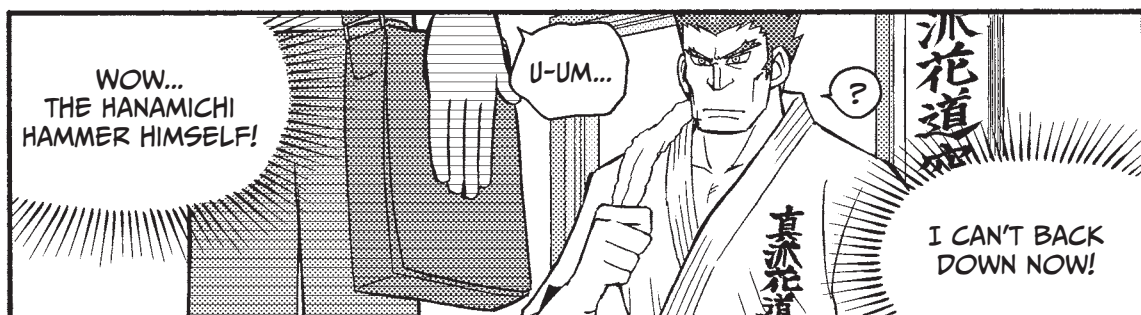
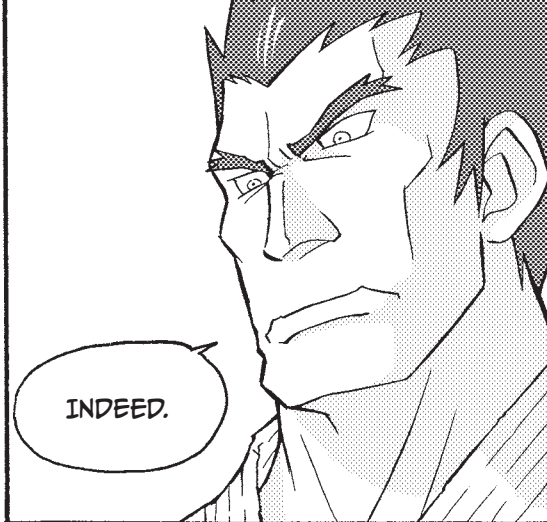
OKAY!

真流花道空手会*

IT'S NOW OR NEVER!

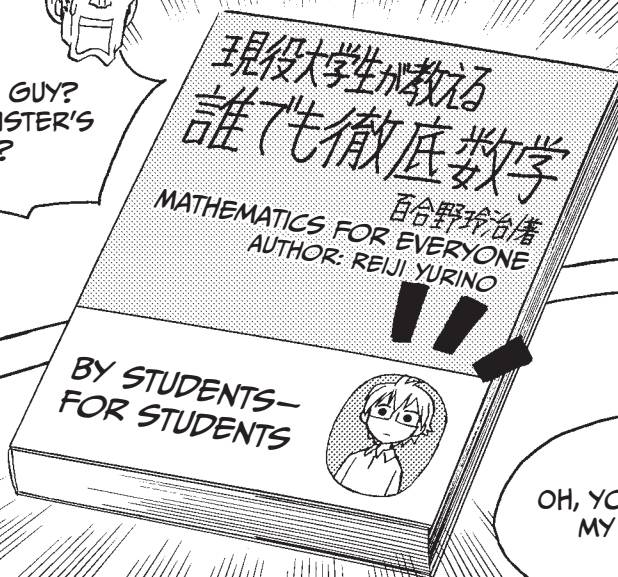
* HANAMICHI KARATE CLUB







AREN'T YOU THAT GUY?
THE ONE ON MY SISTER'S
MATH BOOK?



OH, YOU'VE SEEN
MY BOOK?

SO IT IS YOU!

I MAY NOT BE
THE STRONGEST
GUY...

BUT I'VE ALWAYS
BEEN A WHIZ
WITH NUMBERS.

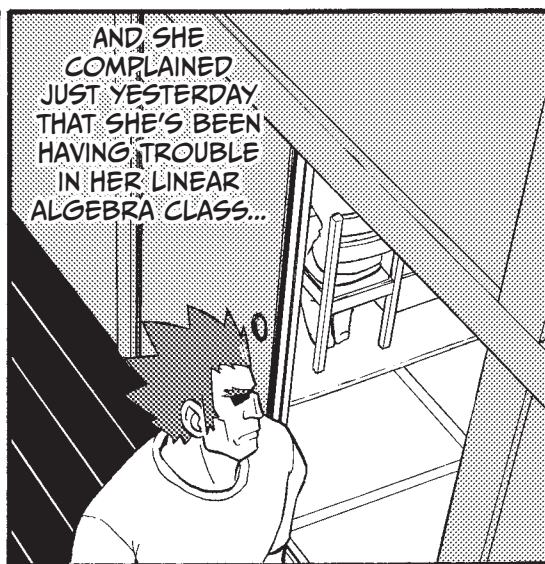
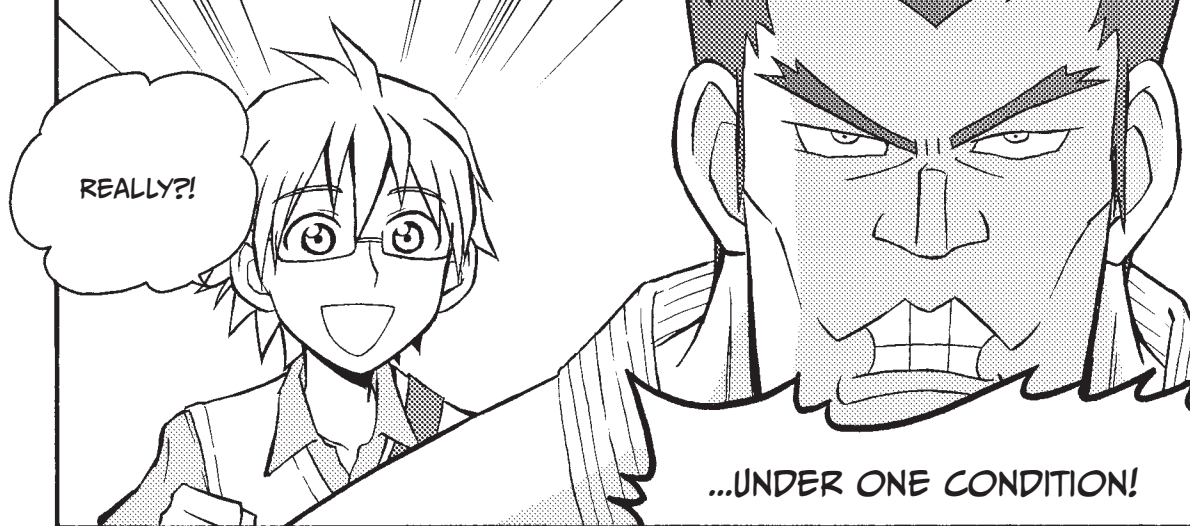
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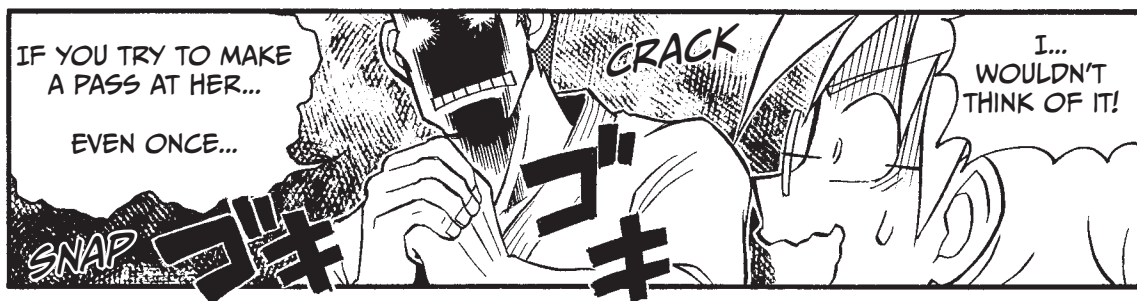
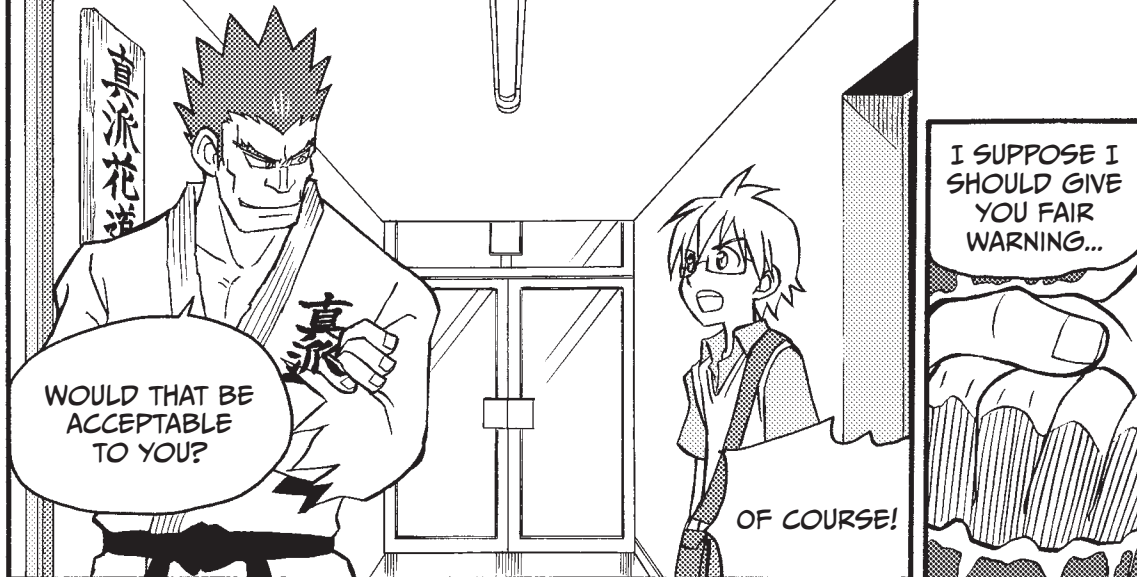
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I MIGHT
CONSIDER
LETTING YOU
INTO THE CLUB...

WHA-?

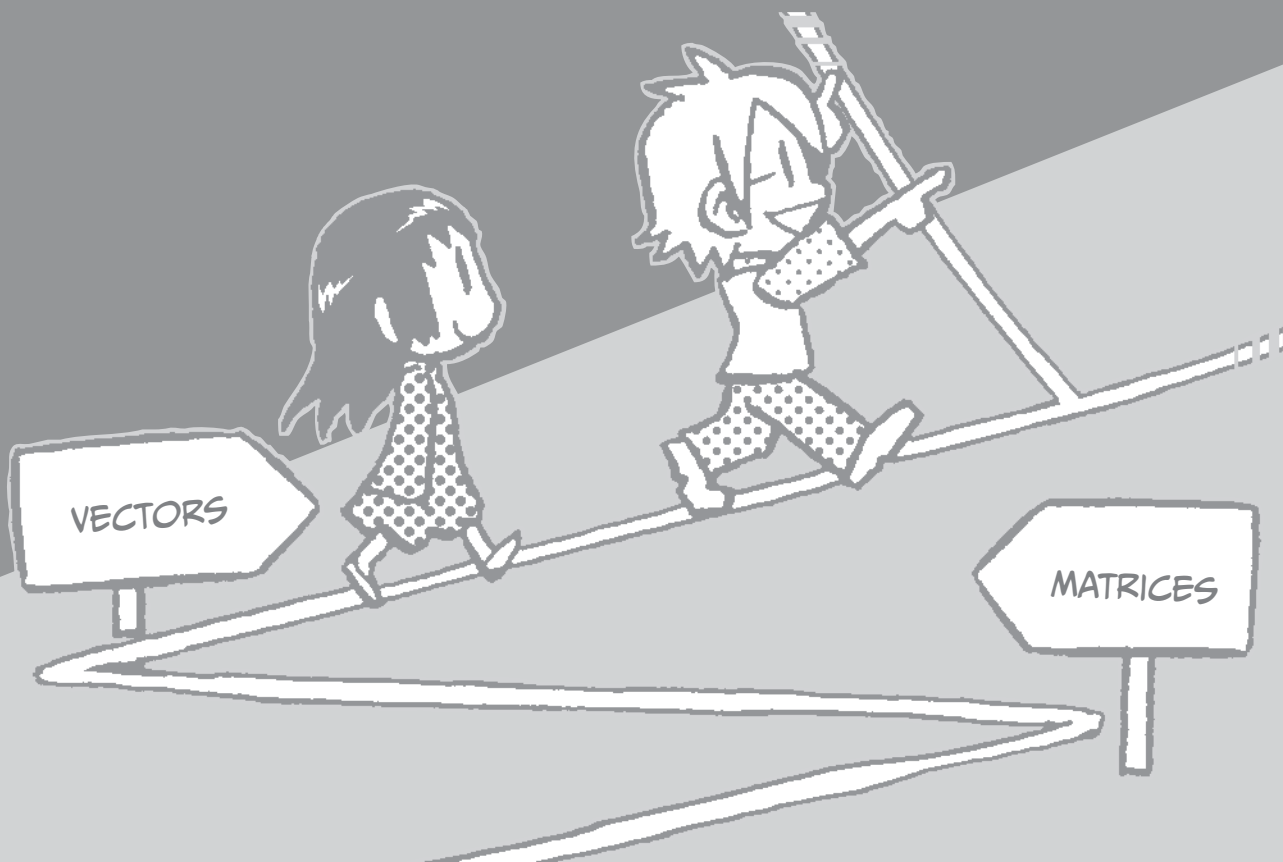
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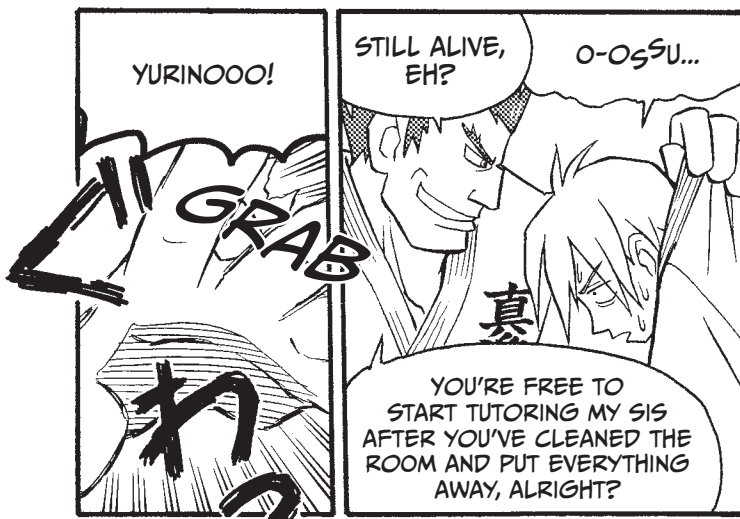
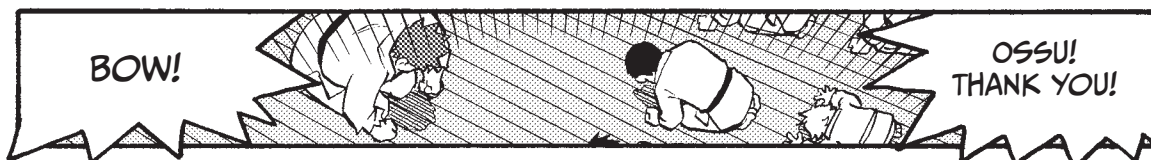
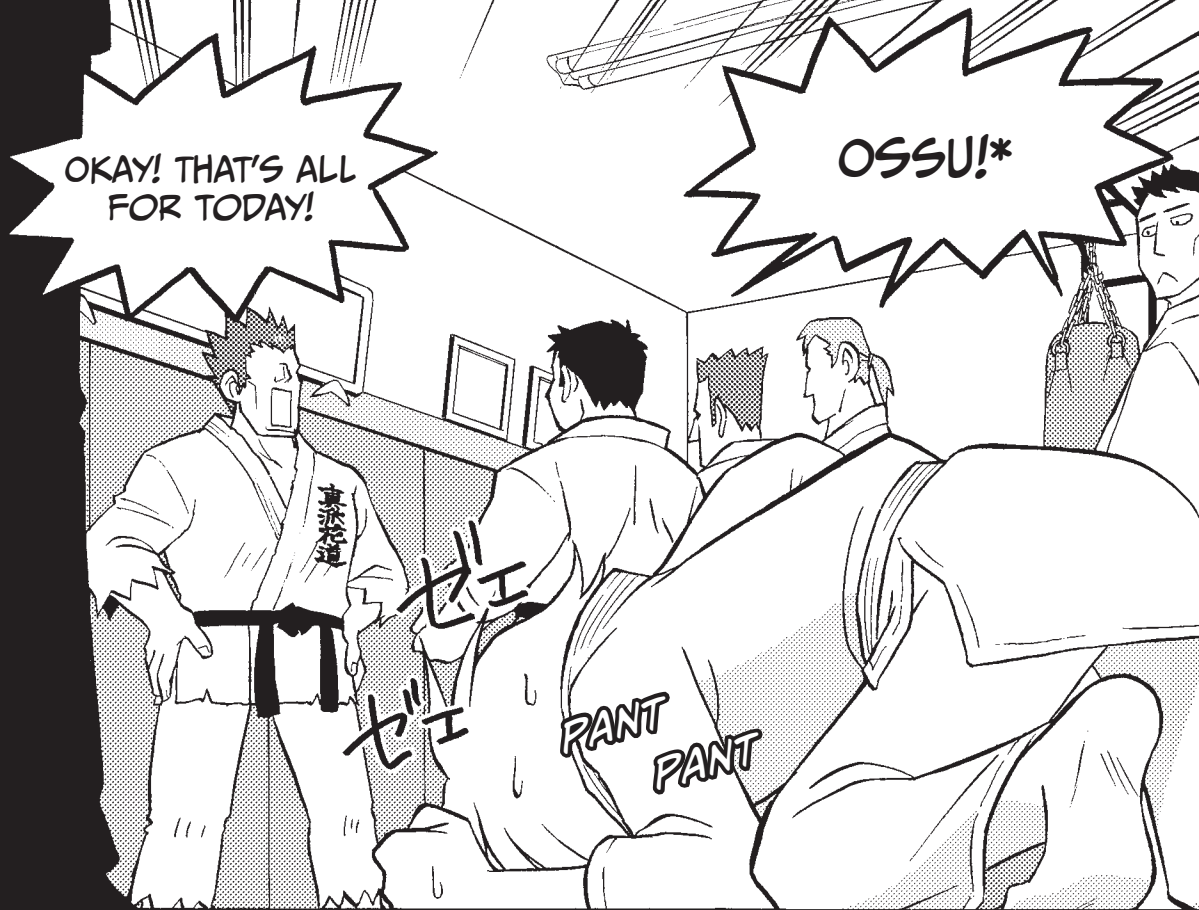




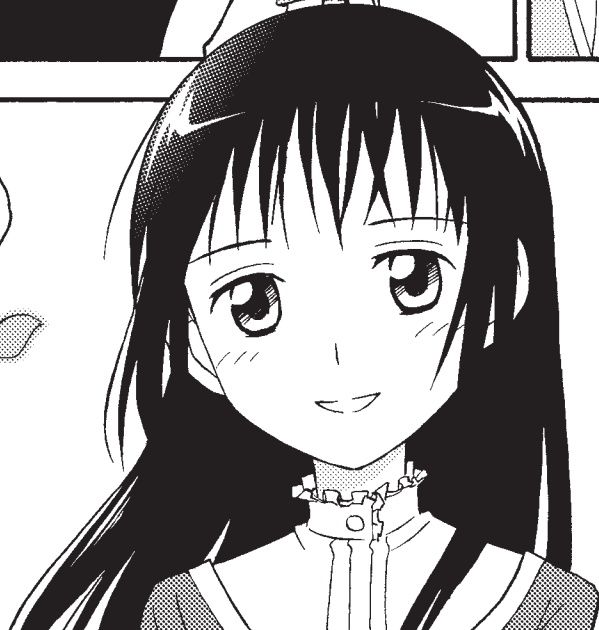
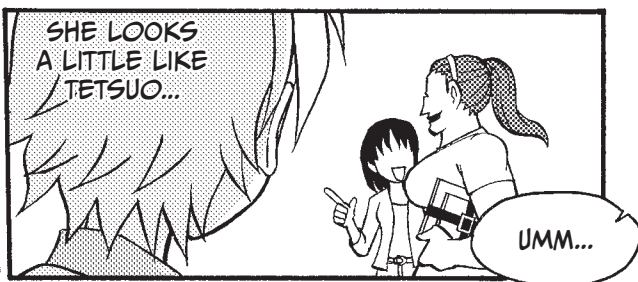
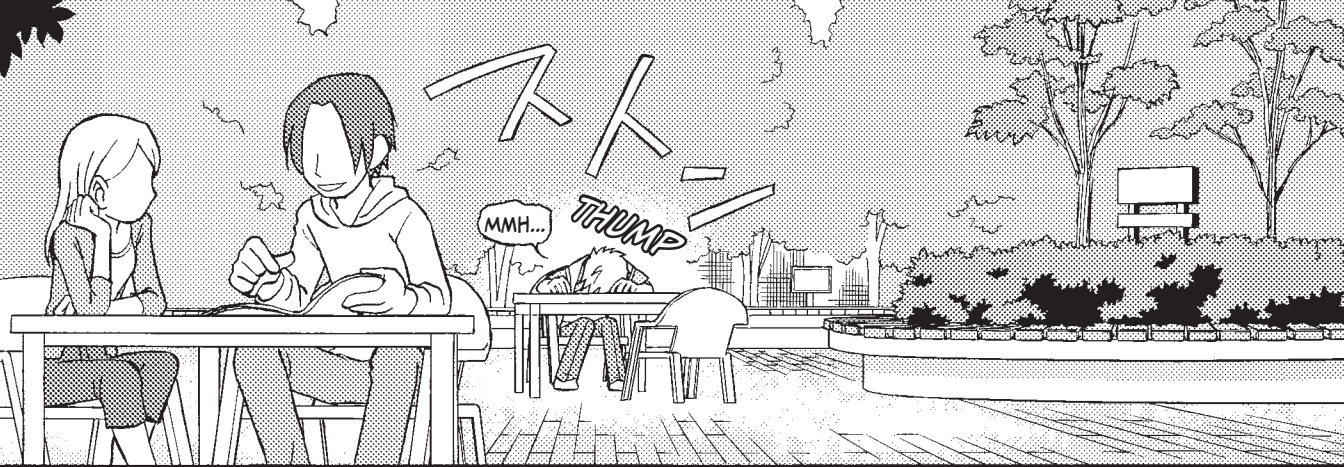
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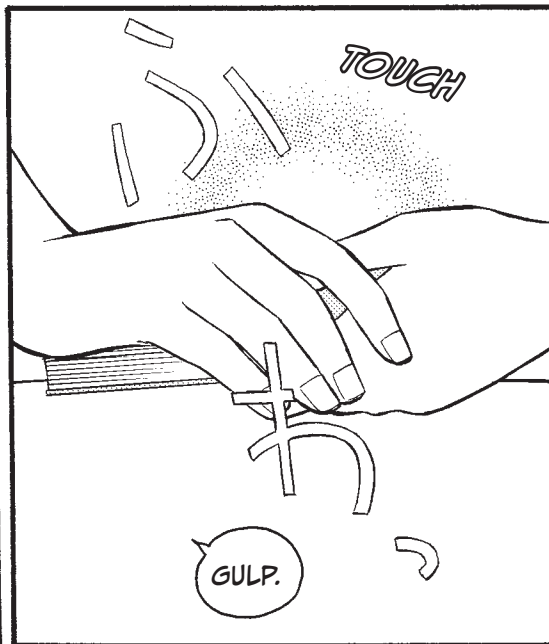
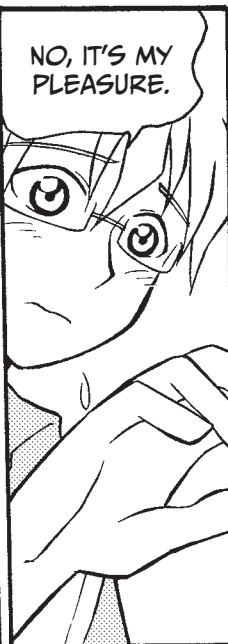
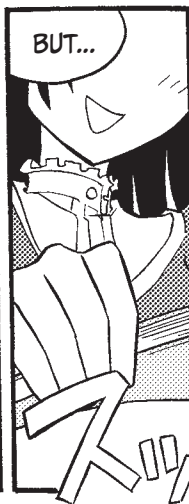
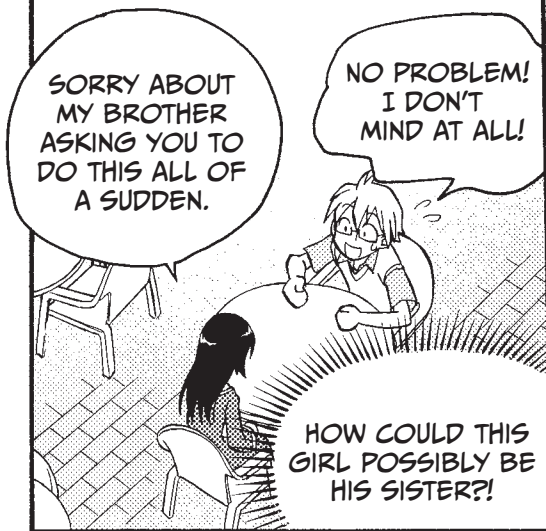
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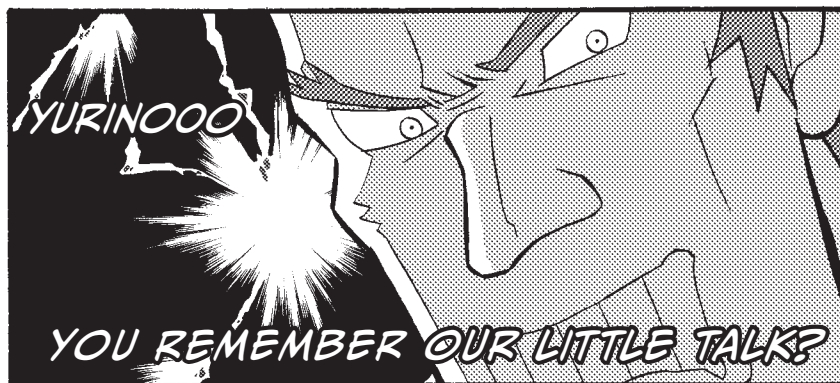
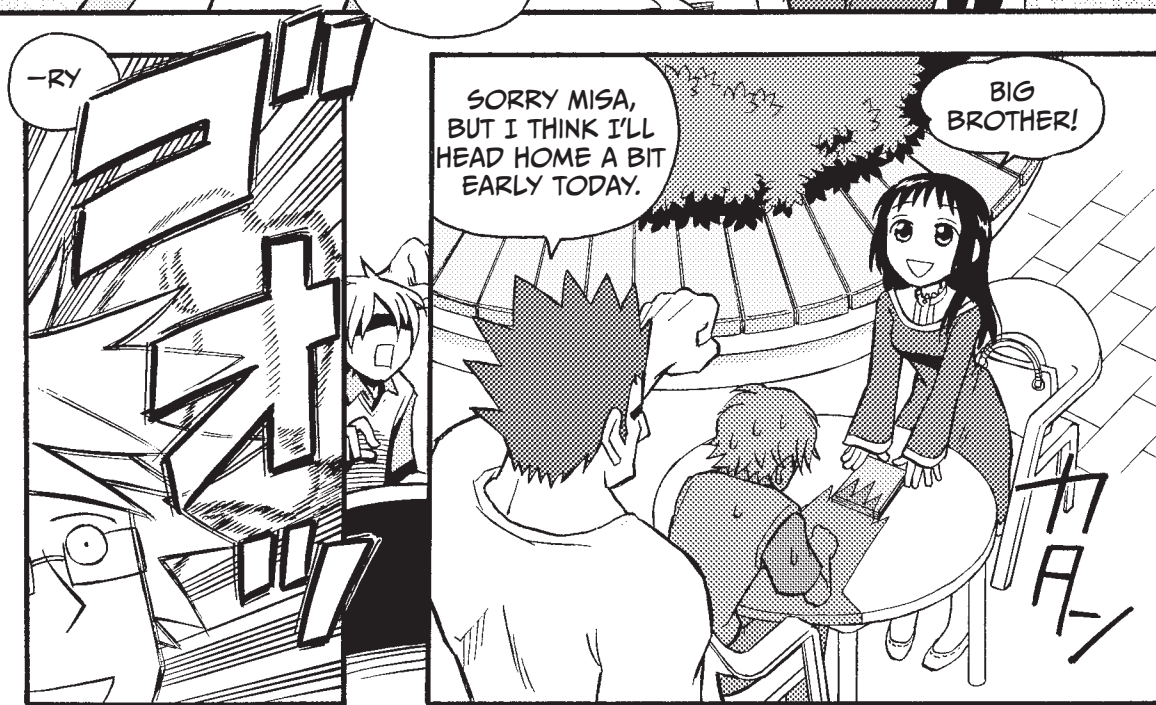
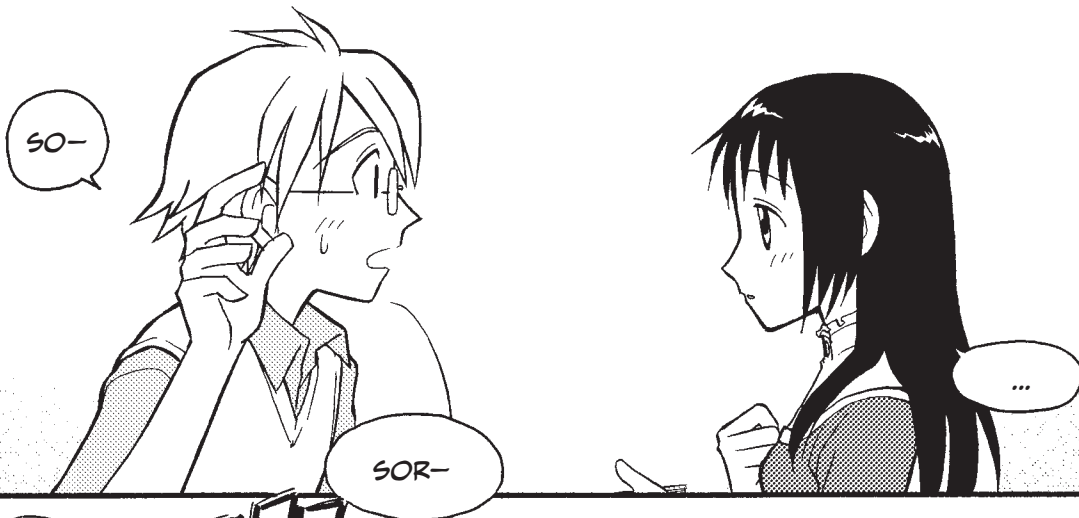




* OSSU IS AN INTERJECTION OFTEN USED IN JAPANESE MARTIAL ARTS TO ENHANCE CONCENTRATION AND INCREASE THE POWER OF ONE'S BLOWS.







AN OVERVIEW OF
LINEAR ALGEBRA

WELL THEN, WHEN
WOULD YOU LIKE
TO START?

HOW ABOUT
RIGHT NOW?

LET'S SEE...

YOUR BROTHER SAID THAT
YOU WERE HAVING
TROUBLE WITH LINEAR
ALGEBRA?

YES.

I DON'T REALLY
UNDERSTAND
THE CONCEPT
OF IT ALL...

AND THE
CALCULATIONS
SEEM WAY OVER
MY HEAD.

IT IS TRUE THAT
LINEAR ALGEBRA
IS A PRETTY
ABSTRACT
SUBJECT,

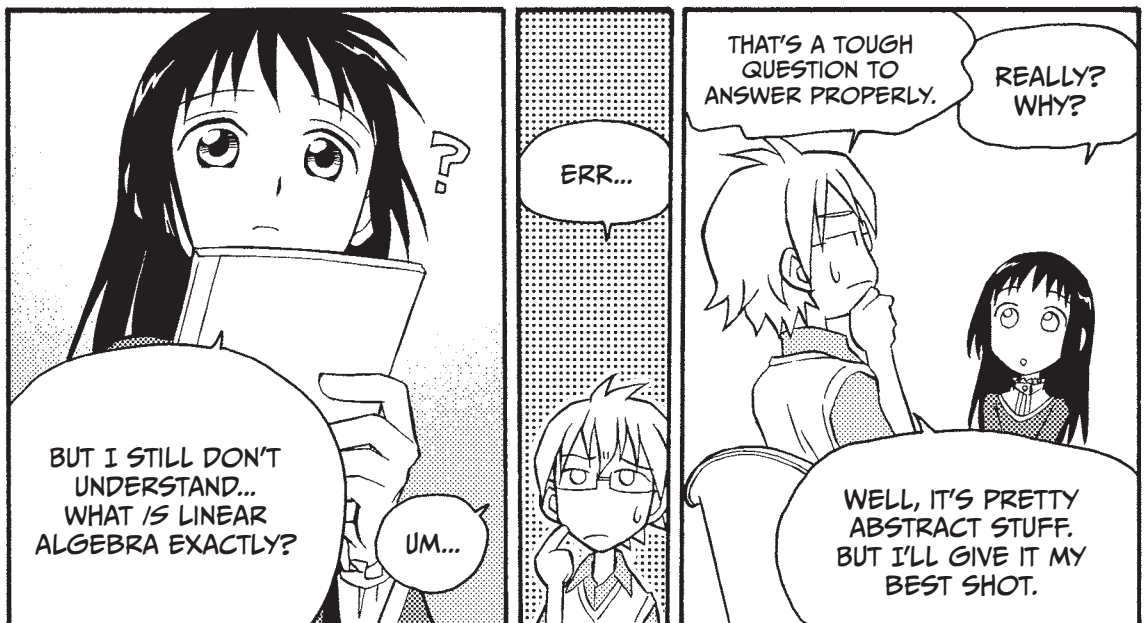
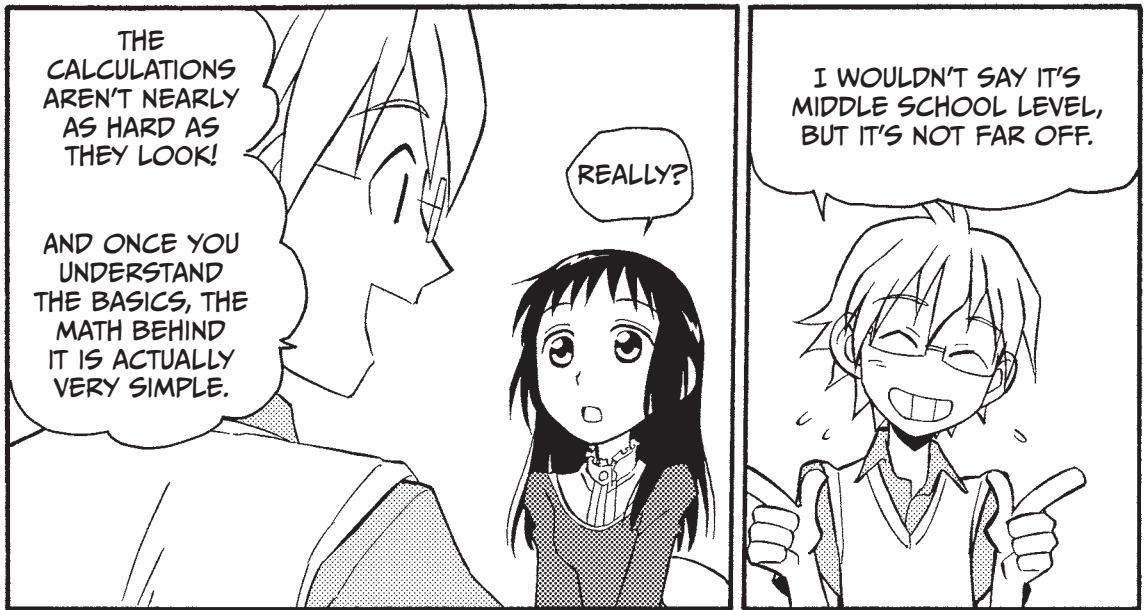
AND THERE ARE
SOME HARD-
TO-UNDERSTAND
CONCEPTS...

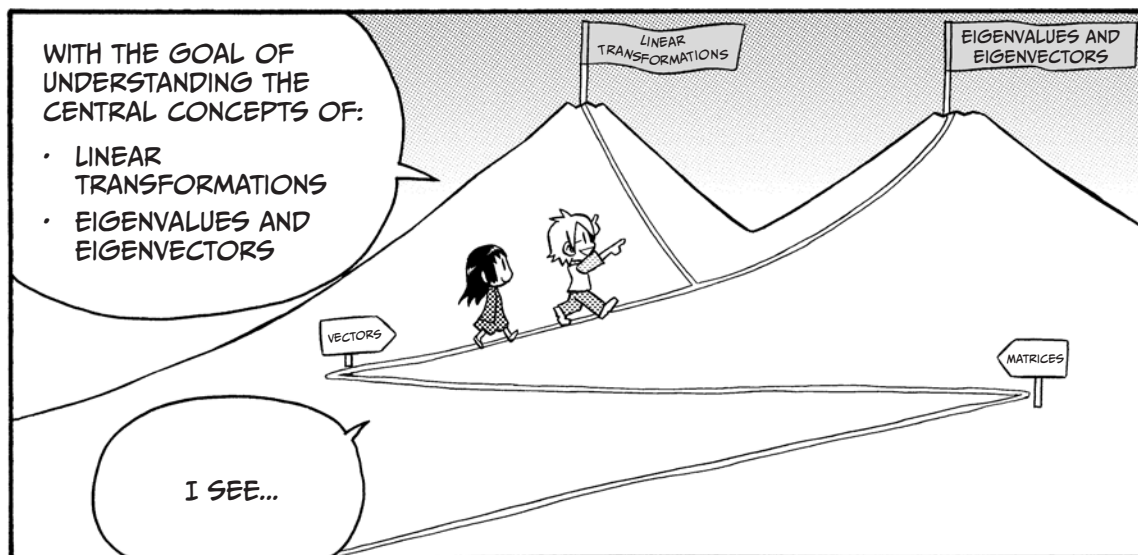
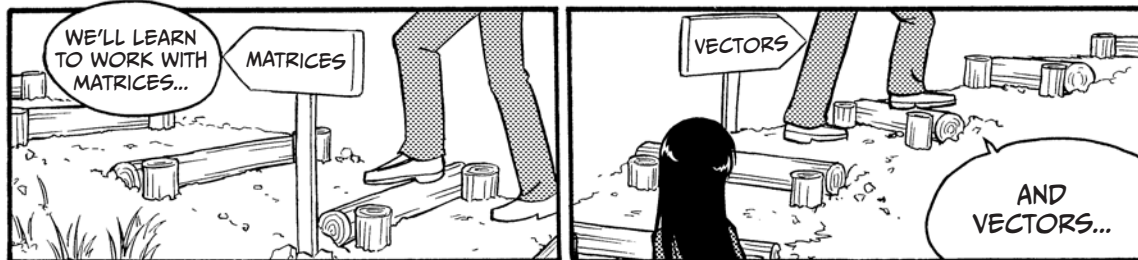
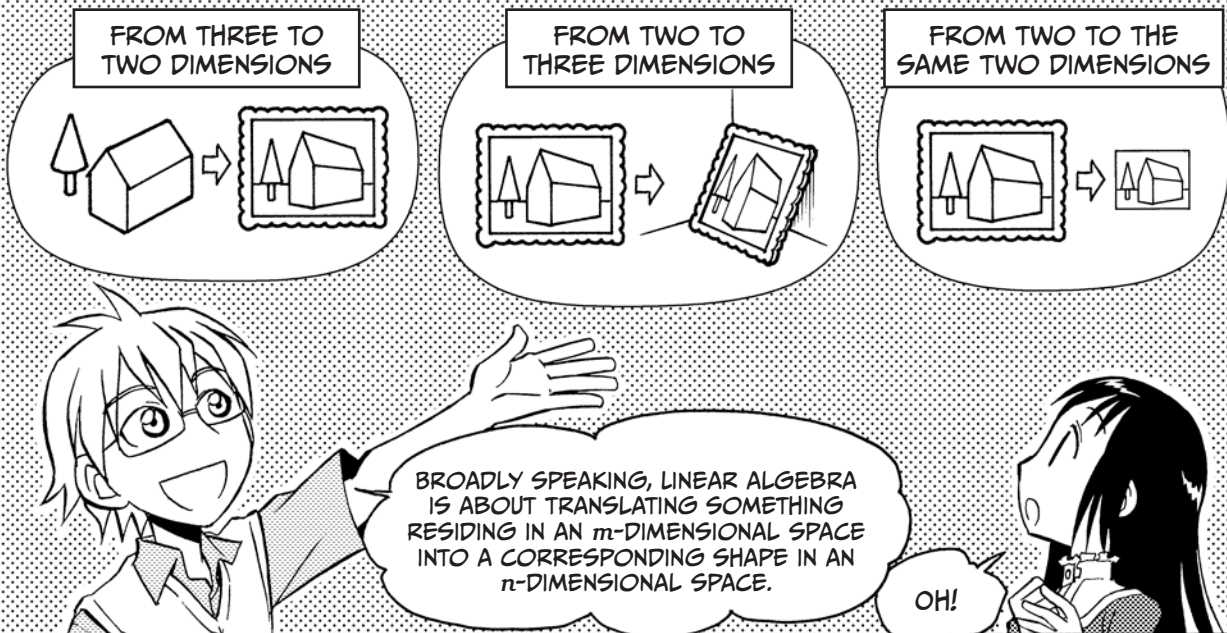
LINEAR
INDEPENDENCE

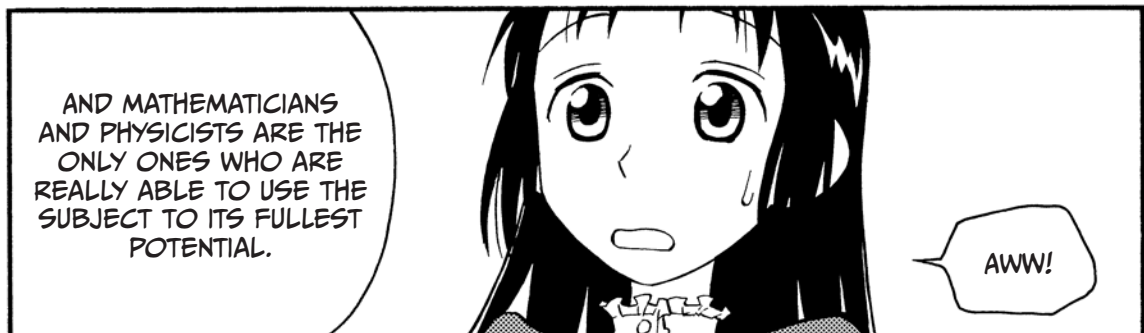
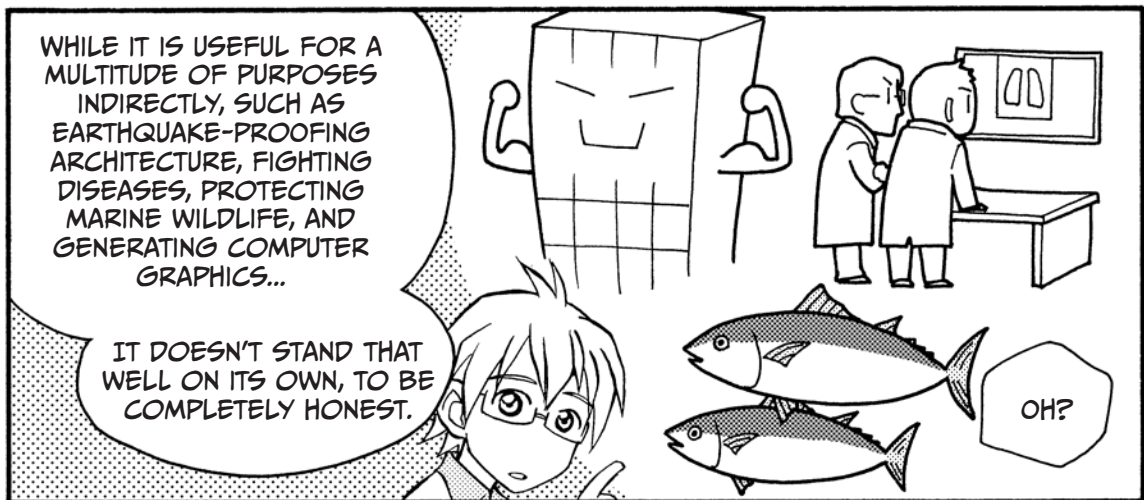
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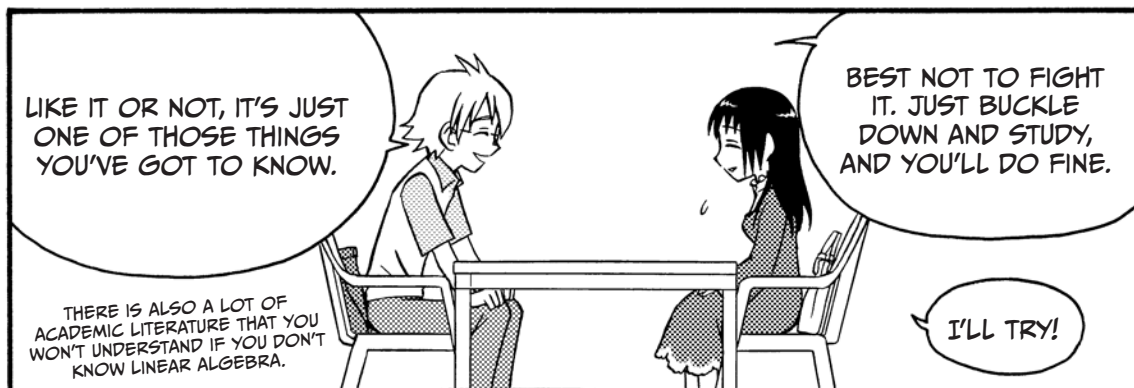
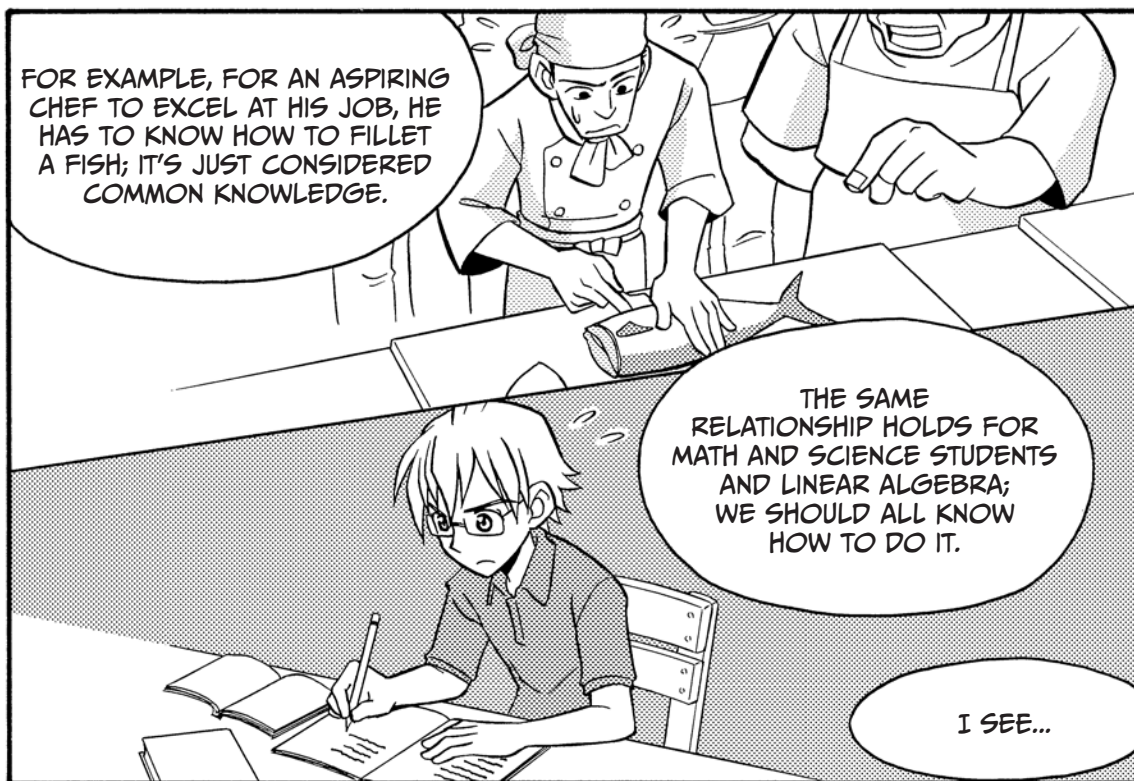
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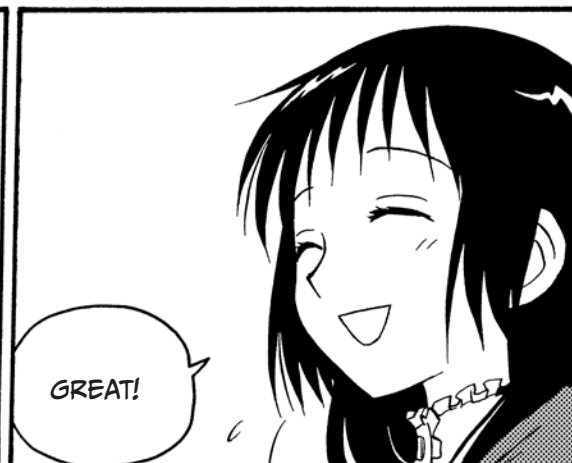
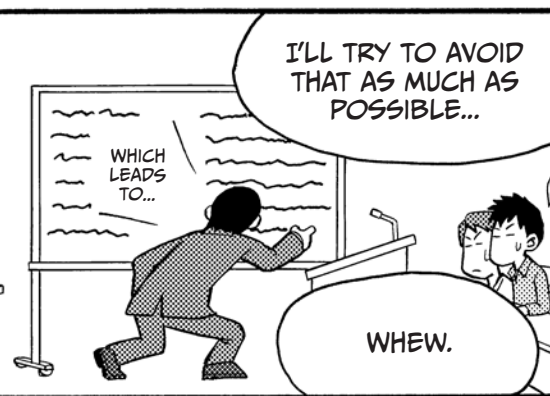
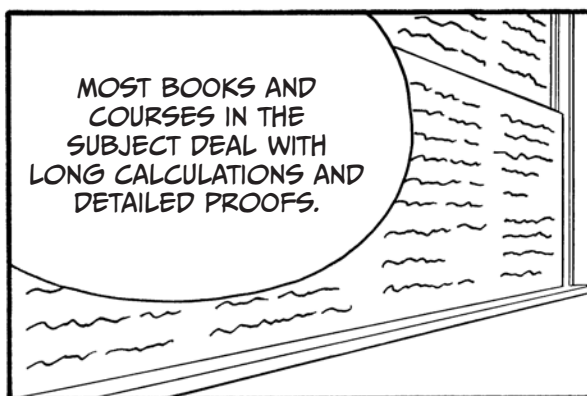
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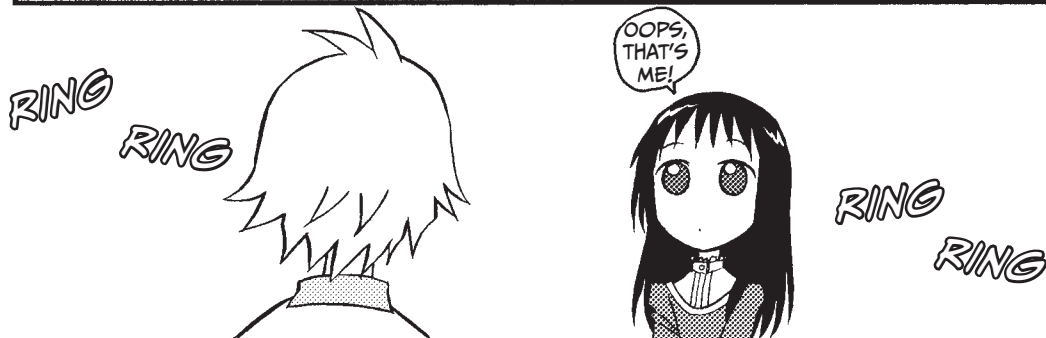
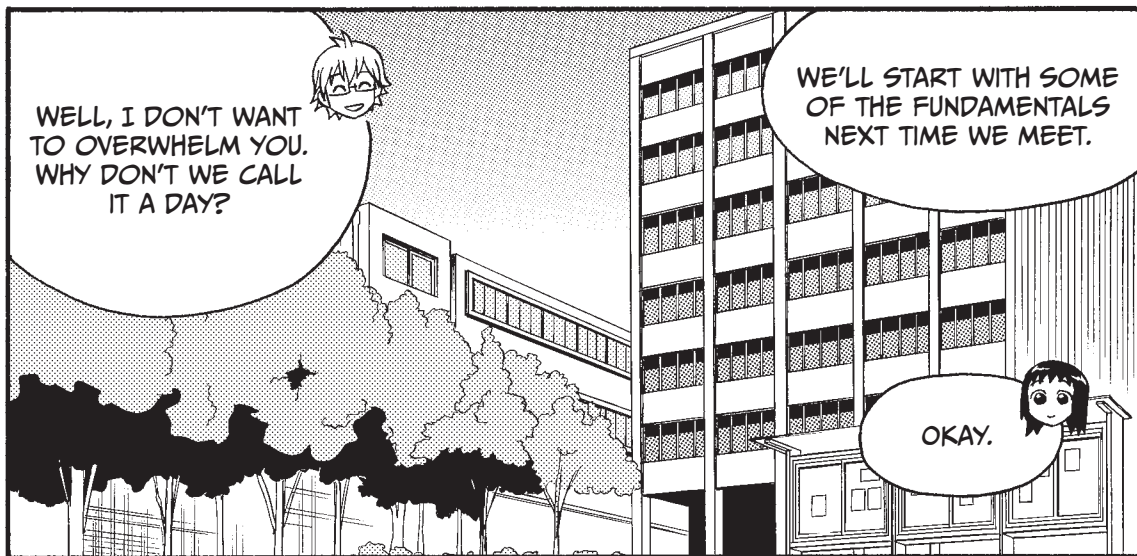








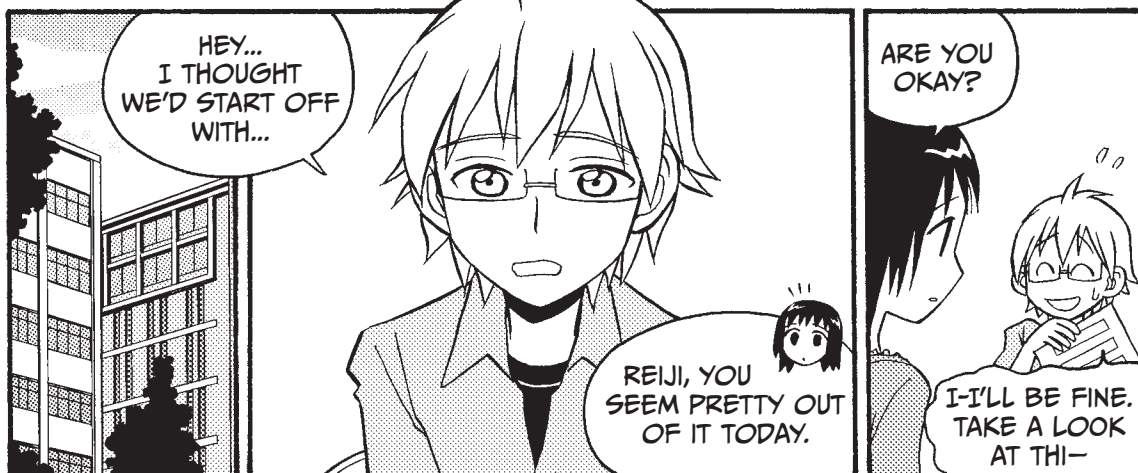


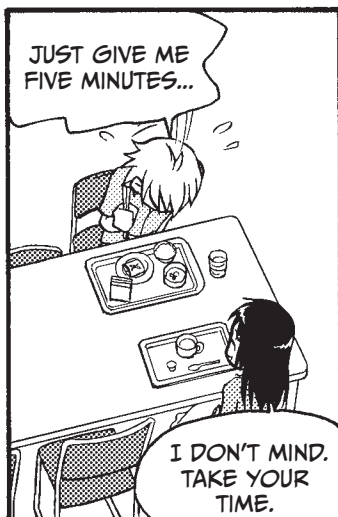
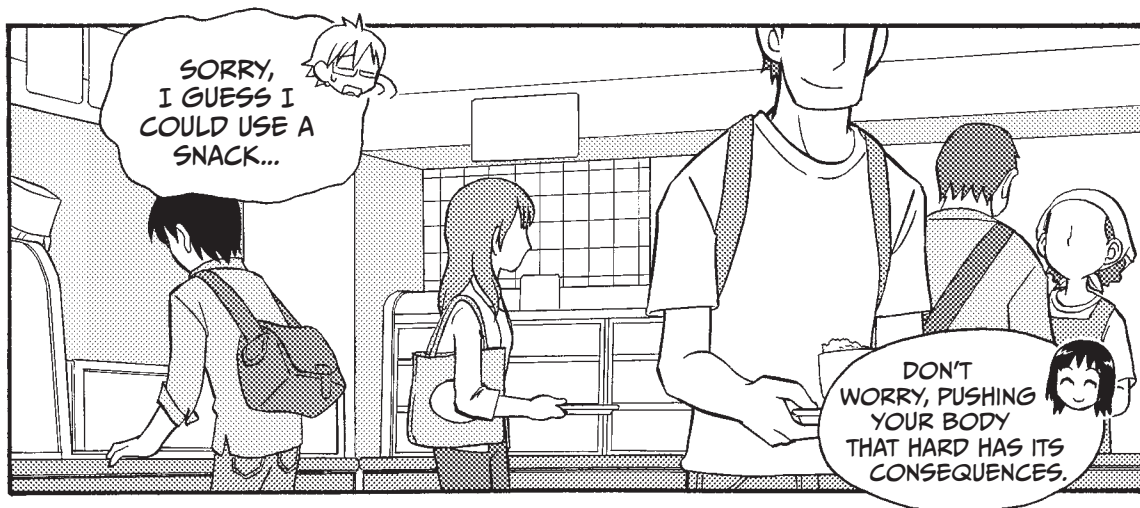
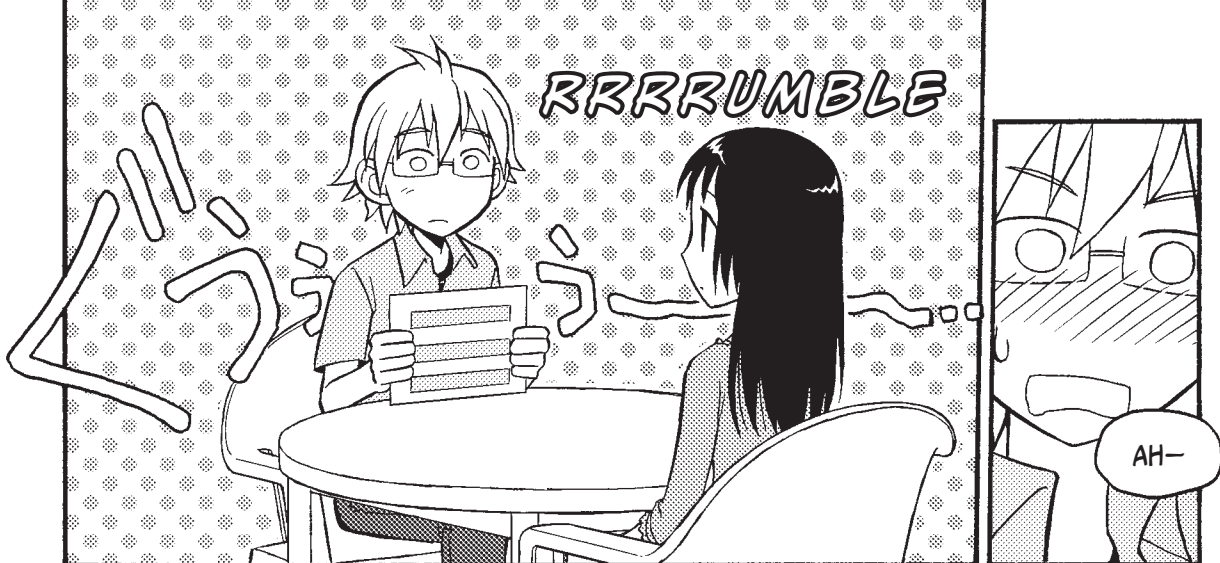


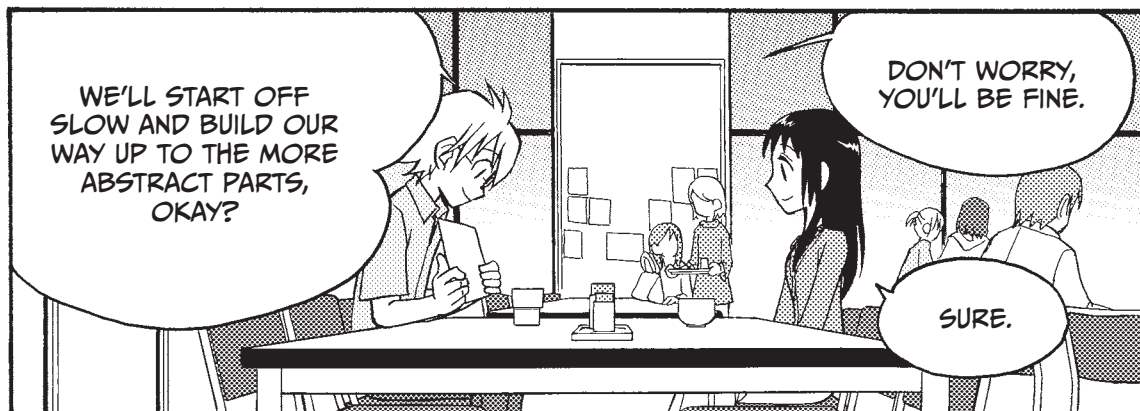
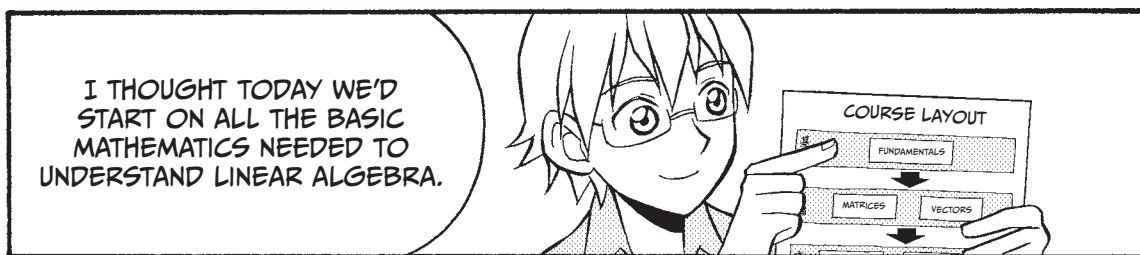
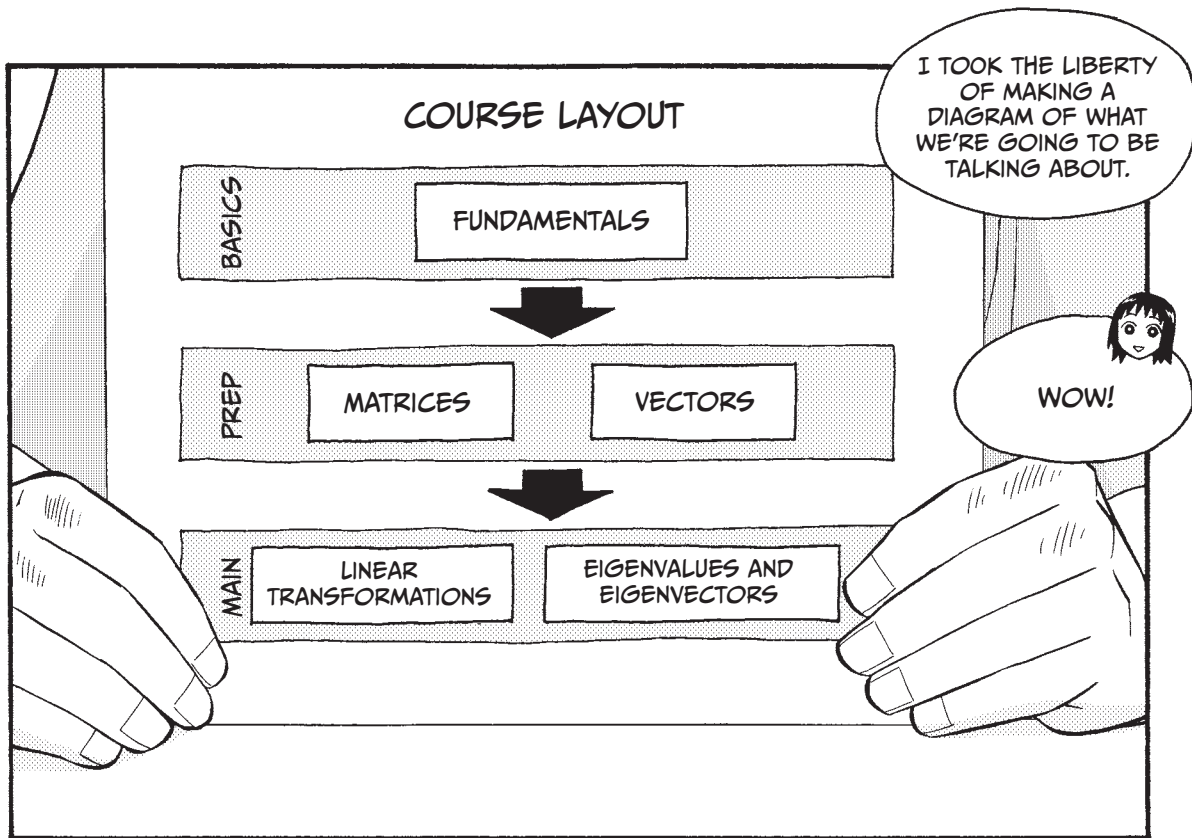
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THE FUNDAMENTALS









COMPLEX NUMBERS

Complex numbers are written in the form

$$a + b \cdot i$$

where a and b are real numbers and i is the *imaginary unit*, defined as $i = \sqrt{-1}$.

REAL NUMBERS

INTEGERS

- Positive natural numbers
- 0
- Negative natural numbers

RATIONAL NUMBERS* (NOT INTEGERS)

- Terminating decimal numbers like 0.3
- Non-terminating decimal numbers like 0.333...

IRRATIONAL NUMBERS

- Numbers like π and $\sqrt{2}$ whose decimals do not follow a pattern and repeat forever

IMAGINARY NUMBERS

- Complex numbers without a real component, like $0 + bi$, where b is a nonzero real number

* NUMBERS THAT CAN BE EXPRESSED IN THE FORM q/p (WHERE q AND p ARE INTEGERS, AND p IS NOT EQUAL TO ZERO) ARE KNOWN AS RATIONAL NUMBERS. INTEGERS ARE JUST SPECIAL CASES OF RATIONAL NUMBERS.



LET'S TALK
ABOUT NUMBER
SYSTEMS FIRST.

THEY'RE ORGANIZED
LIKE THIS.



...

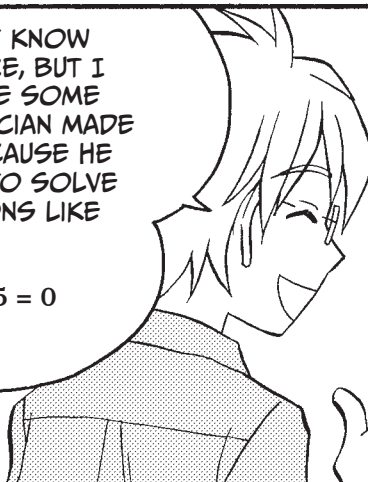
COMPLEX
NUMBERS...I'VE
NEVER REALLY
UNDERSTOOD THE
MEANING OF i ...

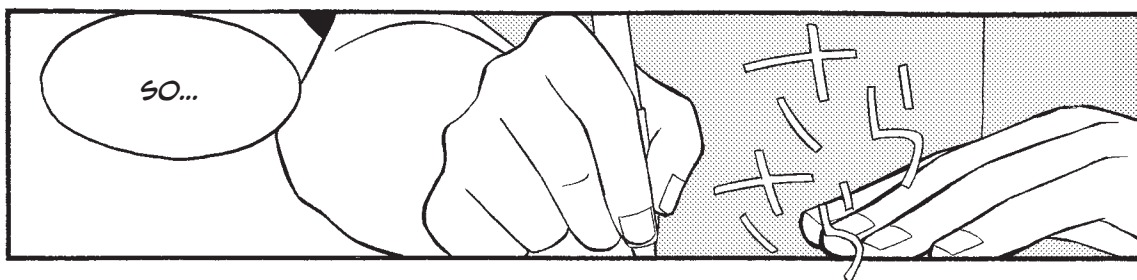
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WELL...

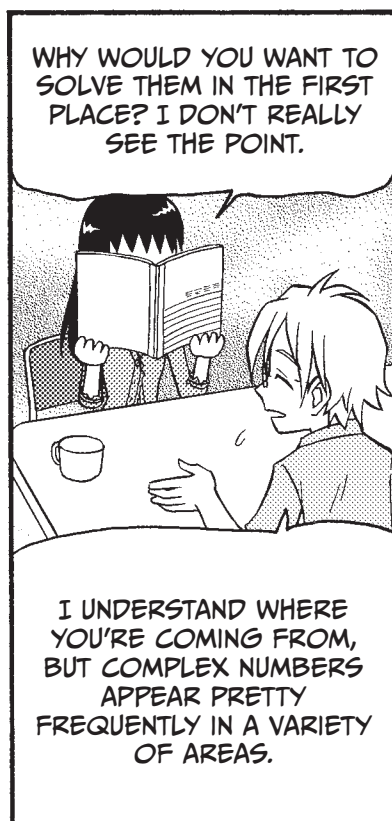
I DON'T KNOW
FOR SURE, BUT I
SUPPOSE SOME
MATHEMATICIAN MADE
IT UP BECAUSE HE
WANTED TO SOLVE
EQUATIONS LIKE

$$x^2 + 5 = 0$$





$$x^2 + 5 = x^2 - (-5) = (x + \sqrt{5}i)(x - \sqrt{5}i) = 0$$



IMPLICATION AND EQUIVALENCE

PROPOSITIONS

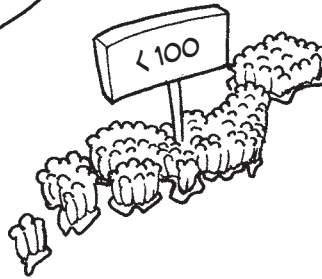
I THOUGHT
WE'D TALK ABOUT
IMPLICATION NEXT.

BUT FIRST,
LET'S DISCUSS
PROPOSITIONS.

A *PROPOSITION* IS A DECLARATIVE
SENTENCE THAT IS EITHER TRUE
OR FALSE, LIKE...

"ONE PLUS ONE EQUALS TWO" OR
"JAPAN'S POPULATION DOES NOT
EXCEED 100 PEOPLE."

$$1 + 1 = 2$$

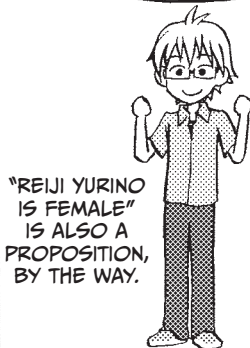


"THAT IS EITHER
TRUE OR FALSE..."

UMM

LET'S LOOK AT A
FEW EXAMPLES.

A SENTENCE LIKE
"REIJI YURINO IS MALE"
IS A PROPOSITION.



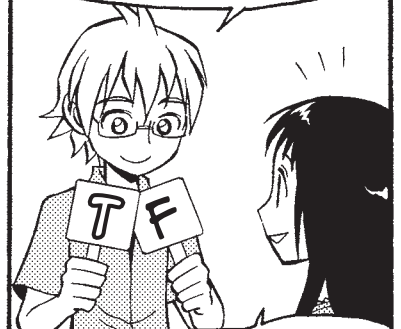
"REIJI YURINO
IS FEMALE"
IS ALSO A
PROPOSITION,
BY THE WAY.

BUT A SENTENCE
LIKE "REIJI YURINO IS
HANDSOME" IS NOT.



MY MOM
SAYS I'M
THE MOST
HANDSOME
GUY IN
SCHOOL...

TO PUT IT SIMPLY,
AMBIGUOUS SENTENCES
THAT PRODUCE DIFFERENT
REACTIONS DEPENDING
ON WHOM YOU ASK ARE
NOT PROPOSITIONS.



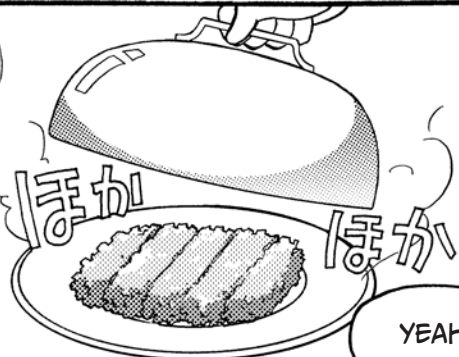
THAT KIND OF
MAKES SENSE.

IMPLICATION

LET'S TRY TO APPLY THIS KNOWLEDGE TO UNDERSTAND THE CONCEPT OF IMPLICATION. THE STATEMENT

"IF THIS DISH IS A SCHNITZEL THEN IT CONTAINS PORK"

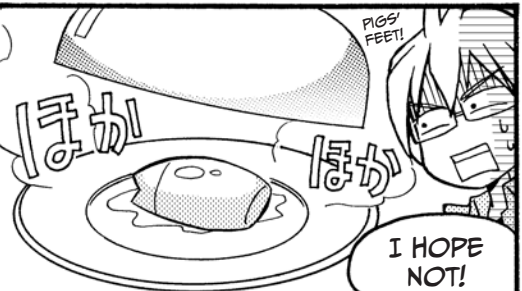
IS ALWAYS TRUE.



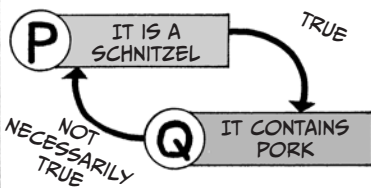
BUT IF WE LOOK AT ITS CONVERSE...

"IF THIS DISH CONTAINS PORK THEN IT IS A SCHNITZEL"

...IT IS NO LONGER NECESSARILY TRUE.



IN SITUATIONS WHERE WE KNOW THAT "IF P THEN Q" IS TRUE, BUT DON'T KNOW ANYTHING ABOUT ITS CONVERSE "IF Q THEN P"...



WE SAY THAT "P ENTAILS Q" AND THAT "Q COULD ENTAIL P."

IT IS A SCHNITZEL

IT CONTAINS PORK

ENTAILS

COULD ENTAIL

IT CONTAINS PORK

IT IS A SCHNITZEL

WHEN A PROPOSITION LIKE "IF P THEN Q" IS TRUE, IT IS COMMON TO WRITE IT WITH THE IMPLICATION SYMBOL, LIKE THIS:

$P \Rightarrow Q$

IF P THEN Q

$P \Rightarrow Q$

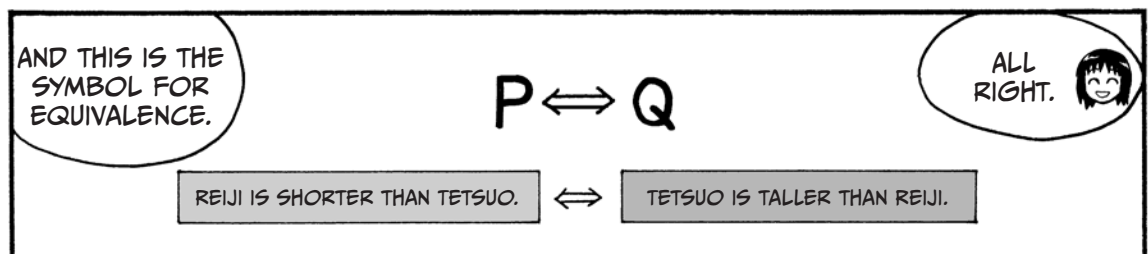
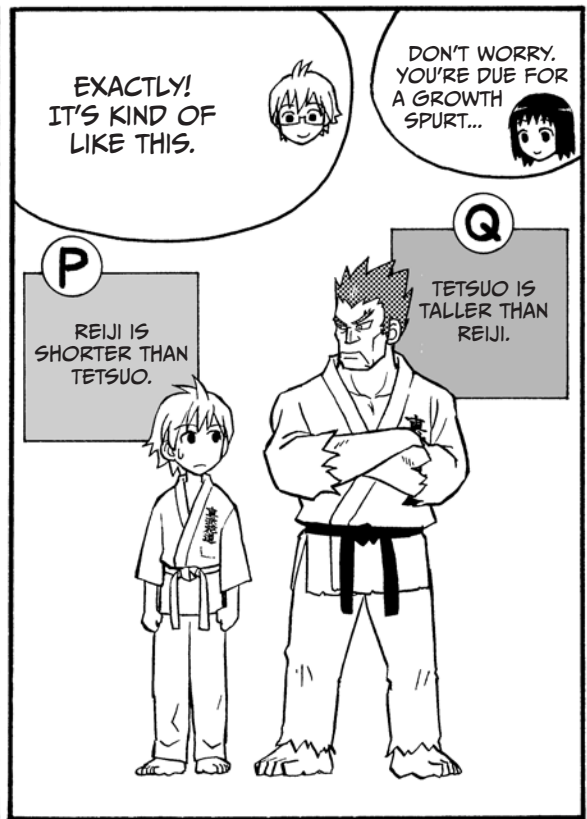
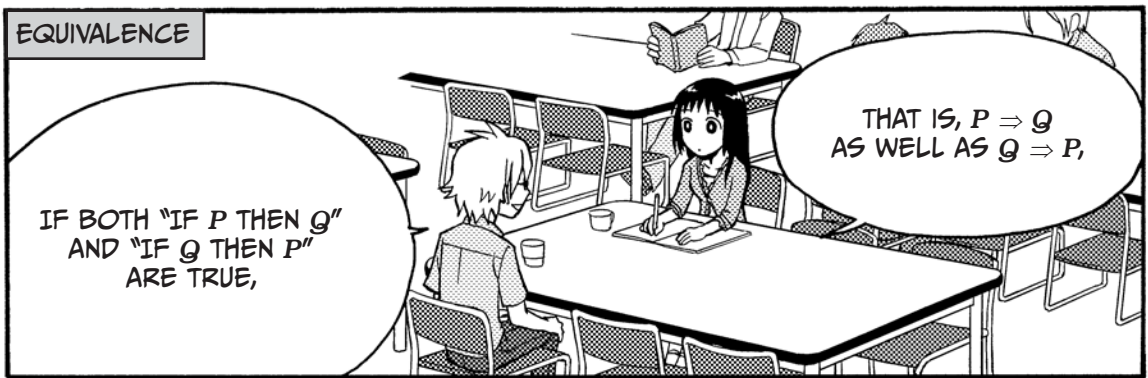
I THINK I GET IT.

THIS IS A SCHNITZEL

\Rightarrow

THIS DISH CONTAINS PORK

EQUIVALENCE



SET THEORY

SETS

ANOTHER
IMPORTANT FIELD
OF MATHEMATICS IS
SET THEORY.

OH YEAH...I THINK
WE COVERED THAT
IN HIGH SCHOOL.

PROBABLY, BUT
LET'S REVIEW IT
ANYWAY.

SLIDE

JUST AS YOU MIGHT THINK,
A *SET* IS A COLLECTION
OF THINGS.

THE THINGS THAT
MAKE UP THE SET ARE
CALLED ITS *ELEMENTS*
OR *OBJECTS*.

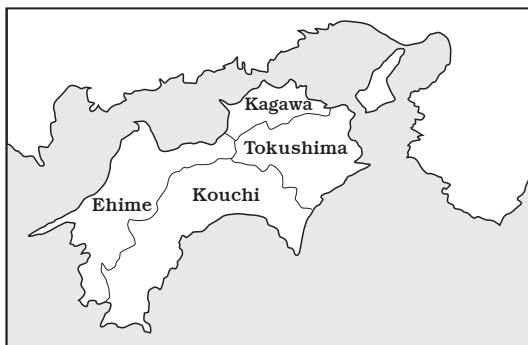
HEHE,
OKAY.

THIS MIGHT
GIVE YOU A
GOOD IDEA OF
WHAT I MEAN.

EXAMPLE 1

The set “Shikoku,” which is the smallest of Japan’s four islands, consists of these four elements:

- Kagawa-ken¹
- Ehime-ken
- Kouchi-ken
- Tokushima-ken



EXAMPLE 2

The set consisting of all even integers from 1 to 10 contains these five elements:

- 2
- 4
- 6
- 8
- 10

1. A Japanese *ken* is kind of like an American state.

SET SYMBOLS

TO ILLUSTRATE, THE SET
CONSISTING OF ALL
EVEN NUMBERS BETWEEN
1 AND 10 WOULD LOOK
LIKE THIS:

ALL EVEN NUMBERS
BETWEEN 1 AND 10

2 4 6
8 10

5

7

9

1

3

THESE
ARE TWO
COMMON
WAYS TO
WRITE OUT
THAT SET:

$\{2, 4, 6, 8, 10\}$

$\{2n | n = 1, 2, 3, 4, 5\}$

MMM...

IT'S ALSO
CONVENIENT TO
GIVE THE SET
A NAME, FOR
EXAMPLE, X .

WITH THAT IN MIND,
OUR DEFINITION
NOW LOOKS
LIKE THIS:

$X = \{2, 4, 6, 8, 10\}$

$X = \{2n | n = 1, 2, 3, 4, 5\}$

X MARKS
THE SET!

THIS IS A GOOD WAY TO EXPRESS
THAT "THE ELEMENT x BELONGS
TO THE SET X ."

OKAY.

$x \in X$

FOR EXAMPLE,
EHIME-KEN \in SHIKOKU

SUBSETS

AND THEN
THERE ARE
SUBSETS.

LET'S SAY THAT ALL
ELEMENTS OF A SET X
ALSO BELONG TO A
SET Y.

SET X
(SHIKOKU)

KAGAWA-KEN
EHIME-KEN
KOUCHI-KEN
TOKUSHIMA-KEN

SET Y
(JAPAN)

HOKKAIDOU
AOMORI-KEN
IWATE-KEN
MIYAGI-KEN
AKITA-KEN
YAMAGATA-KEN
FUKUSHIMA-KEN
IBARAKI-KEN
TOCHIGI-KEN
GUNMA-KEN
SAITAMA-KEN
CHIBA-KEN
TOKYOU-TO
KANAGAWA-KEN
NIIGATA-KEN
TOYAMA-KEN
ISHIKAWA-KEN
FUKUI-KEN

YAMANASHI-KEN
NAGANO-KEN
GIFU-KEN
SHIZUOKA-KEN
AICHI-KEN
MIE-KEN
SHIGA-KEN
KYOTO-FU
OOSAKA-FU
HYOGO-KEN
NARA-KEN
WAKAYAMA-KEN
TOTTORI-KEN
SHIMANE-KEN
OKAYAMA-KEN
HIROSHIMA-KEN
YAMAGUCHI-KEN
FUKUOKA-KEN

X IS A *SUBSET* OF Y
IN THIS CASE.

SAGA-KEN
NAGASAKI-KEN
KUMAMOTO-KEN
OOTA-KEN
MIYAZAKI-KEN
KAGOSHIMA-KEN
OKINAWA-KEN

AND IT'S
WRITTEN LIKE
THIS.

$$X \subset Y$$

FOR EXAMPLE,
SHIKOKU \subset JAPAN

I SEE.

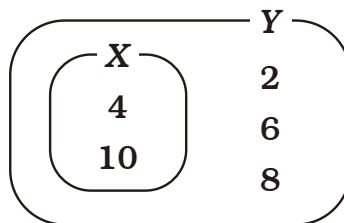
EXAMPLE 1

Suppose we have two sets X and Y :

$$X = \{ 4, 10 \}$$

$$Y = \{ 2, 4, 6, 8, 10 \}$$

X is a subset of Y , since all elements in X also exist in Y .



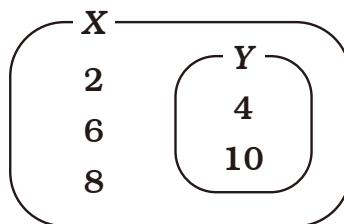
EXAMPLE 2

Suppose we switch the sets:

$$X = \{ 2, 4, 6, 8, 10 \}$$

$$Y = \{ 4, 10 \}$$

Since all elements in X don't exist in Y , X is no longer a subset of Y .



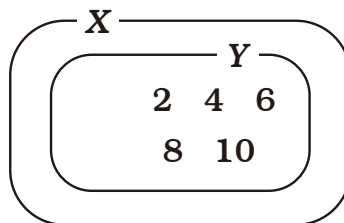
EXAMPLE 3

Suppose we have two equal sets instead:

$$X = \{ 2, 4, 6, 8, 10 \}$$

$$Y = \{ 2, 4, 6, 8, 10 \}$$

In this case, both sets are subsets of each other. So X is a subset of Y , and Y is a subset of X .



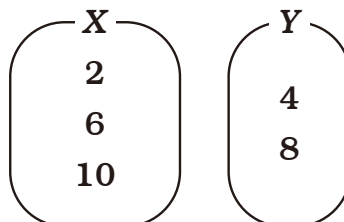
EXAMPLE 4

Suppose we have the two following sets:

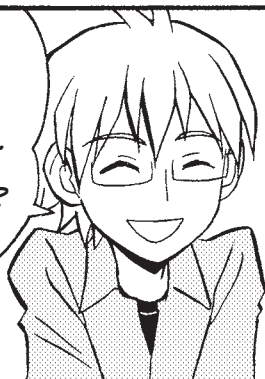
$$X = \{ 2, 6, 10 \}$$

$$Y = \{ 4, 8 \}$$

In this case neither X nor Y is a subset of the other.



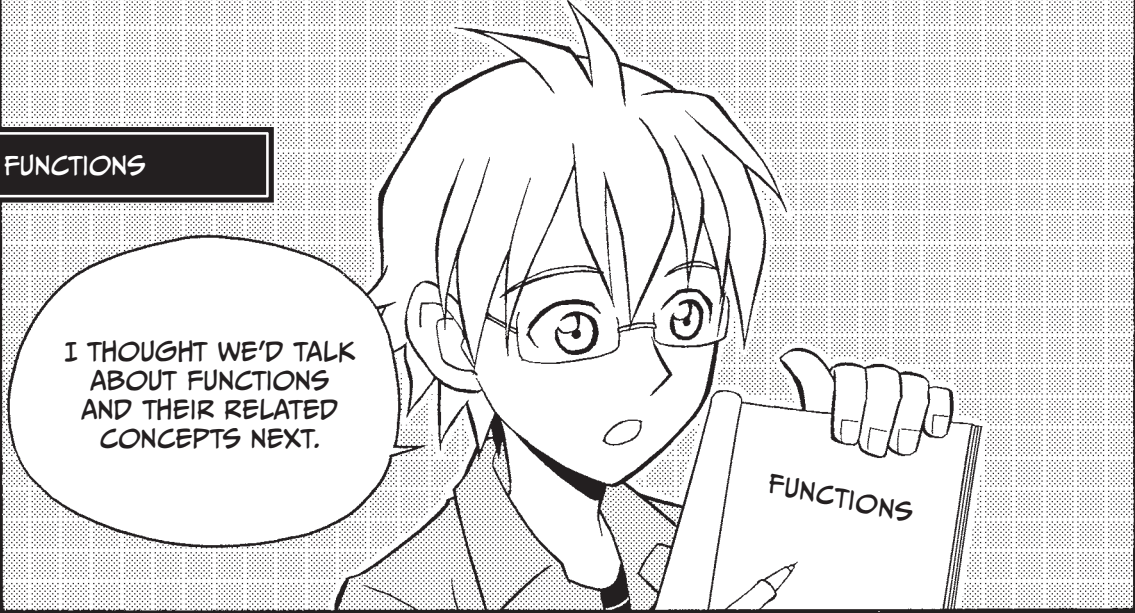
I THINK WE'RE ABOUT HALFWAY DONE FOR TODAY... ARE YOU STILL HANGING IN THERE?



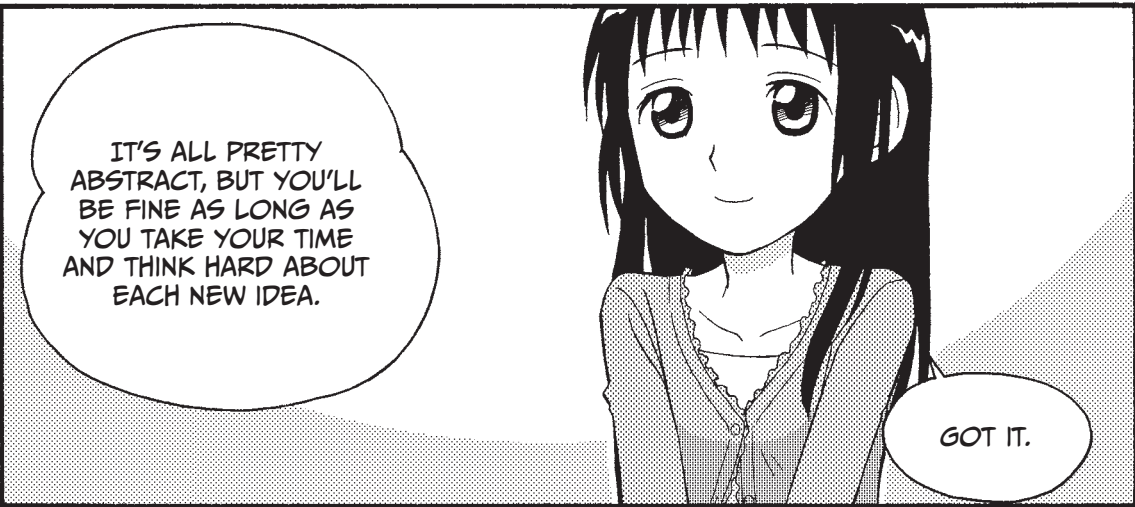
YOU KNOW IT!



FUNCTIONS



I THOUGHT WE'D TALK ABOUT FUNCTIONS AND THEIR RELATED CONCEPTS NEXT.



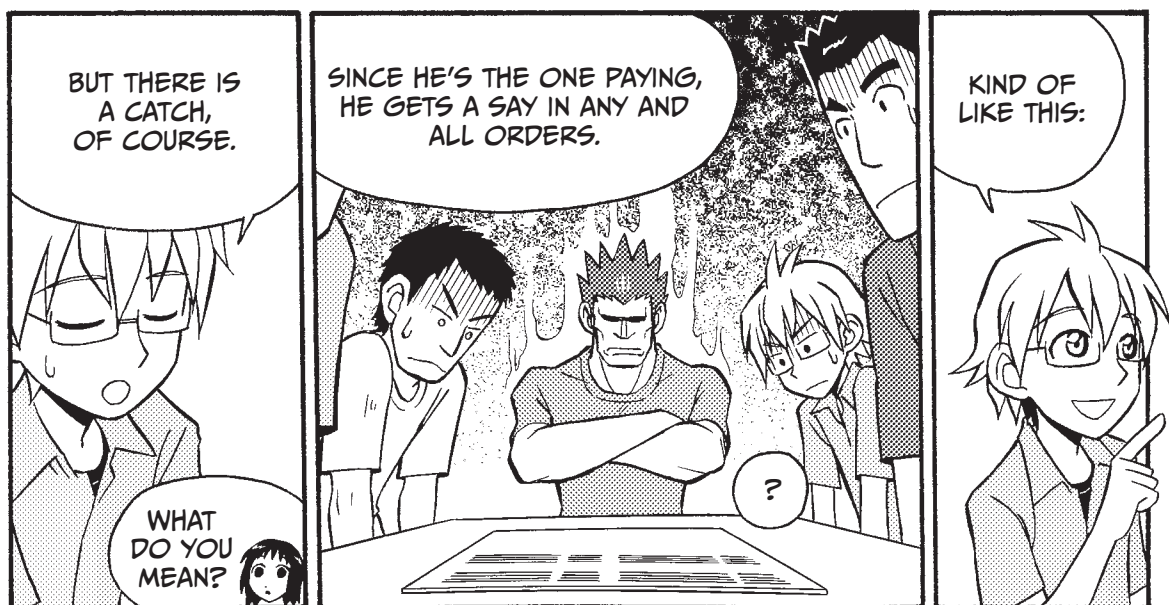
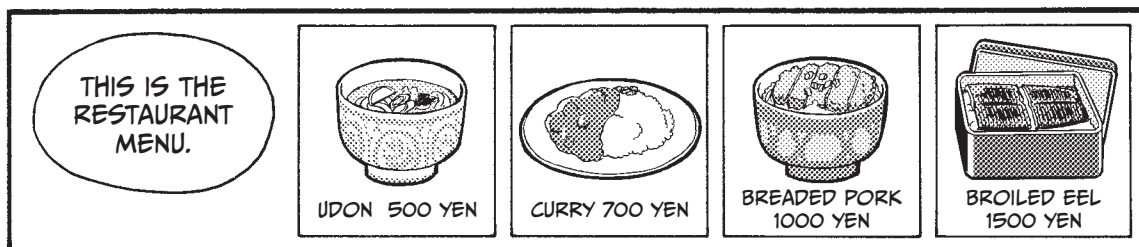
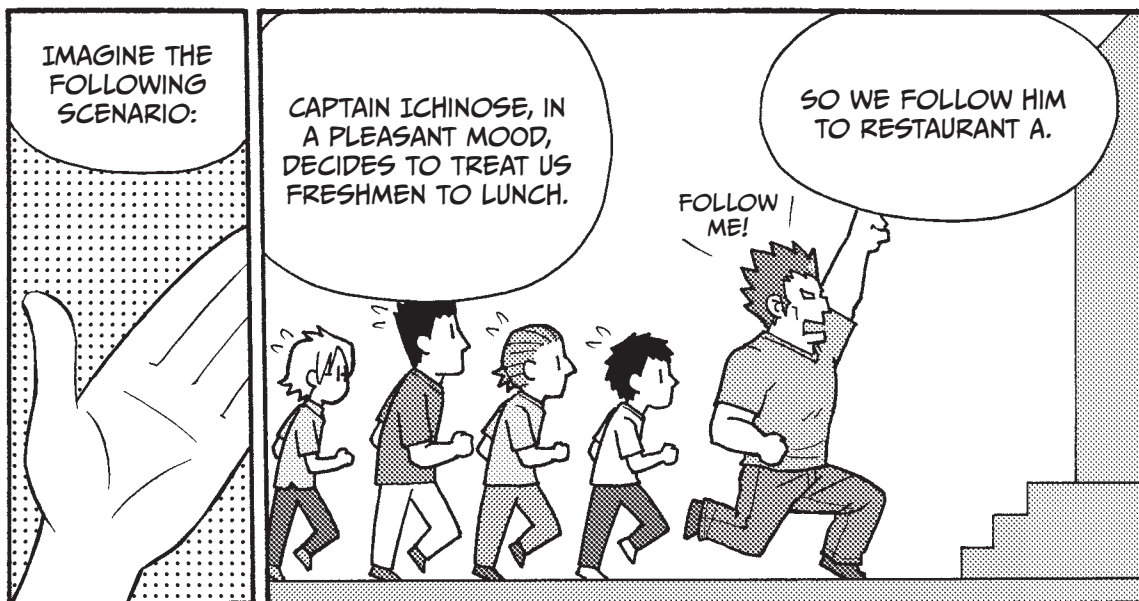
IT'S ALL PRETTY ABSTRACT, BUT YOU'LL BE FINE AS LONG AS YOU TAKE YOUR TIME AND THINK HARD ABOUT EACH NEW IDEA.

GOT IT.



LET'S START BY DEFINING THE CONCEPT ITSELF.

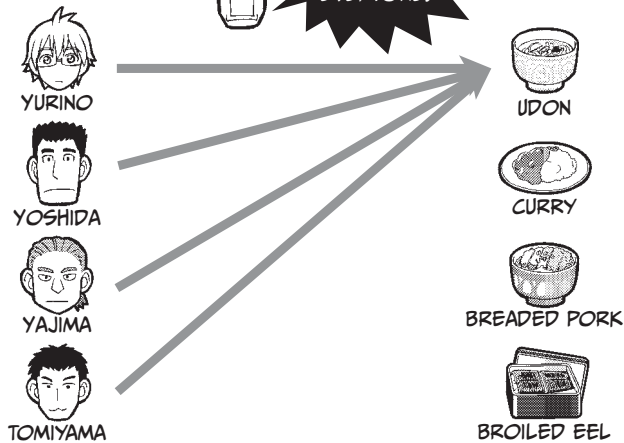
SOUNDS GOOD.



WE WOULDN'T REALLY BE ABLE TO SAY NO IF HE TOLD US TO ORDER THE CHEAPEST DISH, RIGHT?



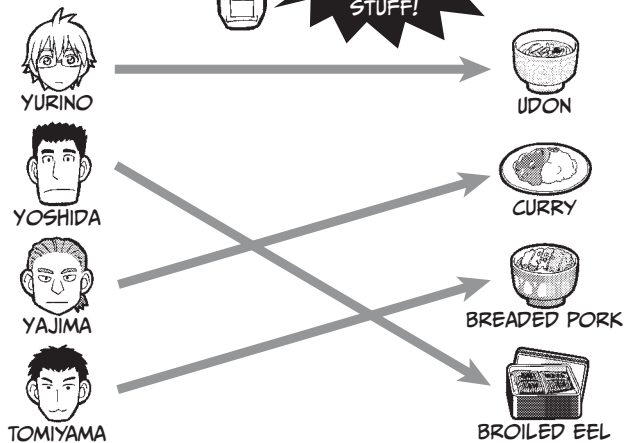
UDON FOR EVERYONE!



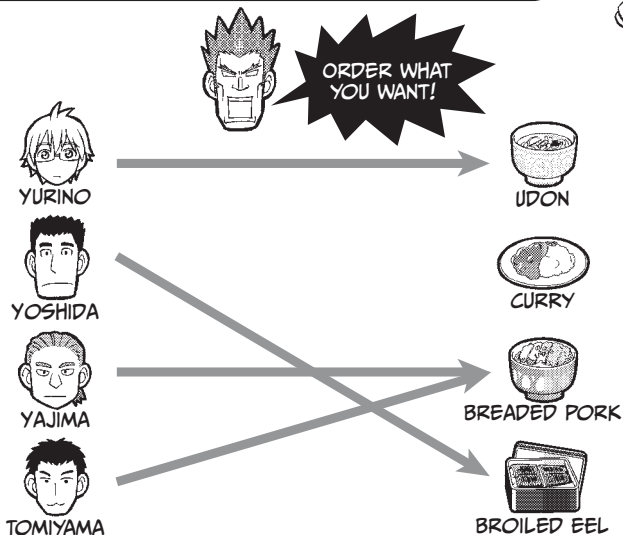
OR SAY, IF HE JUST TOLD US ALL TO ORDER DIFFERENT THINGS.



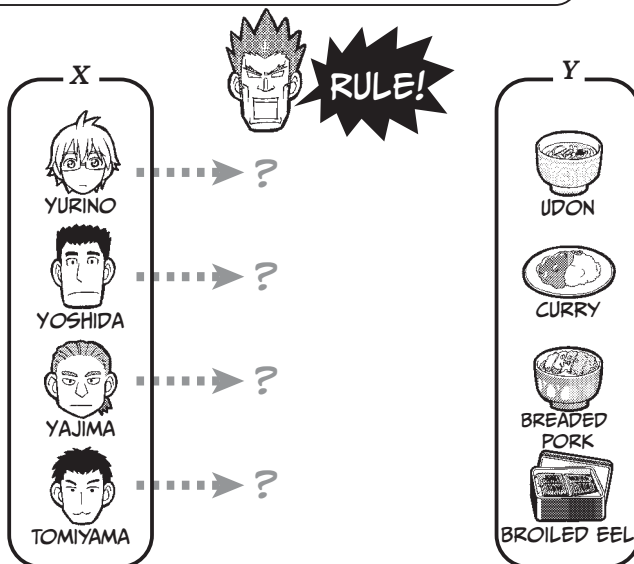
ORDER DIFFERENT STUFF!

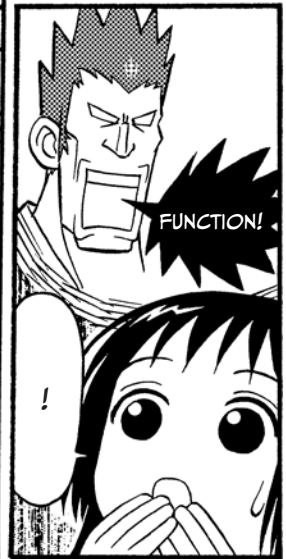
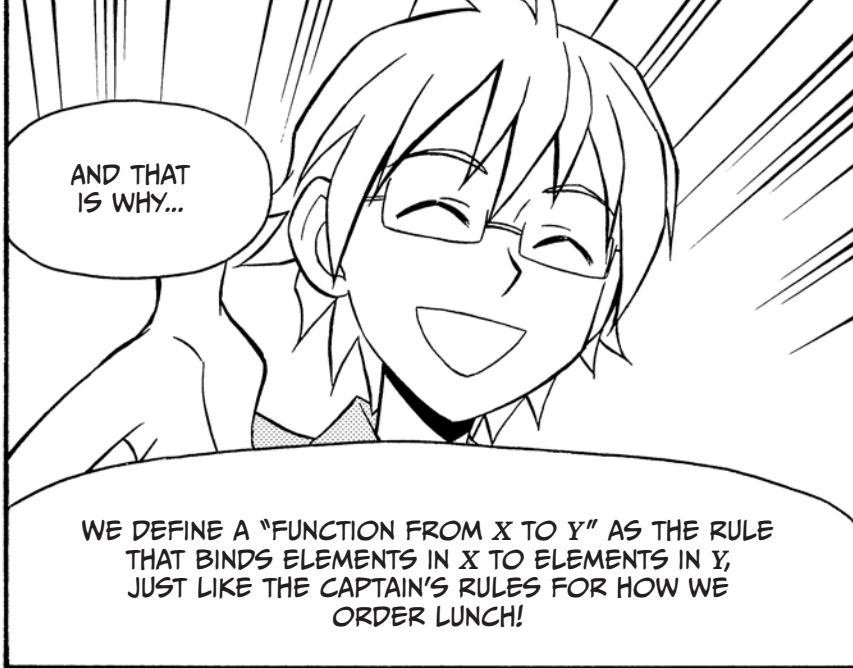


EVEN IF HE TOLD US TO ORDER OUR FAVORITES, WE WOULDN'T REALLY HAVE A CHOICE. THIS MIGHT MAKE US THE MOST HAPPY, BUT THAT DOESN'T CHANGE THE FACT THAT WE HAVE TO OBEY HIM.



YOU COULD SAY THAT THE CAPTAIN'S ORDERING GUIDELINES ARE LIKE A "RULE" THAT BINDS ELEMENTS OF X TO ELEMENTS OF Y.





THIS IS HOW WE WRITE IT:

$$X \xrightarrow{f} Y \quad \text{OR} \quad f: X \rightarrow Y$$

CLUB MEMBER $\xrightarrow{\text{RULE}}$ MENU OR RULE : CLUB MEMBER \longrightarrow MENU



FUNCTIONS

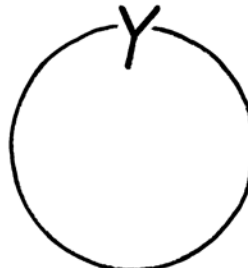
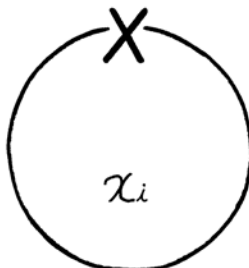
A rule that binds elements of the set X to elements of the set Y is called "a *function from X to Y .*" X is usually called the *domain* and Y the *co-domain* or *target set* of the function.

IMAGES

NEXT UP ARE
IMAGES.

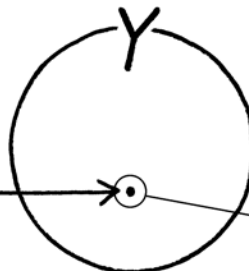
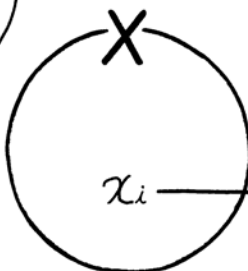
IMAGES?

LET'S ASSUME THAT
 x_i IS AN ELEMENT OF
THE SET X .



THE ELEMENT IN Y THAT
CORRESPONDS TO x_i
WHEN PUT THROUGH f ...

IS CALLED " x_i 'S IMAGE UNDER f IN Y ."

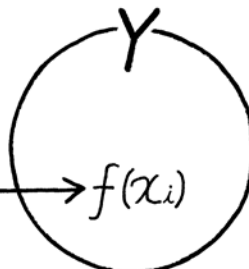
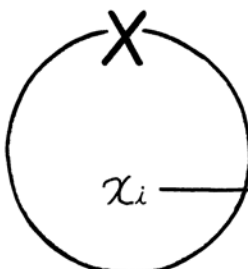


x_i 'S IMAGE
UNDER f
IN Y

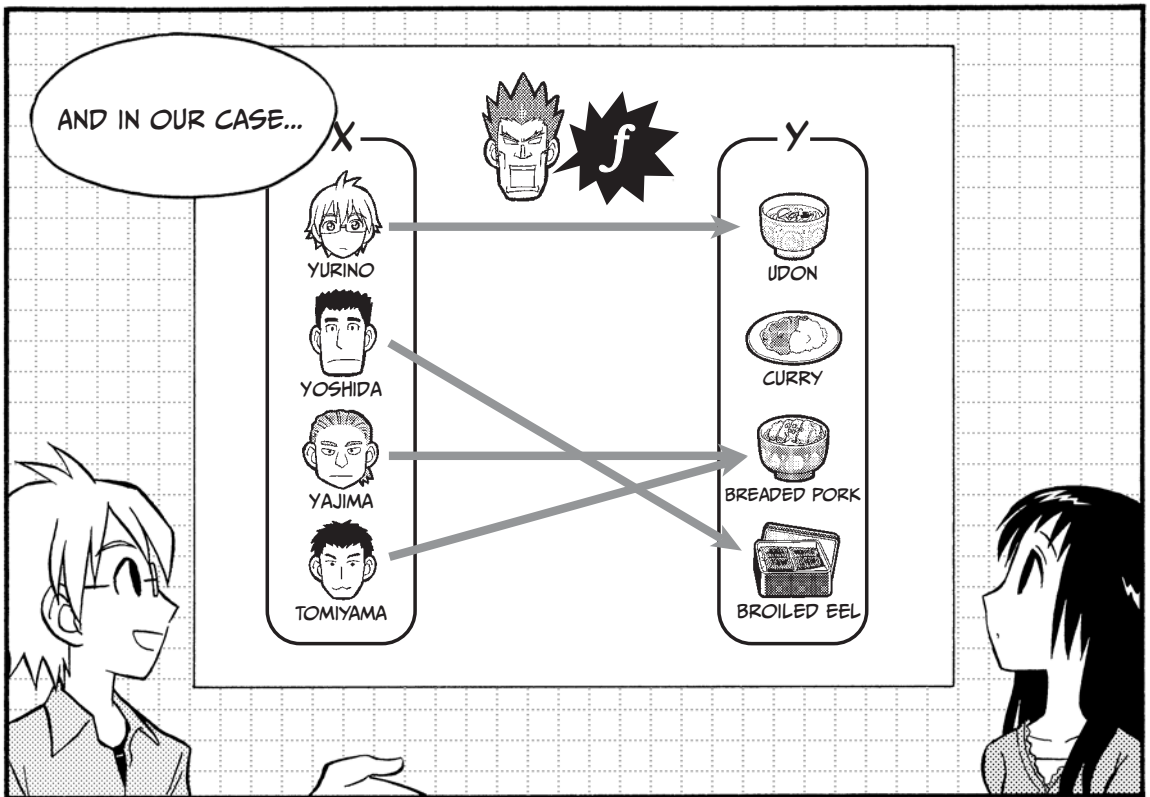
ALSO,

IT'S NOT UNCOMMON
TO WRITE " x_i 'S IMAGE
UNDER f IN Y "...

AS $f(x_i)$.



OKAY!



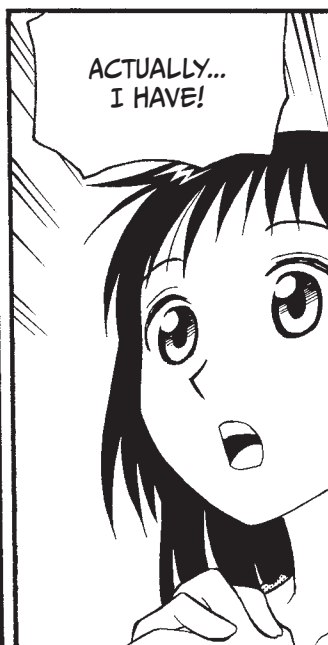
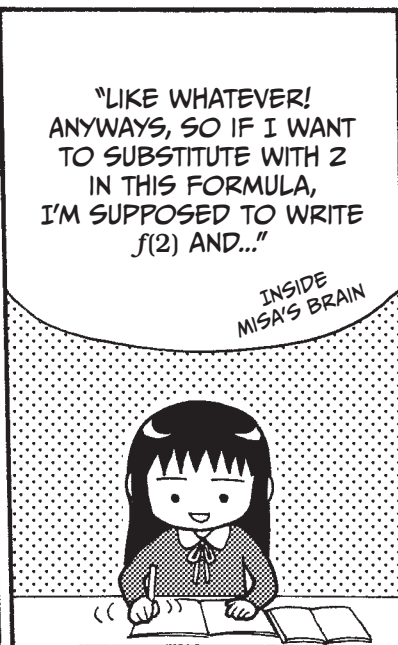
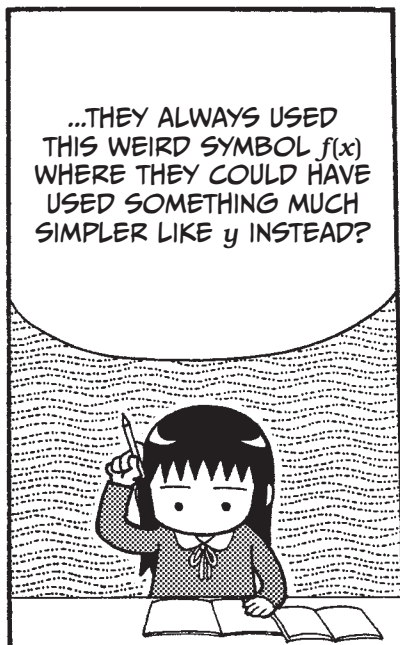
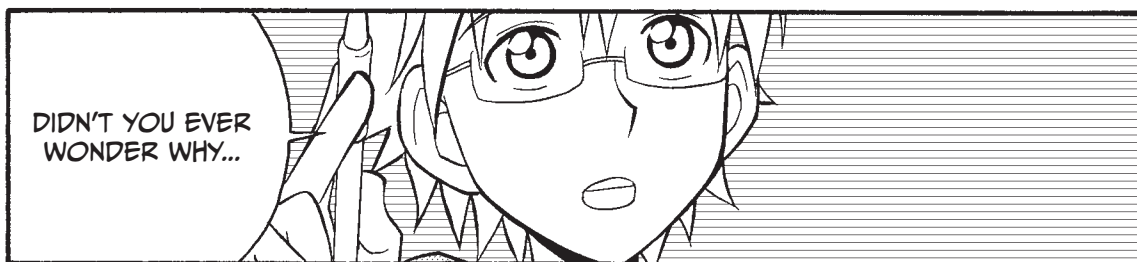
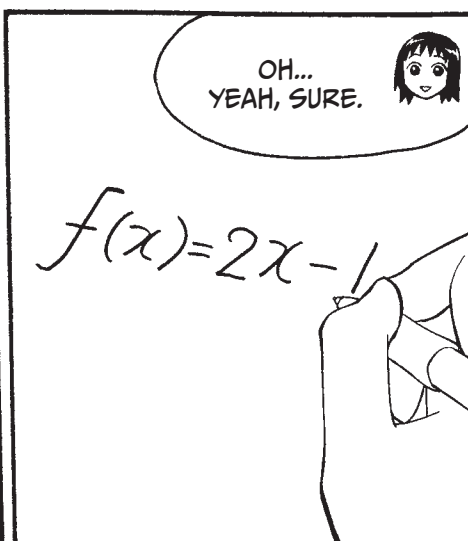
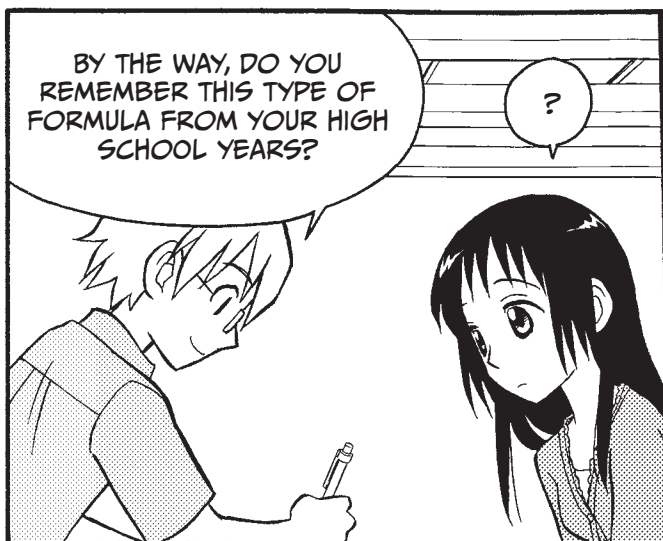
LIKE THIS:

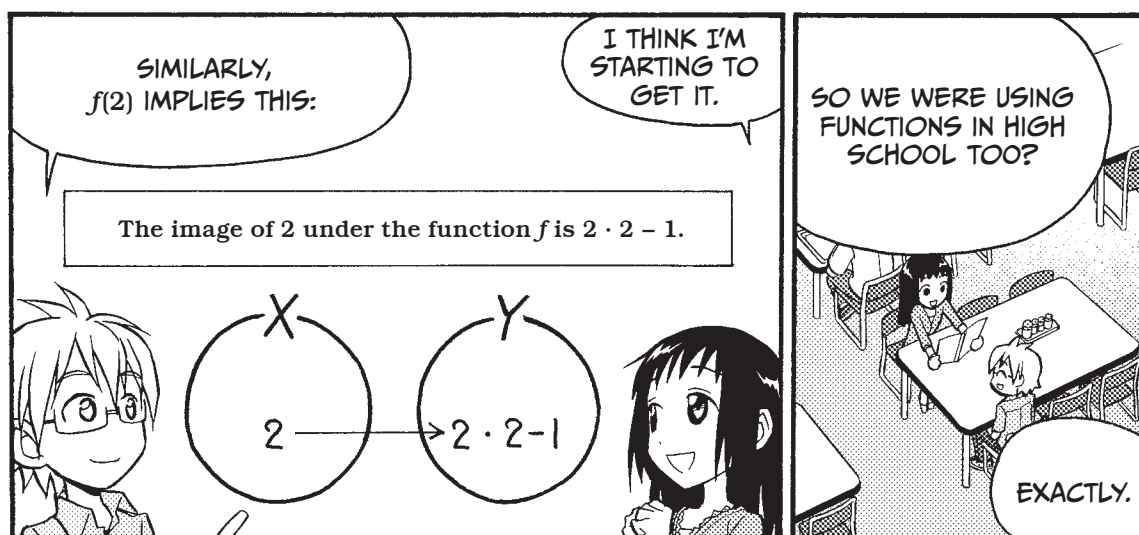
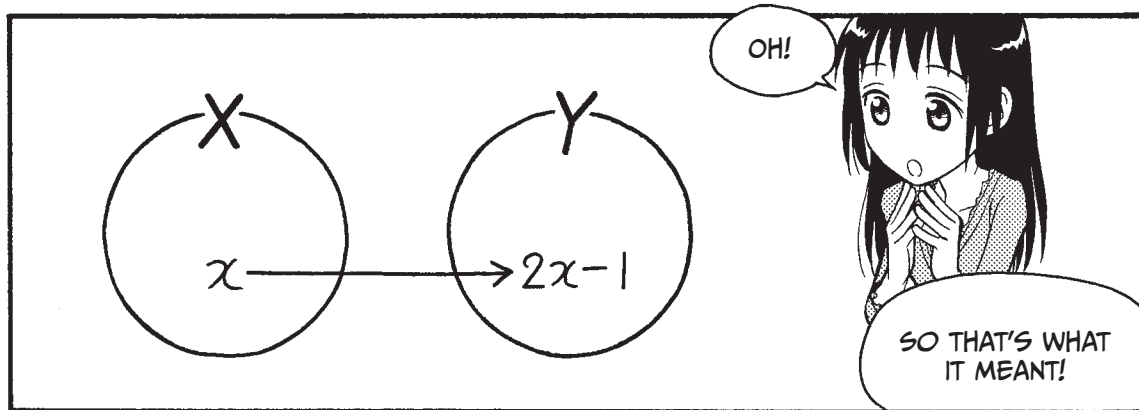
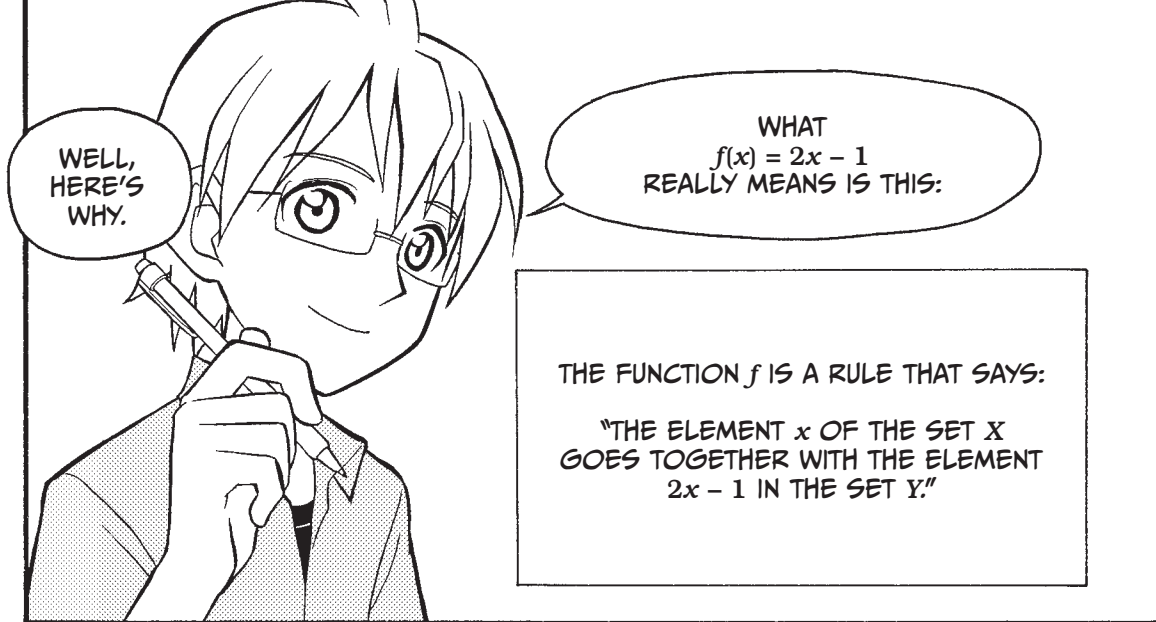
$f(\text{YURINO}) = \text{UDON}$
 $f(\text{YOSHIDA}) = \text{BROILED EEL}$
 $f(\text{YAJIMA}) = \text{BREADED PORK}$
 $f(\text{TOMIYAMA}) = \text{BREADED PORK}$

I HOPE YOU LIKE UDON!

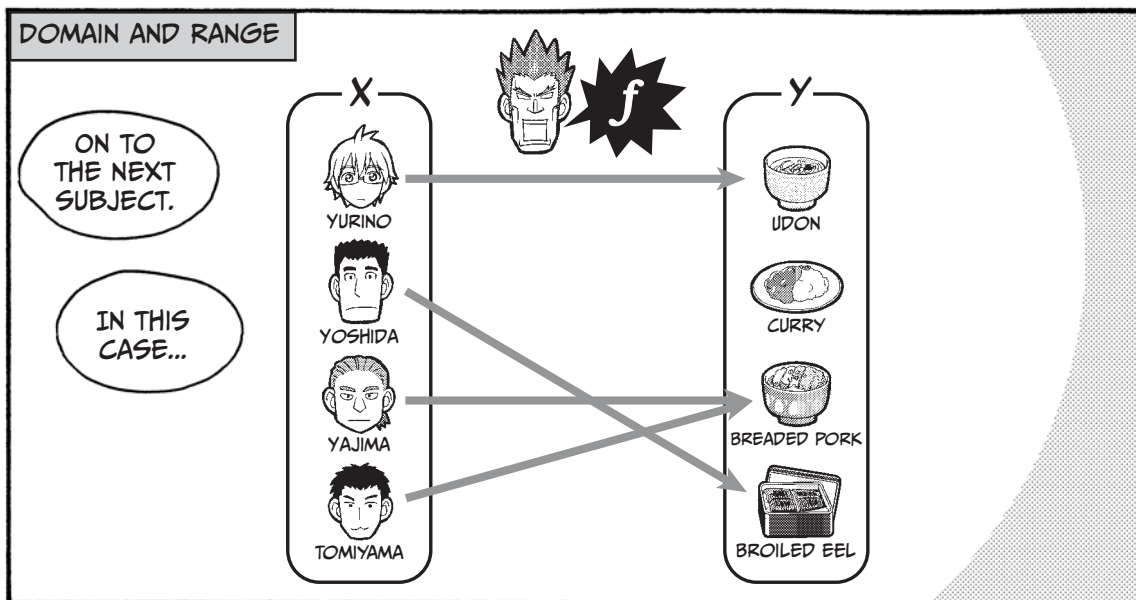
IMAGE

This is the element in Y that corresponds to x_i of the set X , when put through the function f .





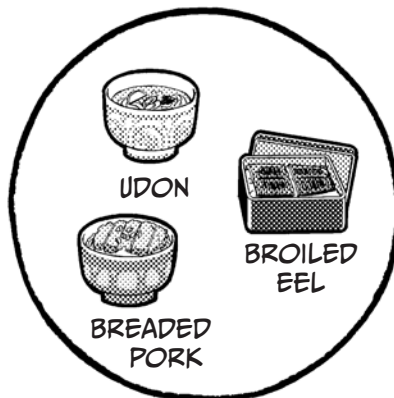
DOMAIN AND RANGE



WE'RE GOING TO WORK
WITH A SET

{UDON, BREADED PORK,
BROILED EEL}

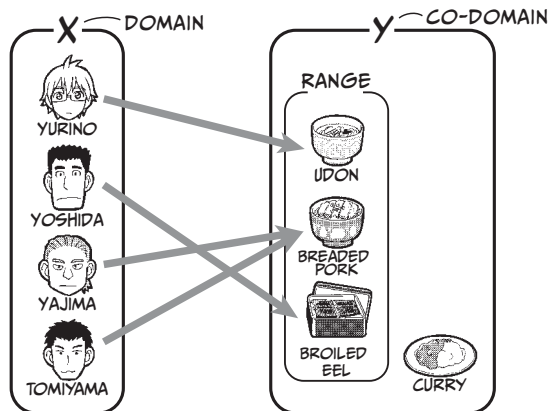
WHICH IS THE IMAGE OF
THE SET X UNDER THE
FUNCTION f .*



THIS SET IS USUALLY CALLED
THE RANGE OF THE FUNCTION f ;
BUT IT IS SOMETIMES ALSO
CALLED THE IMAGE OF f .

KIND OF
CONFUSING...

AND THE SET X IS DENOTED
AS THE *DOMAIN* OF f .



WE COULD EVEN HAVE DESCRIBED THIS FUNCTION AS

$$Y = \{f(\text{Yurino}), f(\text{Yoshida}), f(\text{Yajima}), f(\text{Tomiyama})\}$$

IF WE WANTED TO.

HEHE.

OSSU!



RANGE AND CO-DOMAIN

The set that encompasses the function f 's image $\{f(x_1), f(x_2), \dots, f(x_n)\}$ is called the *range* of f , and the (possibly larger) set being mapped into is called its *co-domain*.

The relationship between the range and the co-domain Y is as follows:

$$\{f(x_1), f(x_2), \dots, f(x_n)\} \subset Y$$

In other words, a function's range is a subset of its co-domain. In the special case where all elements in Y are an image of some element in X , we have

$$\{f(x_1), f(x_2), \dots, f(x_n)\} = Y$$

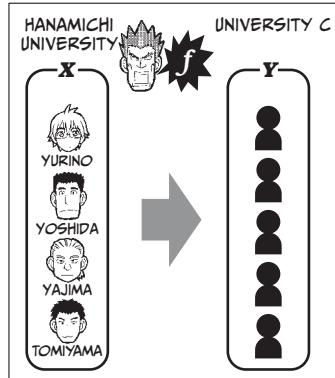
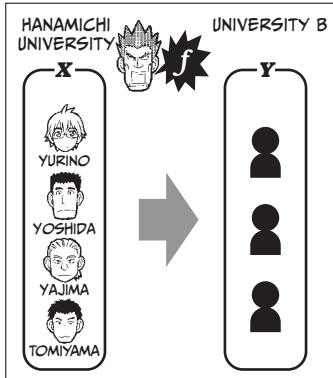
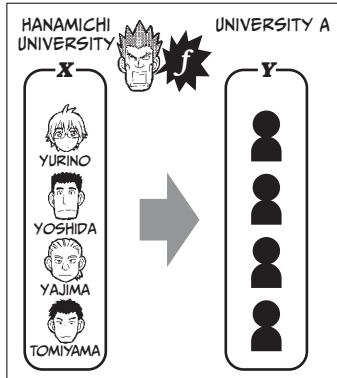
ONTO AND ONE-TO-ONE FUNCTIONS

NEXT WE'LL TALK ABOUT ONTO AND ONE-TO-ONE FUNCTIONS.

RIGHT.

LET'S SAY OUR KARATE CLUB DECIDES TO HAVE A PRACTICE MATCH WITH ANOTHER CLUB...

AND THAT THE CAPTAIN'S MAPPING FUNCTION f IS "FIGHT THAT GUY."

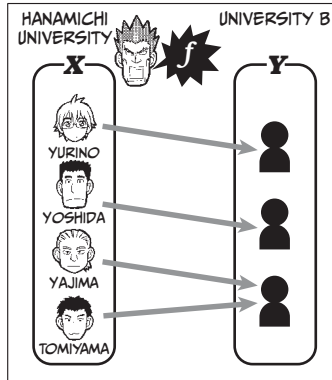
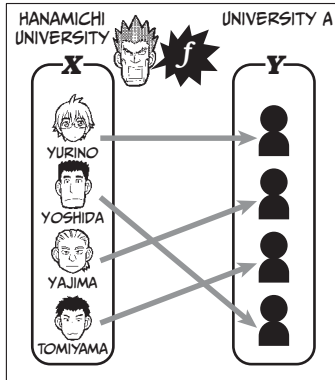


YOU'RE ALREADY DOING PRACTICE MATCHES?

N-NOT REALLY. THIS IS JUST AN EXAMPLE.

STILL WORKING ON THE BASICS!

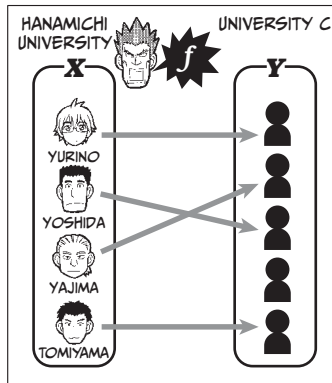
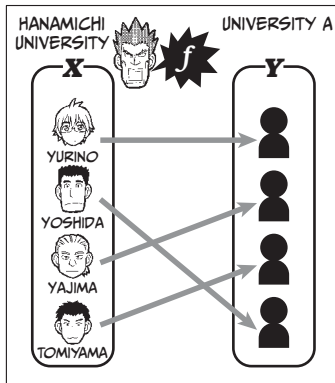
ONTO FUNCTIONS



A FUNCTION IS *ONTO* IF ITS IMAGE IS EQUAL TO ITS CO-DOMAIN. THIS MEANS THAT ALL THE ELEMENTS IN THE CO-DOMAIN OF AN ONTO FUNCTION ARE BEING MAPPED ONTO.



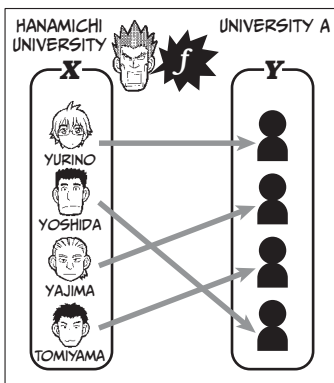
ONE-TO-ONE FUNCTIONS



IF $x_i \neq x_j$ LEADS TO $f(x_i) \neq f(x_j)$, WE SAY THAT THE FUNCTION IS *ONE-TO-ONE*. THIS MEANS THAT NO ELEMENT IN THE CO-DOMAIN CAN BE MAPPED ONTO MORE THAN ONCE.



ONE-TO-ONE AND ONTO FUNCTIONS



IT'S ALSO POSSIBLE FOR A FUNCTION TO BE BOTH ONTO AND ONE-TO-ONE. SUCH A FUNCTION CREATES A "BUDDY SYSTEM" BETWEEN THE ELEMENTS OF THE DOMAIN AND CO-DOMAIN. EACH ELEMENT HAS ONE AND ONLY ONE "PARTNER."



INVERSE FUNCTIONS

NOW WE HAVE INVERSE FUNCTIONS.

INVERSE?

THIS TIME WE'RE GOING TO LOOK AT THE OTHER TEAM CAPTAIN'S ORDERS AS WELL.

HANAMICHI UNIVERSITY
X

g

UNIVERSITY A

Y

YURINO
YOSHIDA
YAJIMA
TOMIYAMA



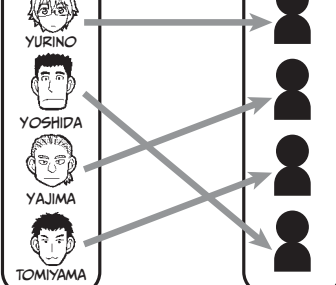
HANAMICHI UNIVERSITY
X

UNIVERSITY A
Y

f

YURINO
YOSHIDA
YAJIMA
TOMIYAMA

YURINO
YOSHIDA
YAJIMA
TOMIYAMA



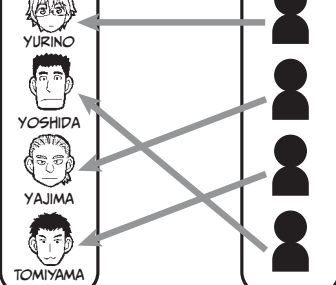
HANAMICHI UNIVERSITY
X

UNIVERSITY A
Y

g

YURINO
YOSHIDA
YAJIMA
TOMIYAMA

YURINO
YOSHIDA
YAJIMA
TOMIYAMA



WE SAY THAT THE FUNCTION g IS f 'S INVERSE WHEN THE TWO CAPTAINS' ORDERS COINCIDE LIKE THIS.

I SEE.

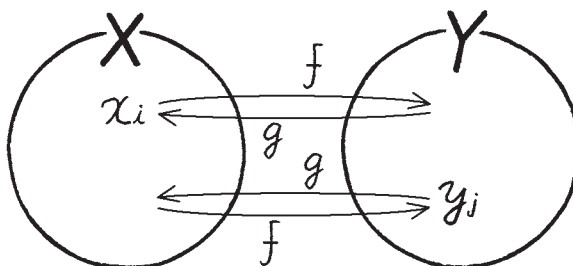
TO SPECIFY
EVEN FURTHER...

f IS AN INVERSE OF g
IF THESE TWO RELATIONS HOLD.

$$\textcircled{1} \ g(f(x_i)) = x_i$$

$$\textcircled{2} \ f(g(y_j)) = y_j$$

OH, IT'S LIKE THE
FUNCTIONS UNDO
EACH OTHER!



THIS IS THE SYMBOL USED TO
INDICATE INVERSE FUNCTIONS.

$$X \xrightarrow{f^{-1}} Y$$

OR

$$f^{-1}: X \rightarrow Y$$

YOU RAISE IT
TO -1 , RIGHT?

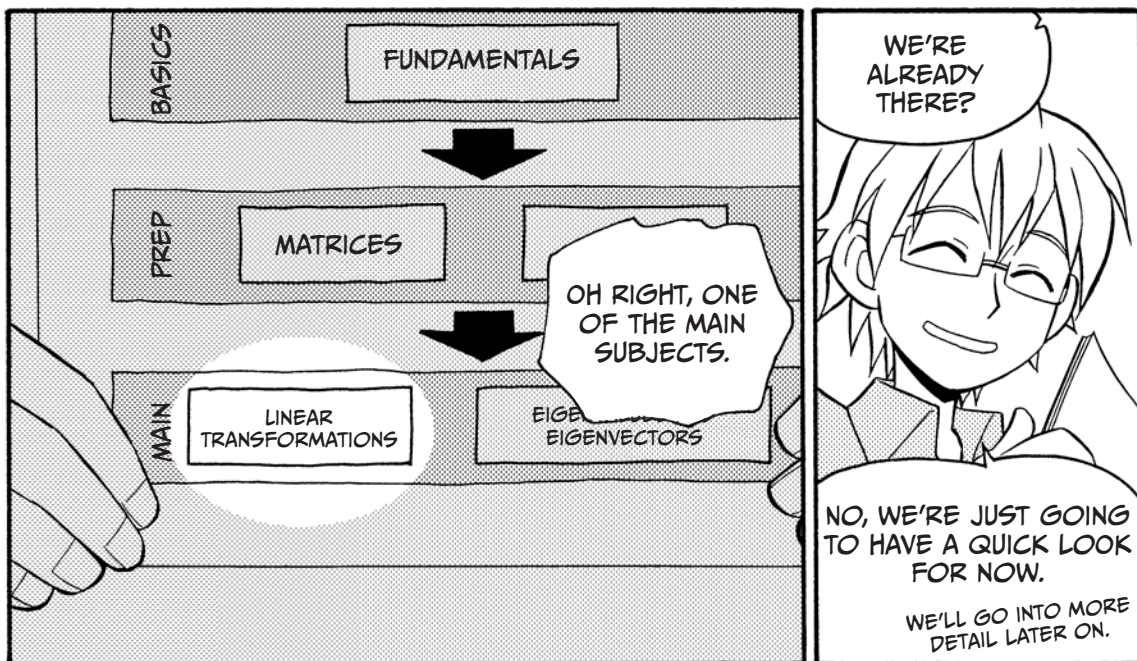
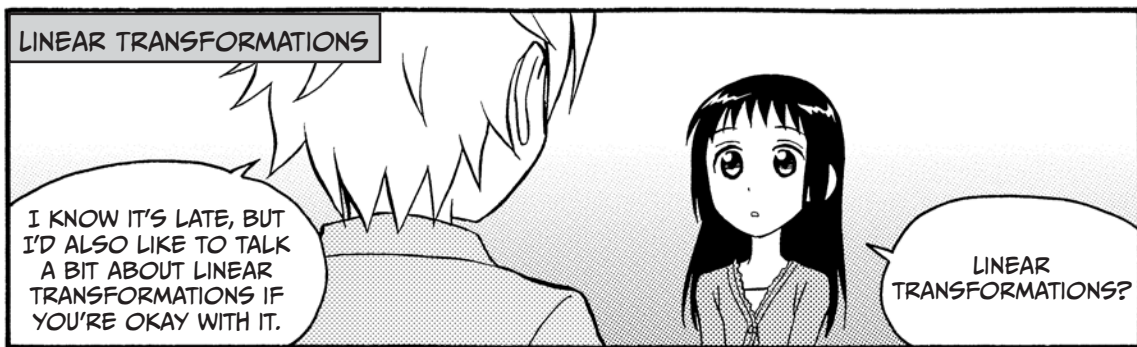
THERE IS ALSO A
CONNECTION BETWEEN
ONE-TO-ONE AND ONTO
FUNCTIONS AND INVERSE
FUNCTIONS. HAVE A
LOOK AT THIS.

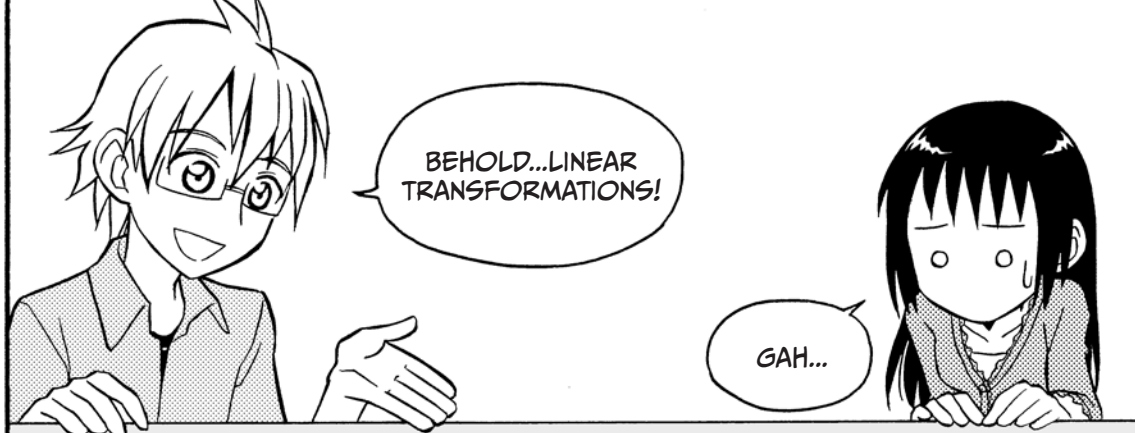
THE FUNCTION f
HAS AN INVERSE.



THE FUNCTION f
IS ONE-TO-ONE
AND ONTO.

SO IF IT'S ONE-TO-
ONE AND ONTO, IT HAS
AN INVERSE, AND VICE
VERSA. GOT IT!

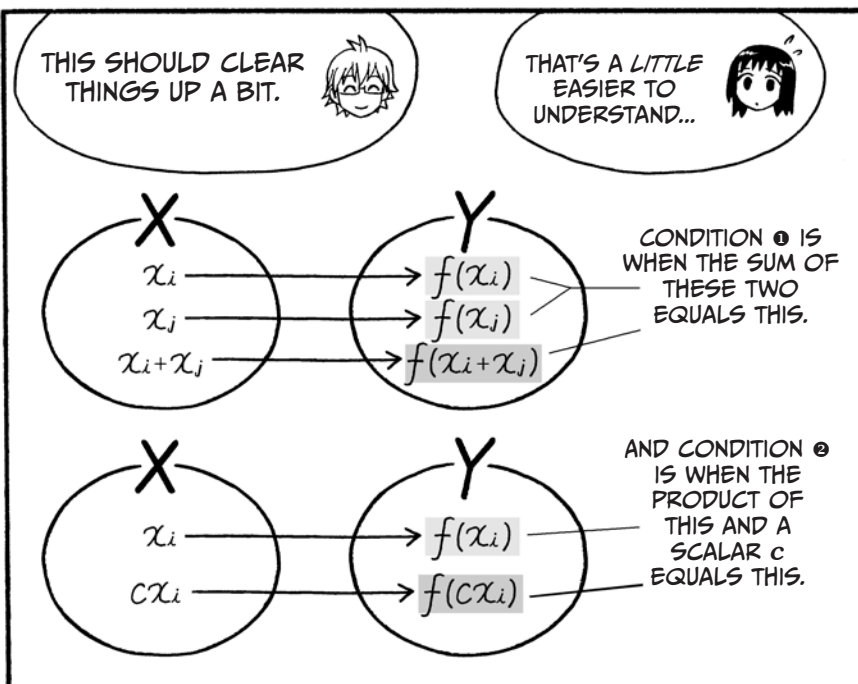




LINEAR TRANSFORMATIONS

Let x_i and x_j be two arbitrary elements of the set X , c be any real number, and f be a function from X to Y . f is called a *linear transformation* from X to Y if it satisfies the following two conditions:

- ① $f(x_i) + f(x_j) = f(x_i + x_j)$
- ② $cf(x_i) = f(cx_i)$



LET'S HAVE A LOOK AT A COUPLE OF EXAMPLES.



AN EXAMPLE OF A LINEAR TRANSFORMATION

The function $f(x) = 2x$ is a linear transformation. This is because it satisfies both ❶ and ❷, as you can see in the table below.

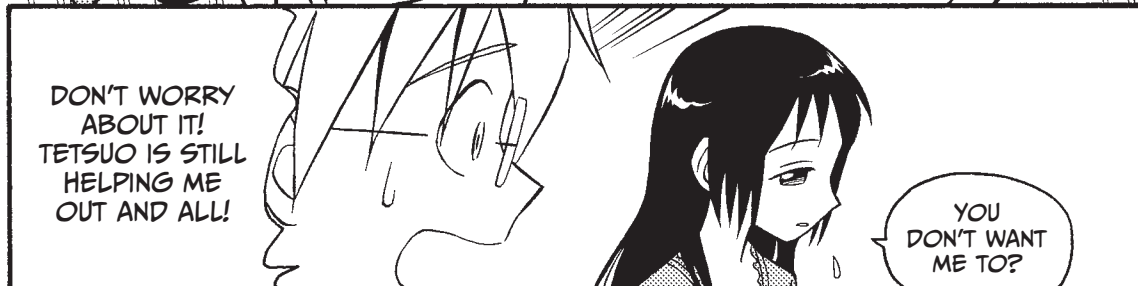
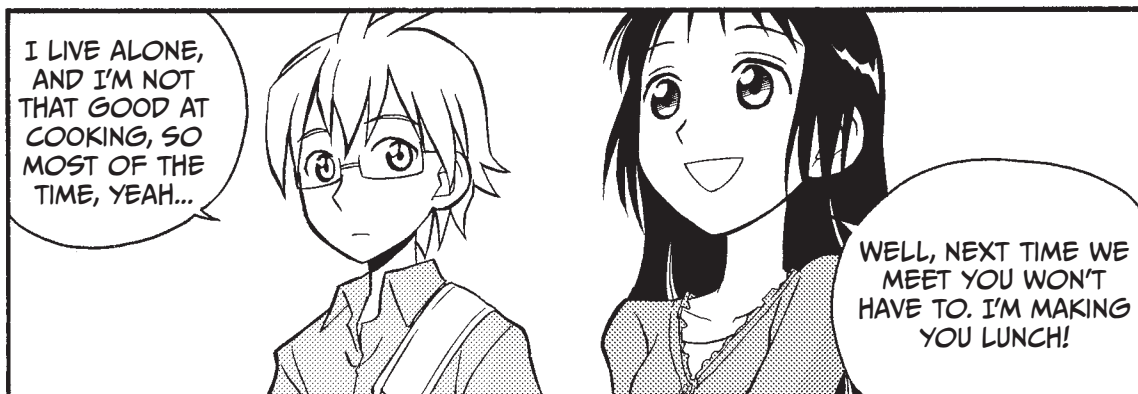
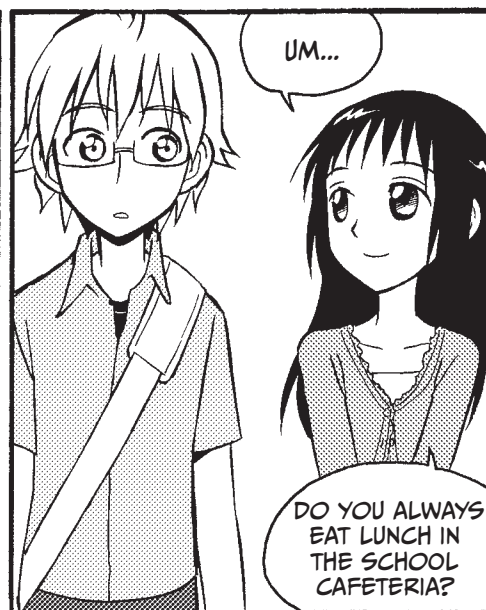
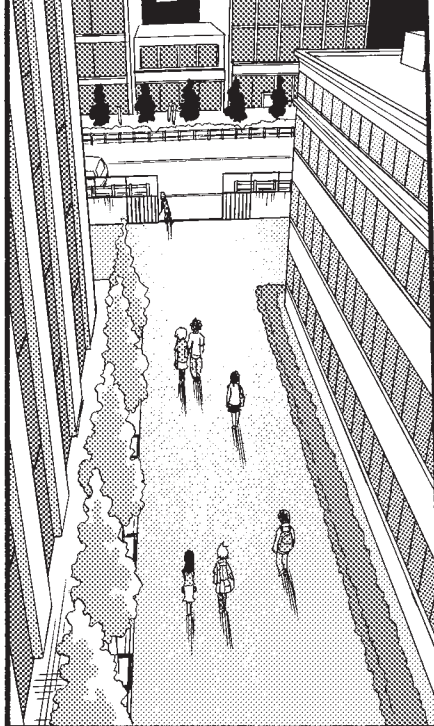
Condition ❶	$\begin{cases} f(x_i) + f(x_j) = 2x_i + 2x_j \\ f(x_i + x_j) = 2(x_i + x_j) = 2x_i + 2x_j \end{cases}$
Condition ❷	$\begin{cases} cf(x_i) = c(2x_i) = 2cx_i \\ f(cx_i) = 2(cx_i) = 2cx_i \end{cases}$

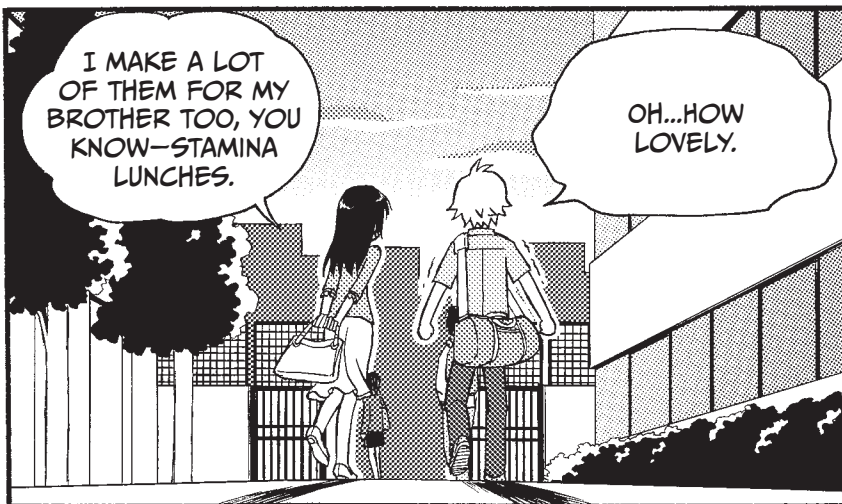
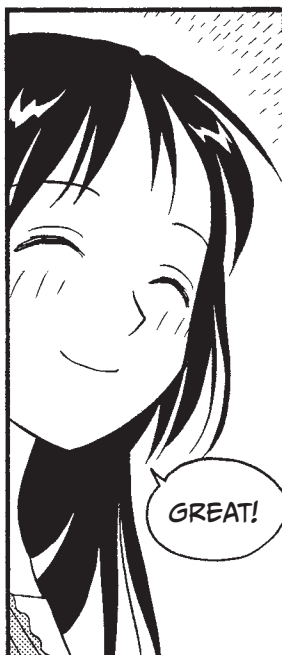
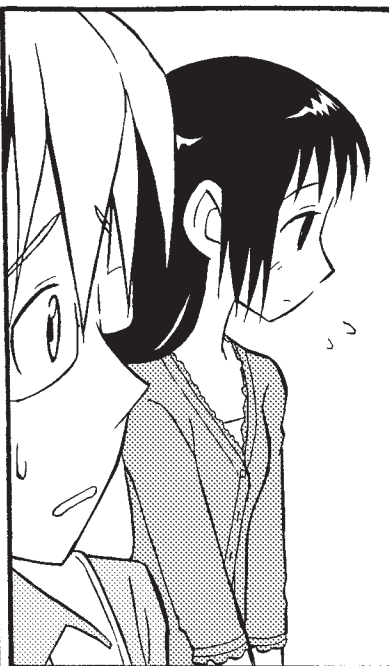
AN EXAMPLE OF A FUNCTION THAT IS NOT A LINEAR TRANSFORMATION

The function $f(x) = 2x - 1$ is not a linear transformation. This is because it satisfies neither ❶ nor ❷, as you can see in the table below.

Condition ❶	$\begin{cases} f(x_i) + f(x_j) = (2x_i - 1) + (2x_j - 1) = 2x_i + 2x_j - 2 \\ f(x_i + x_j) = 2(x_i + x_j) - 1 = 2x_i + 2x_j - 1 \end{cases}$
Condition ❷	$\begin{cases} cf(x_i) = c(2x_i - 1) = 2cx_i - c \\ f(cx_i) = 2(cx_i) - 1 = 2cx_i - 1 \end{cases}$







COMBINATIONS AND PERMUTATIONS

I thought the best way to explain combinations and permutations would be to give a concrete example.

I'll start by explaining the **PROBLEM**, then I'll establish a good **WAY OF THINKING**, and finally I'll present a **SOLUTION**.

PROBLEM

Reiji bought a CD with seven different songs on it a few days ago. Let's call the songs A, B, C, D, E, F, and G. The following day, while packing for a car trip he had planned with his friend Nemoto, it struck him that it might be nice to take the songs along to play during the drive. But he couldn't take all of the songs, since his taste in music wasn't very compatible with Nemoto's. After some deliberation, he decided to make a new CD with only three songs on it from the original seven.

Questions:

1. In how many ways can Reiji select three songs from the original seven?
2. In how many ways can the three songs be arranged?
3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

WAY OF THINKING

It is possible to solve question 3 by dividing it into these two subproblems:

1. Choose three songs out of the seven possible ones.
2. Choose an order in which to play them.

As you may have realized, these are the first two questions. The solution to question 3, then, is as follows:

SOLUTION TO QUESTION 1 · SOLUTION TO QUESTION 2 = SOLUTION TO QUESTION 3		
In how many ways can Reiji select three songs from the original seven?	In how many ways can the three songs be arranged?	In how many ways can a CD be made, where three songs are chosen from a pool of seven?

SOLUTION

1. In how many ways can Reiji select three songs from the original seven?

All 35 different ways to select the songs are in the table below. Feel free to look them over.

Pattern 1	A and B and C	Pattern 16	B and C and D
Pattern 2	A and B and D	Pattern 17	B and C and E
Pattern 3	A and B and E	Pattern 18	B and C and F
Pattern 4	A and B and F	Pattern 19	B and C and G
Pattern 5	A and B and G	Pattern 20	B and D and E
Pattern 6	A and C and D	Pattern 21	B and D and F
Pattern 7	A and C and E	Pattern 22	B and D and G
Pattern 8	A and C and F	Pattern 23	B and E and F
Pattern 9	A and C and G	Pattern 24	B and E and G
Pattern 10	A and D and E	Pattern 25	B and F and G
Pattern 11	A and D and F	Pattern 26	C and D and E
Pattern 12	A and D and G	Pattern 27	C and D and F
Pattern 13	A and E and F	Pattern 28	C and D and G
Pattern 14	A and E and G	Pattern 29	C and E and F
Pattern 15	A and F and G	Pattern 30	C and E and G
		Pattern 31	C and F and G
		Pattern 32	D and E and G
		Pattern 33	D and E and G
		Pattern 34	D and F and G
		Pattern 35	E and F and G

Choosing k among n items without considering the order in which they are chosen is called a *combination*. The number of different ways this can be done is written by using the binomial coefficient notation:

$$\binom{n}{k}$$

which is read “ n choose k .”

In our case,

$$\binom{7}{3} = 35$$

2. In how many ways can the three songs be arranged?

Let's assume we chose the songs A, B, and C. This table illustrates the 6 different ways in which they can be arranged:

Song 1	Song 2	Song 3
A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

Suppose we choose B, E, and G instead:

Song 1	Song 2	Song 3
B	E	G
B	G	E
E	B	G
E	G	B
G	B	E
G	E	B

Trying a few other selections will reveal a pattern: The number of possible arrangements does not depend on which three elements we choose—there are always six of them. Here's why:

Our result (6) can be rewritten as $3 \cdot 2 \cdot 1$, which we get like this:

1. We start out with all three songs and can choose any one of them as our first song.
2. When we're picking our second song, only two remain to choose from.
3. For our last song, we're left with only one choice.

This gives us 3 possibilities \cdot 2 possibilities \cdot 1 possibility = 6 possibilities.

3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

The different possible patterns are

The number of ways
to choose three songs
from seven

· The number of ways
the three songs can
be arranged

$$= \binom{7}{3} \cdot 6$$

$$= 35 \cdot 6$$

$$= 210$$

This means that there are 210 different ways to make the CD.

Choosing three from seven items in a certain order creates a *permutation* of the chosen items. The number of possible permutations of k objects chosen among n objects is written as

$${}_nP_k$$

In our case, this comes to

$${}_7P_3 = 210$$

The number of ways n objects can be chosen among n possible ones is equal to n -factorial:

$${}_nP_n = n! = n \cdot (n - 1) \cdot (n - 2) \cdot \dots \cdot 2 \cdot 1$$

For instance, we could use this if we wanted to know how many different ways seven objects can be arranged. The answer is

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

I've listed all possible ways to choose three songs from the seven original ones (A, B, C, D, E, F, and G) in the table below.

	Song 1	Song 2	Song 3
Pattern 1	A	B	C
Pattern 2	A	B	D
Pattern 3	A	B	E
...
Pattern 30	A	G	F
Pattern 31	B	A	C
...
Pattern 60	B	G	F
Pattern 61	C	A	B
...
Pattern 90	C	G	F
Pattern 91	D	A	B
...
Pattern 120	D	G	F
Pattern 121	E	A	B
...
Pattern 150	E	G	F
Pattern 151	F	A	B
...
Pattern 180	F	G	E
Pattern 181	G	A	B
...
Pattern 209	G	E	F
Pattern 210	G	F	E

We can, analogous to the previous example, rewrite our problem of counting the different ways in which to make a CD as $7 \cdot 6 \cdot 5 = 210$. Here's how we get those numbers:

1. We can choose any of the **7** songs A, B, C, D, E, F, and G as our first song.
2. We can then choose any of the **6** remaining songs as our second song.
3. And finally we choose any of the now **5** remaining songs as our last song.

The definition of the binomial coefficient is as follows:

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-(r-1))}{r \cdot (r-1) \cdots 1} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 1}$$

Notice that

$$\begin{aligned} \binom{n}{r} &= \frac{n \cdot (n-1) \cdots (n-(r-1))}{r \cdot (r-1) \cdots 1} \\ &= \frac{n \cdot (n-1) \cdots (n-(r-1))}{r \cdot (r-1) \cdots 1} \cdot \frac{(n-r) \cdot (n-r+1) \cdots 1}{(n-r) \cdot (n-r+1) \cdots 1} \\ &= \frac{n \cdot (n-1) \cdots (n-(r-1)) \cdot (n-r) \cdot (n-r+1) \cdots 1}{(r \cdot (r-1) \cdots 1) \cdot ((n-r) \cdot (n-r+1) \cdots 1)} \\ &= \frac{n!}{r! \cdot (n-r)!} \end{aligned}$$

Many people find it easier to remember the second version:

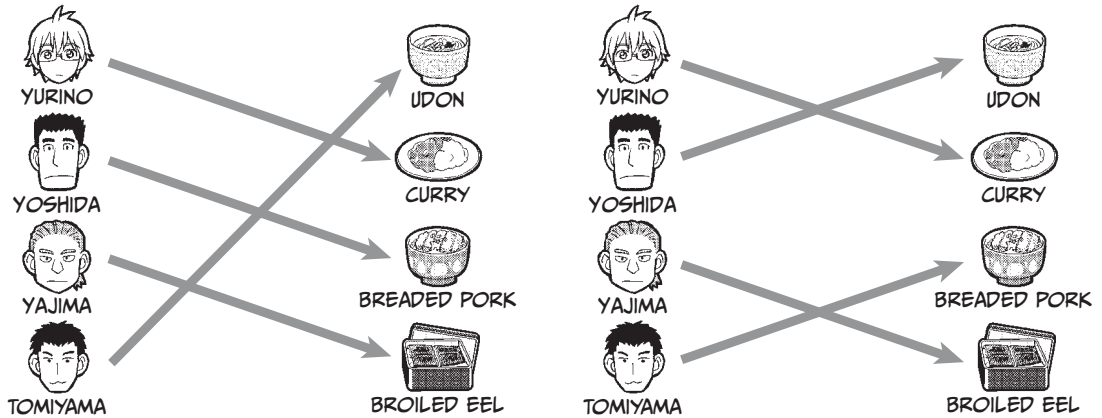
$$\binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}$$

We can rewrite question 3 (how many ways can the CD be made?) like this:

$${}_7P_3 = \binom{7}{3} \cdot 6 = \binom{7}{3} \cdot 3! = \frac{7!}{3! \cdot 4!} \cdot 3! = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 6 \cdot 5 = 210$$

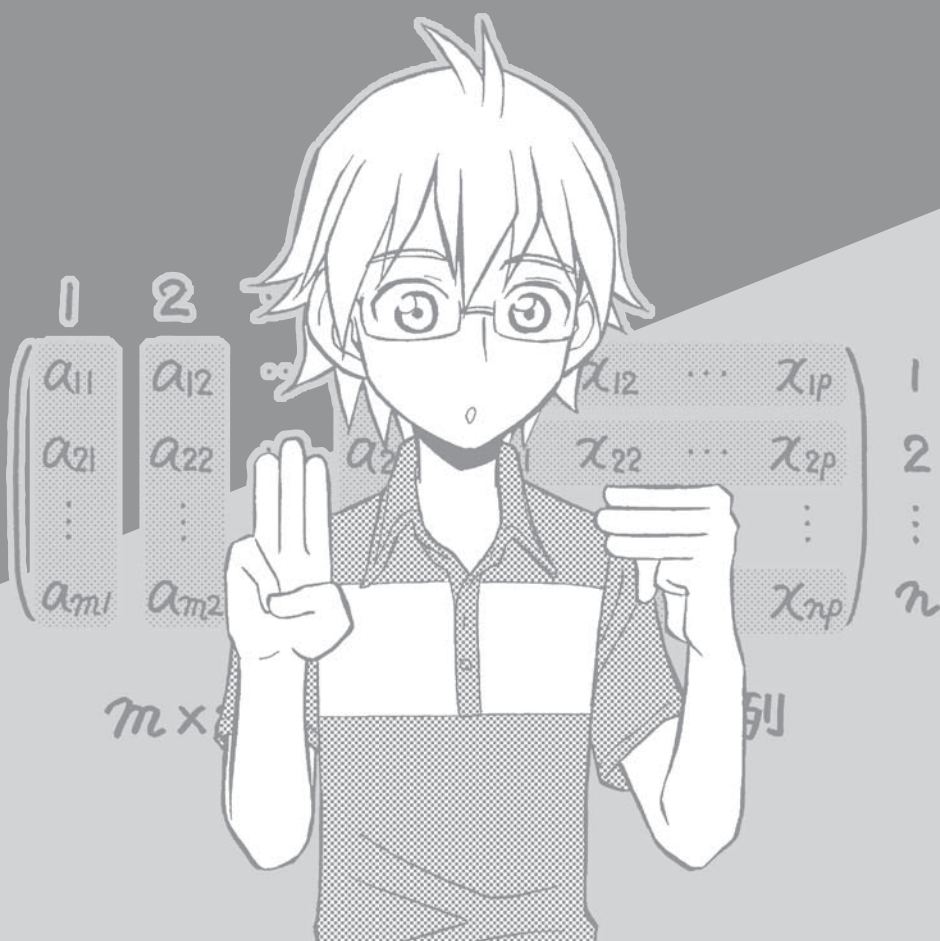
NOT ALL "RULES FOR ORDERING" ARE FUNCTIONS

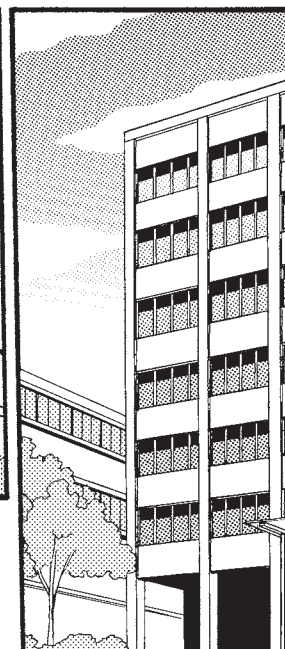
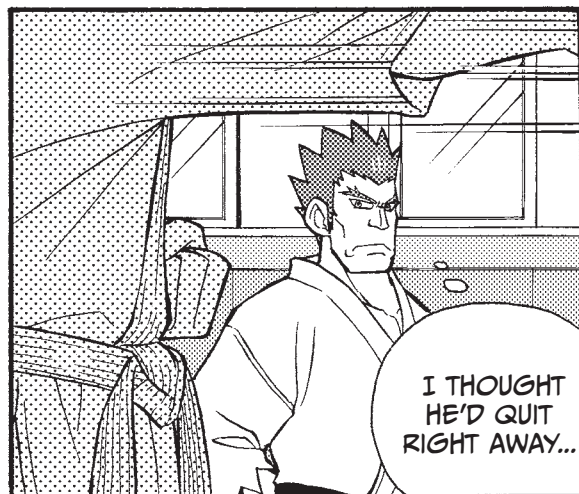
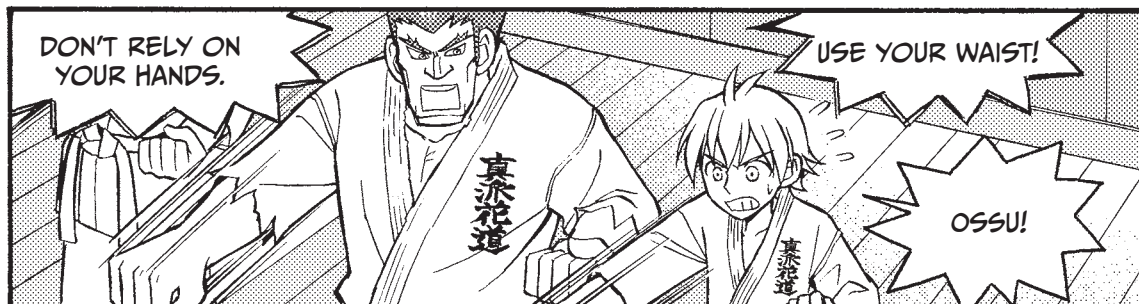
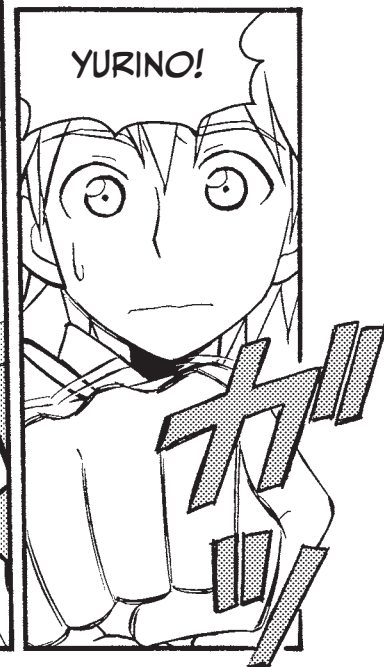
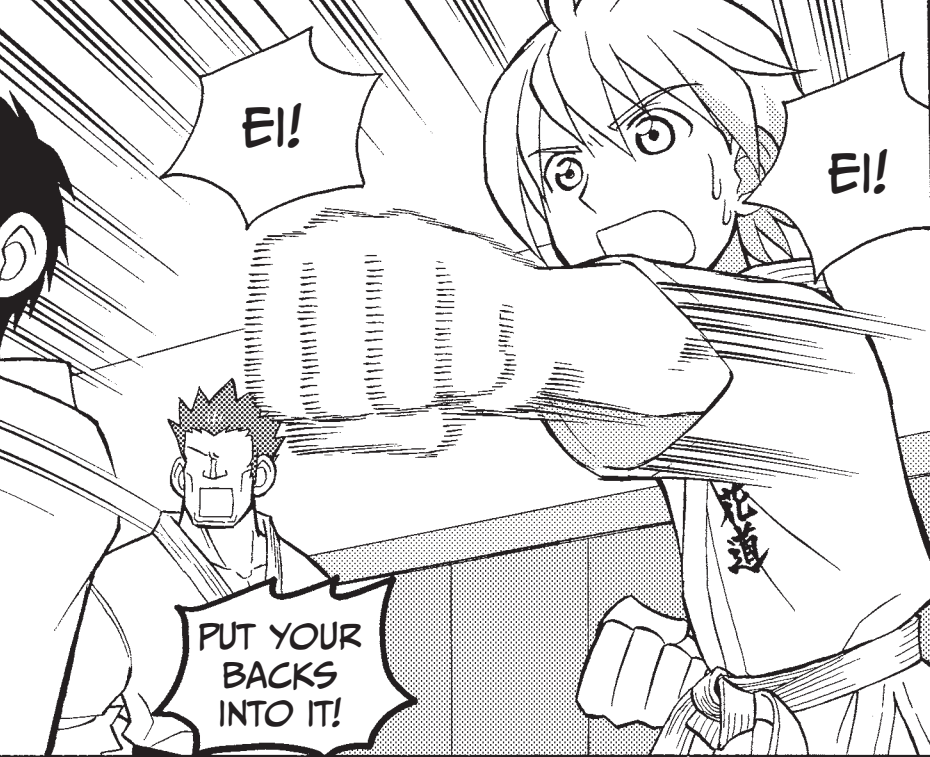
We talked about the three commands "Order the cheapest one!" "Order different stuff!" and "Order what you want!" as functions on pages 37–38. It is important to note, however, that "Order different stuff!" isn't actually a function in the strictest sense, because there are several different ways to obey that command.

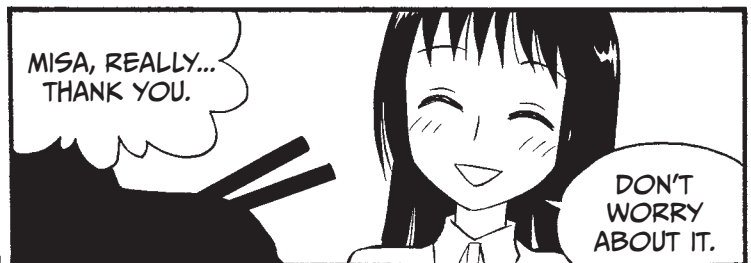
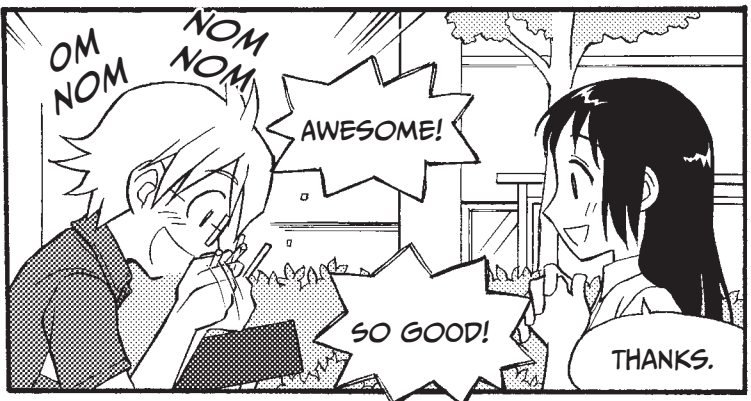
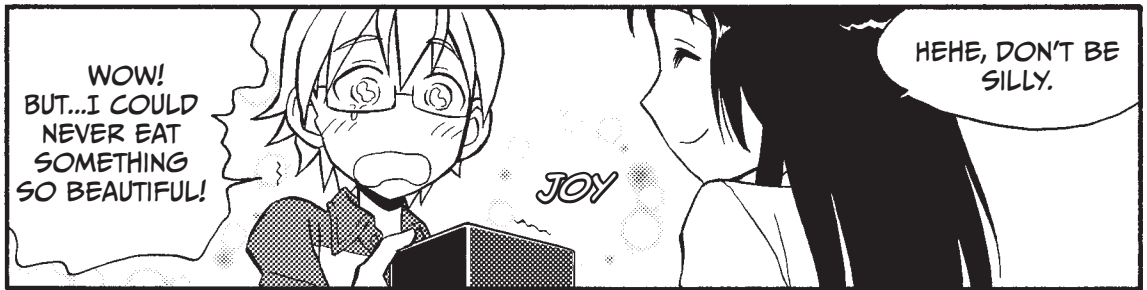
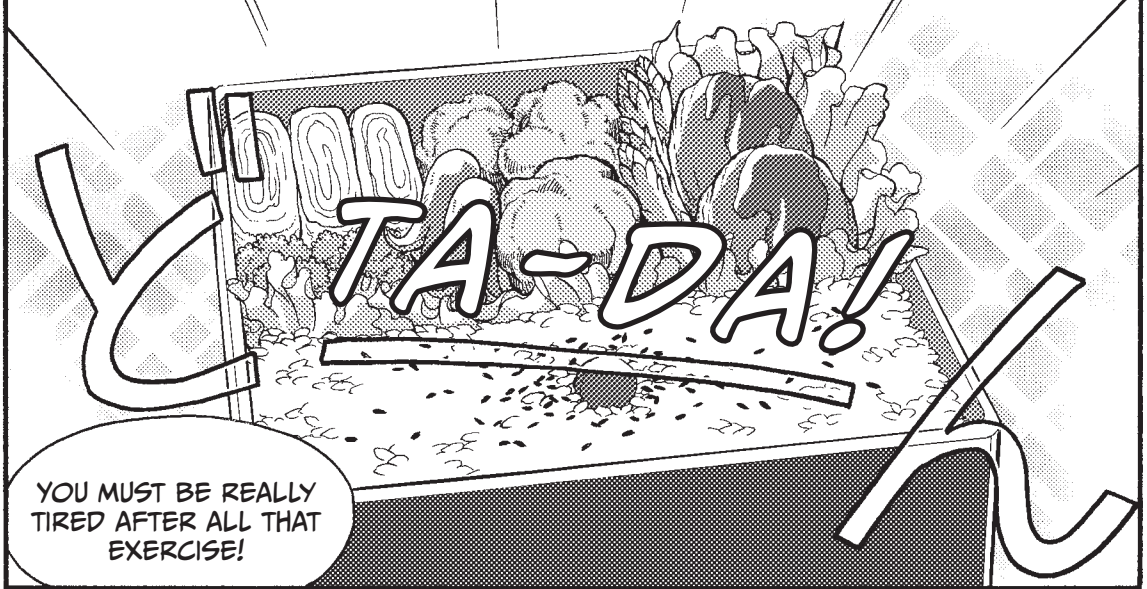


3

INTRO TO MATRICES







AH...

I FEEL A LOT BETTER NOW. ARE YOU READY TO BEGIN?

SURE, WHY NOT.

WE'LL BE TALKING ABOUT MATRICES TODAY.

COURSE LAYOUT

FUNDAMENTALS

MATRICES

VECTO

LINEAR TRANSFORMATIONS

EIGENVALUES
EIGENVECTORS

AND I'D REALLY LIKE TO TAKE MY TIME ON THIS ONE SINCE THEY APPEAR IN MOST PARTS OF LINEAR ALGEBRA.

I DON'T THINK YOU'LL HAVE ANY PROBLEMS WITH THE BASICS THIS TIME AROUND EITHER.

BUT I'LL TALK A LITTLE ABOUT INVERSE MATRICES TOWARD THE END, AND THOSE CAN BE A BIT TRICKY.

OKAY.

WHAT IS A MATRIX?

A MATRIX IS A COLLECTION OF NUMBERS ARRANGED IN m ROWS AND n COLUMNS, BOUNDED BY PARENTHESES, LIKE THIS.

	COLUMN 1	COLUMN 2	...	COLUMN N
ROW 1	a_{11}	a_{12}	\cdots	a_{1n}
ROW 2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\ddots	\vdots
ROW M	a_{m1}	a_{m2}	\cdots	a_{mn}

THESE ARE CALLED SUBSCRIPTS.

A MATRIX WITH
 m ROWS AND
 n COLUMNS IS
 CALLED AN " m BY n
 MATRIX."

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

2x3 MATRIX

$$\begin{pmatrix} -3 \\ 0 \\ 8 \\ -7 \end{pmatrix}$$

4x1 MATRIX

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$m \times n$ MATRIX

AH.

THE ITEMS INSIDE
 A MATRIX ARE
 CALLED ITS
 ELEMENTS.

(ELEMENT)

I'VE MARKED THE (2, 1) ELEMENTS OF
 THESE THREE MATRICES FOR YOU.

	COL 1	COL 2	COL 3
ROW 1	1	2	3
ROW 2	4	5	6

	COL 1
ROW 1	-3
ROW 2	0
ROW 3	8
ROW 4	-7

	COL 1	COL 2	...	COL N
ROW 1	a_{11}	a_{12}	\cdots	a_{1n}
ROW 2	a_{21}	a_{22}	\cdots	a_{2n}
	\vdots	\vdots	\ddots	\vdots
ROW M	a_{m1}	a_{m2}	\cdots	a_{mn}

I SEE.

A MATRIX THAT HAS AN
 EQUAL NUMBER OF ROWS
 AND COLUMNS IS CALLED A
 SQUARE MATRIX.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

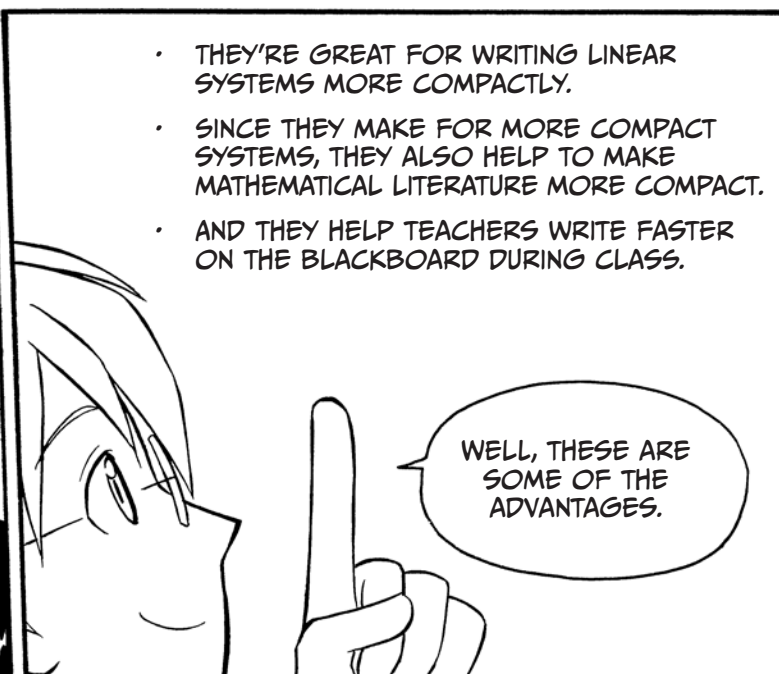
SQUARE MATRIX
 WITH TWO ROWS

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

SQUARE MATRIX
 WITH n ROWS

UH HUH...

THE GRAYED OUT ELEMENTS IN
 THIS MATRIX ARE PART OF WHAT
 IS CALLED ITS MAIN DIAGONAL.



INSTEAD OF WRITING
THIS LINEAR SYSTEM
LIKE THIS...

$$\begin{cases} 1x_1 + 2x_2 = -1 \\ 3x_1 + 4x_2 = 0 \\ 5x_1 + 6x_2 = 5 \end{cases}$$

SKRITCH
SKRITCH

WE COULD WRITE IT LIKE
THIS, USING MATRICES.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$

IT DOES
LOOK A LOT
CLEANER.

EXACTLY!

SO THIS...

$$\begin{cases} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{cases}$$

BECOMES THIS?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

NOT BAD!

WRITING SYSTEMS OF EQUATIONS AS MATRICES

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases} \text{ is written } \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{cases} \text{ is written } \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

MATRIX CALCULATIONS

NOW LET'S
LOOK AT SOME
CALCULATIONS.

THE FOUR RELEVANT
OPERATORS ARE:

- ADDITION
- SUBTRACTION
- SCALAR
MULTIPLICATION
- MATRIX
MULTIPLICATION

ADDITION

LET'S ADD THE 3x2 MATRIX

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

TO THIS 3x2 MATRIX

$$\begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$$

THAT IS:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$$

THE ELEMENTS WOULD BE ADDED
ELEMENTWISE, LIKE THIS:

$$\begin{pmatrix} 1+6 & 2+5 \\ 3+4 & 4+3 \\ 5+2 & 6+1 \end{pmatrix}$$



EXAMPLES

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+6 & 2+5 \\ 3+4 & 4+3 \\ 5+2 & 6+1 \end{pmatrix} = \begin{pmatrix} 7 & 7 \\ 7 & 7 \\ 7 & 7 \end{pmatrix}$$

$$(10, 10) + (-3, -6) = (10 + (-3), 10 + (-6)) = (7, 4)$$

$$\begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 10 + (-3) \\ 10 + (-6) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

NOTE THAT ADDITION AND
SUBTRACTION WORK ONLY
WITH MATRICES THAT HAVE
THE SAME DIMENSIONS.

SUBTRACTION

LET'S SUBTRACT THE 3×2 MATRIX

$$\begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$$

FROM THIS 3×2 MATRIX

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

THAT IS:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix}$$

THE ELEMENTS WOULD SIMILARLY
BE SUBTRACTED ELEMENTWISE,
LIKE THIS:

$$\begin{pmatrix} 1-6 & 2-5 \\ 3-4 & 4-3 \\ 5-2 & 6-1 \end{pmatrix}$$



EXAMPLES

$$\cdot \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 6 & 5 \\ 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1-6 & 2-5 \\ 3-4 & 4-3 \\ 5-2 & 6-1 \end{pmatrix} = \begin{pmatrix} -5 & -3 \\ -1 & 1 \\ 3 & 5 \end{pmatrix}$$

$$\cdot \quad (10, 10) - (-3, -6) = (10 - (-3), 10 - (-6)) = (13, 16)$$

$$\cdot \quad \begin{pmatrix} 10 \\ 10 \end{pmatrix} - \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 10 - (-3) \\ 10 - (-6) \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \end{pmatrix}$$

SCALAR MULTIPLICATION

LET'S MULTIPLY THE 3×2 MATRIX

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

BY 10. THAT IS:

$$10 \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

THE ELEMENTS WOULD EACH BE MULTIPLIED BY 10, LIKE THIS:

$$\begin{pmatrix} 10 \cdot 1 & 10 \cdot 2 \\ 10 \cdot 3 & 10 \cdot 4 \\ 10 \cdot 5 & 10 \cdot 6 \end{pmatrix}$$



EXAMPLES

$$\cdot \quad 10 \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 10 \cdot 1 & 10 \cdot 2 \\ 10 \cdot 3 & 10 \cdot 4 \\ 10 \cdot 5 & 10 \cdot 6 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 30 & 40 \\ 50 & 60 \end{pmatrix}$$

$$\cdot \quad 2 (3, 1) = (2 \cdot 3, 2 \cdot 1) = (6, 2)$$

$$\cdot \quad 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

MATRIX MULTIPLICATION



THE PRODUCT $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 & 1y_1 + 2y_2 \\ 3x_1 + 4x_2 & 3y_1 + 4y_2 \\ 5x_1 + 6x_2 & 5y_1 + 6y_2 \end{pmatrix}$

CAN BE DERIVED BY TEMPORARILY SEPARATING THE TWO COLUMNS $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ AND $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, FORMING THE TWO PRODUCTS

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{pmatrix} \quad \text{AND} \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1y_1 + 2y_2 \\ 3y_1 + 4y_2 \\ 5y_1 + 6y_2 \end{pmatrix}$$

AND THEN REJOINING THE RESULTING COLUMNS:

$$\begin{pmatrix} 1x_1 + 2x_2 & 1y_1 + 2y_2 \\ 3x_1 + 4x_2 & 3y_1 + 4y_2 \\ 5x_1 + 6x_2 & 5y_1 + 6y_2 \end{pmatrix}$$

EXAMPLE

$$\cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 2x_2 & 1y_1 + 2y_2 \\ 3x_1 + 4x_2 & 3y_1 + 4y_2 \\ 5x_1 + 6x_2 & 5y_1 + 6y_2 \end{pmatrix}$$

THERE'S MORE!



AS YOU CAN SEE FROM THE EXAMPLE BELOW,
CHANGING THE ORDER OF FACTORS USUALLY
RESULTS IN A COMPLETELY DIFFERENT PRODUCT.



$$\cdot \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 \cdot 3 + (-3) \cdot 1 & 8 \cdot 1 + (-3) \cdot 2 \\ 2 \cdot 3 + 1 \cdot 1 & 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 24 - 3 & 8 - 6 \\ 6 + 1 & 2 + 2 \end{pmatrix} = \begin{pmatrix} 21 & 2 \\ 7 & 4 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 8 + 1 \cdot 2 & 3 \cdot (-3) + 1 \cdot 1 \\ 1 \cdot 8 + 2 \cdot 2 & 1 \cdot (-3) + 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 24 + 2 & -9 + 1 \\ 8 + 4 & -3 + 2 \end{pmatrix} = \begin{pmatrix} 26 & -8 \\ 12 & -1 \end{pmatrix}$$

AND YOU HAVE
TO WATCH OUT.

$$\begin{matrix} & 1 & 2 & \cdots & n \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} & \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix} \end{matrix}$$

AN $m \times n$ MATRIX TIMES AN $n \times p$ MATRIX
YIELDS AN $m \times p$ MATRIX.

MATRICES CAN BE MULTIPLIED *ONLY* IF THE
NUMBER OF COLUMNS IN THE LEFT FACTOR
MATCHES THE NUMBER OF ROWS IN THE
RIGHT FACTOR.



THIS MEANS WE WOULDN'T BE ABLE TO CALCULATE THE PRODUCT IF WE SWITCHED THE TWO MATRICES IN OUR FIRST EXAMPLE.

HUH, REALLY?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 & 1y_1 \\ 3x_1 & 3y_1 \\ 5x_1 & 5y_1 \end{pmatrix}$$

$$\cancel{\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}}$$

WELL, NOTHING STOPS US FROM TRYING.

PRODUCT OF
3x2 AND 2x2
FACTORS

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$ IS THE SAME AS $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ AND $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ WHICH IS THE SAME AS

$$\begin{cases} 1x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{cases} \quad \begin{cases} 1y_1 + 2y_2 \\ 3y_1 + 4y_2 \\ 5y_1 + 6y_2 \end{cases} \quad \text{IN THE SAME MATRIX.}$$

PRODUCT OF
2x2 AND 3x2
FACTORS

$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ IS THE SAME AS $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$ AND $\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ WHICH IS THE SAME AS

$$\begin{cases} x_1 \cdot 1 + y_1 \cdot 3 + ? \cdot 5 \\ x_2 \cdot 1 + y_2 \cdot 3 + ? \cdot 5 \end{cases} \quad \text{AND} \quad \begin{cases} x_1 \cdot 2 + y_1 \cdot 4 + ? \cdot 6 \\ x_2 \cdot 2 + y_2 \cdot 4 + ? \cdot 6 \end{cases} \quad \text{IN THE SAME MATRIX.}$$

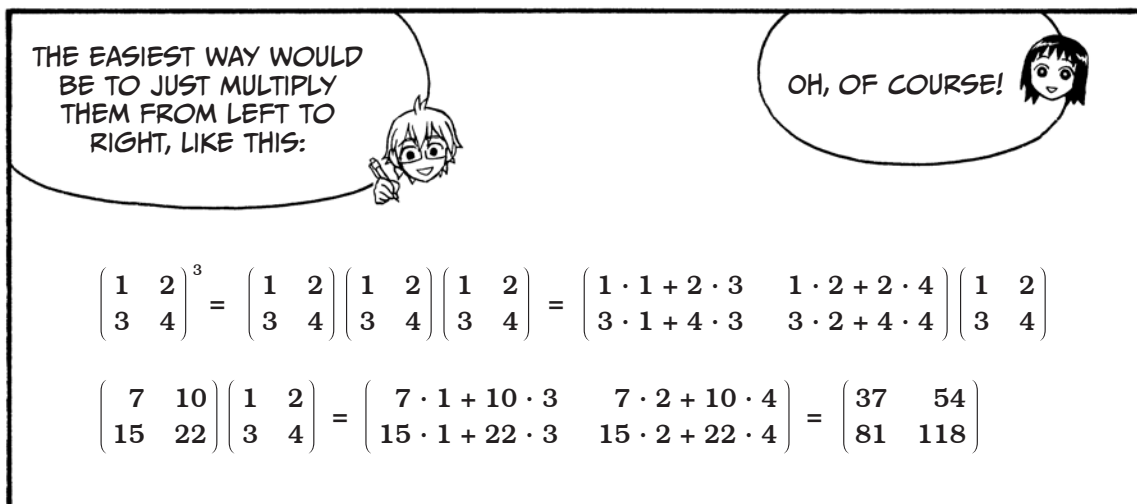
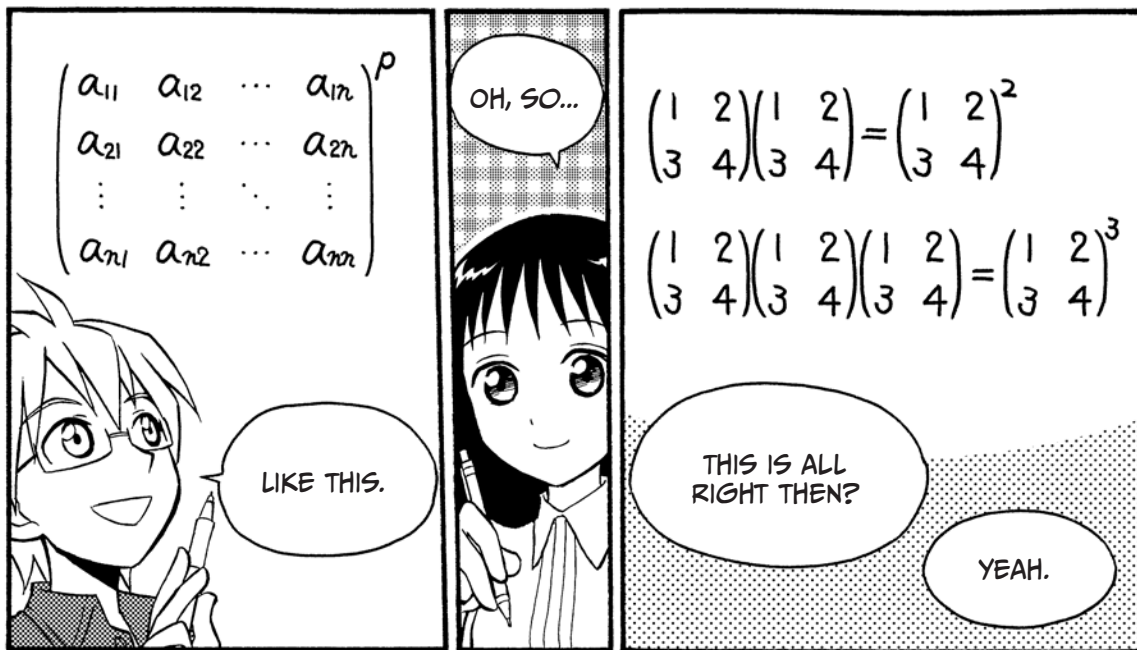
WE RUN INTO A PROBLEM HERE:
THERE ARE NO ELEMENTS
CORRESPONDING TO THESE
POSITIONS!

OOPS...

ONE MORE THING.
IT'S OKAY TO USE
EXPONENTS TO
EXPRESS REPEATED
MULTIPLICATION OF
SQUARE MATRICES.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdots \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

P FACTORS



SPECIAL MATRICES

THERE ARE MANY SPECIAL TYPES OF MATRICES.

TO EXPLAIN THEM ALL WOULD TAKE TOO MUCH TIME...

SO WE'LL LOOK AT ONLY THESE EIGHT TODAY.

- ① ZERO MATRICES
- ② TRANSPOSE MATRICES
- ③ SYMMETRIC MATRICES
- ④ UPPER TRIANGULAR MATRICES
- ⑤ LOWER TRIANGULAR MATRICES
- ⑥ DIAGONAL MATRICES
- ⑦ IDENTITY MATRICES
- ⑧ INVERSE MATRICES

LET'S LOOK AT THEM IN ORDER.

OKAY!

① ZERO MATRICES

0

A zero matrix is a matrix where all elements are equal to zero.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

② TRANSPOSE MATRICES



The easiest way to understand transpose matrices is to just look at an example.

If we transpose the 2×3 matrix $\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

we get the 3×2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

As you can see, the transpose operator switches the rows and columns in a matrix.

The transpose of the $n \times m$ matrix $\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

is consequently $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

The most common way to indicate a transpose is to add a small T at the top-right corner of the matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T$$

AH, T FOR
TRANSPOSE.
I SEE.

For example:

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$



③ SYMMETRIC MATRICES



Symmetric matrices are square matrices that are symmetric around their main diagonals.

$$\begin{pmatrix} 1 & 5 & 6 & 7 \\ 5 & 2 & 8 & 9 \\ 6 & 8 & 3 & 10 \\ 7 & 9 & 10 & 4 \end{pmatrix}$$

Because of this characteristic, a symmetric matrix is always equal to its transpose.

④ UPPER TRIANGULAR AND ⑤ LOWER TRIANGULAR MATRICES



Triangular matrices are square matrices in which the elements either above the main diagonal or below it are all equal to zero.

This is an upper triangular matrix, since all elements *below* the main diagonal are zero.

$$\begin{pmatrix} 1 & 5 & 6 & 7 \\ 0 & 2 & 8 & 9 \\ 0 & 0 & 3 & 10 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

This is a lower triangular matrix—all elements *above* the main diagonal are zero.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 5 & 2 & 0 & 0 \\ 6 & 8 & 3 & 0 \\ 7 & 9 & 10 & 4 \end{pmatrix}$$

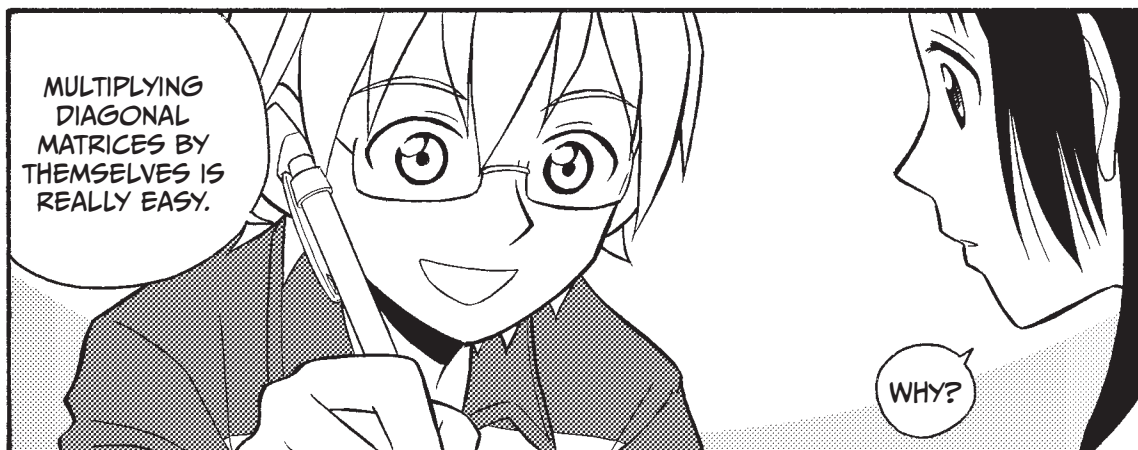
⑥ DIAGONAL MATRICES



A diagonal matrix is a square matrix in which all elements that are not part of its main diagonal are equal to zero.

For example, $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is a diagonal matrix.

Note that this matrix could also be written as $\text{diag}(1,2,3,4)$.



SEE FOR YOURSELF!



$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^P = \begin{pmatrix} a_{11}^P & 0 & \cdots & 0 \\ 0 & a_{22}^P & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^P \end{pmatrix}$$

UH...



TRY CALCULATING

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 \quad \text{AND} \quad \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^3$$

TO SEE WHY.

HMM...

$$\cdot \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 + 0 \cdot 0 & 2 \cdot 0 + 0 \cdot 3 \\ 0 \cdot 2 + 3 \cdot 0 & 0 \cdot 0 + 3 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 \cdot 2 + 0 \cdot 0 & 2^2 \cdot 0 + 0 \cdot 3 \\ 0 \cdot 2 + 3^2 \cdot 0 & 0 \cdot 0 + 3^2 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 3^3 \end{pmatrix}$$

LIKE THIS?

YOU'RE RIGHT!

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^P = \begin{pmatrix} 2^P & 0 \\ 0 & 3^P \end{pmatrix}$$

WEIRD, HUH?

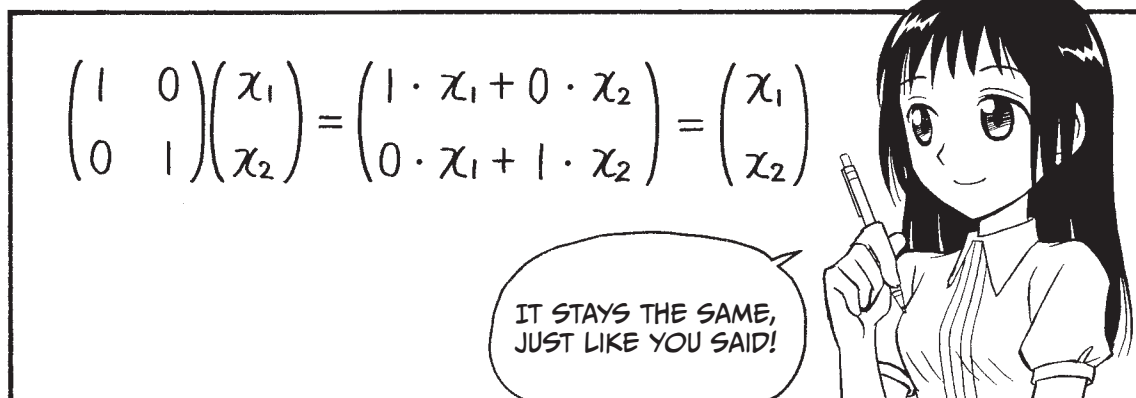
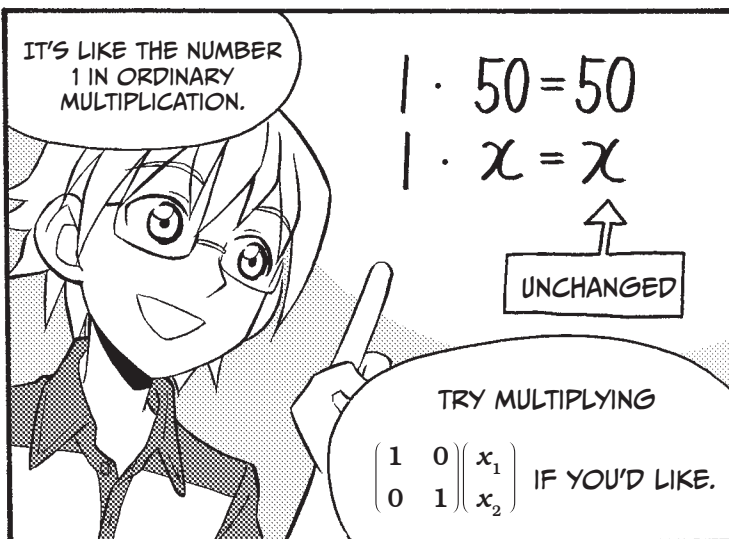
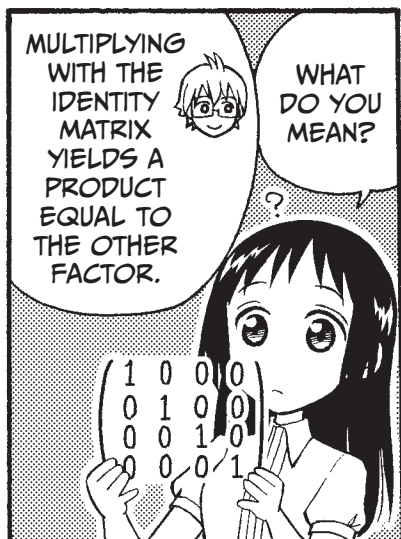
7 IDENTITY MATRICES



Identity matrices are in essence $\text{diag}(1, 1, 1, \dots, 1)$. In other words, they are square matrices with n rows in which all elements on the main diagonal are equal to 1 and all other elements are 0.

For example, an identity matrix with $n = 4$ would look like this:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



LET'S TRY A FEW OTHER EXAMPLES.

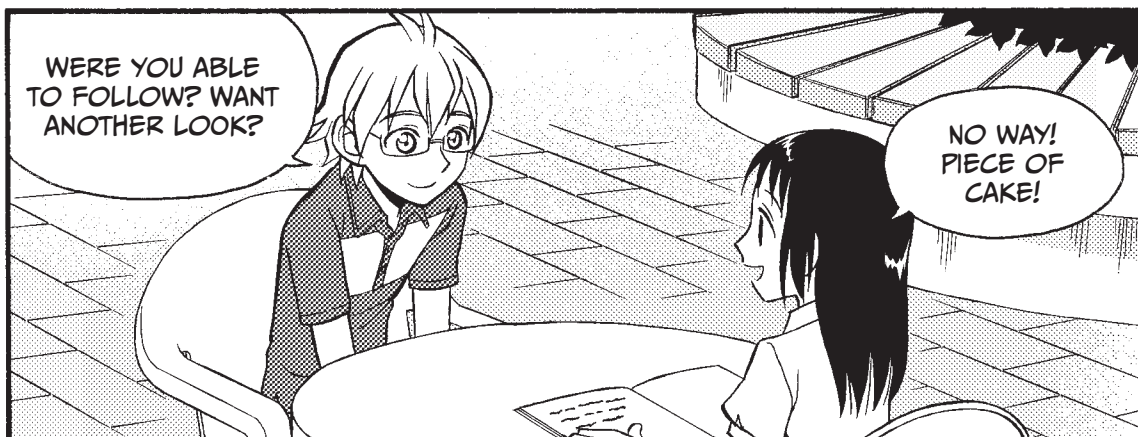


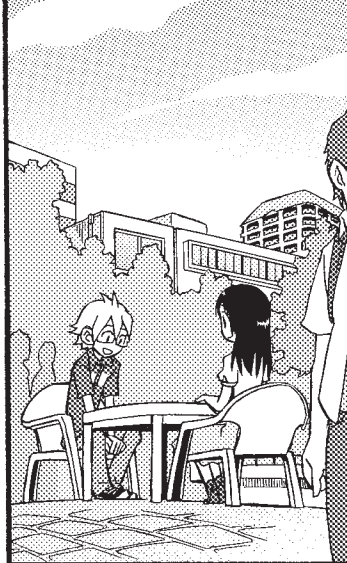
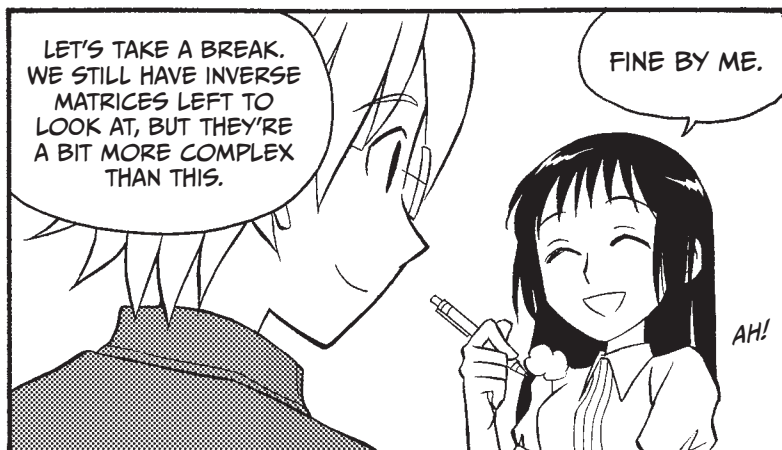
$$\cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n \\ 0 \cdot x_1 + 1 \cdot x_2 + \cdots + 0 \cdot x_n \\ \vdots \\ 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 1 \cdot x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \end{pmatrix} = \begin{pmatrix} 1 \cdot x_{11} + 0 \cdot x_{12} & 1 \cdot x_{21} + 0 \cdot x_{22} & \cdots & 1 \cdot x_{n1} + 0 \cdot x_{n2} \\ 0 \cdot x_{11} + 1 \cdot x_{12} & 0 \cdot x_{21} + 1 \cdot x_{22} & \cdots & 0 \cdot x_{n1} + 1 \cdot x_{n2} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \end{pmatrix}$$

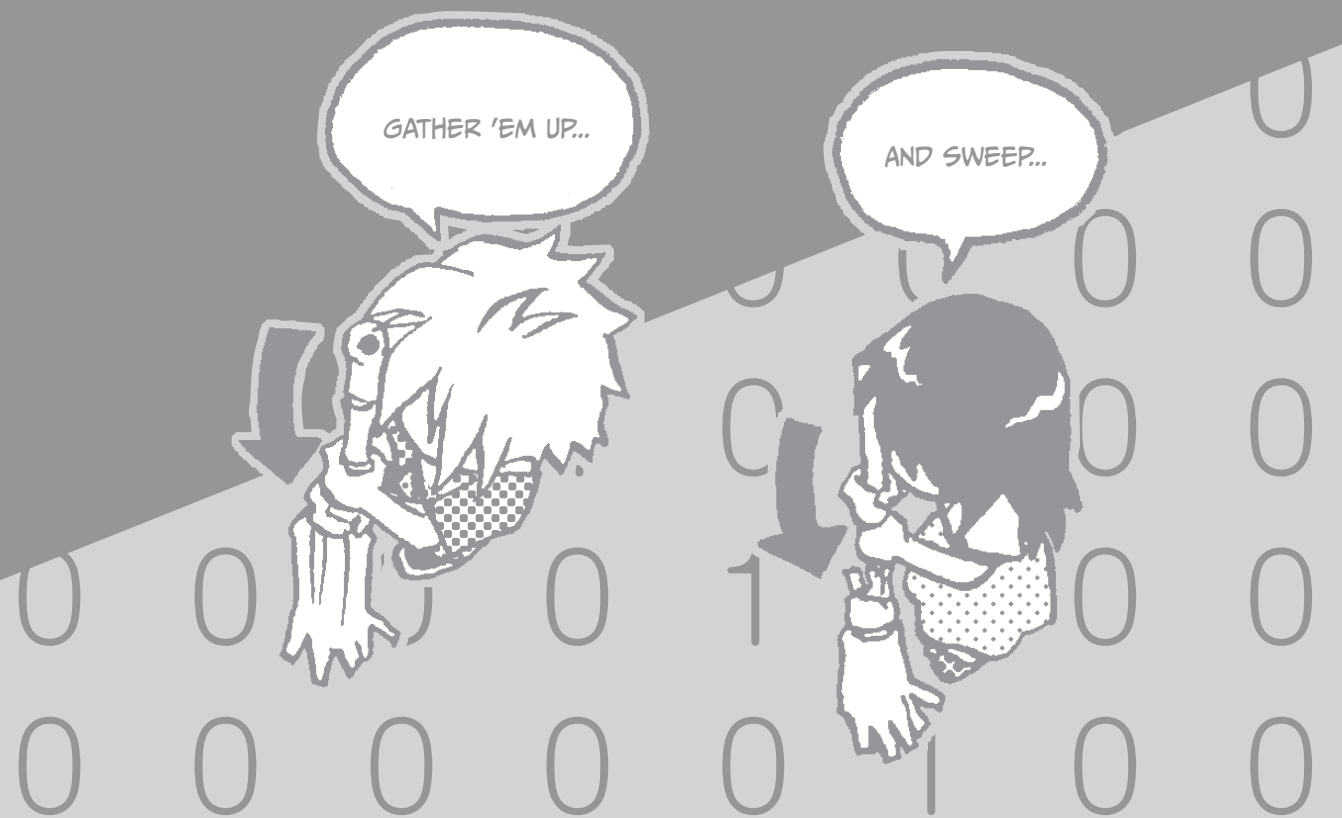
$$\cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ x_{n1} & x_{n2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_{11} \cdot 1 + x_{12} \cdot 0 & x_{11} \cdot 0 + x_{12} \cdot 1 \\ x_{21} \cdot 1 + x_{22} \cdot 0 & x_{21} \cdot 0 + x_{22} \cdot 1 \\ \vdots & \vdots \\ x_{n1} \cdot 1 + x_{n2} \cdot 0 & x_{n1} \cdot 0 + x_{n2} \cdot 1 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ \vdots & \vdots \\ x_{n1} & x_{n2} \end{pmatrix}$$





4

MORE MATRICES



INVERSE MATRICES



8 INVERSE MATRICES

If the product of two square matrices is an identity matrix, then the two factor matrices are *inverses* of each other.

This means that $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is an inverse matrix to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ if

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$





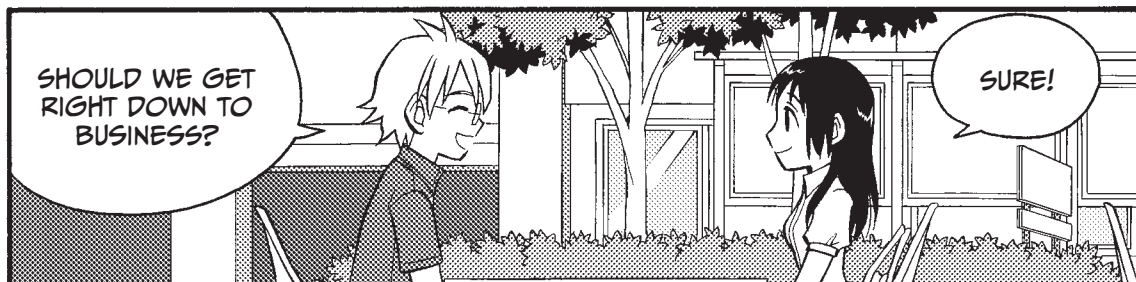
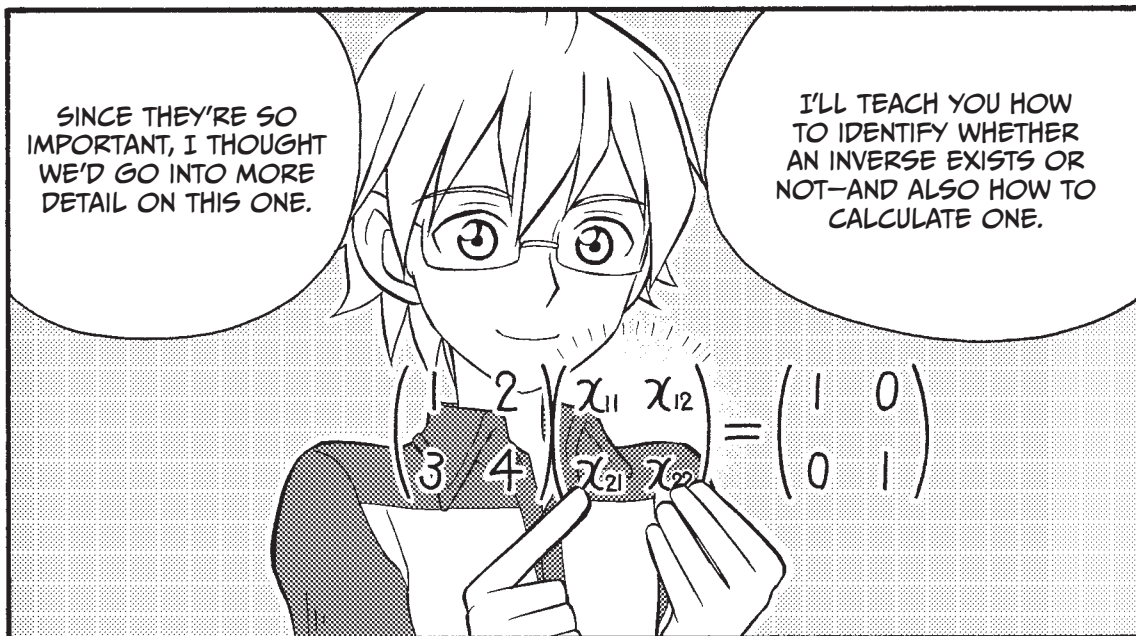
AND THAT'S IT.



DIDN'T YOU SAY
SOMETHING ABOUT
EXAMPLES? WE'RE
DONE ALREADY?



DON'T WORRY, THAT
WAS ONLY THE
DEFINITION!



SHOULD WE GET
RIGHT DOWN TO
BUSINESS?

SURE!

CALCULATING INVERSE MATRICES

COFACTOR
METHOD

GAUSSIAN
ELIMINATION

THERE ARE TWO MAIN WAYS TO
CALCULATE AN INVERSE MATRIX:

USING COFACTORS OR USING
GAUSSIAN ELIMINATION.

THE CALCULATIONS
INVOLVED IN THE
COFACTOR METHOD CAN
VERY EASILY BECOME
CUMBERSOME, SO...

COFACTOR METHOD

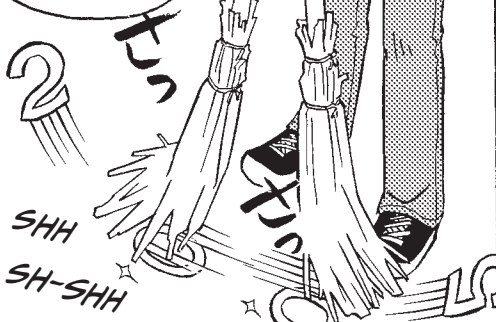
IGNORE IT AS LONG AS
YOU'RE NOT EXPECTING
IT ON A TEST.

CAN
DO.



IN CONTRAST,
GAUSSIAN
ELIMINATION IS
EASY BOTH TO
UNDERSTAND
AND TO
CALCULATE.

IN FACT, IT'S
AS EASY AS
SWEEPING THE
FLOOR!*



ANYWAY, I WON'T TALK
ABOUT COFACTORS AT
ALL TODAY.

GOTCHA.

IN ADDITION TO
FINDING INVERSE
MATRICES, GAUSSIAN
ELIMINATION CAN ALSO
BE USED TO SOLVE
LINEAR SYSTEMS.

LET'S HAVE A
LOOK AT THAT.

COOL!

* THE JAPANESE TERM FOR GAUSSIAN ELIMINATION IS HAKIDASHIHOU, WHICH ROUGHLY TRANSLATES TO "THE SWEEPING OUT METHOD." KEEP THIS IN MIND AS YOU'RE READING THIS CHAPTER!

PROBLEM

Solve the following linear system:

$$\begin{cases} 3x_1 + 1x_2 = 1 \\ 1x_1 + 2x_2 = 0 \end{cases}$$

SOLUTION

THE COMMON METHOD	THE COMMON METHOD EXPRESSED WITH MATRICES	GAUSSIAN ELIMINATION
$\begin{cases} 3x_1 + 1x_2 = 1 \\ 1x_1 + 2x_2 = 0 \end{cases}$ <p>Start by multiplying the top equation by 2.</p>	$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 6x_1 + 2x_2 = 2 \\ 1x_1 + 2x_2 = 0 \end{cases}$ <p>Subtract the bottom equation from the top equation.</p>	$\begin{pmatrix} 6 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 6 & 2 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 1x_1 + 2x_2 = 0 \end{cases}$ <p>Multiply the bottom equation by 5.</p>	$\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 5x_1 + 10x_2 = 0 \end{cases}$ <p>Subtract the top equation from the bottom equation.</p>	$\begin{pmatrix} 5 & 0 \\ 5 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 5 & 10 & 0 \end{pmatrix}$
$\begin{cases} 5x_1 + 0x_2 = 2 \\ 0x_1 + 10x_2 = -2 \end{cases}$ <p>Divide the top equation by 5 and the bottom by 10.</p>	$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 \\ 0 & 10 & -2 \end{pmatrix}$
$\begin{cases} 1x_1 + 0x_2 = \frac{2}{5} \\ 0x_1 + 1x_2 = -\frac{1}{5} \end{cases}$ <p>And we're done!</p>	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{2}{5} \\ 0 & 1 & -\frac{1}{5} \end{pmatrix}$

SO YOU JUST REWRITE THE EQUATIONS AS MATRICES AND CALCULATE AS USUAL?

WELL...

GAUSSIAN ELIMINATION IS ABOUT TRYING TO GET THIS PART HERE TO APPROACH THE IDENTITY MATRIX, NOT ABOUT SOLVING FOR VARIABLES.

HMM...



PROBLEM

Find the inverse of the 2×2 matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$



We're trying to find the inverse of $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

↓

We need to find the matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ that satisfies $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

↓

or $\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ and $\begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ that satisfy $\begin{cases} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

↓

We need to solve the systems $\begin{cases} 3x_{11} + 1x_{21} = 1 \\ 1x_{11} + 2x_{21} = 0 \end{cases}$ and $\begin{cases} 3x_{12} + 1x_{22} = 0 \\ 1x_{12} + 2x_{22} = 1 \end{cases}$

8 SOLUTION

THE COMMON METHOD	THE COMMON METHOD EXPRESSED WITH MATRICES	GAUSSIAN ELIMINATION
$\begin{cases} 3x_{11} + 1x_{21} = 1 \\ 1x_{11} + 2x_{21} = 0 \end{cases} \quad \begin{cases} 3x_{12} + 1x_{22} = 0 \\ 1x_{12} + 2x_{22} = 1 \end{cases}$ <p>Multiply the top equation by 2.</p>	$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$
$\begin{cases} 6x_{11} + 2x_{21} = 2 \\ 1x_{11} + 2x_{21} = 0 \end{cases} \quad \begin{cases} 6x_{12} + 2x_{22} = 0 \\ 1x_{12} + 2x_{22} = 1 \end{cases}$ <p>Subtract the bottom equation from the top.</p>	$\begin{pmatrix} 6 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 & 2 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$
$\begin{cases} 5x_{11} + 0x_{21} = 2 \\ 1x_{11} + 2x_{21} = 0 \end{cases} \quad \begin{cases} 5x_{12} + 0x_{22} = -1 \\ 1x_{12} + 2x_{22} = 1 \end{cases}$ <p>Multiply the bottom equation by 5.</p>	$\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 & -1 \\ 1 & 2 & 0 & 1 \end{pmatrix}$
$\begin{cases} 5x_{11} + 0x_{21} = 2 \\ 5x_{11} + 10x_{21} = 0 \end{cases} \quad \begin{cases} 5x_{12} + 0x_{22} = -1 \\ 5x_{12} + 10x_{22} = 5 \end{cases}$ <p>Subtract the top equation from the bottom.</p>	$\begin{pmatrix} 5 & 0 \\ 5 & 10 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 5 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 & -1 \\ 5 & 10 & 0 & 5 \end{pmatrix}$
$\begin{cases} 5x_{11} + 0x_{21} = 2 \\ 0x_{11} + 10x_{21} = -2 \end{cases} \quad \begin{cases} 5x_{12} + 0x_{22} = -1 \\ 0x_{12} + 10x_{22} = 6 \end{cases}$ <p>Divide the top by 5 and the bottom by 10.</p>	$\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & 6 \end{pmatrix}$	$\begin{pmatrix} 5 & 0 & 2 & -1 \\ 0 & 10 & -2 & 6 \end{pmatrix}$
$\begin{cases} 1x_{11} + 0x_{21} = \frac{2}{5} \\ 0x_{11} + 1x_{21} = -\frac{1}{5} \end{cases} \quad \begin{cases} 1x_{12} + 0x_{22} = -\frac{1}{5} \\ 0x_{12} + 1x_{22} = \frac{3}{5} \end{cases}$ <p>This is our inverse matrix; we're done!</p>	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$



LET'S MAKE SURE THAT THE PRODUCT OF THE ORIGINAL AND CALCULATED MATRICES REALLY IS THE IDENTITY MATRIX.



The product of the original and inverse matrix is

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 3 \cdot \frac{2}{5} + 1 \cdot \left(-\frac{1}{5}\right) & 3 \cdot \left(-\frac{1}{5}\right) + 1 \cdot \frac{3}{5} \\ 1 \cdot \frac{2}{5} + 2 \cdot \left(-\frac{1}{5}\right) & 1 \cdot \left(-\frac{1}{5}\right) + 2 \cdot \frac{3}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product of the inverse and original matrix is

$$\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \cdot 3 + \left(-\frac{1}{5}\right) \cdot 1 & \frac{2}{5} \cdot 1 + \left(-\frac{1}{5}\right) \cdot 2 \\ \left(-\frac{1}{5}\right) \cdot 3 + \frac{3}{5} \cdot 1 & \left(-\frac{1}{5}\right) \cdot 1 + \frac{3}{5} \cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



IT SEEMS LIKE THEY BOTH BECOME THE IDENTITY MATRIX...

THAT'S AN IMPORTANT POINT: THE ORDER OF THE FACTORS DOESN'T MATTER. THE PRODUCT IS ALWAYS THE IDENTITY MATRIX! REMEMBERING THIS TEST IS VERY USEFUL. YOU SHOULD USE IT AS OFTEN AS YOU CAN TO CHECK YOUR CALCULATIONS.



BY THE WAY...



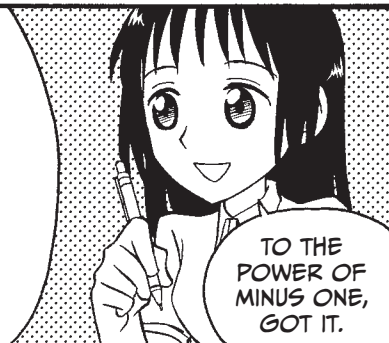
THE SYMBOL USED TO DENOTE INVERSE MATRICES IS THE SAME AS ANY INVERSE IN MATHEMATICS, SO...

THE INVERSE OF

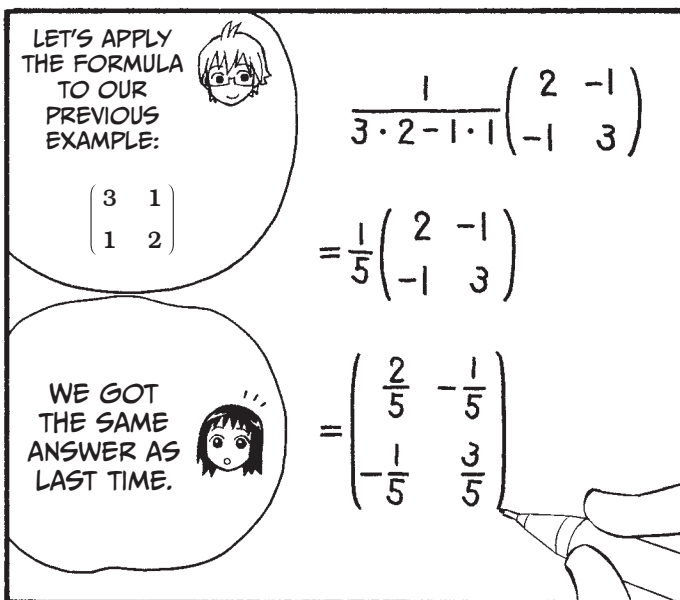
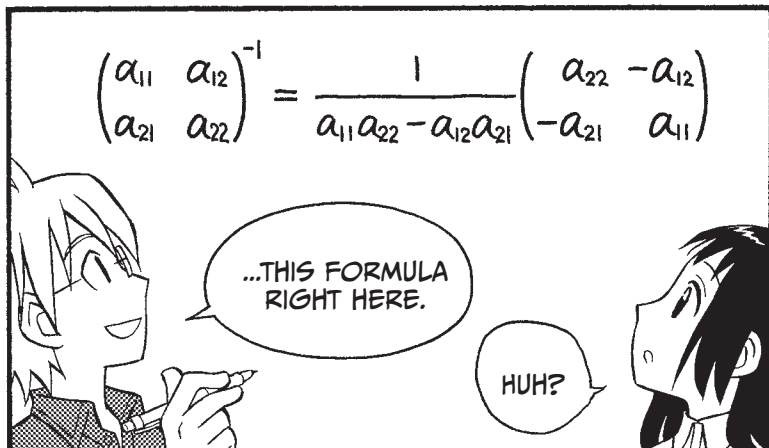
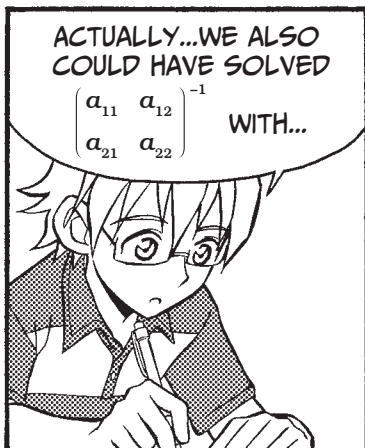
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

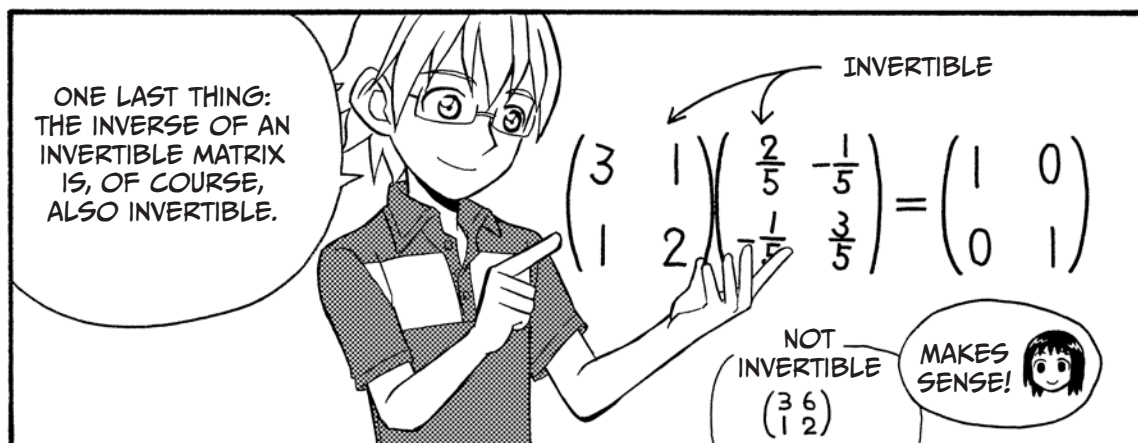
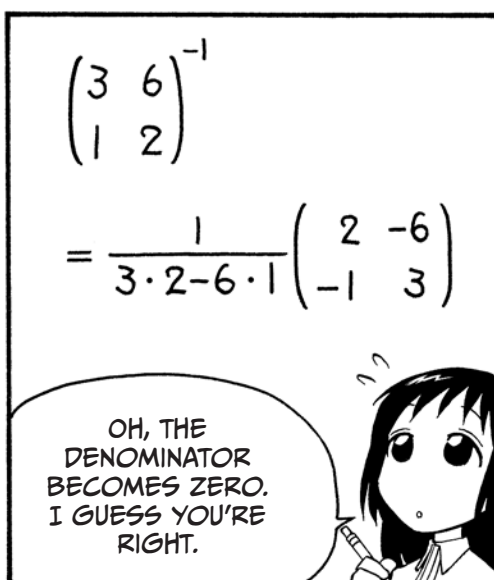
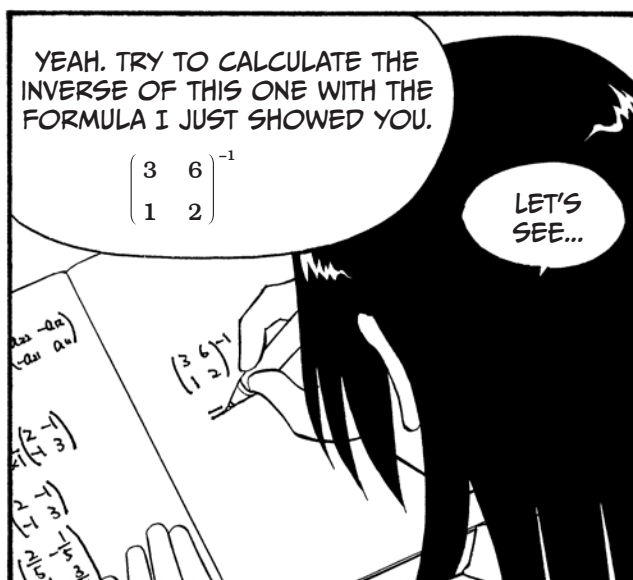
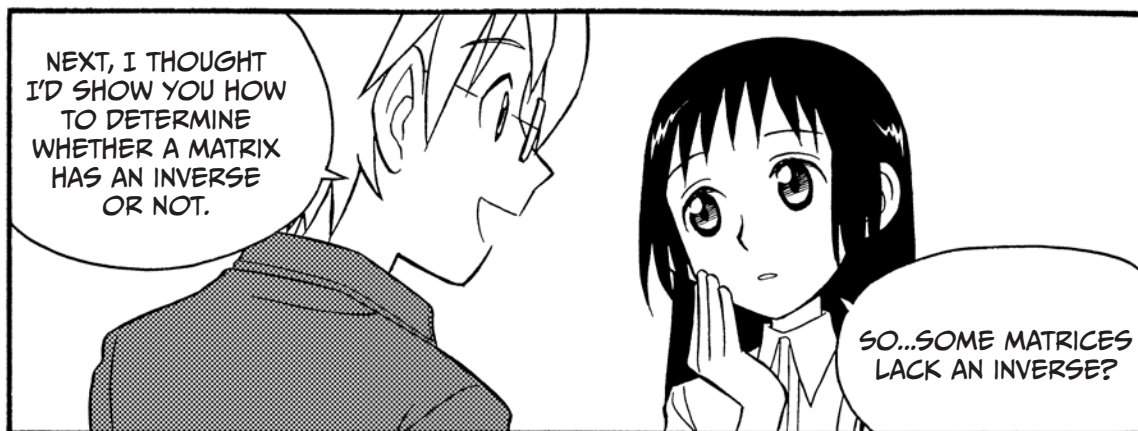
IS WRITTEN AS

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}^{-1}$$



TO THE POWER OF MINUS ONE, GOT IT.





DETERMINANTS

NOW FOR
THE TEST TO
SEE WHETHER
A MATRIX IS
INVERTIBLE
OR NOT.

WE'LL BE USING
THIS FUNCTION.

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

IT'S ALSO
WRITTEN
WITH
STRAIGHT
BARS, LIKE
THIS:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

DET?



determinant

IT'S SHORT FOR
DETERMINANT.



DOES A GIVEN MATRIX HAVE AN INVERSE?

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \neq 0 \text{ means that } \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \text{ exists.}$$

THE INVERSE OF
A MATRIX EXISTS
AS LONG AS ITS
DETERMINANT
ISN'T ZERO.

HMM.

CALCULATING DETERMINANTS

$n=2$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$n=3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

THERE ARE SEVERAL DIFFERENT WAYS TO CALCULATE A DETERMINANT. WHICH ONE'S BEST DEPENDS ON THE SIZE OF THE MATRIX.

LET'S START WITH THE FORMULA FOR TWO-DIMENSIONAL MATRICES AND WORK OUR WAY UP.

SOUNDS GOOD.

TO FIND THE DETERMINANT OF A 2×2 MATRIX, JUST SUBSTITUTE THE EXPRESSION LIKE THIS.



$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

HOLDING YOUR FINGERS LIKE THIS MAKES FOR A GOOD TRICK TO REMEMBER THE FORMULA.



$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Diagram showing the calculation of the determinant for a 2x2 matrix. Above the matrix, there are two columns of signs: ① + and ② -. A diagonal line crosses from the top-left element a_{11} to the bottom-right element a_{22} , and another diagonal line crosses from the top-right element a_{12} to the bottom-left element a_{21} .

OH, COOL!



LET'S SEE WHETHER $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ HAS AN INVERSE OR NOT.



$$\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = 3 \cdot 2 - 0 \cdot 0 = 6$$



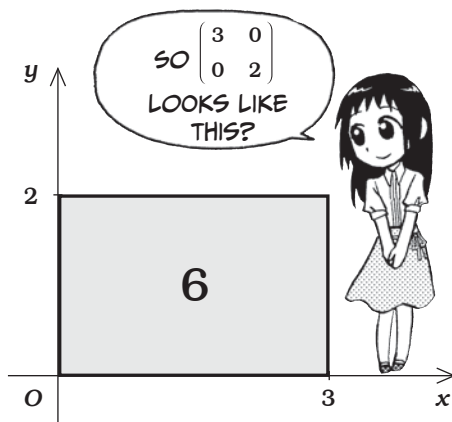
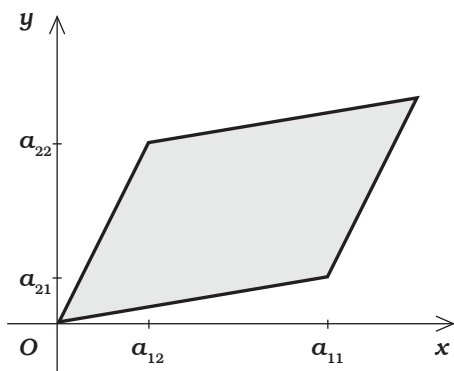
IT DOES, SINCE $\det \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \neq 0$.

INCIDENTALLY, THE AREA OF THE PARALLELOGRAM SPANNED BY THE FOLLOWING FOUR POINTS...

- THE ORIGIN
- THE POINT (a_{11}, a_{21})
- THE POINT (a_{12}, a_{22})
- THE POINT $(a_{11} + a_{12}, a_{21} + a_{22})$

...COINCIDES WITH THE ABSOLUTE VALUE OF

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



SO $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$
LOOKS LIKE
THIS?



TO FIND THE DETERMINANT OF
A 3×3 MATRIX, JUST USE THE
FOLLOWING FORMULA.

THIS IS SOMETIMES CALLED
SARRUS' RULE.

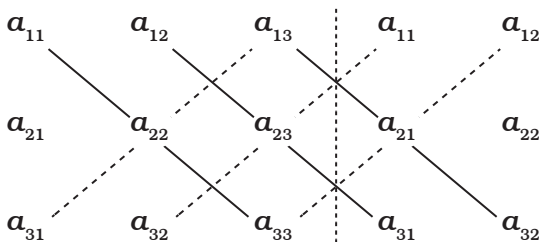
$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

I'M SUPPOSED
TO MEMORIZE
THIS?

DON'T WORRY,
THERE'S A NICE
TRICK FOR THIS
ONE TOO.

SARRUS' RULE

Write out the matrix, and then write its first two columns again after the third column, giving you a total of five columns. Add the products of the diagonals going from top to bottom (indicated by the solid lines) and subtract the products of the diagonals going from bottom to top (indicated by dotted lines). This will generate the formula for Sarrus' Rule, and it's much easier to remember!



LET'S SEE IF $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$ HAS AN INVERSE.



$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} = 1 \cdot 1 \cdot 3 + 0 \cdot (-1) \cdot (-2) + 0 \cdot 1 \cdot 0 - 0 \cdot 1 \cdot (-2) - 0 \cdot 1 \cdot 3 - 1 \cdot (-1) \cdot 0$$

$$= 3 + 0 + 0 - 0 - 0 - 0$$

$$= 3$$



$$\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} \neq 0$$

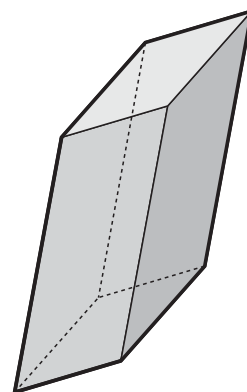
SO THIS ONE HAS AN INVERSE TOO!

AND THE VOLUME OF THE PARALLELEPIPED* SPANNED BY THE FOLLOWING EIGHT POINTS...

- THE ORIGIN
- THE POINT (a_{11}, a_{21}, a_{31})
- THE POINT (a_{12}, a_{22}, a_{32})
- THE POINT (a_{13}, a_{23}, a_{33})
- THE POINT $(a_{11} + a_{12}, a_{21} + a_{22}, a_{31} + a_{32})$
- THE POINT $(a_{11} + a_{13}, a_{21} + a_{23}, a_{31} + a_{33})$
- THE POINT $(a_{12} + a_{13}, a_{22} + a_{23}, a_{32} + a_{33})$
- THE POINT $(a_{11} + a_{12} + a_{13}, a_{21} + a_{22} + a_{23}, a_{31} + a_{32} + a_{33})$

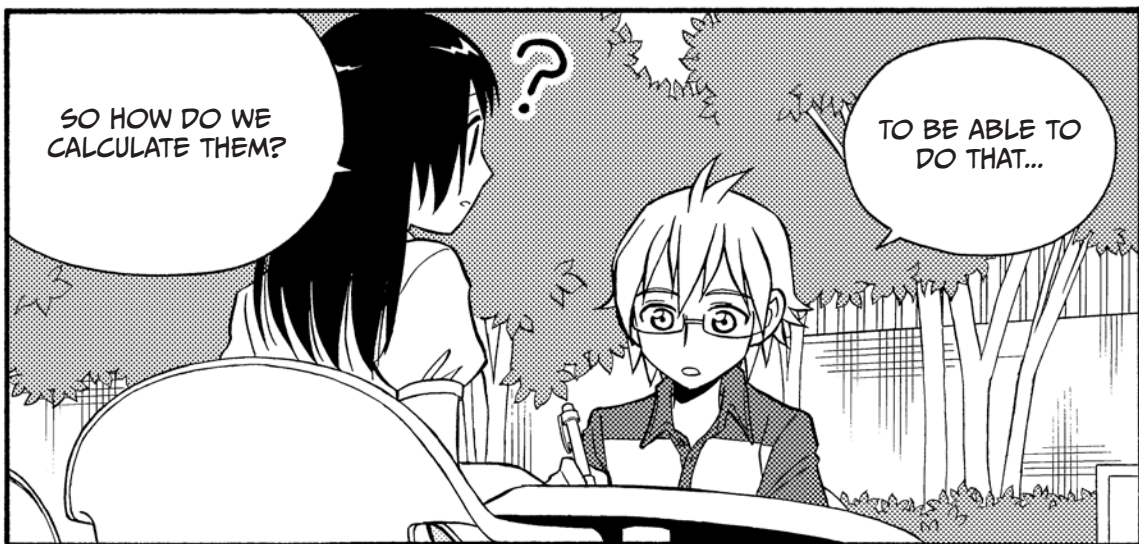
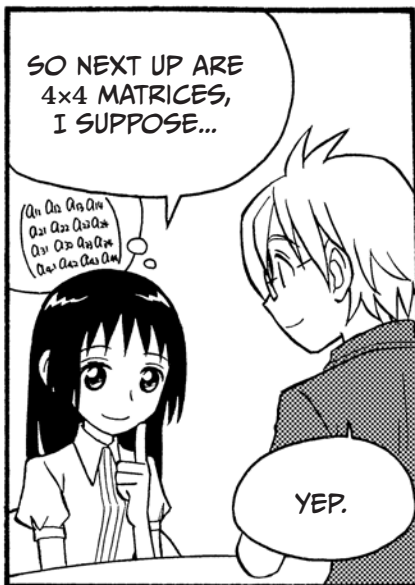
...ALSO COINCIDES WITH THE ABSOLUTE VALUE OF

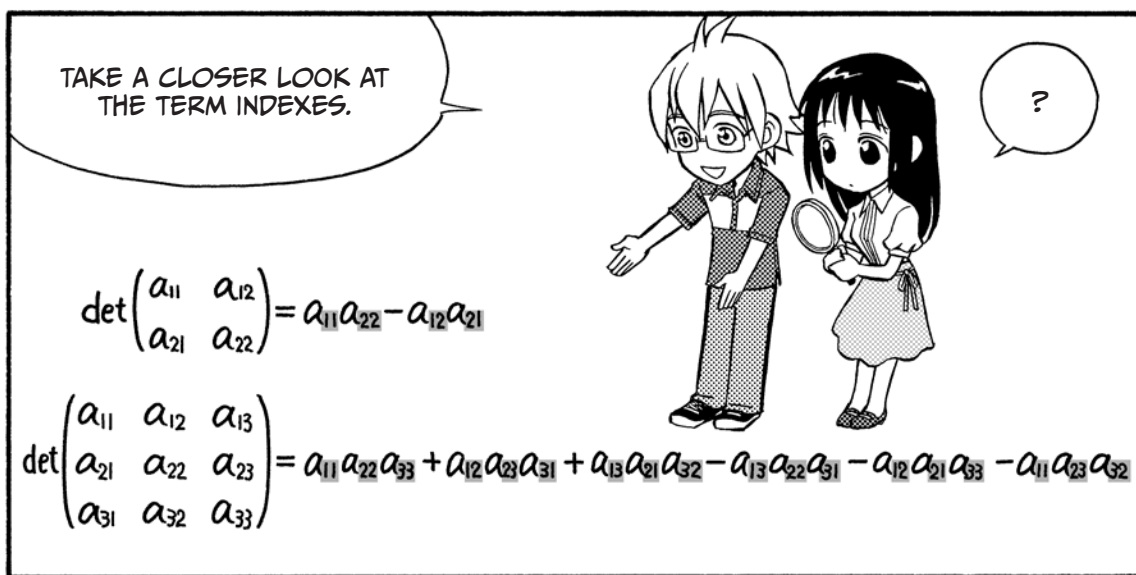
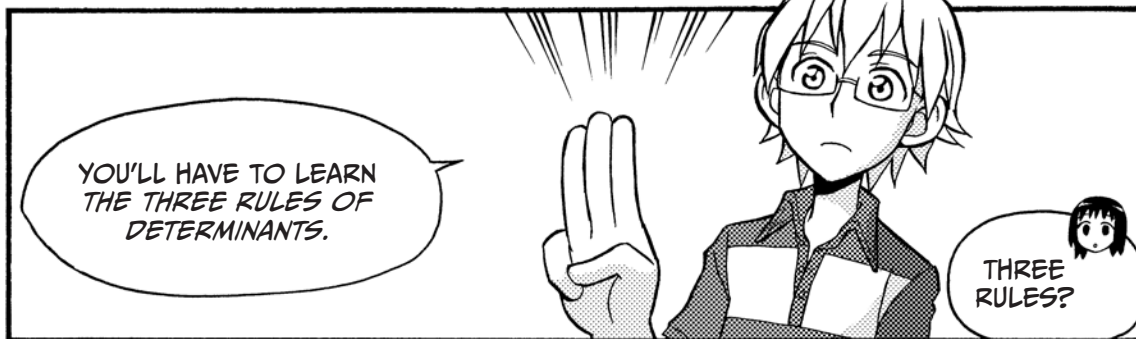
$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



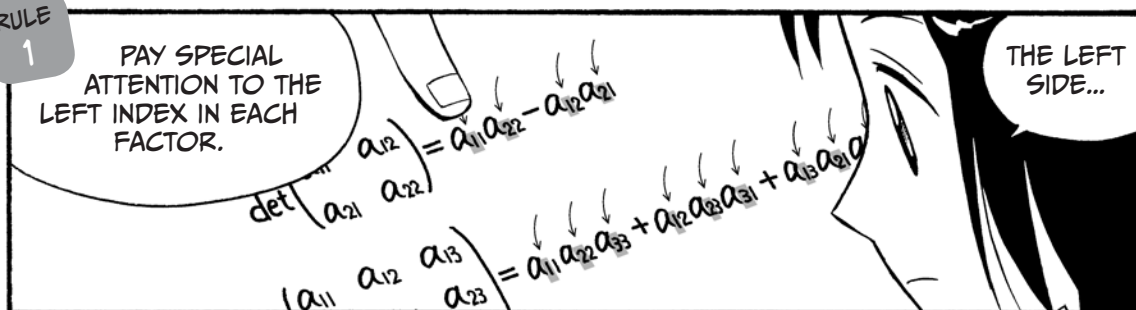
EACH PAIR OF OPPOSITE FACES ON THE PARALLELEPIPED ARE PARALLEL AND HAVE THE SAME AREA.

* A PARALLELEPIPED IS A THREE-DIMENSIONAL FIGURE FORMED BY SIX PARALLELOGRAMS.





RULE
1



OH, THEY ALL GO FROM ONE TO THE NUMBER OF DIMENSIONS!

EXACTLY.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11}a_{22}}{1 \ 2} - \frac{a_{12}a_{21}}{1 \ 2}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \frac{a_{11}a_{22}a_{33}}{1 \ 2 \ 3} + \frac{a_{12}a_{23}a_{31}}{1 \ 2 \ 3} + \frac{a_{13}a_{21}a_{32}}{1 \ 2 \ 3} - \frac{a_{13}a_{22}a_{31}}{1 \ 2 \ 3} - \frac{a_{12}a_{21}a_{33}}{1 \ 2 \ 3} - \frac{a_{11}a_{23}a_{32}}{1 \ 2 \ 3}$$

AND THAT'S RULE NUMBER ONE!

RULE 2

NOW FOR THE RIGHT INDEXES.

HMM... THEY SEEM A BIT MORE RANDOM.

ACTUALLY, THEY'RE NOT. THEIR ORDERS ARE ALL PERMUTATIONS OF 1, 2, AND 3—LIKE IN THE TABLE TO THE RIGHT. THIS IS RULE NUMBER TWO.

I SEE IT NOW!

PERMUTATIONS OF 1-2

PATTERN 1	1 2
PATTERN 2	2 1

PERMUTATIONS OF 1-3

PATTERN 1	1 2 3
PATTERN 2	1 3 2
PATTERN 3	2 1 3
PATTERN 4	2 3 1
PATTERN 5	3 1 2
PATTERN 6	3 2 1

$$\frac{a_{11}a_{22}}{1 \ 2} - \frac{a_{12}a_{21}}{2 \ 1}$$

$$\frac{a_{11}a_{22}a_{33}}{1 \ 2 \ 3} + \frac{a_{12}a_{23}a_{31}}{2 \ 3 \ 1} + \frac{a_{13}a_{21}a_{32}}{3 \ 1 \ 2} - \frac{a_{13}a_{22}a_{31}}{3 \ 2 \ 1} - \frac{a_{12}a_{21}a_{33}}{2 \ 1 \ 3} - \frac{a_{11}a_{23}a_{32}}{1 \ 3 \ 2}$$

RULE 3

THE THIRD RULE IS A BIT TRICKY, SO DON'T LOSE CONCENTRATION.

SQUEEZE

OKAY!

LET'S START BY MAKING AN AGREEMENT.

WE WILL SAY THAT THE RIGHT INDEX IS IN ITS NATURAL ORDER IF

$$a_{?1} a_{?2}$$

$$a_{?1} a_{?2} a_{?3}$$

THAT IS, INDEXES HAVE TO BE IN AN INCREASING ORDER.

?

THE NEXT STEP IS TO FIND ALL THE PLACES WHERE TWO TERMS AREN'T IN THE NATURAL ORDER—MEANING THE PLACES WHERE TWO INDEXES HAVE TO BE SWITCHED FOR THEM TO BE IN AN INCREASING ORDER.

$$-a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32}$$

WE GATHER ALL THIS INFORMATION INTO A TABLE LIKE THIS.

WHOA.

	PERMUTATIONS OF 1-2	CORRESPONDING TERM IN THE DETERMINANT	SWITCHES		
PATTERN 1	1 2	$a_{11} a_{22}$			
PATTERN 2	2 1	$a_{12} a_{21}$	2 AND 1		
	PERMUTATIONS OF 1-3	CORRESPONDING TERM IN THE DETERMINANT	SWITCHES		
PATTERN 1	1 2 3	$a_{11} a_{22} a_{33}$			
PATTERN 2	1 3 2	$a_{11} a_{23} a_{32}$			3 AND 2
PATTERN 3	2 1 3	$a_{12} a_{21} a_{33}$	2 AND 1		
PATTERN 4	2 3 1	$a_{12} a_{23} a_{31}$	2 AND 1	3 AND 1	
PATTERN 5	3 1 2	$a_{13} a_{21} a_{32}$		3 AND 1	3 AND 2
PATTERN 6	3 2 1	$a_{13} a_{22} a_{31}$	2 AND 1	3 AND 1	3 AND 2

THEN WE COUNT HOW MANY SWITCHES WE NEED FOR EACH TERM.

IF THE NUMBER IS EVEN, WE WRITE THE TERM AS POSITIVE. IF IT IS ODD, WE WRITE IT AS NEGATIVE.

	PERMUTATIONS OF 1-2	CORRESPONDING TERM IN THE DETERMINANT	SWITCHES	NUMBER OF SWITCHES	SIGN
PATTERN 1	1 2	$a_{11} a_{22}$		0	+
PATTERN 2	2 1	$a_{12} a_{21}$	2 AND 1	1	-

	PERMUTATIONS OF 1-3	CORRESPONDING TERM IN THE DETERMINANT	SWITCHES	NUMBER OF SWITCHES	SIGN
PATTERN 1	1 2 3	$a_{11} a_{22} a_{33}$		0	+
PATTERN 2	1 3 2	$a_{11} a_{23} a_{32}$		1	-
PATTERN 3	2 1 3	$a_{12} a_{21} a_{33}$	2 AND 1	1	-
PATTERN 4	2 3 1	$a_{12} a_{23} a_{31}$	2 AND 1	2	+
PATTERN 5	3 1 2	$a_{13} a_{21} a_{32}$	3 AND 1	2	+
PATTERN 6	3 2 1	$a_{13} a_{22} a_{31}$	2 AND 1	3	-

LIKE THIS.

HMM...

TRY COMPARING OUR EARLIER
DETERMINANT FORMULAS WITH THE
COLUMNS "CORRESPONDING TERM IN
THE DETERMINANT" AND "SIGN."

AH!

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

CORRESPONDING TERM IN THE DETERMINANT	SIGN
$a_{11} a_{22}$	+
$a_{12} a_{21}$	-

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

CORRESPONDING TERM IN THE DETERMINANT	SIGN
$a_{11} a_{22} a_{33}$	+
$a_{11} a_{23} a_{32}$	-
$a_{12} a_{21} a_{33}$	-
$a_{12} a_{23} a_{31}$	+
$a_{13} a_{21} a_{32}$	+
$a_{13} a_{22} a_{31}$	-

WOW,
THEY'RE
THE SAME!

EXACTLY, AND THAT'S
THE THIRD RULE.

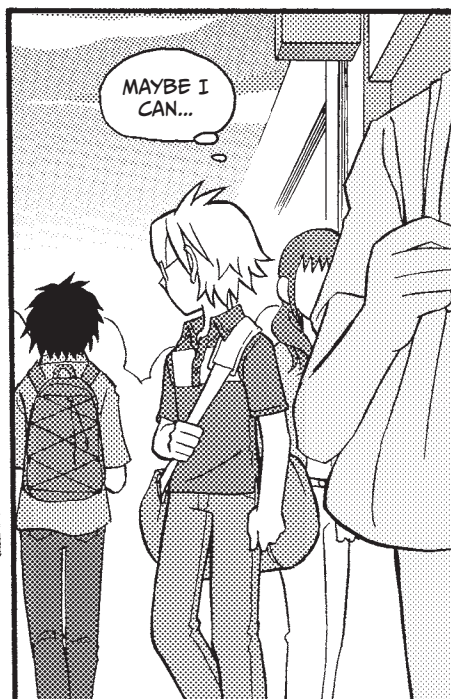
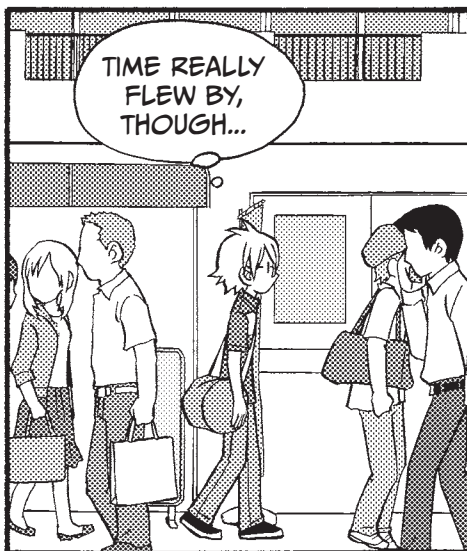
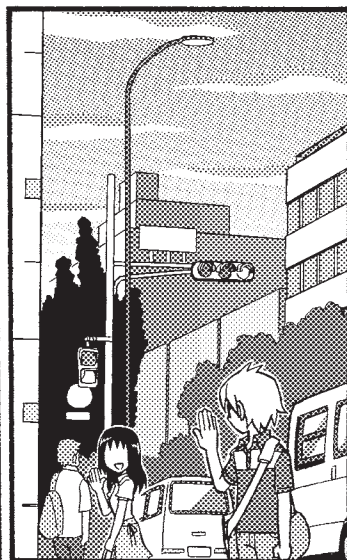
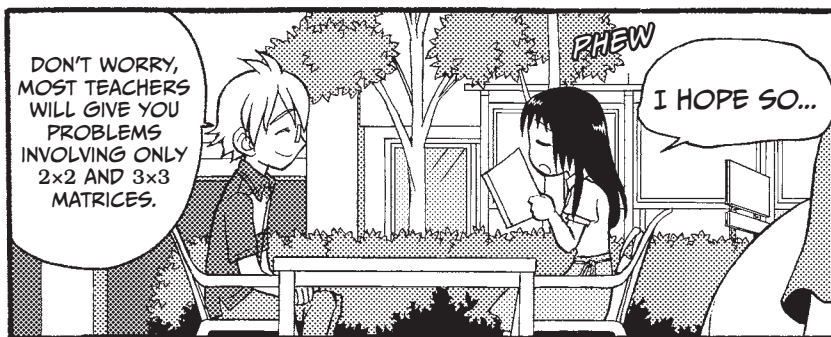


SO, SAY WE WANTED TO CALCULATE THE DETERMINANT OF THIS 4x4 MATRIX:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} =$$

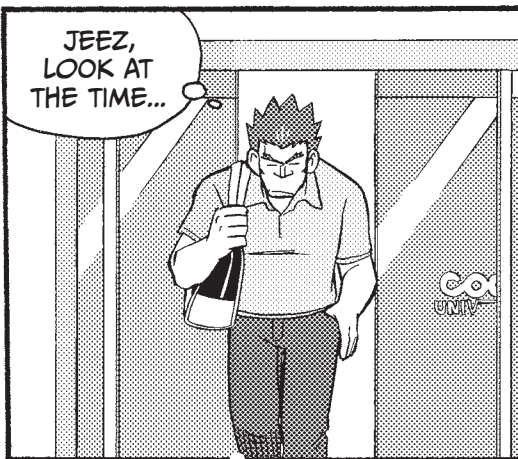
	PERMUTATIONS OF 1-4	CORRESPONDING TERM IN THE DETERMINANT	SWITCHES				NUM. OF SWITCHES	SIGN
PATTERN 1	1 2 3 4	$a_{11} a_{22} a_{33} a_{44}$					0	+
PATTERN 2	1 2 4 3	$a_{11} a_{22} a_{34} a_{43}$	4 & 3				1	-
PATTERN 3	1 3 2 4	$a_{11} a_{23} a_{32} a_{44}$	3 & 2				1	-
PATTERN 4	1 3 4 2	$a_{11} a_{23} a_{34} a_{42}$	3 & 2 4 & 2				2	+
PATTERN 5	1 4 2 3	$a_{11} a_{24} a_{32} a_{43}$	4 & 2 4 & 3				2	+
PATTERN 6	1 4 3 2	$a_{11} a_{24} a_{33} a_{42}$	3 & 2 4 & 2 4 & 3				3	-
PATTERN 7	2 1 3 4	$a_{12} a_{21} a_{33} a_{44}$	2 & 1				1	-
PATTERN 8	2 1 4 3	$a_{12} a_{21} a_{34} a_{43}$	2 & 1 4 & 3				2	+
PATTERN 9	2 3 1 4	$a_{12} a_{23} a_{31} a_{44}$	2 & 1 3 & 1				2	+
PATTERN 10	2 3 4 1	$a_{12} a_{23} a_{34} a_{41}$	2 & 1 3 & 1 4 & 1				3	-
PATTERN 11	2 4 1 3	$a_{12} a_{24} a_{31} a_{43}$	2 & 1 4 & 1 4 & 3				3	-
PATTERN 12	2 4 3 1	$a_{12} a_{24} a_{33} a_{41}$	2 & 1 3 & 1 4 & 1 4 & 3				4	+
PATTERN 13	3 1 2 4	$a_{13} a_{21} a_{32} a_{44}$	3 & 1 3 & 2				2	+
PATTERN 14	3 1 4 2	$a_{13} a_{21} a_{34} a_{42}$	3 & 1 3 & 2 4 & 2				3	-
PATTERN 15	3 2 1 4	$a_{13} a_{22} a_{31} a_{44}$	2 & 1 3 & 1 3 & 2				3	-
PATTERN 16	3 2 4 1	$a_{13} a_{22} a_{34} a_{41}$	2 & 1 3 & 1 3 & 2 4 & 1				4	+
PATTERN 17	3 4 1 2	$a_{13} a_{24} a_{31} a_{42}$	3 & 1 3 & 2 4 & 1 4 & 2				4	+
PATTERN 18	3 4 2 1	$a_{13} a_{24} a_{32} a_{41}$	2 & 1 3 & 1 3 & 2 4 & 1 4 & 2				5	-
PATTERN 19	4 1 2 3	$a_{14} a_{21} a_{32} a_{43}$	4 & 1 4 & 2 4 & 3				3	-
PATTERN 20	4 1 3 2	$a_{14} a_{21} a_{33} a_{42}$	3 & 2 4 & 1 4 & 2 4 & 3				4	+
PATTERN 21	4 2 1 3	$a_{14} a_{22} a_{31} a_{43}$	2 & 1 4 & 1 4 & 2 4 & 3				4	+
PATTERN 22	4 2 3 1	$a_{14} a_{22} a_{33} a_{41}$	2 & 1 3 & 1 4 & 1 4 & 2 4 & 3				5	-
PATTERN 23	4 3 1 2	$a_{14} a_{23} a_{31} a_{42}$	3 & 1 3 & 2 4 & 1 4 & 2 4 & 3				5	-
PATTERN 24	4 3 2 1	$a_{14} a_{23} a_{32} a_{41}$	2 & 1 3 & 1 3 & 2 4 & 1 4 & 2 4 & 3				6	+



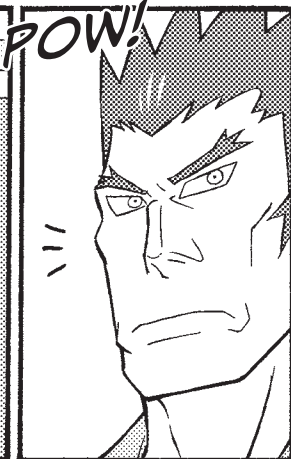




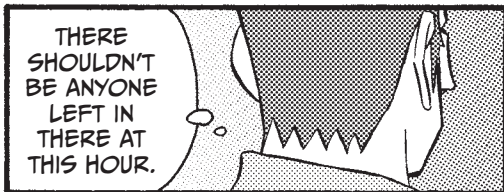
DO YOU
NEED A BAG
FOR THAT?



JEEZ,
LOOK AT
THE TIME...



POW!

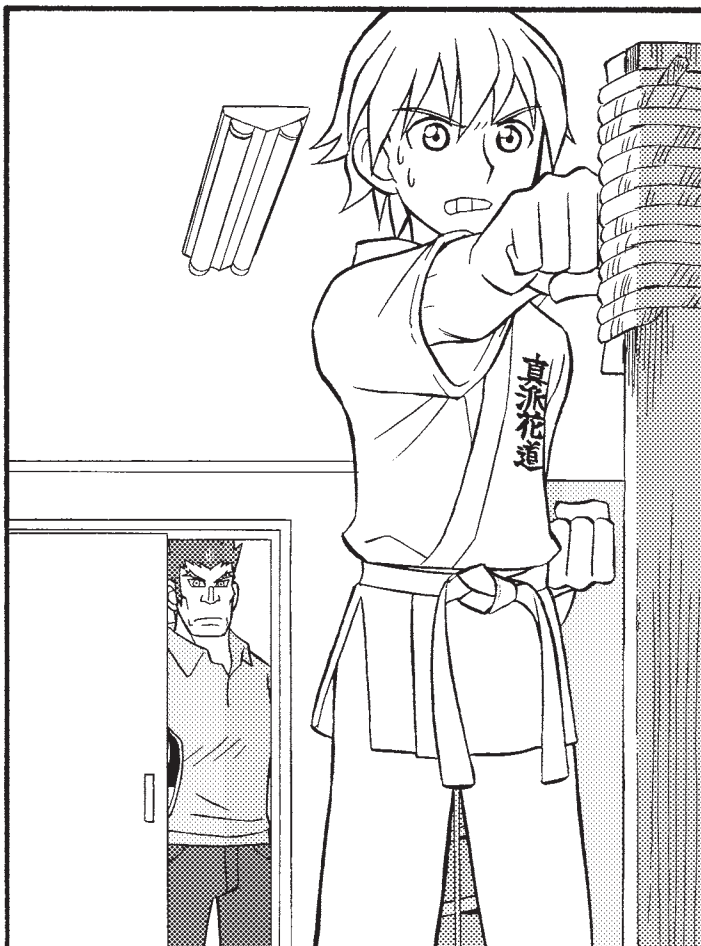


THERE
SHOULDN'T
BE ANYONE
LEFT IN
THERE AT
THIS HOUR.

ガニ
ガニ

SMACK!

BIFF!



CALCULATING INVERSE MATRICES USING COFACTORS

There are two practical ways to calculate inverse matrices, as mentioned on page 88.

- Using cofactors
- Using Gaussian elimination

Since the cofactor method involves a lot of cumbersome calculations, we avoided using it in this chapter. However, since most books seem to introduce the method, here's a quick explanation.

To use this method, you first have to understand these two concepts:

- The (i, j) -minor, written as M_{ij}
- The (i, j) -cofactor, written as C_{ij}

So first we'll have a look at these.

$M_{i,j}$

The (i, j) -minor is the determinant produced when we remove row i and column j from the $n \times n$ matrix A :

$$M_{ij} = \det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

All the minors of the 3×3 matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$ are listed on the next page.

$M_{11}(1, 1)$ $\det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = 3$	$M_{12}(1, 2)$ $\det \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} = 1$	$M_{13}(1, 3)$ $\det \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} = 2$
$M_{21}(2, 1)$ $\det \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix} = 0$	$M_{22}(2, 2)$ $\det \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = 3$	$M_{23}(2, 3)$ $\det \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} = 0$
$M_{31}(3, 1)$ $\det \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} = 0$	$M_{32}(3, 2)$ $\det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} = -1$	$M_{33}(3, 3)$ $\det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1$

C_{ij}

If we multiply the (i, j) -minor by $(-1)^{i+j}$, we get the (i, j) -cofactor. The standard way to write this is C_{ij} . The table below contains all cofactors of the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$$

$C_{11}(1, 1)$ $= (-1)^{1+1} \cdot \det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$ $= 1 \cdot 3$ $= 3$	$C_{12}(1, 2)$ $= (-1)^{1+2} \cdot \det \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ $= (-1) \cdot 1$ $= -1$	$C_{13}(1, 3)$ $= (-1)^{1+3} \cdot \det \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ $= 1 \cdot 2$ $= 2$
$C_{21}(2, 1)$ $= (-1)^{2+1} \cdot \det \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$ $= (-1) \cdot 0$ $= 0$	$C_{22}(2, 2)$ $= (-1)^{2+2} \cdot \det \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$ $= 1 \cdot 3$ $= 3$	$C_{23}(2, 3)$ $= (-1)^{2+3} \cdot \det \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}$ $= (-1) \cdot 0$ $= 0$
$C_{31}(3, 1)$ $= (-1)^{3+1} \cdot \det \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ $= 1 \cdot 0$ $= 0$	$C_{32}(3, 2)$ $= (-1)^{3+2} \cdot \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ $= (-1) \cdot (-1)$ $= 1$	$C_{33}(3, 3)$ $= (-1)^{3+3} \cdot \det \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $= 1 \cdot 1$ $= 1$

The $n \times n$ matrix

$$\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

which at place (i, j) has the (j, i) -cofactor¹ of the original matrix is called a *cofactor matrix*.

The sum of any row or column of the $n \times n$ matrix

$$\begin{pmatrix} a_{11}C_{11} & a_{21}C_{21} & \cdots & a_{n1}C_{n1} \\ a_{12}C_{12} & a_{22}C_{22} & \cdots & a_{n2}C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}C_{1n} & a_{2n}C_{2n} & \cdots & a_{nn}C_{nn} \end{pmatrix}$$

is equal to the determinant of the original $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

CALCULATING INVERSE MATRICES

The inverse of a matrix can be calculated using the following formula:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

1. This is not a typo. (j, i) -cofactor is the correct index order. This is the transpose of the matrix with the cofactors in the expected positions.

For example, the inverse of the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$$

is equal to

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}^{-1} = \frac{1}{\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}} \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

USING DETERMINANTS

The method presented in this chapter only defines the determinant and does nothing to explain what it is used for. A typical application (in image processing, for example) can easily reach determinant sizes in the $n = 100$ range, which with the approach used here would produce insurmountable numbers of calculations.

Because of this, determinants are usually calculated by first simplifying them with Gaussian elimination-like methods and then using these three properties, which can be derived using the definition presented in the book:

- If a row (or column) in a determinant is replaced by the sum of the row (column) and a multiple of another row (column), the value stays unchanged.
- If two rows (or columns) switch places, the values of the determinant are multiplied by -1 .
- The value of an upper or lower triangular determinant is equal to the product of its main diagonal.

The difference between the two methods is so extreme that determinants that would be practically impossible to calculate (even using modern computers) with the first method can be done in a jiffy with the second one.

SOLVING LINEAR SYSTEMS WITH CRAMER'S RULE

Gaussian elimination, as presented on page 89, is only one of many methods you can use to solve linear systems. Even though Gaussian elimination is one of the best ways to solve them by hand, it is always good to know about alternatives, which is why we'll cover the *Cramer's rule* method next.

PROBLEM

Use Cramer's rule to solve the following linear system:

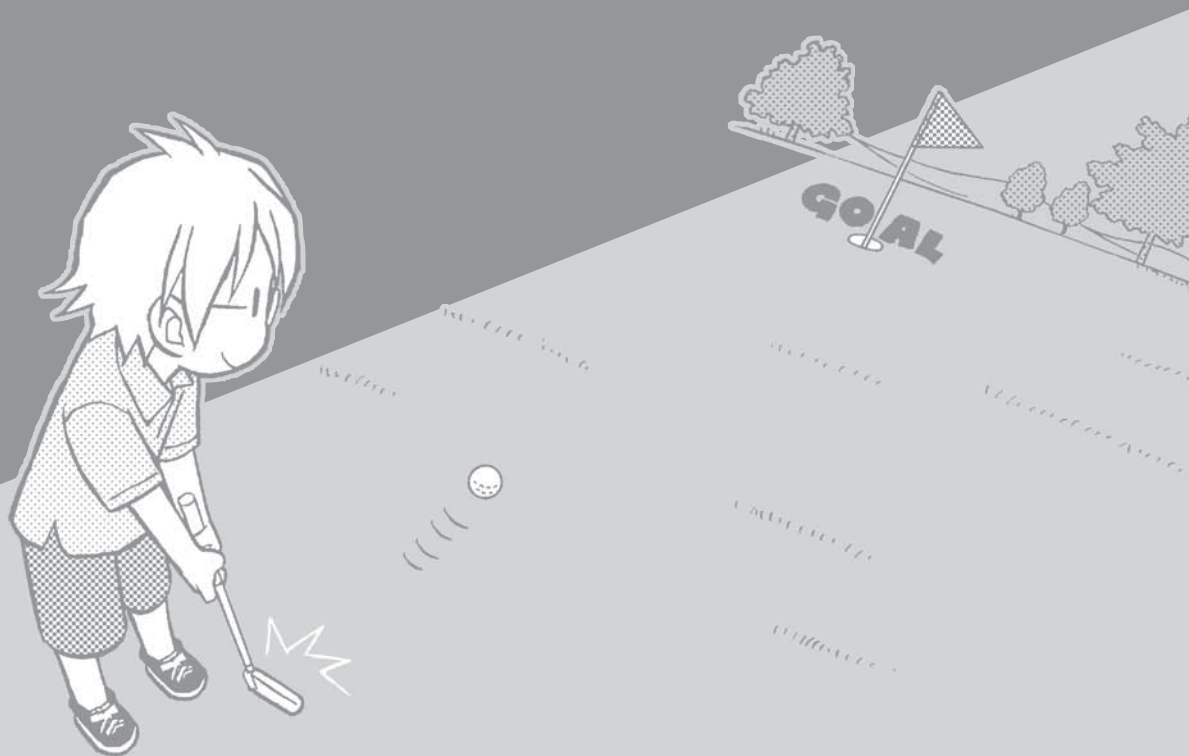
$$\begin{cases} 3x_1 + 1x_2 = 1 \\ 1x_1 + 2x_2 = 0 \end{cases}$$

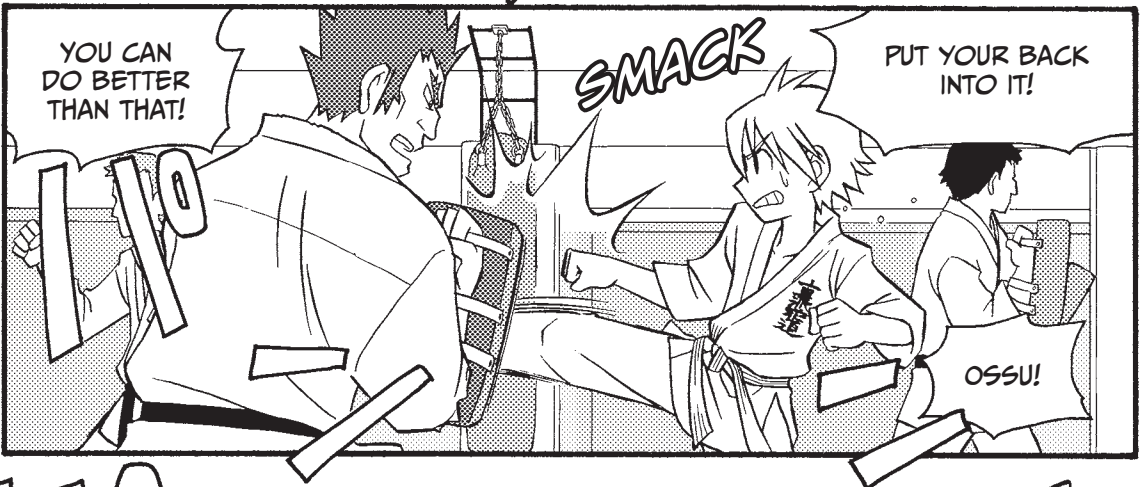
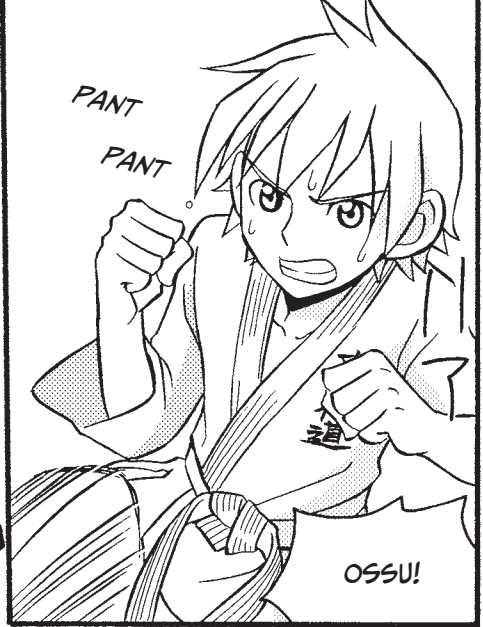
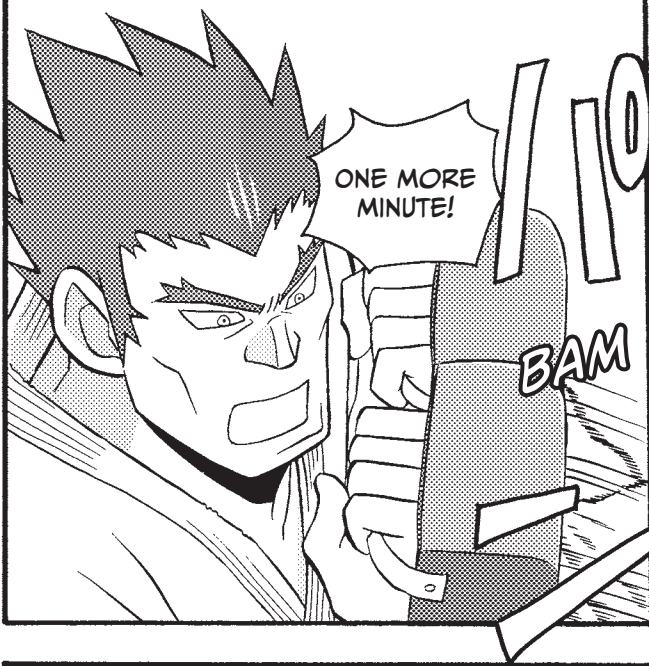
SOLUTION

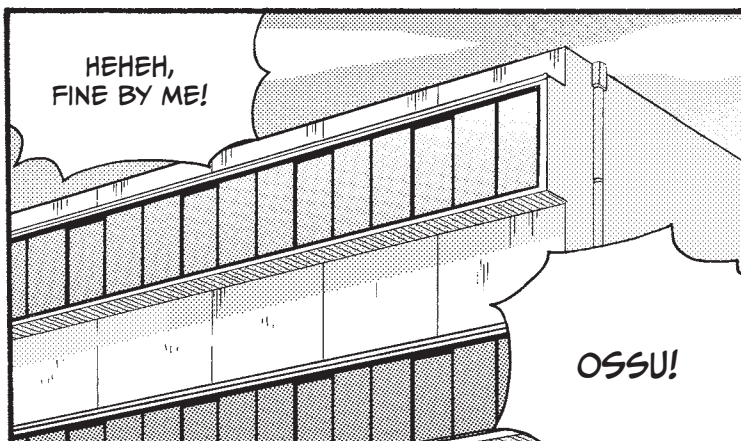
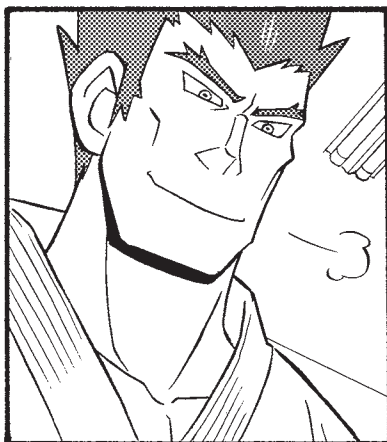
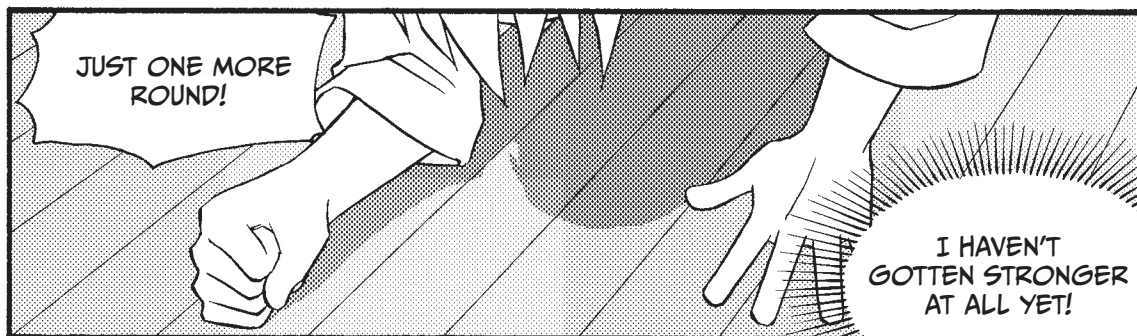
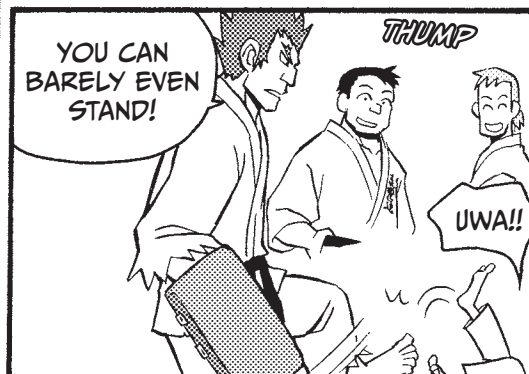
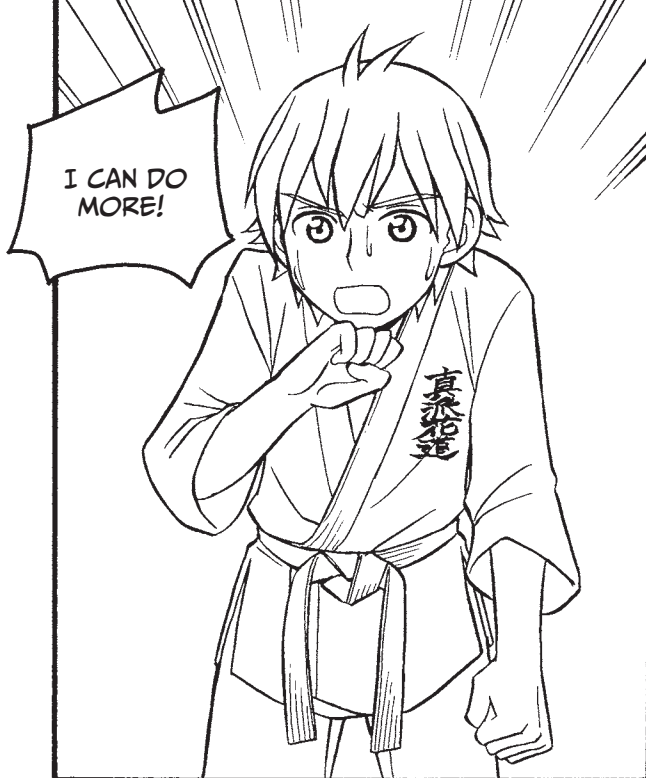
[illegible]

5

INTRODUCTION TO VECTORS







WE'RE GOING TO TAKE A LOOK AT VECTORS TODAY.

THEY APPEAR QUITE FREQUENTLY IN LINEAR ALGEBRA, SO PAY CLOSE ATTENTION.

OF COURSE!

```

graph TD
    subgraph BASICS
        F[FUNDAMENTALS]
    end
    F --> M[MATRICES]
    F --> V[VECTORS]
    M --> LT[LINEAR TRANSFORMATION]
    V --> EE[EIGENVALUES AND EIGENVECTORS]
  
```

FWUMP

REIJI, ARE YOU OKAY?

DO YOU WANT TO MEET TOMORROW INSTEAD?

NO, I'M OKAY. JUST GIVE ME FIVE MINUTES TO DIGEST THIS DELICIOUS LUNCH, AND I'LL BE GREAT!

WHAT ARE VECTORS?

5 MINUTES LATER

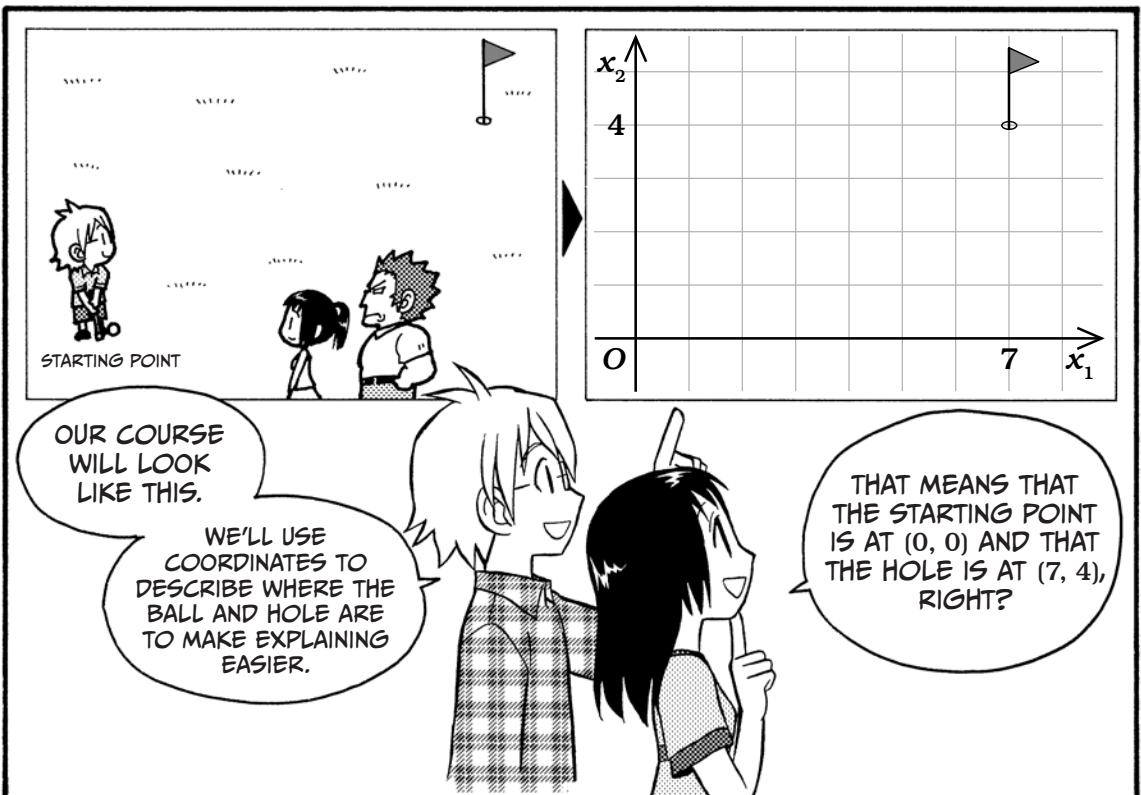
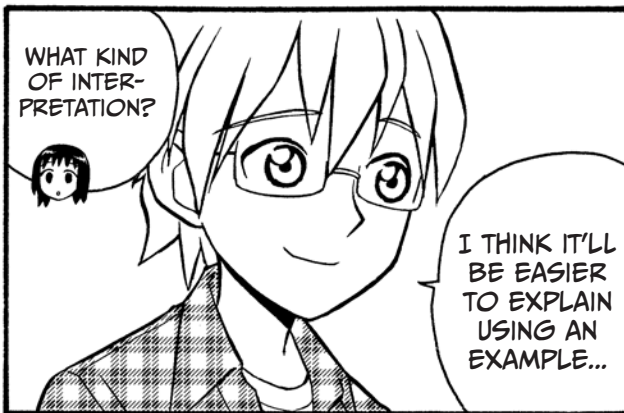
SORRY ABOUT THAT! READY?

LET'S TALK VECTORS!

VECTORS ARE ACTUALLY JUST A SPECIAL INTERPRETATION OF MATRICES.

REALLY?

REVIVED





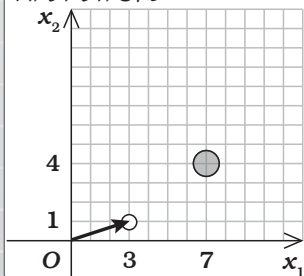
PLAYER 1 REIJI YURINO

I WENT FIRST.
I PLAYED CONSERVATIVELY
AND PUT THE BALL IN WITH
THREE STROKES.

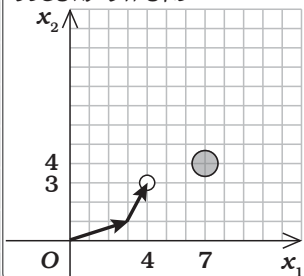


REPLAY

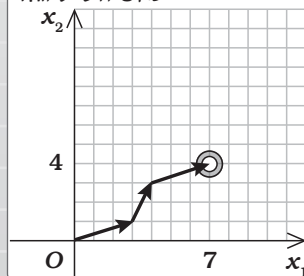
FIRST STROKE



SECOND STROKE



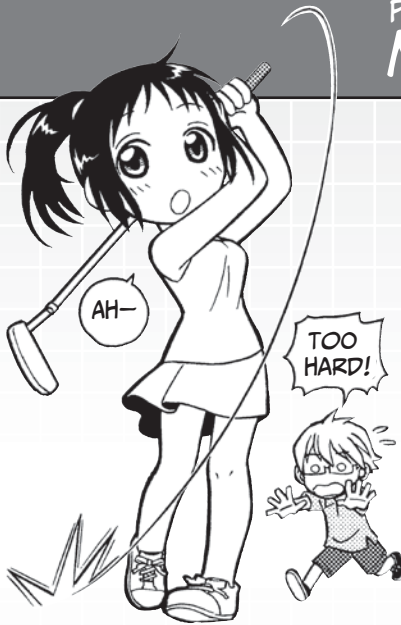
THIRD STROKE



STROKE INFORMATION

	First stroke	Second stroke	Third stroke
Ball position	Point (3, 1)	Point (4, 3)	Point (7, 4)
Ball position relative to its last position	3 to the right and 1 up relative to (0, 0)	1 to the right and 2 up relative to (3, 1)	3 to the right and 1 up relative to (4, 3)
Ball movement expressed in the form (to the right, up)	(3, 1)	$(3, 1) + (1, 2)$ $= (4, 3)$	$(3, 1) + (1, 2) + (3, 1)$ $= (7, 4)$

PLAYER 2 MISA ICHINOSE

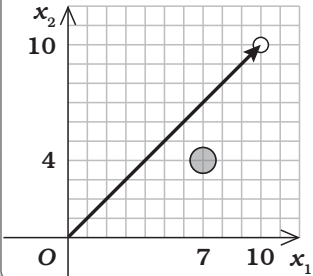


YOU GAVE THE BALL A GOOD WALLOP AND PUT THE BALL IN WITH TWO STROKES.

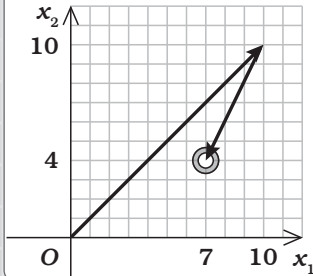


REPLAY

FIRST STROKE



SECOND STROKE



STROKE INFORMATION

	First stroke	Second stroke
Ball position	Point (10, 10)	Point (7, 4)
Ball position relative to its last position	10 to the right and 10 up relative to (0, 0)	-3 to the right and -6 up relative to (10, 10)
Ball movement expressed in the form (to the right, up)	(10, 10)	$(10, 10) + (-3, -6) = (7, 4)$

PLAYER 3 TETSUO ICHINOSE

OBSERVE
MY EXERCISE
IN SKILL!



YAY
BIG BRO!

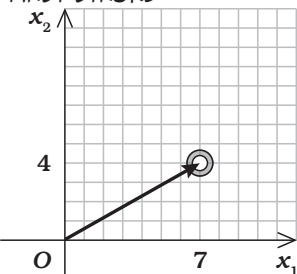


AND YOUR BROTHER GOT
A HOLE-IN-ONE...OF COURSE.



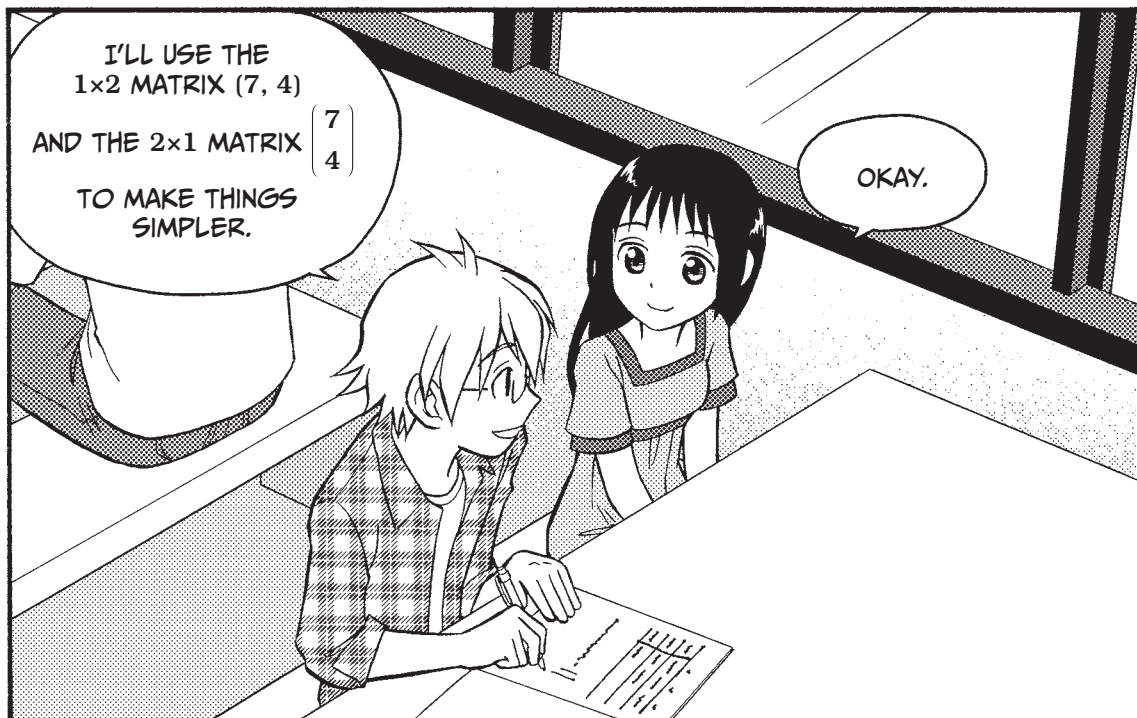
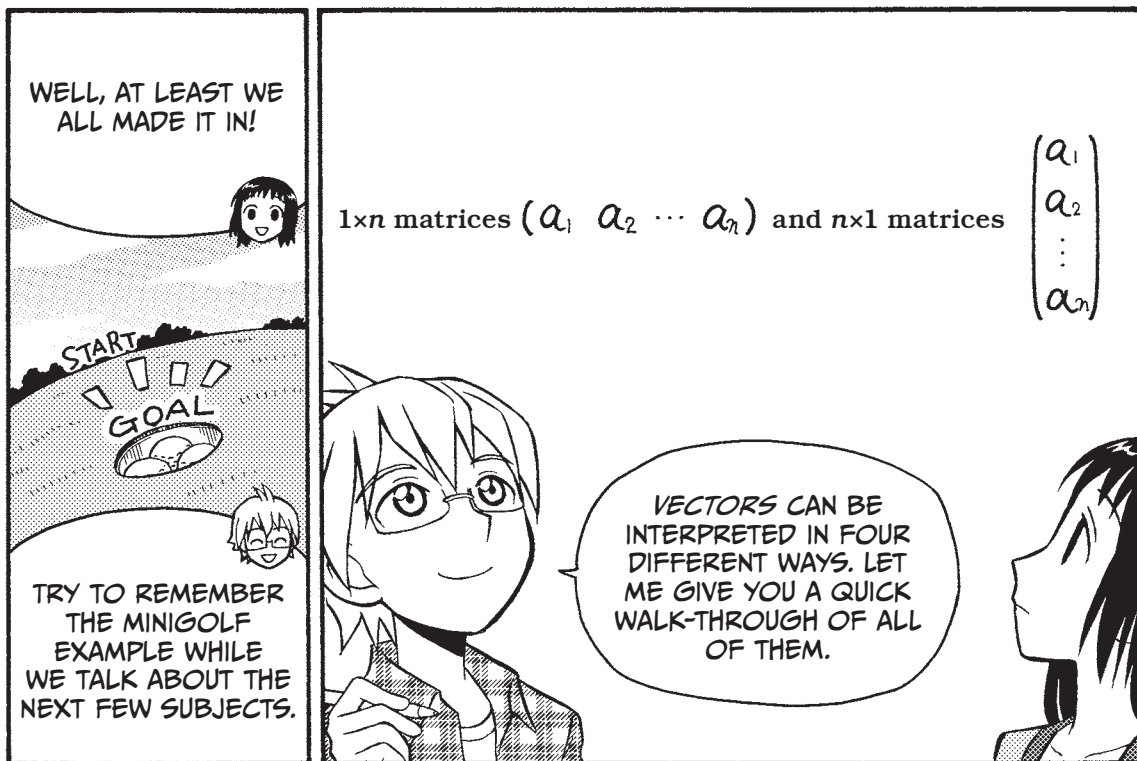
REPLAY

FIRST STROKE

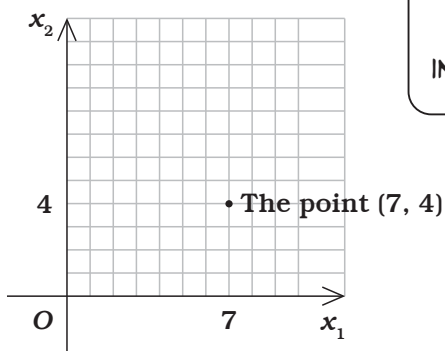


STROKE INFORMATION

	First stroke
Ball position	Point (7, 4)
Ball position relative to its last position	7 to the right and 4 up relative to (0, 0)
Ball movement expressed in the form (to the right, up)	(7, 4)



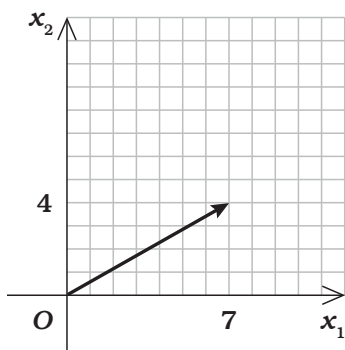
INTERPRETATION 1



$(7, 4)$ AND $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ ARE SOMETIMES INTERPRETED AS A POINT IN SPACE.



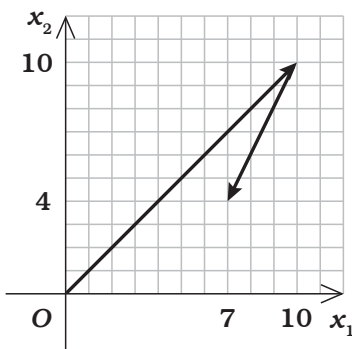
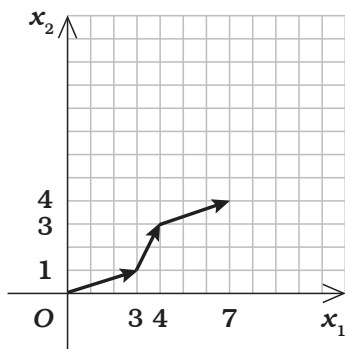
INTERPRETATION 2



IN OTHER CASES, $(7, 4)$ AND $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ ARE INTERPRETED AS THE "ARROW" FROM THE ORIGIN TO THE POINT $(7, 4)$.



INTERPRETATION 3

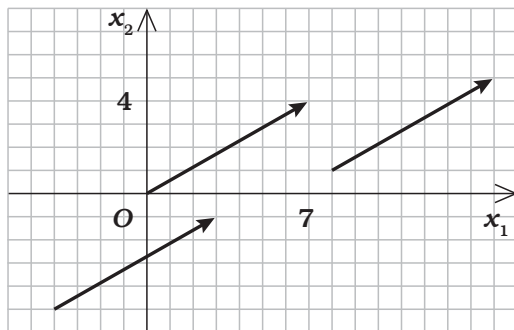


AND IN YET OTHER CASES,

$(7, 4)$ AND $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ CAN MEAN THE SUM OF SEVERAL ARROWS EQUAL TO $(7, 4)$.



INTERPRETATION 4



FINALLY, $(7, 4)$ AND $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$ CAN ALSO
BE INTERPRETED AS ANY OF
THE ARROWS ON MY LEFT, OR
ALL OF THEM AT THE SAME TIME!



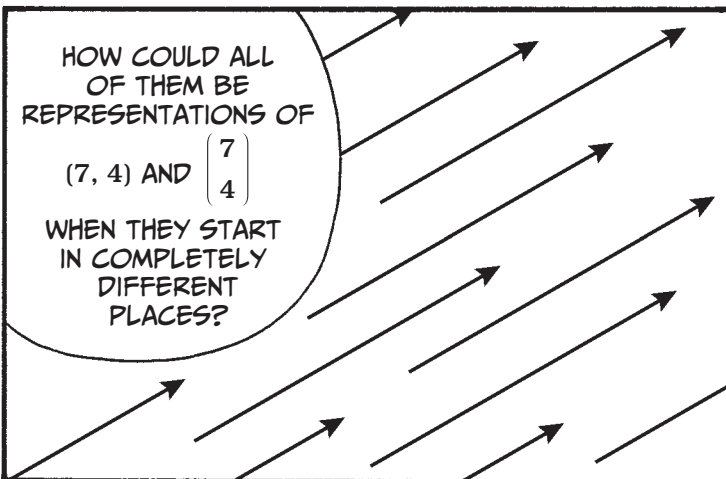
HANG ON A SECOND.
I WAS WITH YOU UNTIL
THAT LAST ONE...



HOW COULD ALL
OF THEM BE
REPRESENTATIONS OF

$$(7, 4) \text{ AND } \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

WHEN THEY START
IN COMPLETELY
DIFFERENT
PLACES?

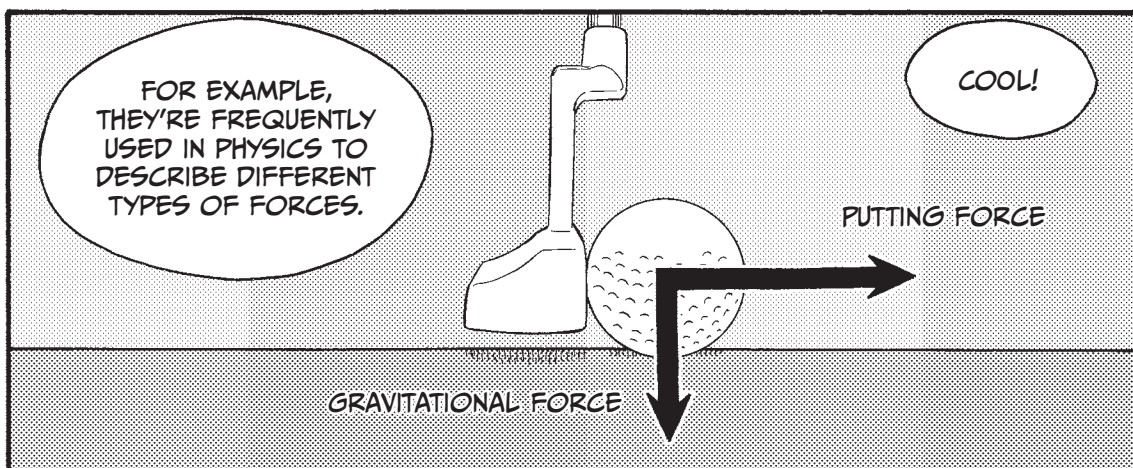


WHILE THEY DO START
IN DIFFERENT PLACES,
THEY'RE ALL THE SAME
IN THAT THEY GO "SEVEN TO
THE RIGHT AND FOUR UP,"
RIGHT?



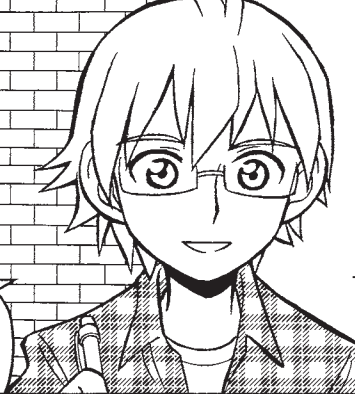
YEAH, I GUESS
THAT'S TRUE!





VECTOR CALCULATIONS

EVEN THOUGH VECTORS
HAVE A FEW SPECIAL
INTERPRETATIONS, THEY'RE ALL
JUST $1 \times n$ AND $n \times 1$ MATRICES...



AND THEY'RE
CALCULATED IN THE
EXACT SAME WAY.

ADDITION

$$\cdot (10, 10) + (-3, -6) = (10 + (-3), 10 + (-6)) = (7, 4)$$

$$\cdot \begin{pmatrix} 10 \\ 10 \end{pmatrix} + \begin{pmatrix} -3 \\ -6 \end{pmatrix} = \begin{pmatrix} 10 + (-3) \\ 10 + (-6) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

SUBTRACTION

$$\cdot (10, 10) - (3, 6) = (10 - 3, 10 - 6) = (7, 4)$$

$$\cdot \begin{pmatrix} 10 \\ 10 \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 10 - 3 \\ 10 - 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

SCALAR MULTIPLICATION

$$\cdot 2(3, 1) = (2 \cdot 3, 2 \cdot 1) = (6, 2)$$

$$\cdot 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 3 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

MATRIX MULTIPLICATION

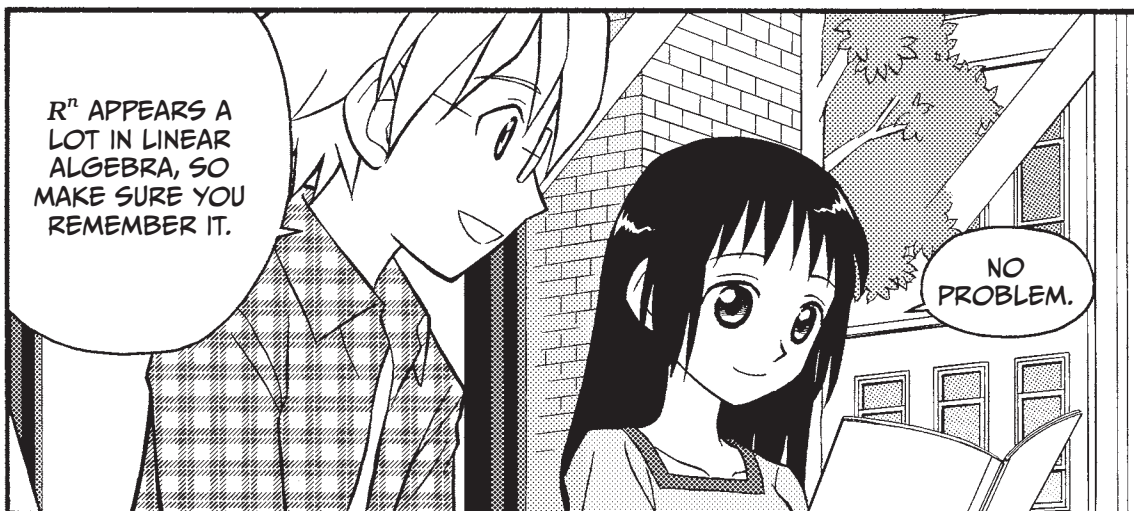
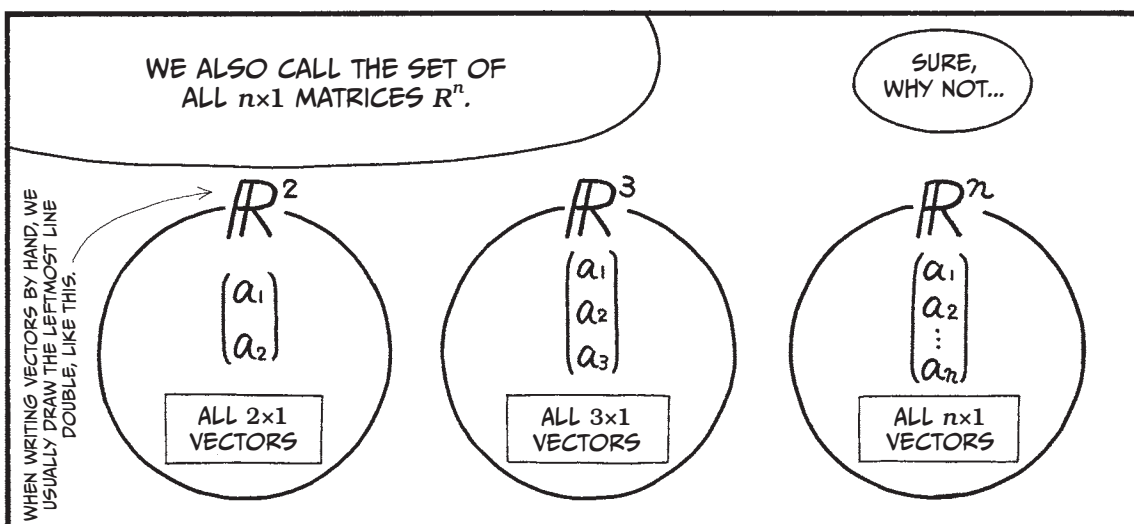
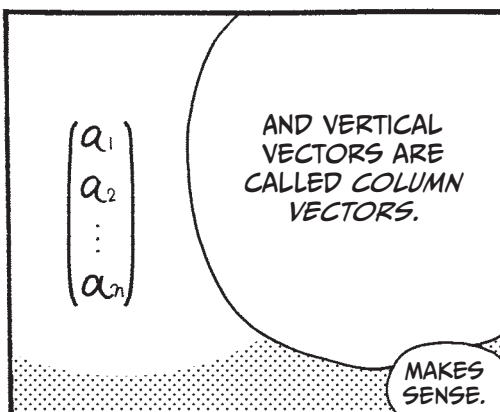
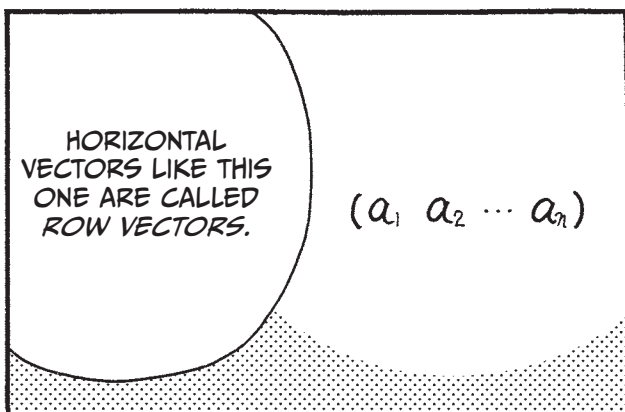
$$\cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} (1, 2) = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 1 \cdot 1 & 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$$

$$\cdot (3, 1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (3 \cdot 1 + 1 \cdot 2) = 5$$

$$\cdot \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \cdot 3 + (-3) \cdot 1 \\ 2 \cdot 3 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 21 \\ 7 \end{pmatrix} = 7 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

SIMPLE!





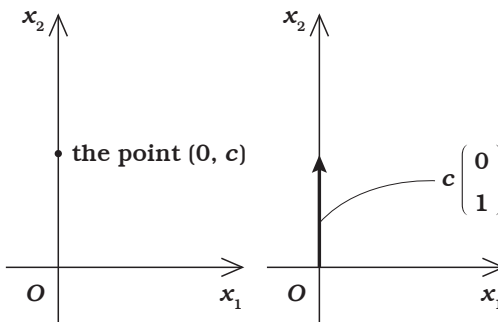
GEOMETRIC INTERPRETATIONS

LET'S HAVE A LOOK AT HOW TO EXPRESS POINTS, LINES, AND SPACES WITH VECTORS.

THE NOTATION MIGHT LOOK A BIT WEIRD AT FIRST, BUT YOU'LL GET USED TO IT.

A POINT

LET'S SAY THAT c IS AN ARBITRARY REAL NUMBER. CAN YOU SEE HOW THE POINT $(0, c)$ AND THE VECTOR $c \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ARE RELATED?



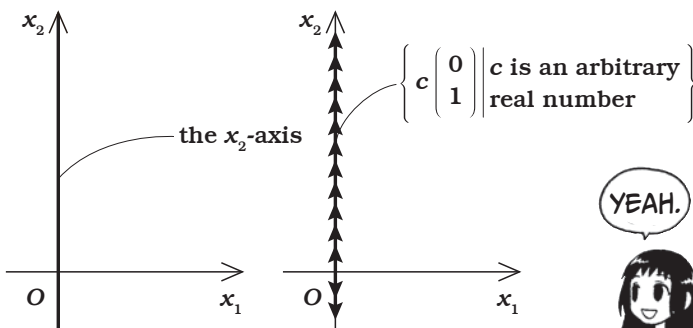
YUP.

AN AXIS

DO YOU UNDERSTAND THIS NOTATION?

$\left\{ c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid c \text{ is an arbitrary real number} \right\}$

"I" CAN BE READ AS "WHERE."

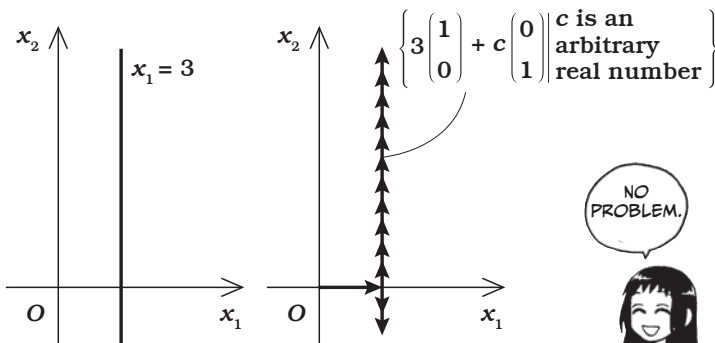


YEAH.

A STRAIGHT LINE

EVEN THE STRAIGHT LINE $x_1 = 3$ CAN BE EXPRESSED AS:

$\left\{ 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid c \text{ is an arbitrary real number} \right\}$

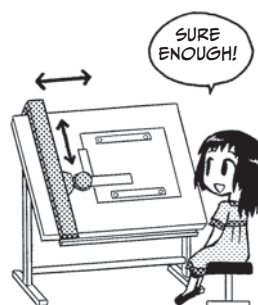
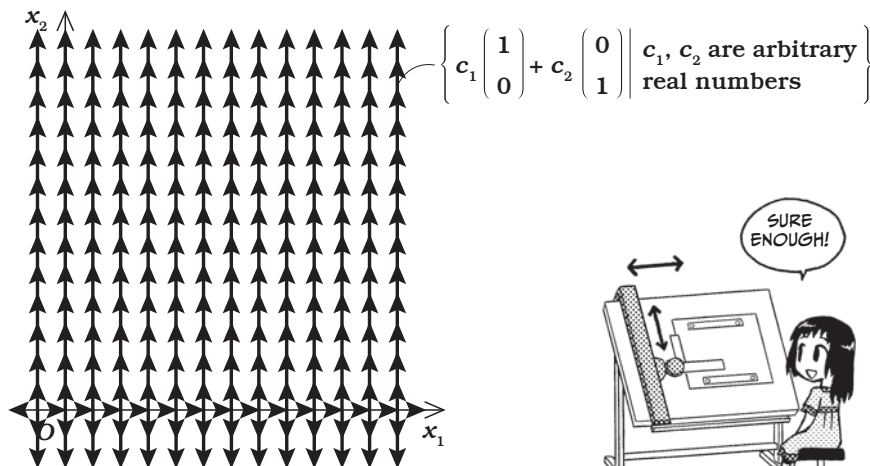


NO PROBLEM.

A PLANE

AND THE x_1x_2 PLANE R^2 CAN BE EXPRESSED AS:

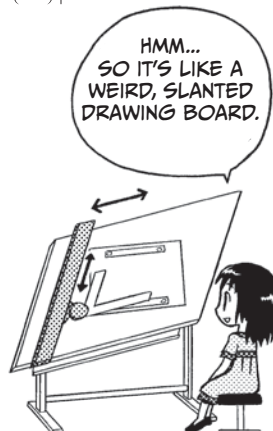
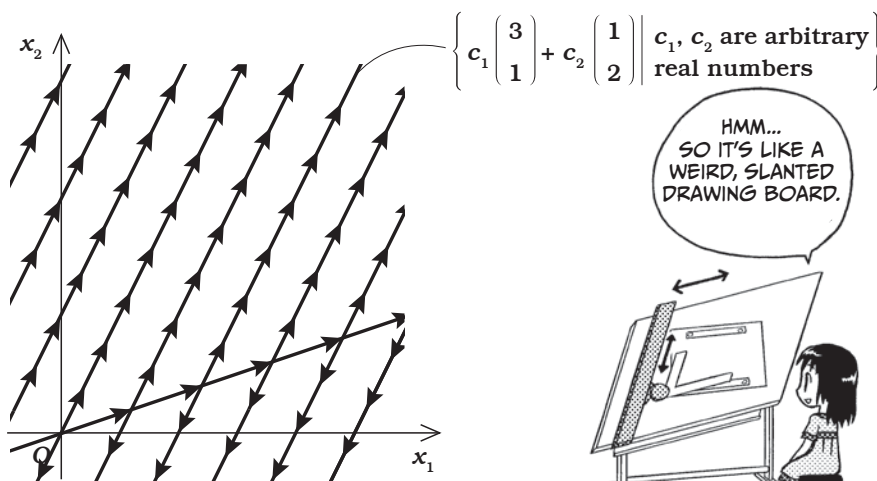
$$\left\{ c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid c_1, c_2 \text{ are arbitrary real numbers} \right\}$$



ANOTHER PLANE

IT CAN ALSO BE WRITTEN ANOTHER WAY:

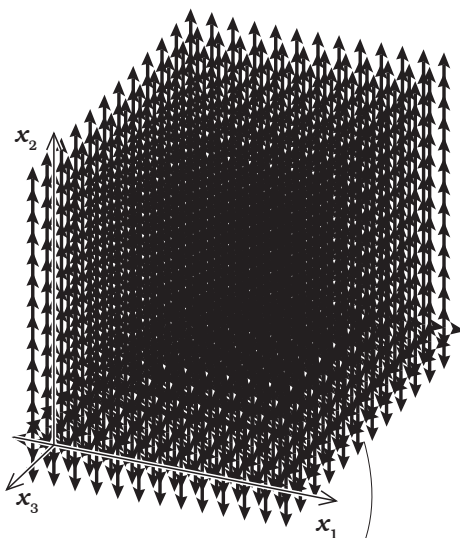
$$\left\{ c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid c_1, c_2 \text{ are arbitrary real numbers} \right\}$$



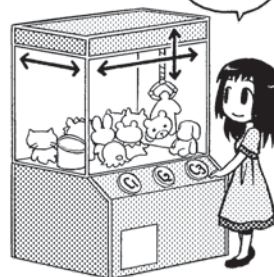
A VECTOR SPACE

THE THREE-DIMENSIONAL SPACE R^3 IS THE NATURAL NEXT STEP. IT IS SPANNED BY x_1, x_2 , AND x_3 LIKE THIS:

$$\left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid c_1, c_2, c_3 \text{ are arbitrary real numbers} \right\}$$



$$\left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid c_1, c_2, c_3 \text{ are arbitrary real numbers} \right\}$$



SOUNDS FAMILIAR.

ANOTHER VECTOR SPACE

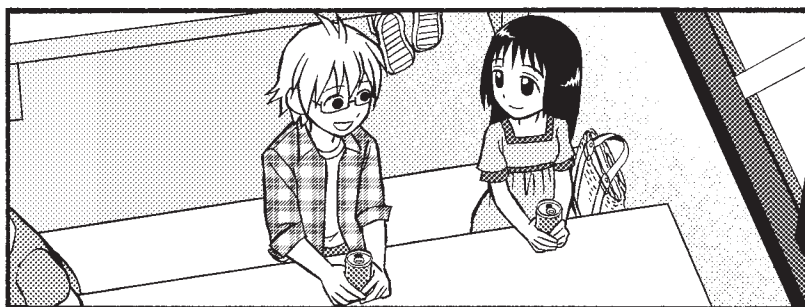
NOW TRY TO IMAGINE THE n -DIMENSIONAL SPACE R^n , SPANNED BY x_1, x_2, \dots, x_n :

$$\left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \mid c_1, c_2, \dots, c_n \text{ are arbitrary real numbers} \right\}$$



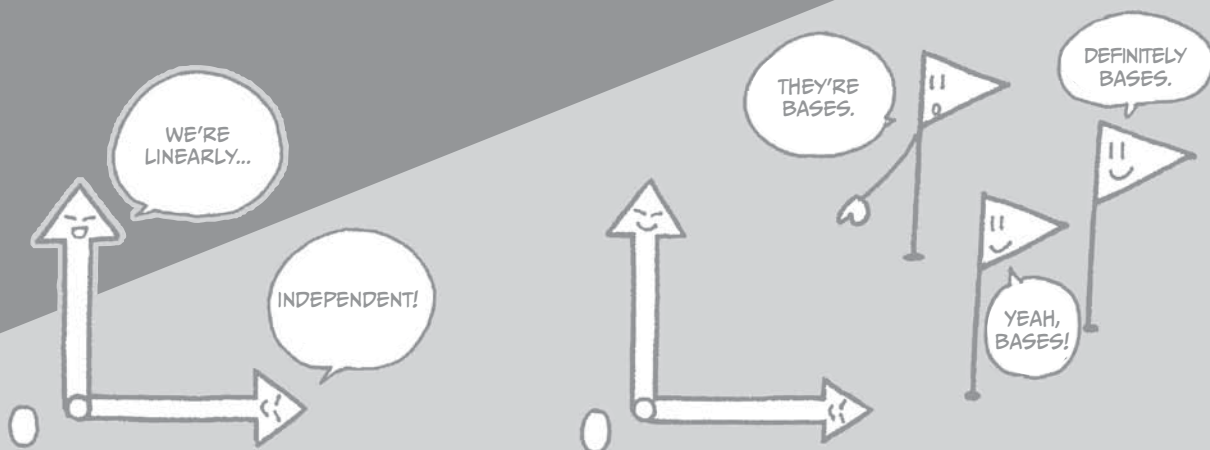
I UNDERSTAND THE FORMULA, BUT THIS ONE'S A LITTLE HARDER TO VISUALIZE...

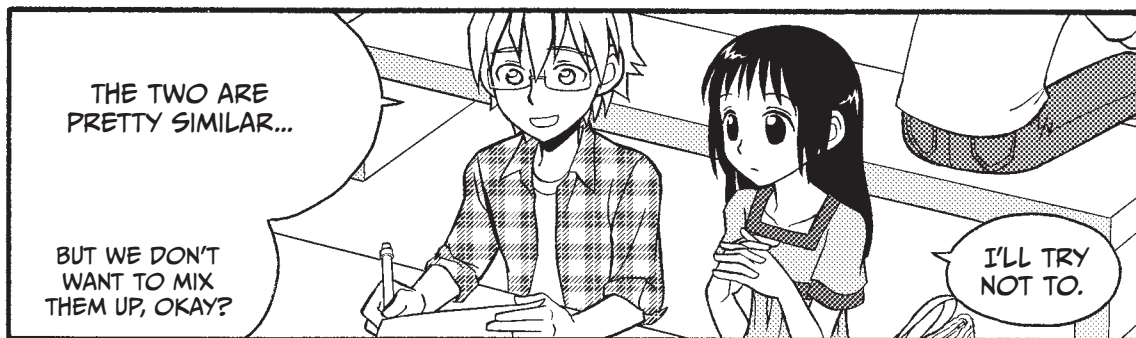
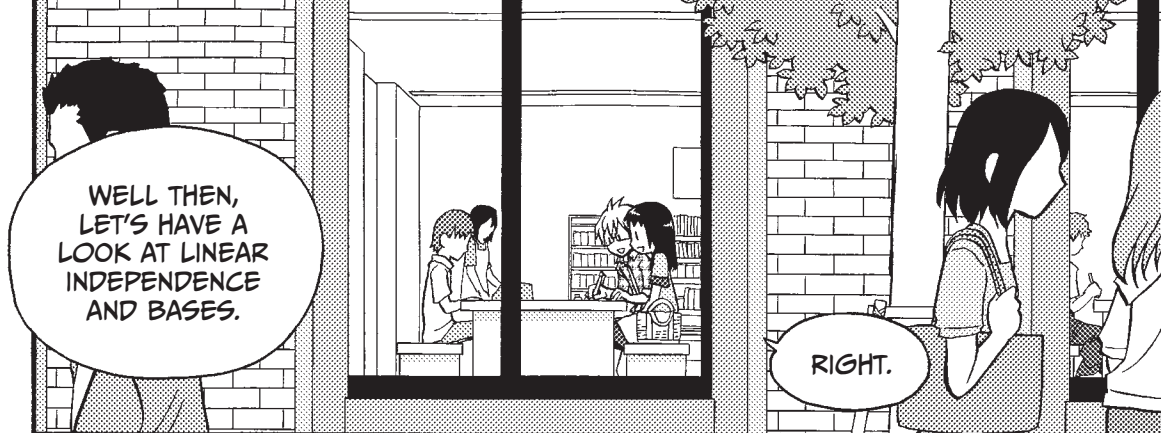




6

MORE VECTORS





LINEAR INDEPENDENCE



PROBLEM 1

Find the constants c_1 and c_2 satisfying this equation:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



THAT'S EASY.

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

CORRECT!

PROBLEM 2

Find the constants c_1 and c_2 satisfying this equation:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

WELL THEN,
QUESTION TWO.

ISN'T THAT ALSO

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$$

IT IS.

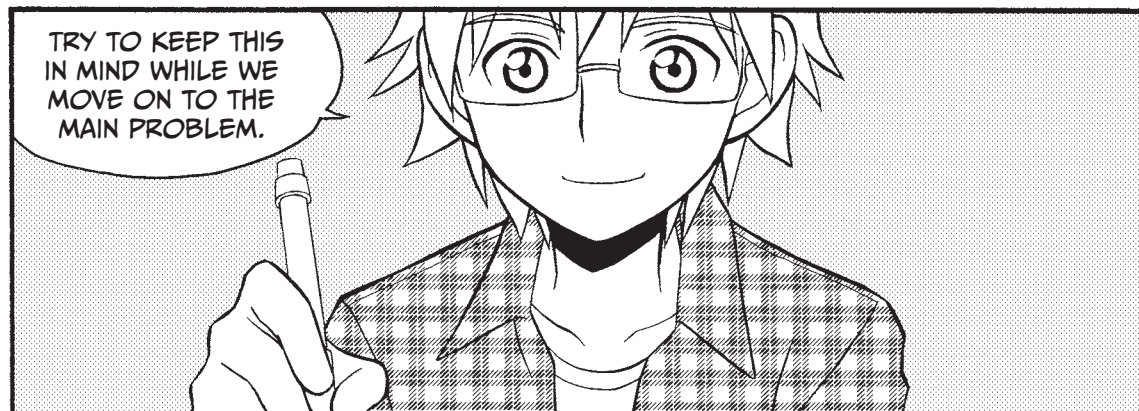
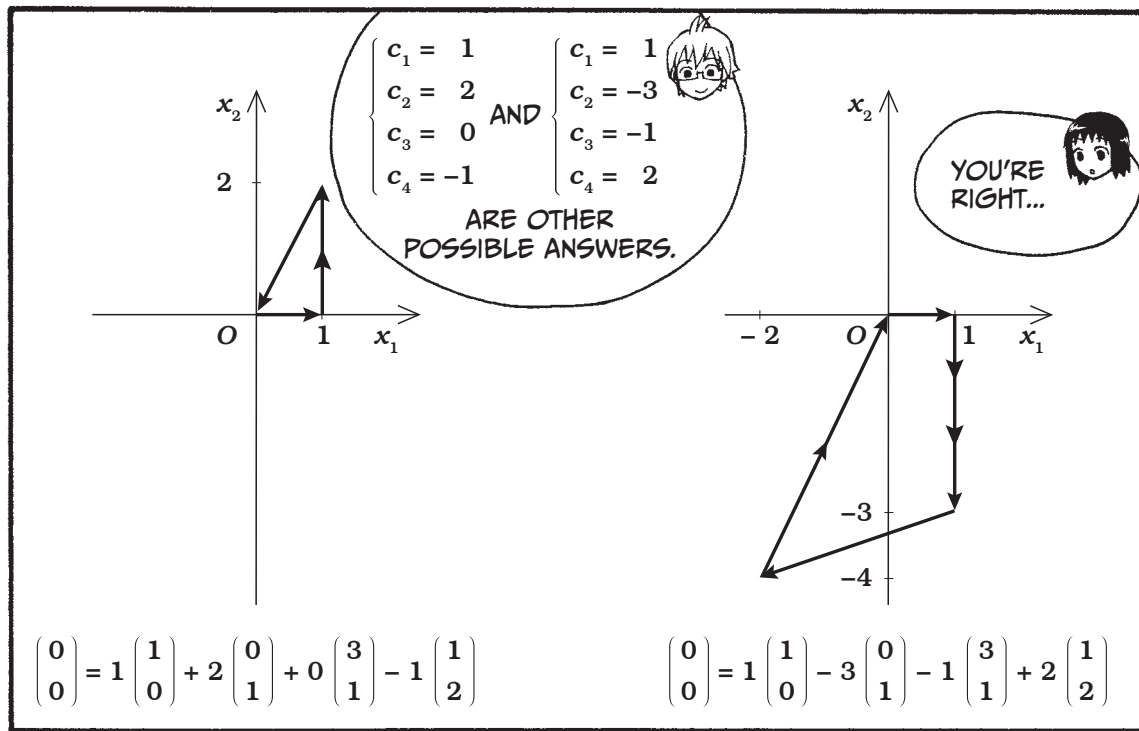
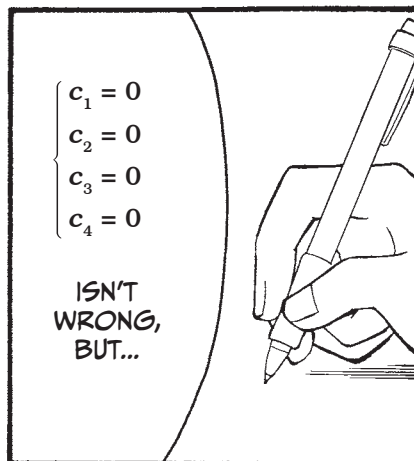
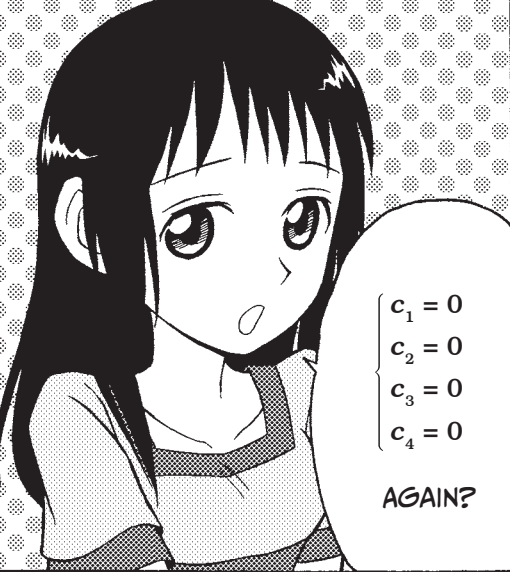
PROBLEM 3

Find the constants c_1 , c_2 , c_3 , and c_4 satisfying this equation:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

LAST ONE.

...

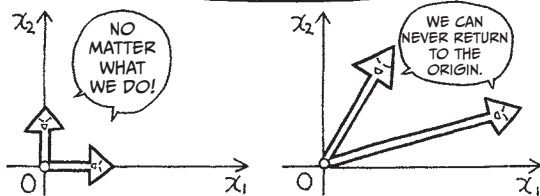


AS LONG AS THERE IS
ONLY ONE UNIQUE SOLUTION

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_n = 0 \end{cases}$$

TO PROBLEMS SUCH AS
THE FIRST OR SECOND EXAMPLES:

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$



LINEAR INDEPENDENCE

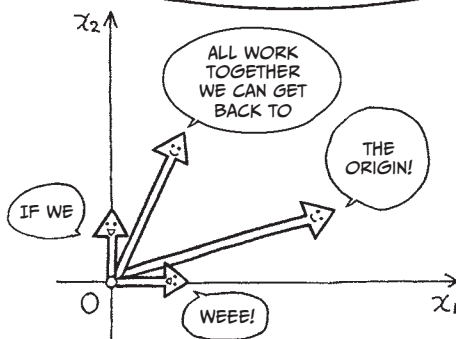
WE SAY THAT ITS VECTORS

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \text{ AND } \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

ARE LINEARLY INDEPENDENT.

AS FOR PROBLEMS LIKE THE
THIRD EXAMPLE, WHERE THERE
ARE SOLUTIONS OTHER THAN

$$\begin{cases} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_n = 0 \end{cases}$$



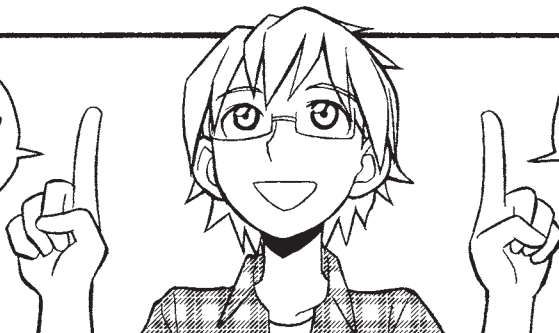
LINEAR DEPENDENCE

THEIR VECTORS

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \text{ AND } \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

ARE CALLED LINEARLY DEPENDENT.

LINEAR
INDEPENDENCE IS
SOMETIMES CALLED
ONE-DIMENSIONAL
INDEPENDENCE...



AND LINEAR
DEPENDENCE
IS SIMILARLY
SOMETIMES CALLED
ONE-DIMENSIONAL
DEPENDENCE.

AH...

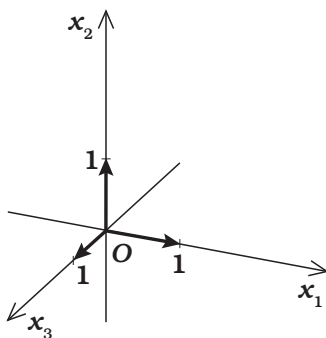


HERE ARE SOME EXAMPLES. LET'S LOOK AT LINEAR INDEPENDENCE FIRST.



EXAMPLE 1

The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



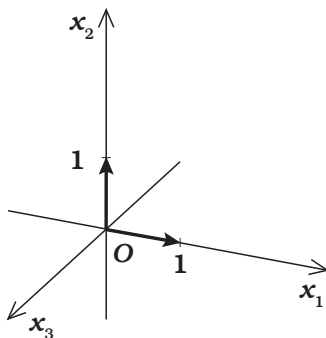
give us the equation $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

which has the unique solution $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$

The vectors are therefore linearly independent.

EXAMPLE 2

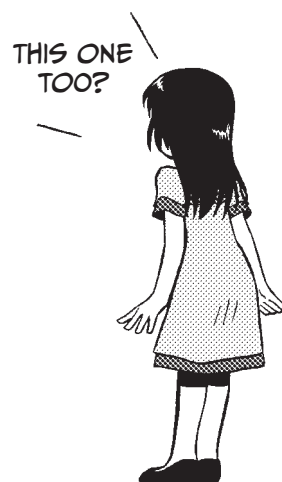
The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$



give us the equation $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

which has the unique solution $\begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$

These vectors are therefore also linearly independent.

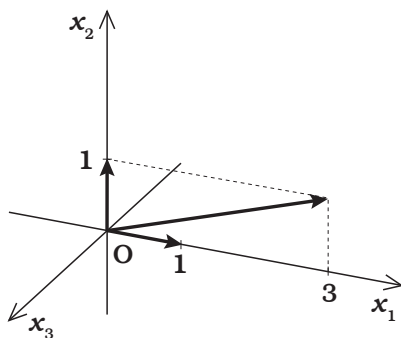


AND NOW WE'LL LOOK AT LINEAR DEPENDENCE.



EXAMPLE 1

The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$



give us the equation $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

which has several solutions, for example $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$ and $\begin{cases} c_1 = 3 \\ c_2 = 1 \\ c_3 = -1 \end{cases}$

This means that the vectors are linearly dependent.

EXAMPLE 2

Suppose we have the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

as well as the equation $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

The vectors are linearly dependent because there are several solutions to the system—

$$\text{for example, } \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \\ c_4 = 0 \end{cases} \text{ and } \begin{cases} c_1 = a_1 \\ c_2 = a_2 \\ c_3 = a_3 \\ c_4 = -1 \end{cases}$$

The vectors $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, and $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$

are similarly linearly dependent because there are several solutions to the equation

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \dots + c_m \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + c_{m+1} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

$$\text{Among them is } \begin{cases} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_m = 0 \\ c_{m+1} = 0 \end{cases} \text{ but also } \begin{cases} c_1 = a_1 \\ c_2 = a_2 \\ \vdots \\ c_m = a_m \\ c_{m+1} = -1 \end{cases}$$

HERE ARE THREE MORE PROBLEMS.

MHMM.

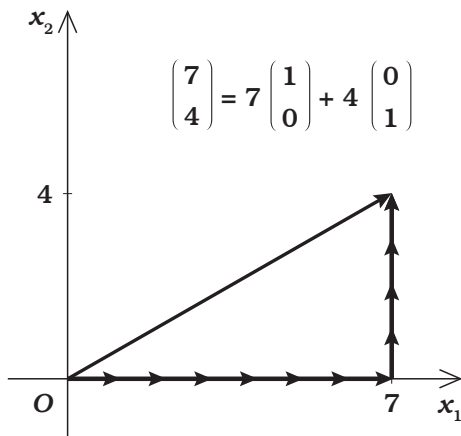
FIRST ONE.

? PROBLEM 4

Find the constants c_1 and c_2 satisfying this equation:

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

IT KINDA LOOKS LIKE THE OTHER PROBLEMS...



$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{cases} c_1 = 7 \\ c_2 = 4 \end{cases}$$

SHOULD WORK.

CORRECT!

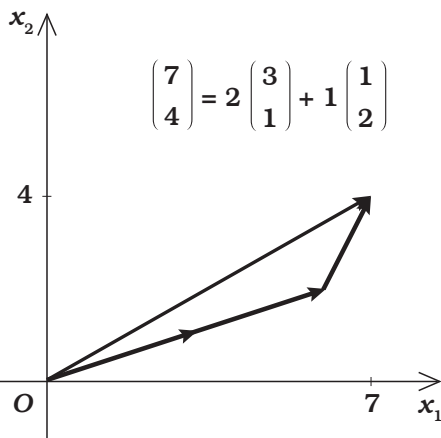
HERE'S THE
SECOND ONE.

PROBLEM 5

Find the constants c_1 and c_2 satisfying this equation:

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

LET'S SEE...



$$\begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$$

RIGHT?

CORRECT
AGAIN!

YOU'RE
REALLY
GOOD AT
THIS!



WELL THOSE WERE
PRETTY EASY...

LAST ONE.

PROBLEM 6

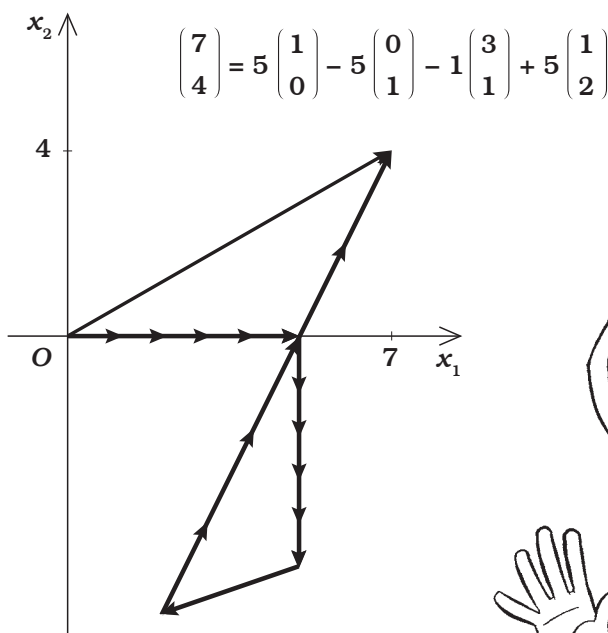
Find the constants c_1 , c_2 , c_3 , and c_4 satisfying this equation:

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

AH, IT HAS LOTS OF POSSIBLE SOLUTIONS, DOESN'T IT?



THERE'S $\begin{cases} c_1 = 7 \\ c_2 = 4 \\ c_3 = 0 \\ c_4 = 0 \end{cases}$ AND $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 2 \\ c_4 = 1 \end{cases}$ AND OF COURSE $\begin{cases} c_1 = 5 \\ c_2 = -5 \\ c_3 = -1 \\ c_4 = 5 \end{cases} \dots$



THAT'S ENOUGH.



LINEAR DEPENDENCE AND INDEPENDENCE ARE CLOSELY RELATED TO THE CONCEPT OF A BASIS. HAVE A LOOK AT THE FOLLOWING EQUATION:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

WHERE THE LEFT SIDE OF THE EQUATION IS AN ARBITRARY VECTOR IN R^m AND THE RIGHT SIDE IS A NUMBER OF n VECTORS OF THE SAME DIMENSION, AS WELL AS THEIR COEFFICIENTS.

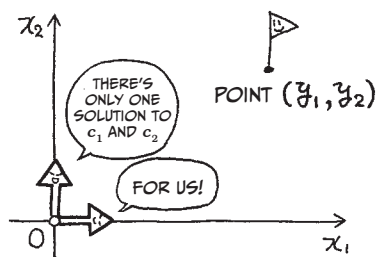
IF THERE'S ONLY ONE SOLUTION

$$c_1 = c_2 = \dots = c_n = 0$$

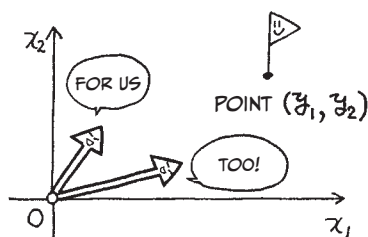
TO THE EQUATION, THEN OUR VECTORS

$$\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$$

MAKE UP A BASIS FOR R^n .



BASIS



DOES THAT MEAN THAT THE SOLUTION

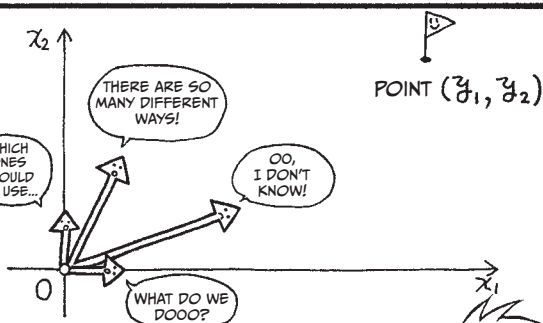
$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ FOR PROBLEM 4}$$

$$\text{AND THE SOLUTION } \left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

FOR PROBLEM 5 ARE BASES, BUT THE SOLUTION

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

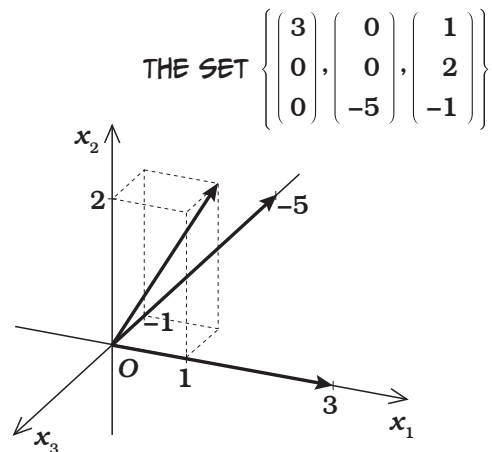
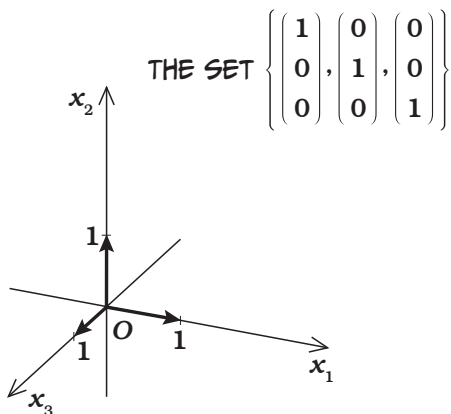
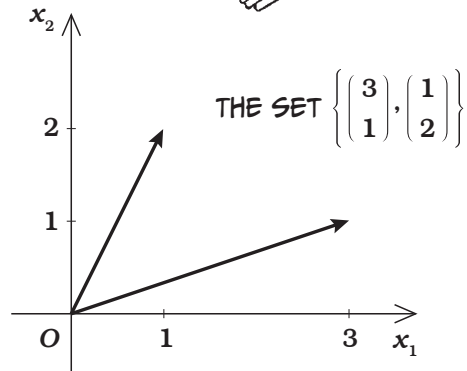
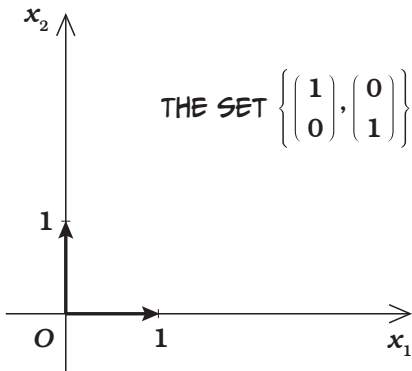
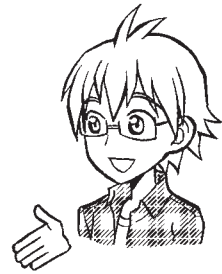
FOR PROBLEM 6 ISN'T?



HERE ARE SOME EXAMPLES OF WHAT IS AND WHAT IS NOT A BASIS.

OKAY.

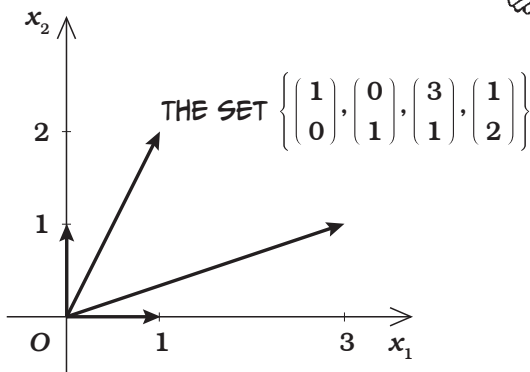
ALL THESE VECTOR SETS MAKE UP BASES FOR THEIR GRAPHS.



IN OTHER WORDS, A **BASES** IS A MINIMAL SET OF VECTORS NEEDED TO EXPRESS AN ARBITRARY VECTOR IN \mathbb{R}^m . ANOTHER IMPORTANT FEATURE OF BASES IS THAT THEY'RE ALL LINEARLY INDEPENDENT.



THE VECTORS OF THE FOLLOWING SET DO NOT FORM A BASIS.



TO UNDERSTAND WHY THEY DON'T FORM A BASIS, HAVE A LOOK AT THE FOLLOWING EQUATION:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

WHERE $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ IS AN ARBITRARY VECTOR IN \mathbb{R}^2 .

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ CAN BE FORMED IN MANY DIFFERENT WAYS

(USING DIFFERENT CHOICES FOR c_1 , c_2 , c_3 , AND c_4).

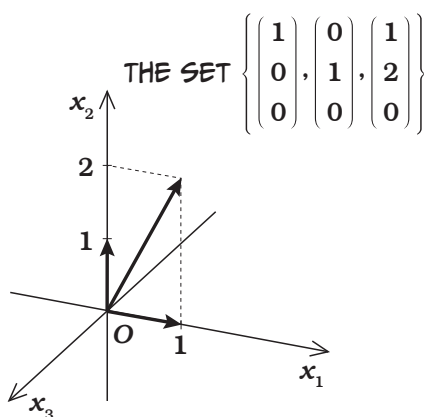
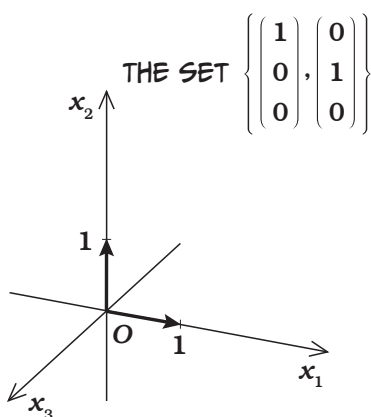
BECAUSE OF THIS, THE SET DOES NOT FORM "A MINIMAL SET OF VECTORS NEEDED TO EXPRESS AN ARBITRARY VECTOR IN \mathbb{R}^m ."



NEITHER OF THE TWO VECTOR SETS BELOW IS ABLE

TO DESCRIBE THE VECTOR $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, AND IF THEY CAN'T

DESCRIBE THAT VECTOR, THEN THERE'S NO WAY THAT THEY COULD DESCRIBE "AN ARBITRARY VECTOR IN \mathbb{R}^3 ." BECAUSE OF THIS, THEY'RE NOT BASES.



JUST BECAUSE A SET OF VECTORS IS LINEARLY INDEPENDENT DOESN'T MEAN THAT IT FORMS A BASIS.

FOR INSTANCE, THE SET $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ FORMS A BASIS,

WHILE THE SET $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ DOES NOT, EVEN THOUGH

THEY'RE BOTH LINEARLY INDEPENDENT.



SINCE BASES AND LINEAR INDEPENDENCE ARE CONFUSINGLY SIMILAR, I THOUGHT I'D TALK A BIT ABOUT THE DIFFERENCES BETWEEN THE TWO.



LINEAR INDEPENDENCE

We say that a set of vectors $\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$ is linearly independent

if there's only one solution $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ \vdots \\ c_n = 0 \end{cases}$

to the equation $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

where the left side is the zero vector of R^m .

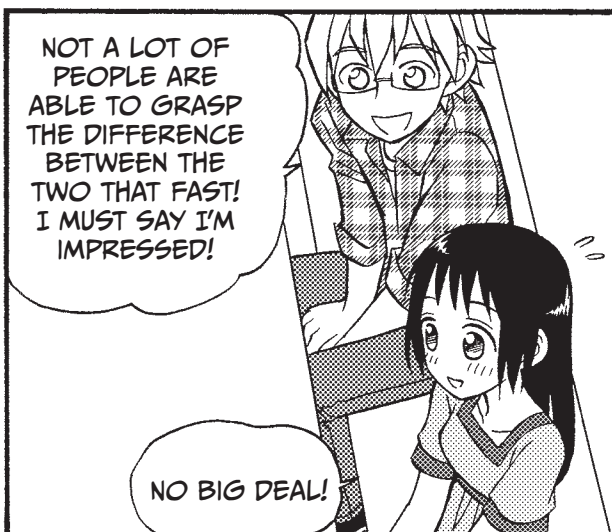
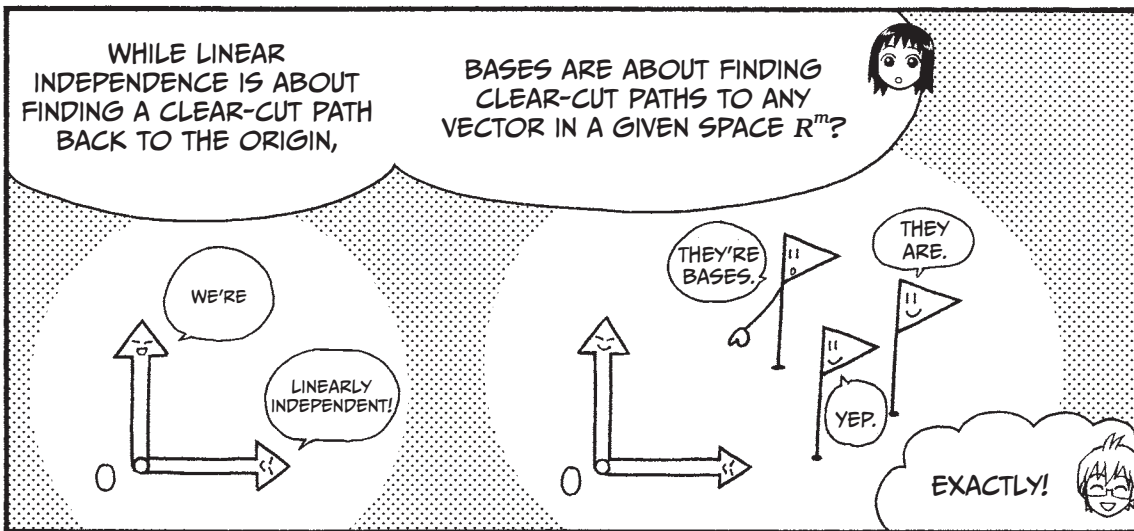
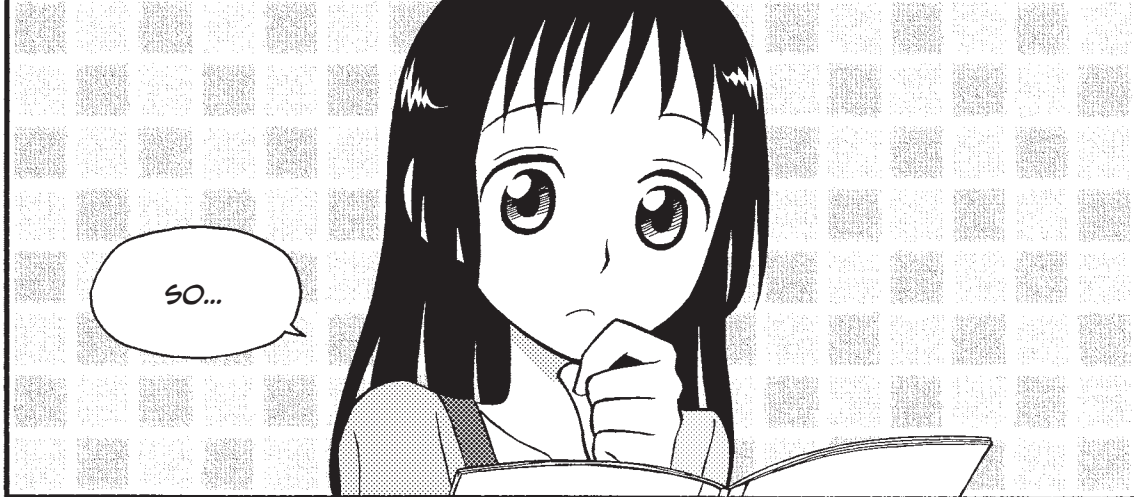
BASES

A set of vectors $\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$ forms a basis if there's only

one solution to the equation $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$

where the left side is an arbitrary vector $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ in R^m . And once again, a basis

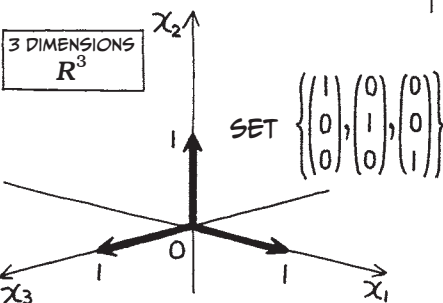
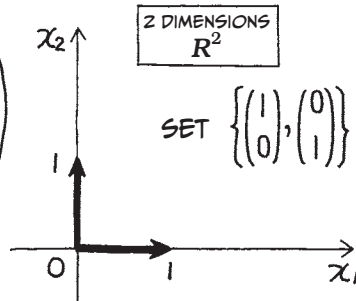
is a minimal set of vectors needed to express an arbitrary vector in R^m .



DIMENSION



IT'S KIND OF OBVIOUS THAT A BASIS IS MADE UP OF TWO VECTORS WHEN IN R^2 AND THREE VECTORS WHEN IN R^3 .



BUT WHY IS IT THAT THE BASIS OF AN m -DIMENSIONAL SPACE CONSISTS OF n VECTORS AND NOT m ?

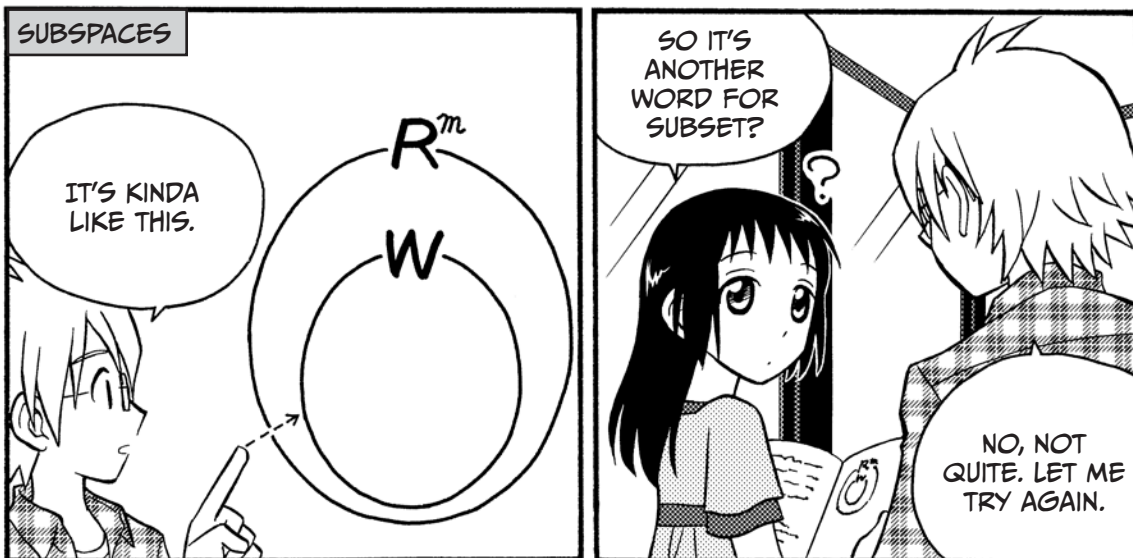
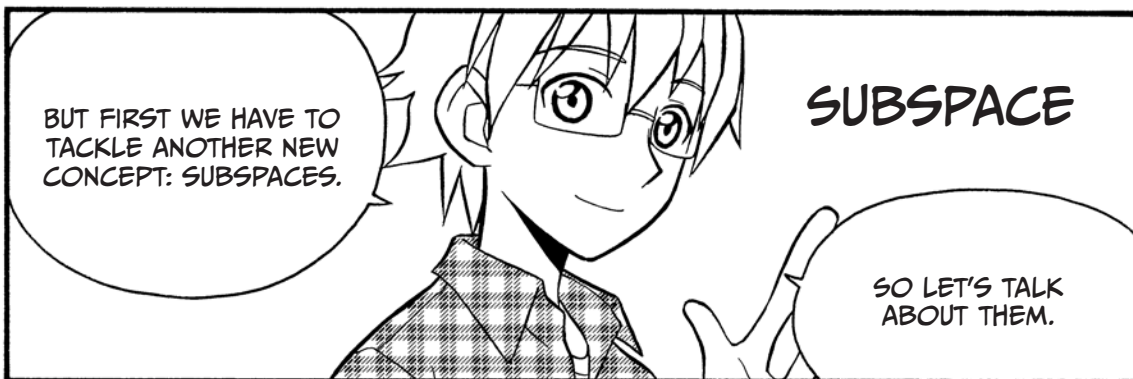
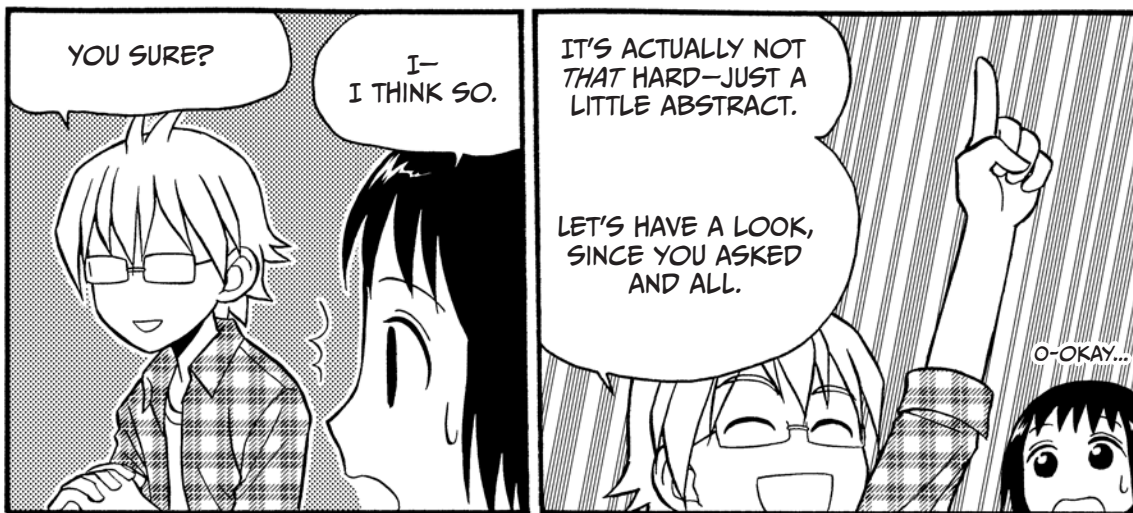
$$\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$$

OH, WOW... I DIDN'T THINK YOU'D NOTICE...

TO ANSWER THAT, WE'LL HAVE TO TAKE A LOOK AT ANOTHER, MORE PRECISE DEFINITION OF BASES.

THERE'S ALSO A MORE PRECISE DEFINITION OF VECTORS, WHICH CAN BE HARD TO UNDERSTAND.

I'M UP FOR IT!



WHAT IS A SUBSPACE?

Let c be an arbitrary real number and W be a nonempty subset of R^m satisfying these two conditions:

① An element in W multiplied by c is still an element in W . (Closed under scalar multiplication.)

$$\text{If } \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \in W, \text{ then } c \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \in W$$

② The sum of two arbitrary elements in W is still an element in W . (Closed under addition.)

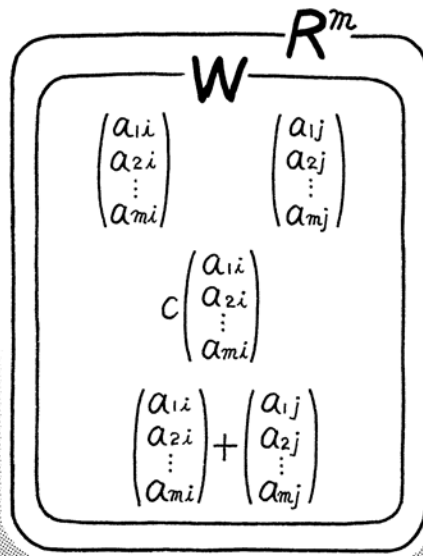
$$\text{If } \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} \in W \text{ and } \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in W, \text{ then } \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{pmatrix} + \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \in W$$

If both of these conditions hold, then W is a subspace of R^m .

THIS IS THE DEFINITION.



THIS PICTURE ILLUSTRATES THE RELATIONSHIP.



UMM...

IT'S PRETTY ABSTRACT, SO YOU MIGHT HAVE TO READ IT A FEW TIMES BEFORE IT STARTS TO SINK IN.

ANOTHER, MORE CONCRETE WAY TO LOOK AT ONE-DIMENSIONAL SUBSPACES IS AS LINES THROUGH THE ORIGIN. TWO-DIMENSIONAL SUBSPACES ARE SIMILARLY PLANES THROUGH THE ORIGIN. OTHER SUBSPACES CAN ALSO BE VISUALIZED, BUT NOT AS EASILY.

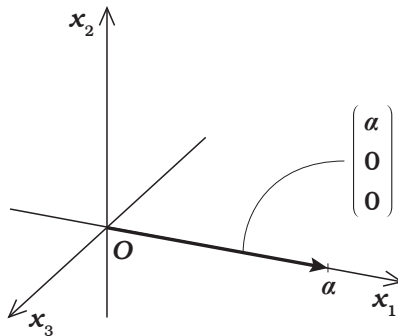
I MADE SOME EXAMPLES OF SPACES THAT ARE SUBSPACES—AND OF SOME THAT ARE NOT. HAVE A LOOK!



THIS IS A SUBSPACE

Let's have a look at the subspace in R^3 defined by the set

$$\left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \mid a \text{ is an arbitrary real number} \right\}, \text{ in other words, the } x\text{-axis.}$$



If it really is a subspace, it should satisfy the two conditions we talked about before.

$$\begin{aligned} \textcircled{1} \quad c \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} ca_1 \\ 0 \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \mid a \text{ is an arbitrary real number} \right\} \\ \textcircled{2} \quad \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} a_1 + a_2 \\ 0 \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \mid a \text{ is an arbitrary real number} \right\} \end{aligned}$$

It seems like they do! This means it actually is a subspace.

THIS IS NOT A SUBSPACE

The set $\left\{ \begin{pmatrix} a \\ a^2 \\ 0 \end{pmatrix} \middle| \begin{array}{l} a \text{ is an} \\ \text{arbitrary} \\ \text{real number} \end{array} \right\}$ is not a subspace of R^3 .

Let's use our conditions to see why:

$$\textcircled{1} \quad c \begin{pmatrix} a_1 \\ a_1^2 \\ 0 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_1^2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} ca_1 \\ (ca_1)^2 \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} a \\ a^2 \\ 0 \end{pmatrix} \middle| \begin{array}{l} a \text{ is an} \\ \text{arbitrary} \\ \text{real number} \end{array} \right\}$$

$$\textcircled{2} \quad \begin{pmatrix} a_1 \\ a_1^2 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ a_2^2 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_1^2 + a_2^2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} a_1 + a_2 \\ (a_1 + a_2)^2 \\ 0 \end{pmatrix} \in \left\{ \begin{pmatrix} a \\ a^2 \\ 0 \end{pmatrix} \middle| \begin{array}{l} a \text{ is an} \\ \text{arbitrary} \\ \text{real number} \end{array} \right\}$$

The set doesn't seem to satisfy either of the two conditions, and therefore it is not a subspace!

I'D IMAGINE YOU MIGHT THINK THAT "BOTH $\textcircled{1}$ AND $\textcircled{2}$ HOLD IF WE USE $a_1 = a_2 = 0$, SO IT SHOULD BE A SUBSPACE!"

IT'S TRUE THAT THE CONDITIONS HOLD FOR THOSE VALUES, BUT SINCE THE CONDITIONS HAVE TO HOLD FOR ARBITRARY REAL VALUES—THAT IS, ALL REAL VALUES—IT'S JUST NOT ENOUGH TO TEST WITH A FEW CHOSEN NUMERICAL EXAMPLES. THE VECTOR SET IS A SUBSPACE ONLY IF BOTH CONDITIONS HOLD FOR ALL KINDS OF VECTORS.

IF THIS STILL DOESN'T MAKE SENSE, DON'T GIVE UP! THIS IS HARD!



I THINK
I GET IT...

IT'LL MAKE
MORE SENSE AFTER
SOLVING A FEW
PROBLEMS.

THE FOLLOWING SUBSPACES ARE CALLED *LINEAR SPANS* AND ARE A BIT SPECIAL.



WHAT IS A LINEAR SPAN?

We say that a set of m -dimensional vectors

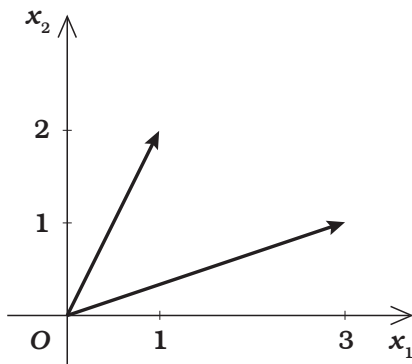
$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ span the following subspace in R^m :

$$\left\{ c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \mid c_1, c_2, \text{ and } c_n \text{ are arbitrary numbers} \right\}$$

This set forms a subspace and is called the *linear span* of the n original vectors.

EXAMPLE 1

The x_1x_2 -plane is a subspace of R^2 and can, for example, be spanned by using the two vectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ like so: $\left\{ c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mid c_1 \text{ and } c_2 \text{ are arbitrary numbers} \right\}$

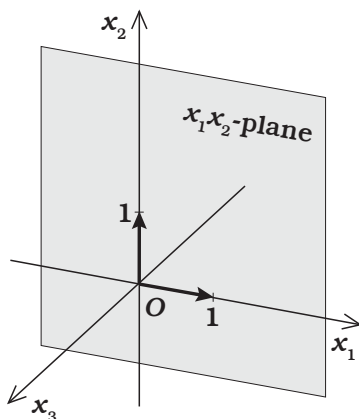


EXAMPLE 2

The x_1x_2 -plane could also be a subspace of R^3 , and we could span it using the

vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, creating this set:

$$\left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid c_1 \text{ and } c_2 \text{ are arbitrary numbers} \right\}$$



R^m IS ALSO A SUBSPACE OF ITSELF, AS YOU MIGHT HAVE GUESSED FROM EXAMPLE 1.

ALL SUBSPACES CONTAIN THE ZERO VECTOR, WHICH YOU COULD PROBABLY TELL FROM LOOKING AT THE EXAMPLE ON PAGE 152. REMEMBER, THEY MUST PASS THROUGH THE ORIGIN!

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$



SORRY FOR
THE WAIT.

HERE ARE THE
DEFINITIONS OF *BASIS*
AND *DIMENSION*.

WHAT ARE BASIS AND DIMENSION?

Suppose that W is a subspace of R^m and that it is spanned by the

linearly independent vectors $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$, $\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$, and $\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$.

This could also be written as follows:

$$W = \left\{ c_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + c_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + c_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \mid c_1, c_2, \text{ and } c_n \text{ are arbitrary numbers} \right\}$$

When this equality holds, we say that the set

forms a *basis* to the subspace W . $\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}$

The *dimension* of the subspace W is equal to the number of vectors in any basis for W .

"THE DIMENSION OF THE
SUBSPACE W " IS USUALLY
WRITTEN AS $\dim W$.

I'M A LITTLE
LOST...

HMM...

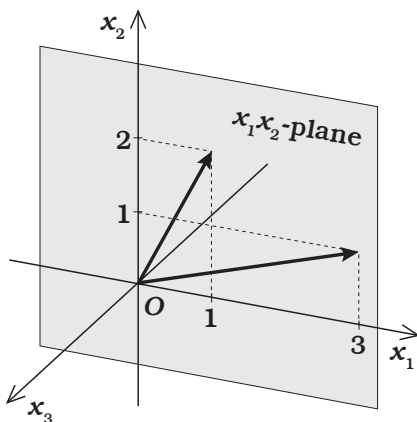
THIS EXAMPLE MIGHT CLEAR THINGS UP A LITTLE.



EXAMPLE

Let's call the x_1x_2 -plane W , for simplicity's sake. So suppose that W is a subspace of R^3 and is spanned by the linearly independent vectors

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$



We have this:

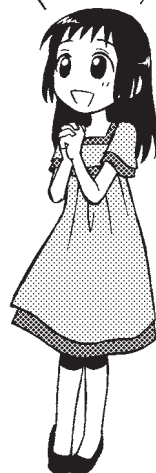
$$W = \left\{ c_1 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \mid c_1 \text{ and } c_2 \text{ are arbitrary numbers} \right\}$$

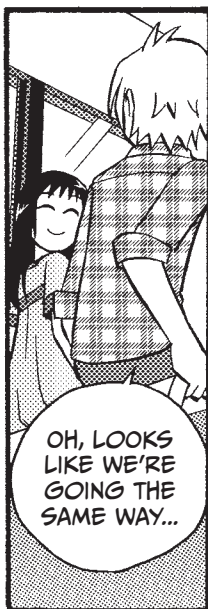
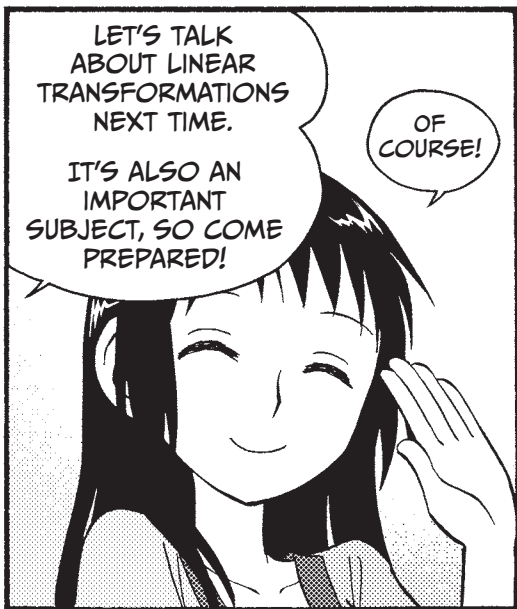
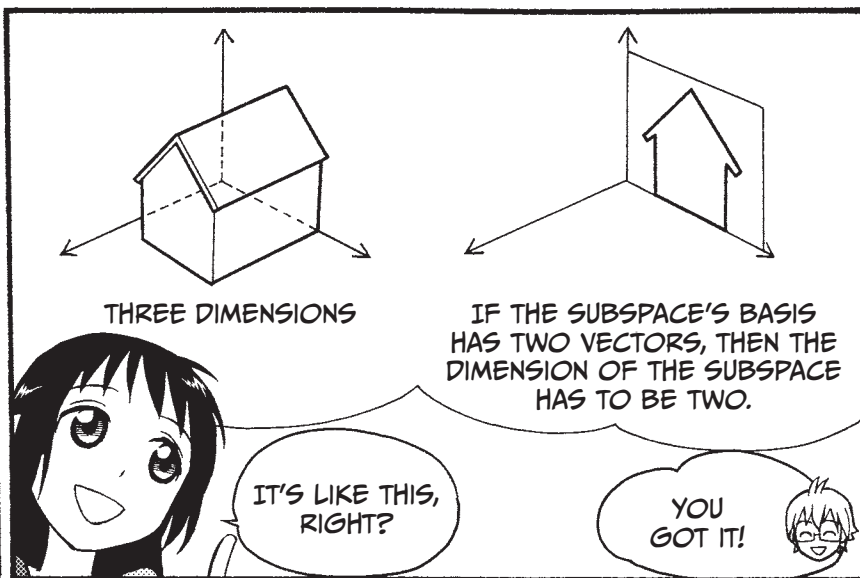
The fact that this equality holds means that the vector set

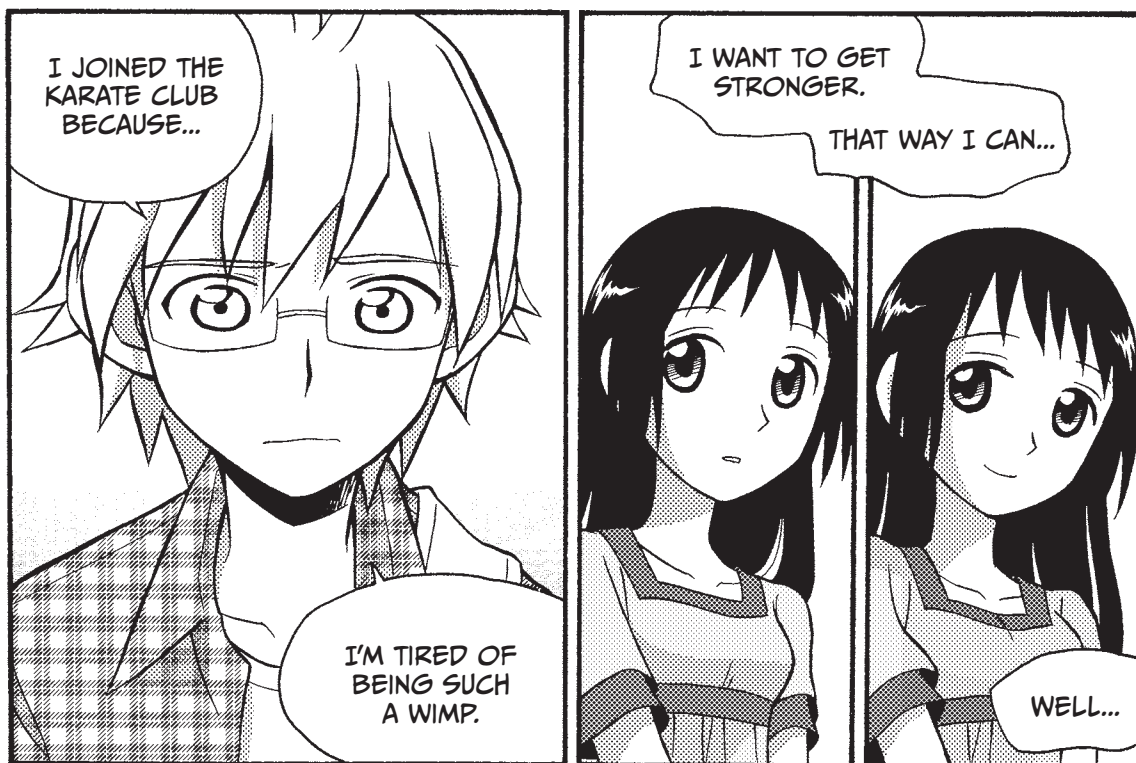
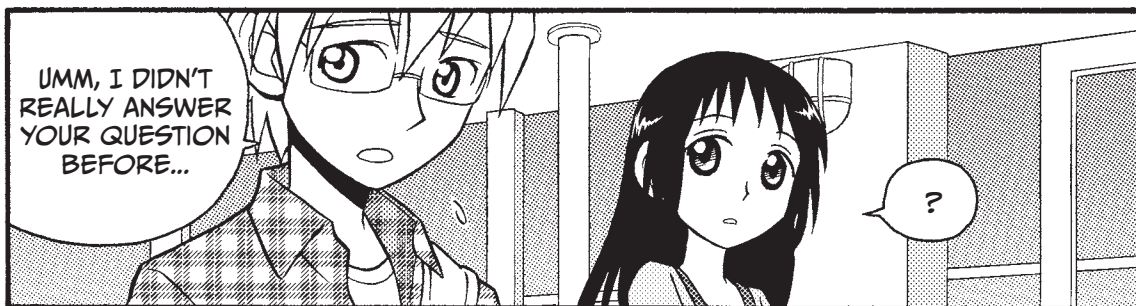
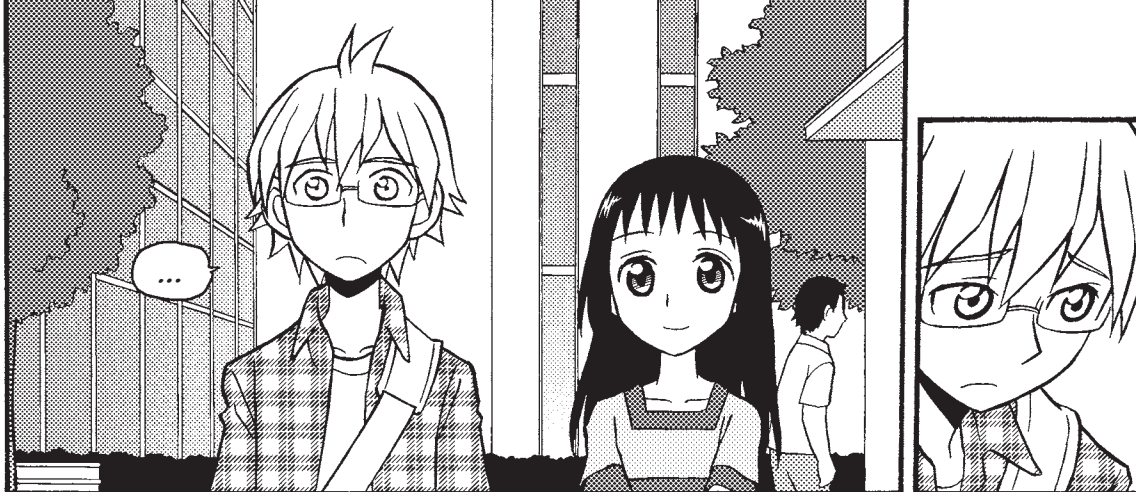
$$\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

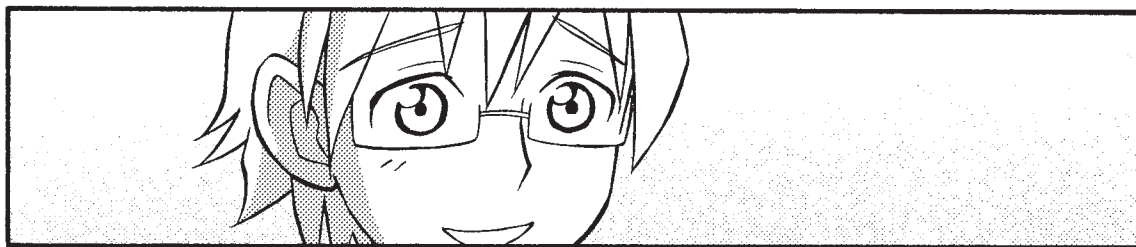
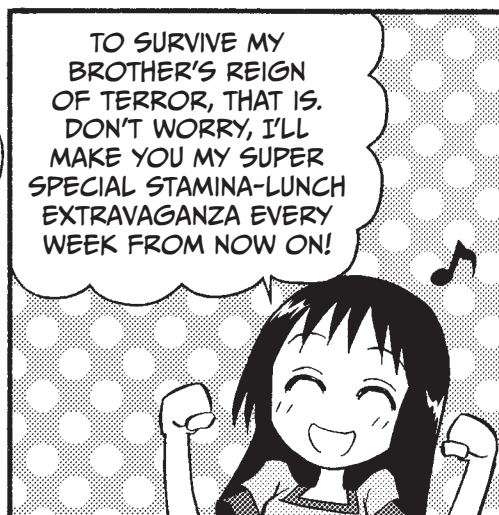
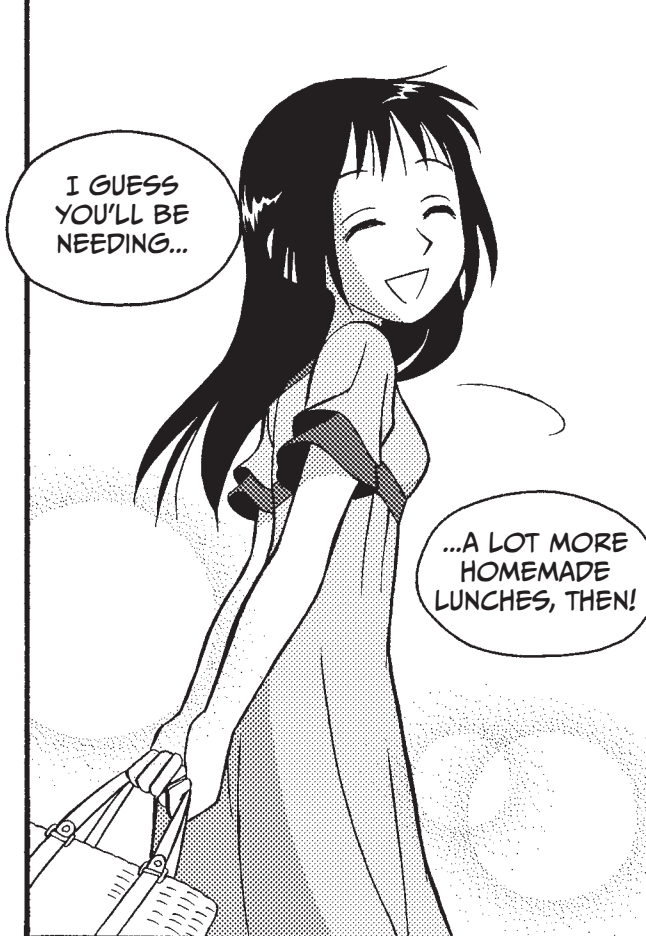
is a basis of the subspace W . Since the base contains two vectors, $\dim W = 2$.

I SEE!



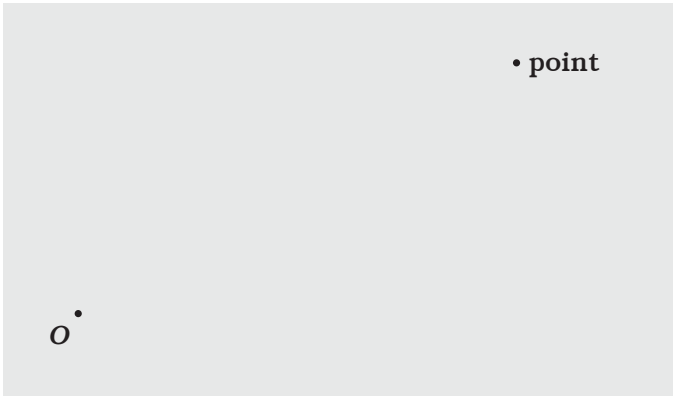






COORDINATES

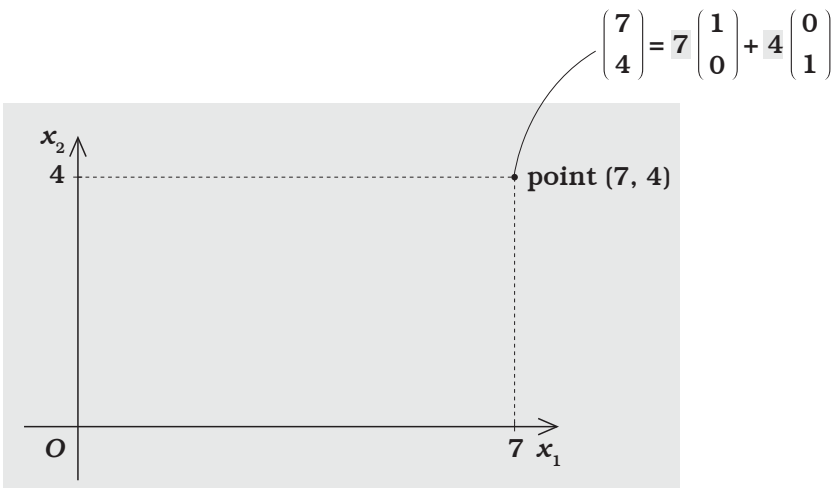
Coordinates in linear algebra are a bit different from the coordinates explained in high school. I'll try explaining the difference between the two using the image below.



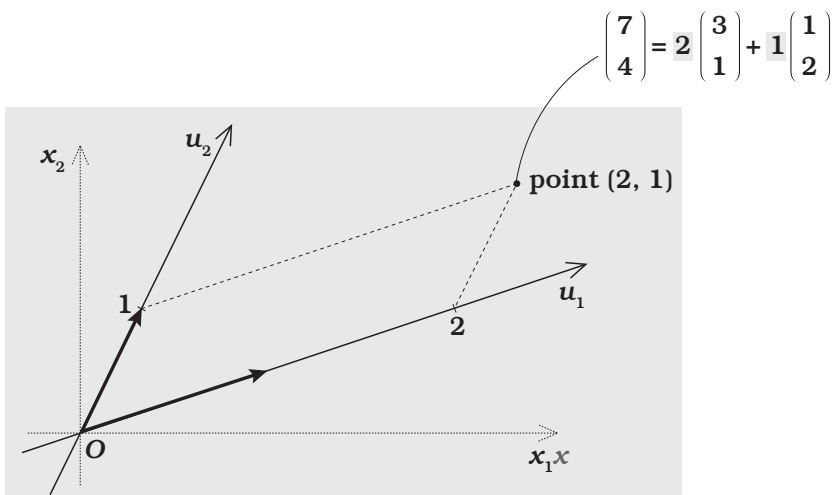
When working with coordinates and coordinate systems at the high school level, it's much easier to use only the trivial basis:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

In this kind of system, the relationship between the origin and the point in the top right is interpreted as follows:



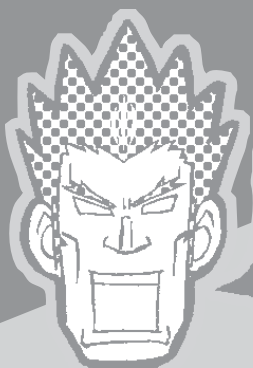
It is important to understand that the trivial basis is only one of many bases when we move into the realm of linear algebra—and that using other bases produces other relationships between the origin and a given point. The image below illustrates the point $(2, 1)$ in a system using the nontrivial basis consisting of the two vectors $u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.



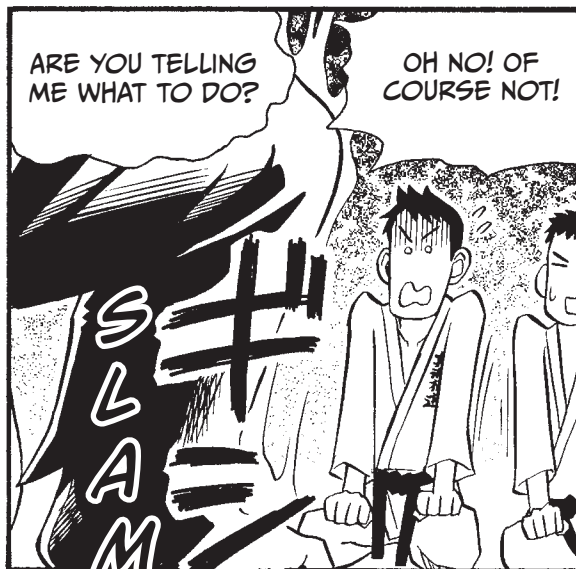
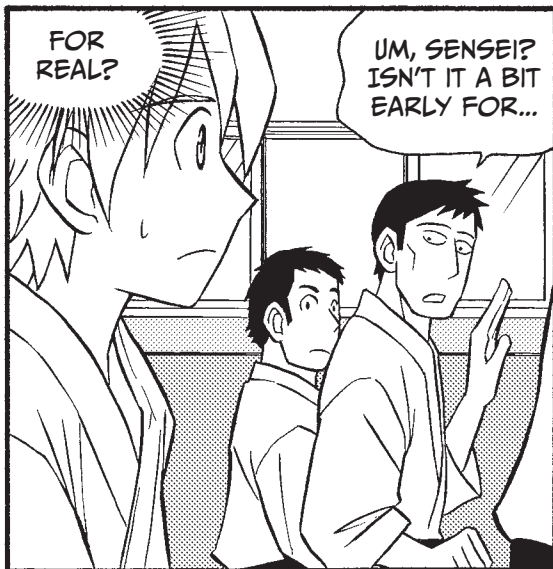
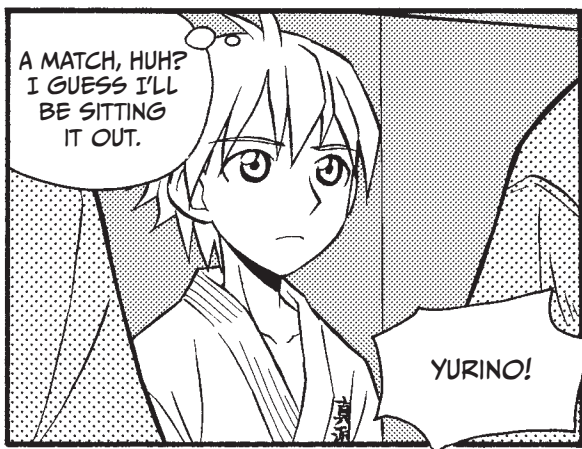
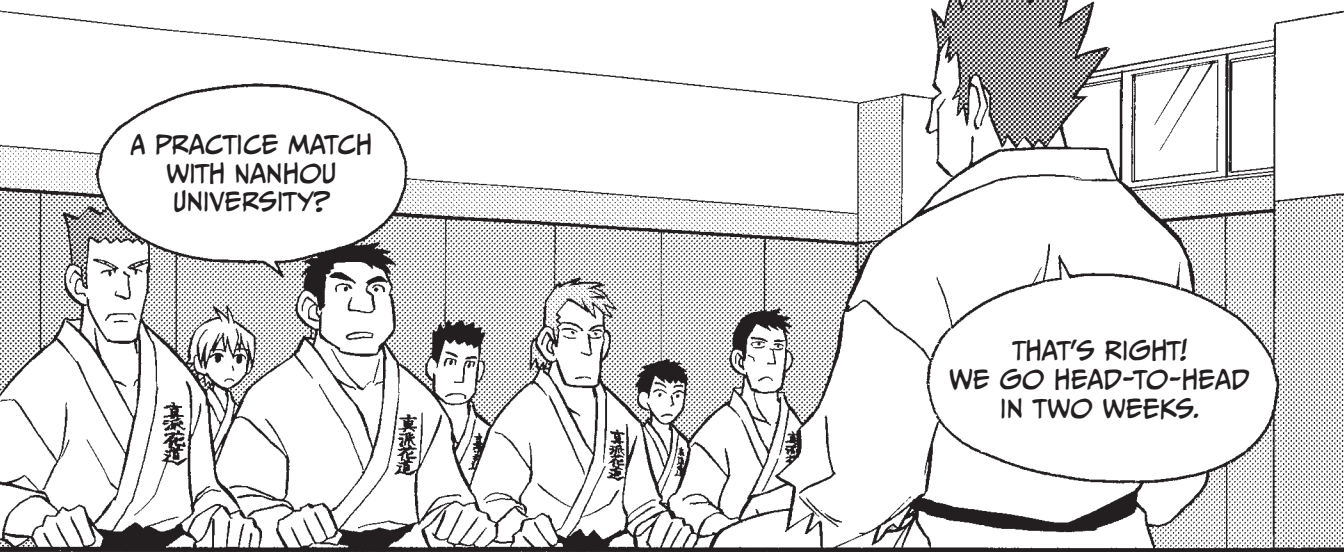
This alternative way of thinking about coordinates is very useful in factor analysis, for example.

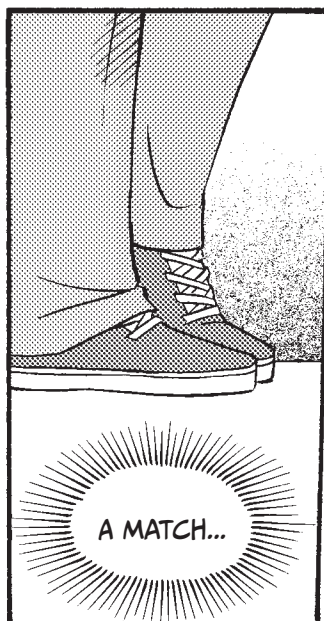
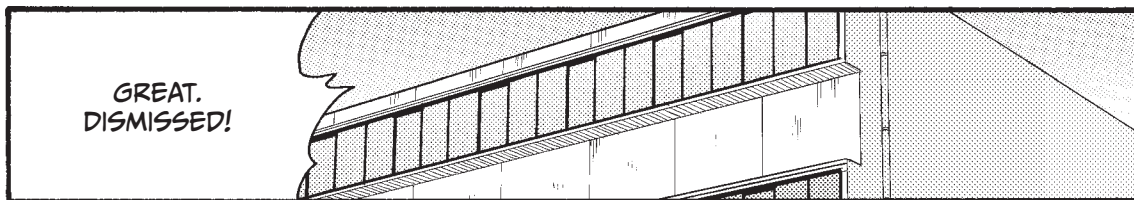
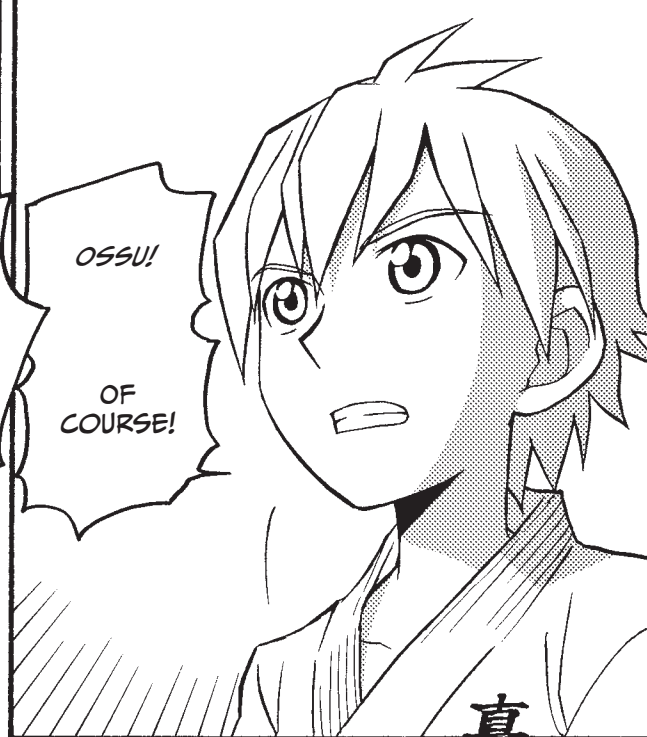
7

LINEAR TRANSFORMATIONS



$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$





WHAT IS A LINEAR TRANSFORMATION?

IT SEEMS WE'VE FINALLY ARRIVED AT LINEAR TRANSFORMATIONS!

COURSE LAYOUT

FUNDAMENTALS

PREP

MATRICES

VECTORS

LINEAR TRANSFORMATIONS

EIGENVALUES AND EIGENVECTORS

LET'S START WITH THE DEFINITION.

SOUNDS GOOD.

WE TOUCHED ON THIS A BIT IN CHAPTER 2.

YEAH...

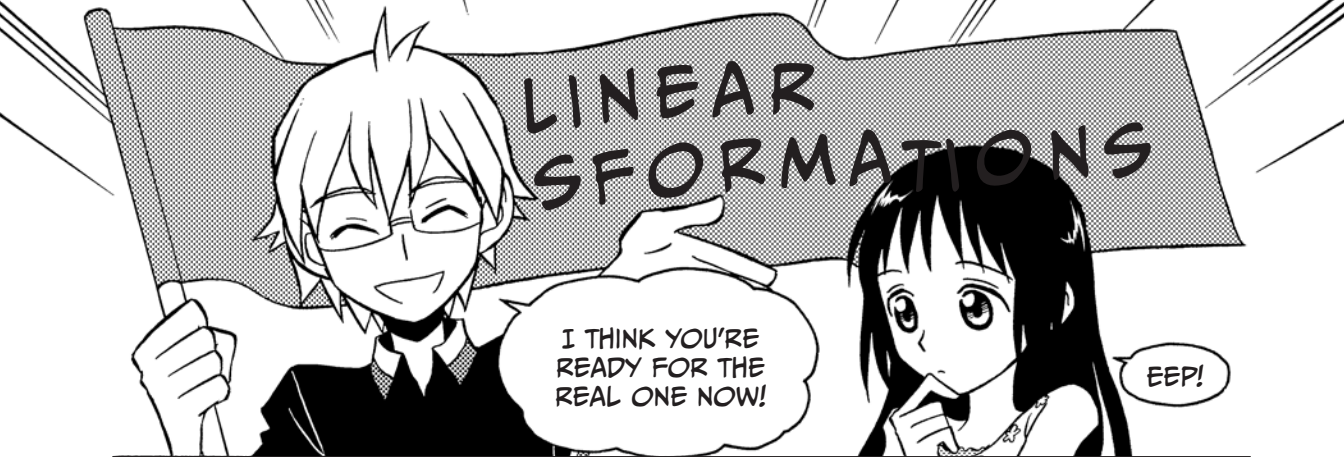
LINEAR TRANSFORMATIONS

Let x_i and x_j be two arbitrary elements, c an arbitrary real number, and f a function from X to Y .

We say that f is a linear transformation from X to Y if it satisfies the following two conditions:

- ① $f(x_i) + f(x_j)$ and $f(x_i + x_j)$ are equal
- ② $cf(x_i)$ and $f(cx_i)$ are equal

BUT THIS DEFINITION IS ACTUALLY INCOMPLETE.



LINEAR TRANSFORMATIONS

Let $\begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$ and $\begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix}$ be two arbitrary elements from R^n , c an arbitrary real number, and f a function from R^n to R^m .

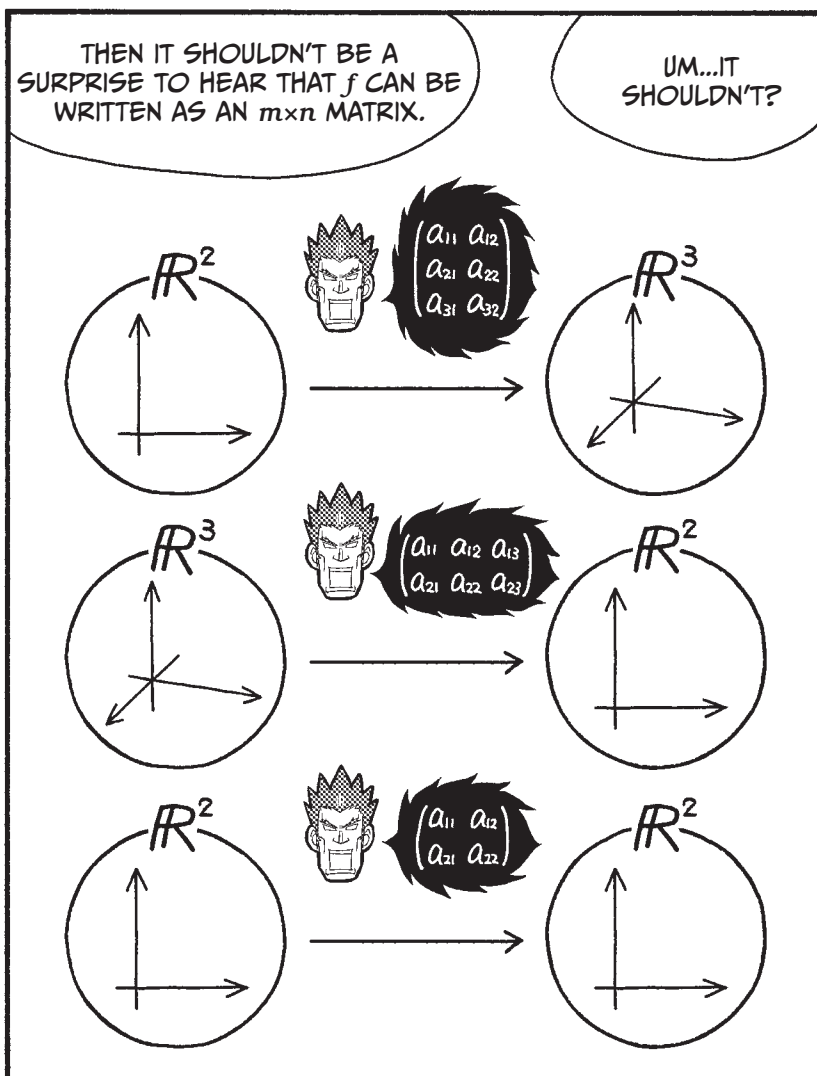
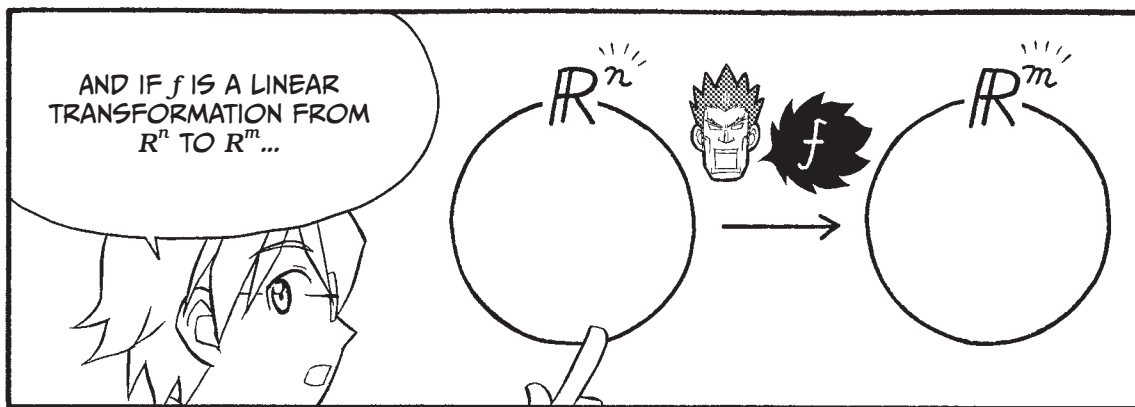
We say that f is a linear transformation from R^n to R^m if it satisfies the following two conditions:

$$\textcircled{1} f \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} + f \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} \text{ and } f \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{pmatrix} \text{ are equal.}$$

$$\textcircled{2} cf \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \text{ and } f \begin{pmatrix} cx_{1i} \\ cx_{2i} \\ \vdots \\ cx_{ni} \end{pmatrix} \text{ are equal.}$$

A linear transformation from R^n to R^m is sometimes called a *linear map* or *linear operation*.





❶ We'll verify the first rule first:

$$f \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} + f \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} = f \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{pmatrix}$$

We just replace f with a matrix, then simplify:

$$\begin{aligned} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}x_{1i} + a_{12}x_{2i} + \cdots + a_{1n}x_{ni} \\ a_{21}x_{1i} + a_{22}x_{2i} + \cdots + a_{2n}x_{ni} \\ \vdots \\ a_{m1}x_{1i} + a_{m2}x_{2i} + \cdots + a_{mn}x_{ni} \end{pmatrix} + \begin{pmatrix} a_{11}x_{1j} + a_{12}x_{2j} + \cdots + a_{1n}x_{nj} \\ a_{21}x_{1j} + a_{22}x_{2j} + \cdots + a_{2n}x_{nj} \\ \vdots \\ a_{m1}x_{1j} + a_{m2}x_{2j} + \cdots + a_{mn}x_{nj} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}(x_{1i} + x_{1j}) + a_{12}(x_{2i} + x_{2j}) + \cdots + a_{1n}(x_{ni} + x_{nj}) \\ a_{21}(x_{1i} + x_{1j}) + a_{22}(x_{2i} + x_{2j}) + \cdots + a_{2n}(x_{ni} + x_{nj}) \\ \vdots \\ a_{m1}(x_{1i} + x_{1j}) + a_{m2}(x_{2i} + x_{2j}) + \cdots + a_{mn}(x_{ni} + x_{nj}) \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \\ \vdots \\ x_{ni} + x_{nj} \end{pmatrix} \end{aligned}$$

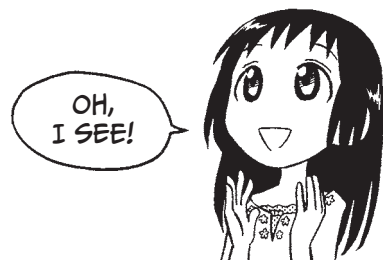
UH-HUH.



② Now for the second rule:
$$cf \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} = f \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$$

Again, just replace f with a matrix and simplify:

$$\begin{aligned} & c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \\ &= c \begin{pmatrix} a_{11}x_{1i} + a_{12}x_{2i} + \cdots + a_{1n}x_{ni} \\ a_{21}x_{1i} + a_{22}x_{2i} + \cdots + a_{2n}x_{ni} \\ \vdots \\ a_{m1}x_{1i} + a_{m2}x_{2i} + \cdots + a_{mn}x_{ni} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}(cx_{1i}) + a_{12}(cx_{2i}) + \cdots + a_{1n}(cx_{ni}) \\ a_{21}(cx_{1i}) + a_{22}(cx_{2i}) + \cdots + a_{2n}(cx_{ni}) \\ \vdots \\ a_{m1}(cx_{1i}) + a_{m2}(cx_{2i}) + \cdots + a_{mn}(cx_{ni}) \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} cx_{1i} \\ cx_{2i} \\ \vdots \\ cx_{ni} \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \left[c \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix} \right] \end{aligned}$$



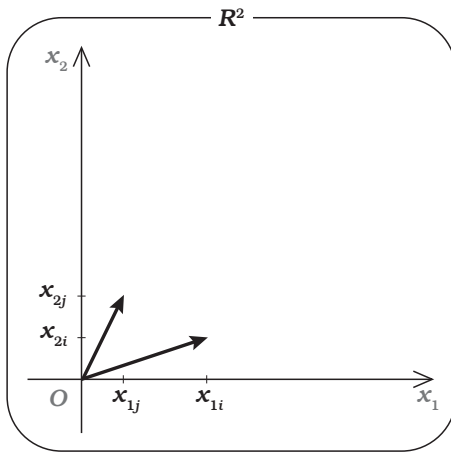
WE CAN DEMONSTRATE THE SAME THING VISUALLY.

WE'LL USE THE 2×2 MATRIX $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ AS f .

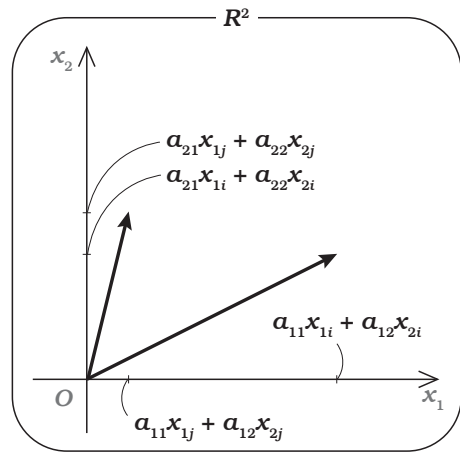


❶ We'll show that the first rule holds:

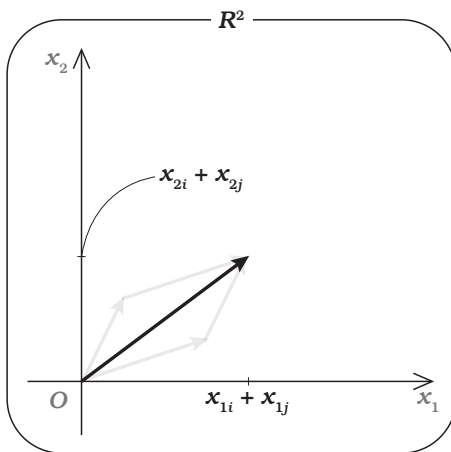
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1j} \\ x_{2j} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} + x_{1j} \\ x_{2i} + x_{2j} \end{pmatrix}$$



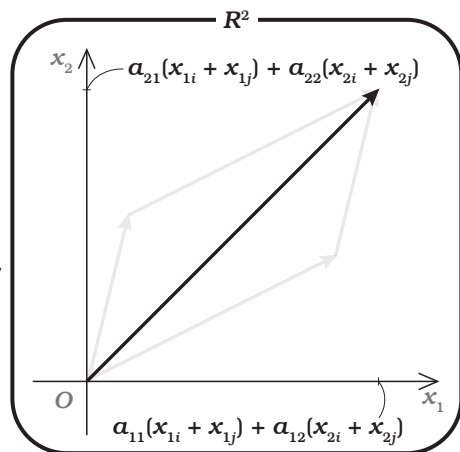
IF YOU
MULTIPLY
FIRST...



IF YOU
ADD FIRST...

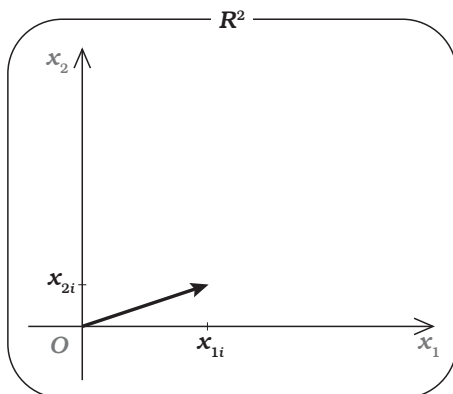


THEN
MULTIPLY...

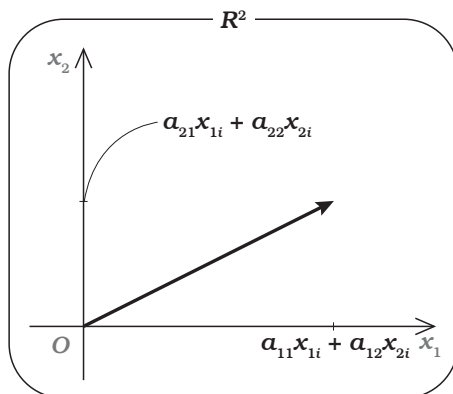


YOU GET THE SAME FINAL RESULT!

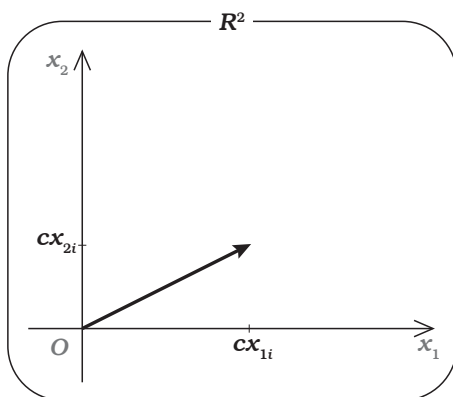
② And the second rule, too: $c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \left[c \begin{pmatrix} x_{1i} \\ x_{2i} \end{pmatrix} \right]$



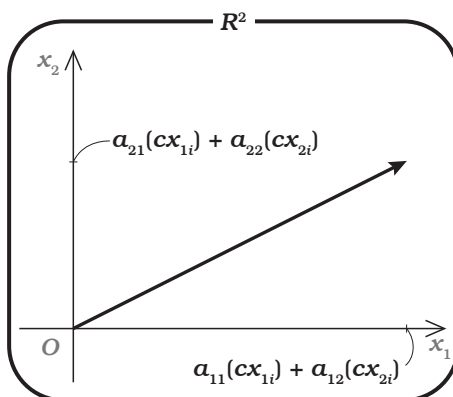
IF YOU
MULTIPLY
BY THE
MATRIX
FIRST...



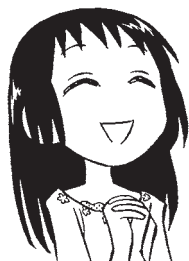
IF YOU MULTIPLY
BY c FIRST...



THEN
MULTIPLY
BY THE
MATRIX...



YOU GET THE SAME FINAL RESULT!



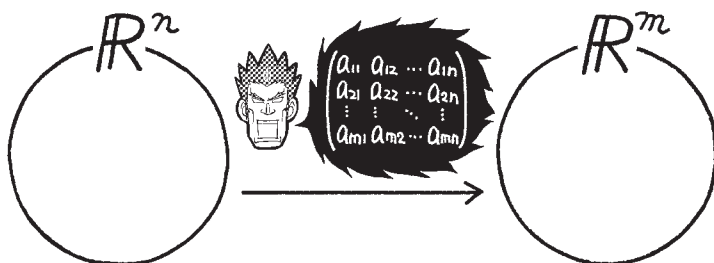
AWESOME!

SO WHEN f IS A LINEAR TRANSFORMATION FROM \mathbb{R}^n TO \mathbb{R}^m , WE CAN ALSO SAY THAT f IS EQUIVALENT TO THE $m \times n$ MATRIX THAT DEFINES THE LINEAR TRANSFORMATION FROM \mathbb{R}^n TO \mathbb{R}^m .



$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

NOW I GET IT!



WHY WE STUDY LINEAR TRANSFORMATIONS

SO...WHAT ARE LINEAR TRANSFORMATIONS GOOD FOR, EXACTLY?

THEY SEEM PRETTY IMPORTANT. I GUESS WE'LL BE USING THEM A LOT FROM NOW ON?

WELL, IT'S NOT REALLY A QUESTION OF IMPORTANCE...

SO WHY DO WE HAVE TO STUDY THEM?

WELL...

THAT'S EXACTLY WHAT I WANTED TO TALK ABOUT NEXT.

CONSIDER THE LINEAR TRANSFORMATION FROM R^n TO R^m
DEFINED BY THE FOLLOWING $m \times n$ MATRIX:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

IF $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ IS THE IMAGE OF $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ UNDER THIS LINEAR TRANSFORMATION,

THEN THE FOLLOWING EQUATION IS TRUE:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

IMAGE?

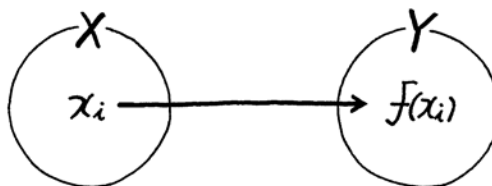
YEP. HERE'S A
DEFINITION.

IMAGES

Suppose x_i is an element from X .

WE TALKED
A BIT ABOUT
THIS BEFORE,
DIDN'T WE?

The element in Y corresponding to x_i under f is called
“ x_i 's image under f .”



YEAH, IN
CHAPTER 2.

BUT THAT DEFINITION IS A BIT VAGUE. TAKE A LOOK AT THIS.

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

OKAY.

DOESN'T IT KIND OF LOOK LIKE A COMMON ONE-DIMENSIONAL EQUATION $y = ax$ TO YOU?

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

a

MAYBE IF I SQUINT...

WHAT IF I PUT IT LIKE THIS?

I GUESS THAT MAKES SENSE.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$



$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$



$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

Multiplying an n -dimensional space by an $m \times n$ matrix...

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

turns it m -dimensional!

WE STUDY LINEAR TRANSFORMATIONS IN AN EFFORT TO BETTER UNDERSTAND THE CONCEPT OF IMAGE, USING MORE VISUAL MEANS THAN SIMPLE FORMULAE.

TA-DA!

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

HUH?

I HAVE TO LEARN THIS STUFF BECAUSE OF...THAT?

OOH, BUT "THAT" IS A LOT MORE SIGNIFICANT THAN YOU MIGHT THINK!

TAKE THIS LINEAR TRANSFORMATION FROM THREE TO TWO DIMENSIONS, FOR EXAMPLE.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

YOU COULD WRITE IT AS THIS LINEAR SYSTEM OF EQUATIONS INSTEAD, IF YOU WANTED TO.

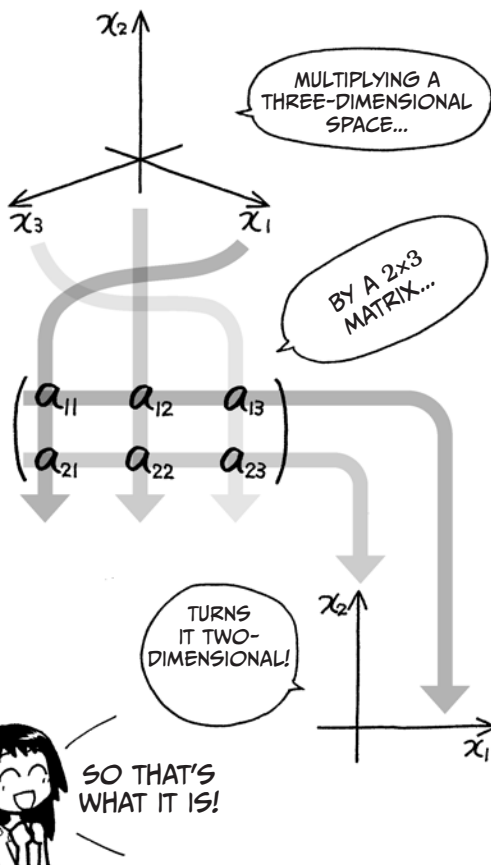
$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{cases}$$

BUT YOU HAVE TO AGREE THAT THIS DOESN'T REALLY CONVEY THE FEELING OF "TRANSFORMING A THREE-DIMENSIONAL SPACE INTO A TWO-DIMENSIONAL ONE," RIGHT?

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{cases} \text{ IS THE SAME AS...}$$

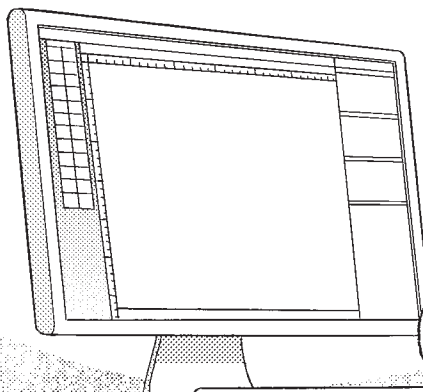


THIS!
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



SPECIAL TRANSFORMATIONS

I WOULDN'T WANT YOU THINKING THAT LINEAR TRANSFORMATIONS LACK PRACTICAL USES, THOUGH. COMPUTER GRAPHICS, FOR EXAMPLE, RELY HEAVILY ON LINEAR ALGEBRA AND LINEAR TRANSFORMATIONS IN PARTICULAR.



REALLY?

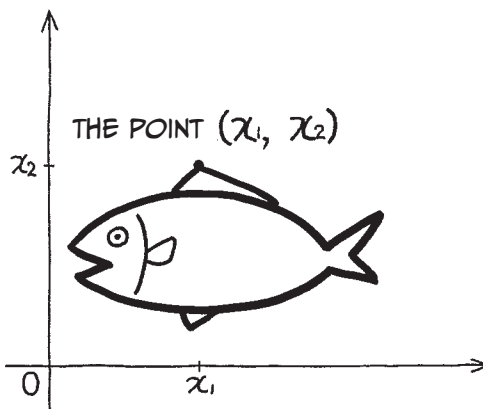
YEAH.
AS WE'RE ALREADY ON THE SUBJECT, LET'S HAVE A LOOK AT SOME OF THE TRANSFORMATIONS THAT LET US DO THINGS LIKE SCALING, ROTATION, TRANSLATION, AND 3-D PROJECTION.

click

AWW!
CUTE!

LET'S USE ONE OF MY DRAWINGS.

LET (x_1, x_2) BE SOME POINT ON THE DRAWING. THE TOP OF THE DORSAL FIN WILL DO!



SCALING

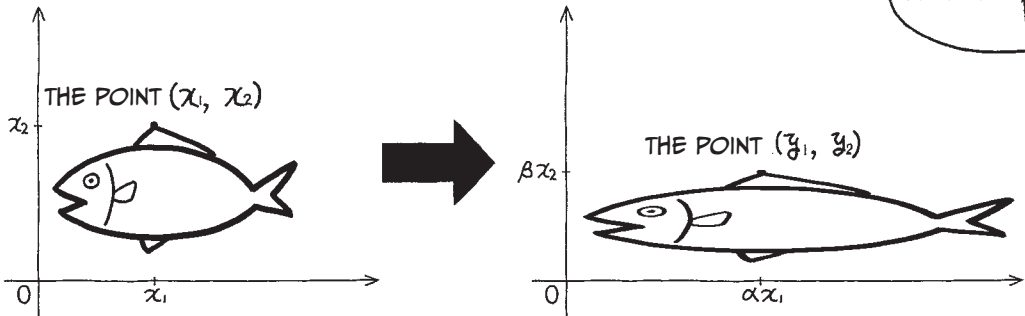
LET'S SAY WE DECIDE TO

- Multiply all x_1 values by α
- Multiply all x_2 values by β

THIS GIVES RISE TO THE INTERESTING RELATIONSHIP

$$\begin{cases} y_1 = \alpha x_1 \\ y_2 = \beta x_2 \end{cases}$$


UH-HUH...



AND

$$\begin{cases} y_1 = \alpha x_1 \\ y_2 = \beta x_2 \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \beta x_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

COULD BE
REWRITTEN LIKE
THIS, RIGHT?

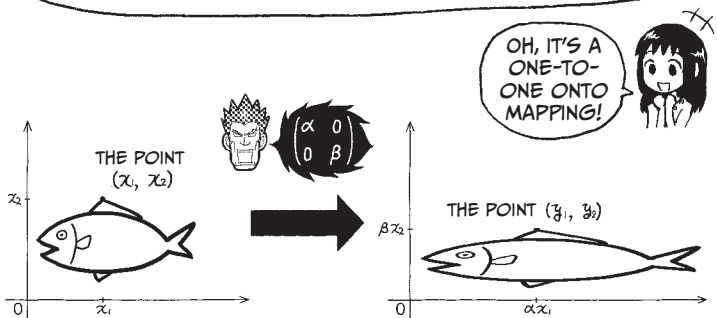
YEAH,
SURE.

SO THAT MEANS THAT APPLYING THE SET OF RULES

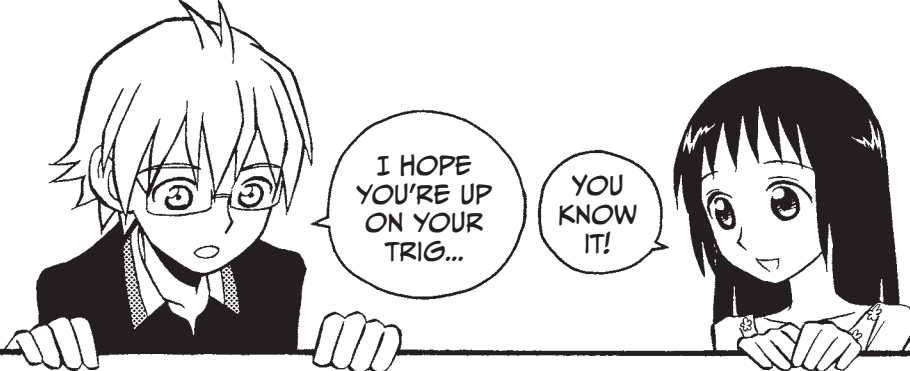
- Multiply all x_1 values by α
- Multiply all x_2 values by β

ONTO AN ARBITRARY IMAGE IS BASICALLY THE
SAME THING AS PASSING THE IMAGE THROUGH A
LINEAR TRANSFORMATION IN \mathbb{R}^2 EQUAL TO THE
FOLLOWING MATRIX!

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$



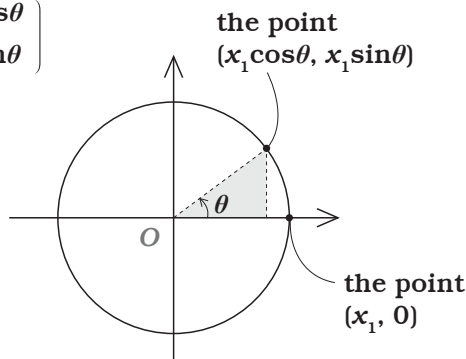
OH, IT'S A
ONE-TO-
ONE ONTO
MAPPING!



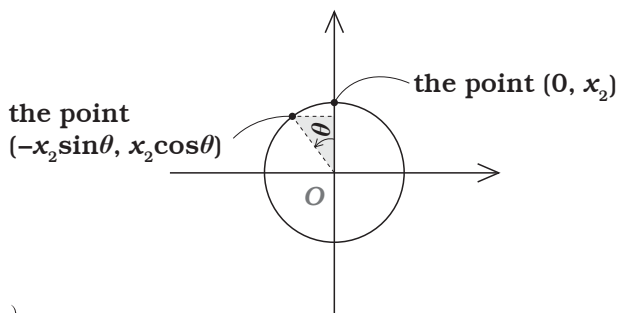
I HOPE
YOU'RE UP
ON YOUR
TRIG...

YOU
KNOW
IT!

- Rotating $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ by θ^* degrees gets us $\begin{pmatrix} x_1 \cos \theta \\ x_1 \sin \theta \end{pmatrix}$



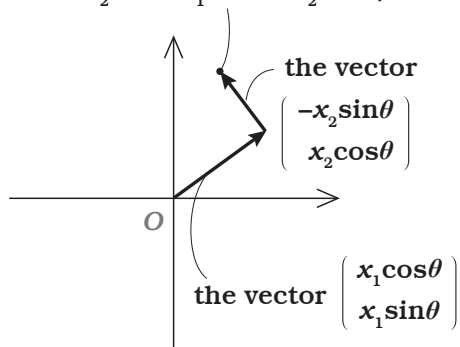
- Rotating $\begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ by θ degrees gets us $\begin{pmatrix} -x_2 \sin \theta \\ x_2 \cos \theta \end{pmatrix}$



- Rotating $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, that is $\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, by θ degrees gets us

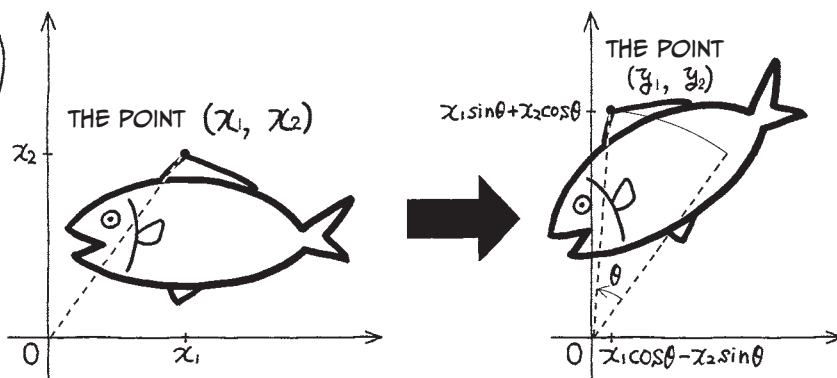
$$\begin{pmatrix} x_1 \cos \theta \\ x_1 \sin \theta \end{pmatrix} + \begin{pmatrix} -x_2 \sin \theta \\ x_2 \cos \theta \end{pmatrix} \\ = \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}$$

the point
 $(x_1 \cos \theta - x_2 \sin \theta, x_1 \sin \theta + x_2 \cos \theta)$



* θ is the Greek letter *theta*.

SO IF WE WANTED TO ROTATE THE ENTIRE PICTURE BY θ DEGREES, WE'D GET...



...DUE TO THIS RELATIONSHIP.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{pmatrix}$$

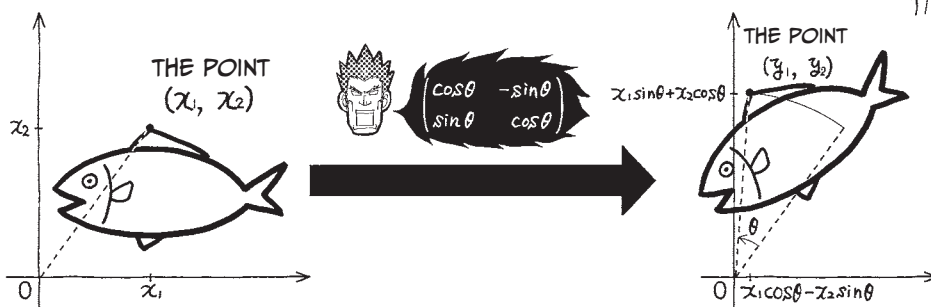
$$= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

AHA.

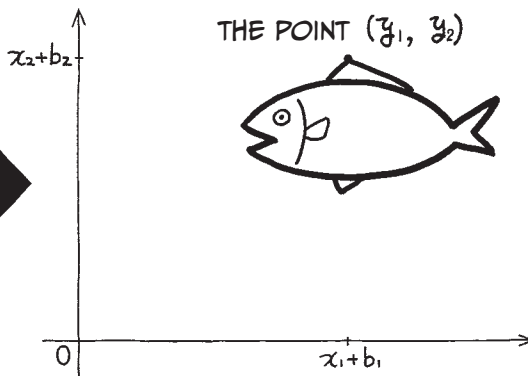
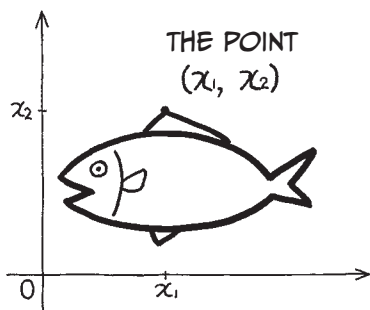
ROTATING AN ARBITRARY IMAGE BY θ DEGREES CONSEQUENTLY MEANS WE'RE USING A LINEAR TRANSFORMATION IN \mathbb{R}^2 EQUAL TO THIS MATRIX:

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

ANOTHER ONE-TO-ONE ONTO MAPPING!



TRANSLATION



IF WE INSTEAD DECIDE TO $\begin{cases} \text{Translate all } x_1 \text{ values by } b_1 \\ \text{Translate all } x_2 \text{ values by } b_2 \end{cases}$

WE GET ANOTHER INTERESTING RELATIONSHIP: $\begin{cases} y_1 = x_1 + b_1 \\ y_2 = x_2 + b_2 \end{cases}$

AND THIS CAN
ALSO BE
REWRITTEN
LIKE SO:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ x_2 + b_2 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

THAT'S TRUE.

IF WE WANTED
TO, WE COULD
ALSO REWRITE
IT LIKE THIS:

$$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} x_1 + b_1 \\ x_2 + b_2 \\ 1 \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$$

SEEMS SILLY,
BUT OKAY.

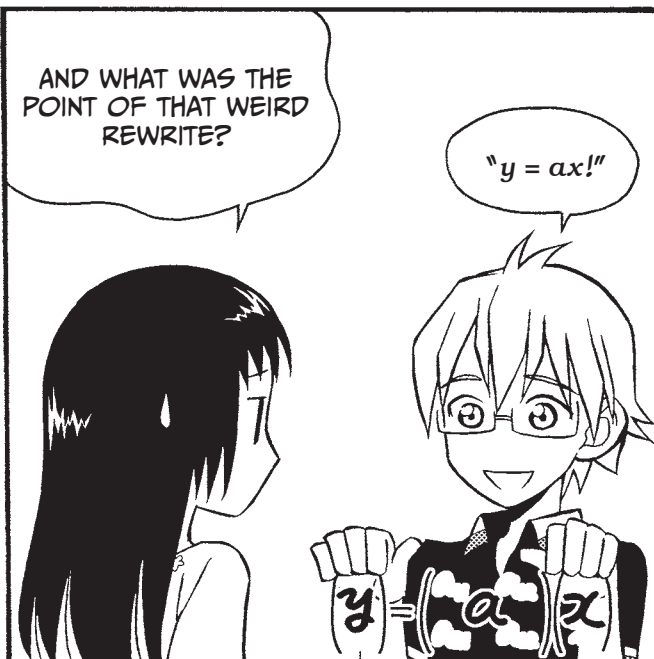
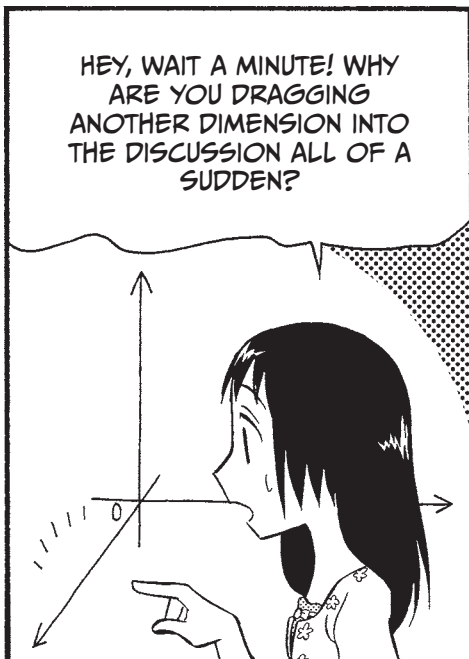
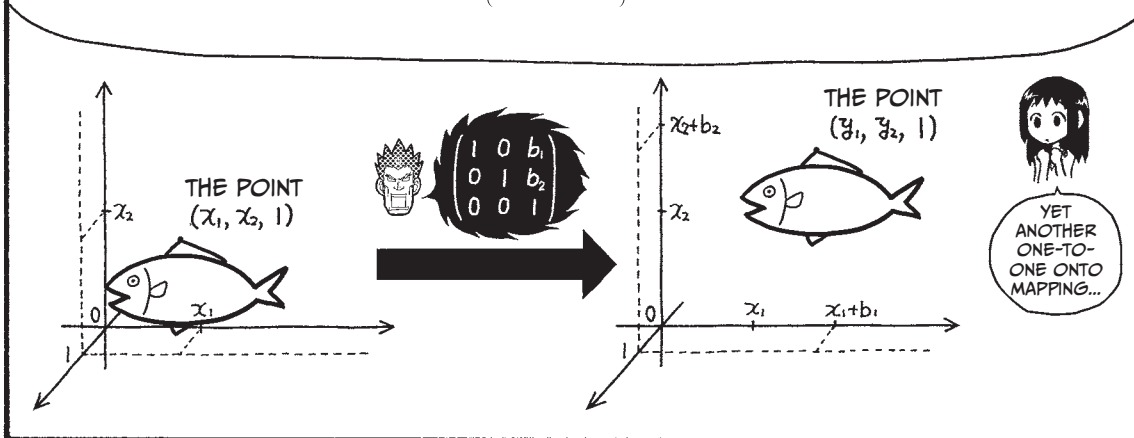
?



SO APPLYING THE SET OF RULES $\begin{cases} \text{Translate all } x_1 \text{ values by } b_1 \\ \text{Translate all } x_2 \text{ values by } b_2 \end{cases}$

ONTO AN ARBITRARY IMAGE IS BASICALLY THE SAME THING AS PASSING THE IMAGE THROUGH A LINEAR TRANSFORMATION IN \mathbb{R}^3 EQUAL TO THE FOLLOWING MATRIX:

$$\begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}$$



WE'D LIKE TO EXPRESS
TRANSLATIONS IN THE
SAME WAY AS ROTATIONS
AND SCALE OPERATIONS,
WITH

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

INSTEAD OF WITH

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

THE FIRST FORMULA
IS MORE PRACTICAL THAN
THE SECOND, ESPECIALLY
WHEN DEALING WITH
COMPUTER GRAPHICS.

ERRR...

A COMPUTER
STORES ALL
TRANSFORMATIONS
AS 3x3 MATRICES...

...EVEN ROTATIONS AND
SCALING OPERATIONS.



NOT TOO
DIFFERENT,
I GUESS.



	CONVENTIONAL LINEAR TRANSFORMATIONS	LINEAR TRANSFORMATIONS USED BY COMPUTER GRAPHICS SYSTEMS
SCALING	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$
ROTATION	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$
TRANSLATION	$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}^*$	$\begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}$

* NOTE: THIS ONE ISN'T ACTUALLY A LINEAR TRANSFORMATION. YOU CAN VERIFY THIS BY SETTING b_1 AND b_2 TO 1 AND CHECKING THAT BOTH LINEAR TRANSFORMATION CONDITIONS FAIL.

3-D PROJECTION

NEXT WE'LL VERY BRIEFLY TALK ABOUT A 3-D PROJECTION TECHNIQUE CALLED PERSPECTIVE PROJECTION.

DON'T WORRY TOO MUCH ABOUT THE DETAILS.

THE POINT (S_1, S_2, S_3)

PERSPECTIVE PROJECTION PROVIDES US WITH A WAY TO PROJECT THREE-DIMENSIONAL OBJECTS ONTO A NEAR PLANE BY TRACING OUR WAY FROM EACH POINT ON THE OBJECT TOWARD A COMMON OBSERVATION POINT AND NOTING WHERE THESE LINES INTERSECT WITH THE NEAR PLANE.

THE POINT (x_1, x_2, x_3)

THE POINT $(y_1, y_2, 0)$

x_2

0

x_3

x_1, x_2 - THE PLANE

OH, AN ONTO MAPPING!

THE MATH IS A BIT MORE COMPLEX THAN WHAT WE'VE SEEN SO FAR.

SO I'LL CHEAT A LITTLE BIT AND SKIP RIGHT TO THE END!

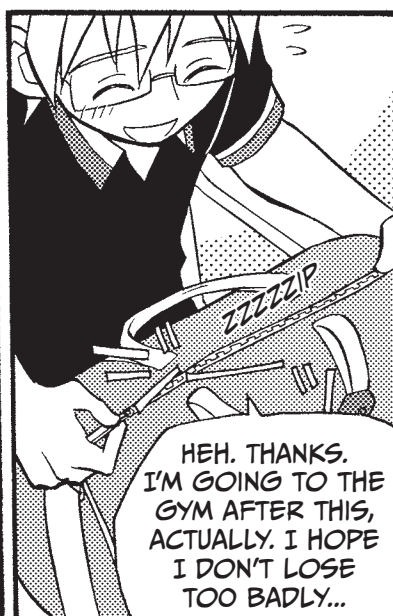
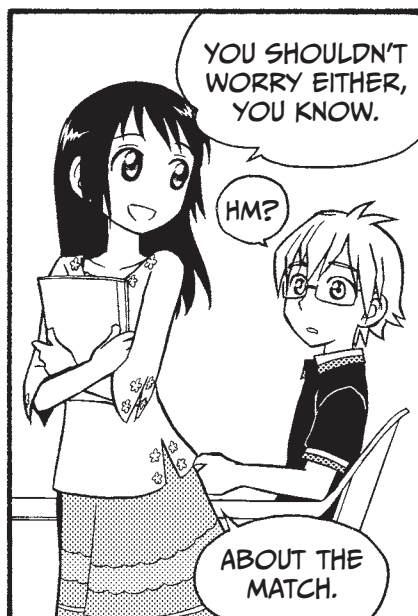
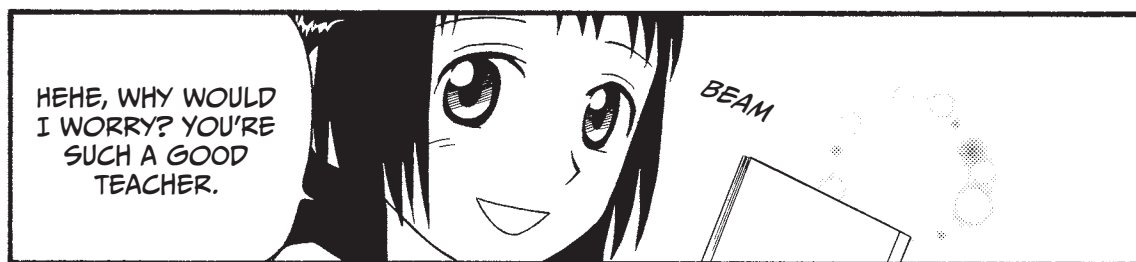
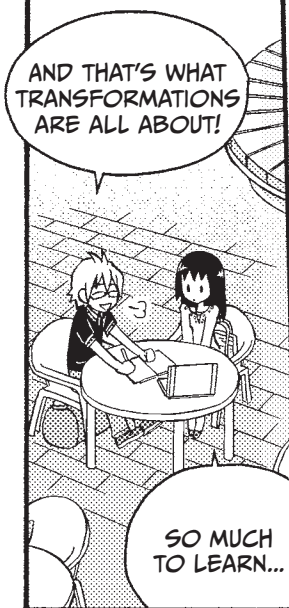
THE LINEAR TRANSFORMATION WE USE FOR PERSPECTIVE PROJECTION IS IN R^4 AND CAN BE WRITTEN AS THE FOLLOWING MATRIX:

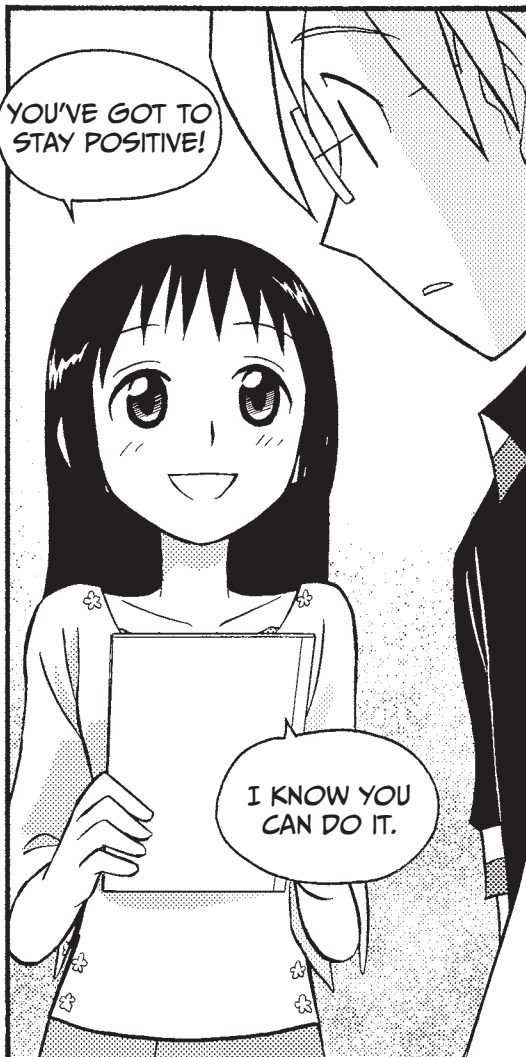
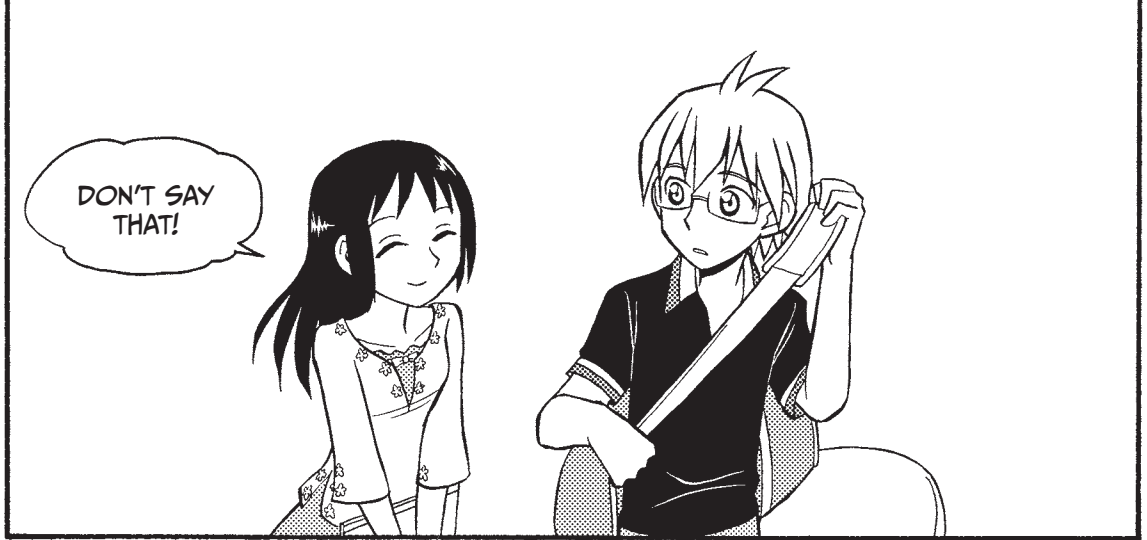
$$\frac{1}{x_3 - s_3} \begin{pmatrix} -s_3 & 0 & s_1 & 0 \\ 0 & -s_3 & s_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -s_3 \end{pmatrix}$$



$$\frac{1}{x_3 - s_3} \begin{pmatrix} -s_3 & 0 & s_1 & 0 \\ 0 & -s_3 & s_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -s_3 \end{pmatrix}$$

COOOL.





SOME PRELIMINARY TIPS

Before we dive into kernel, rank, and the other advanced topics we're going to cover in the remainder of this chapter, there's a little mathematical trick that you may find handy while working some of these problems out.

The equation

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

can be rewritten like this:

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \left[x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right] \\ &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \end{aligned}$$

As you can see, the product of the matrix M and the vector \mathbf{x} can be viewed as a linear combination of the columns of M with the entries of \mathbf{x} as the weights.

Also note that the function f referred to throughout this chapter is the linear transformation from R^n to R^m corresponding to the following $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

KERNEL, IMAGE, AND THE DIMENSION THEOREM FOR LINEAR TRANSFORMATIONS

The set of vectors whose images are the zero vector, that is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\}$$

is called the *kernel* of the linear transformation f and is written $\text{Ker } f$.

The *image* of f (written $\text{Im } f$) is also important in this context. The image of f is equal to the set of vectors that is made up of all of the possible output values of f , as you can see in the following relation:

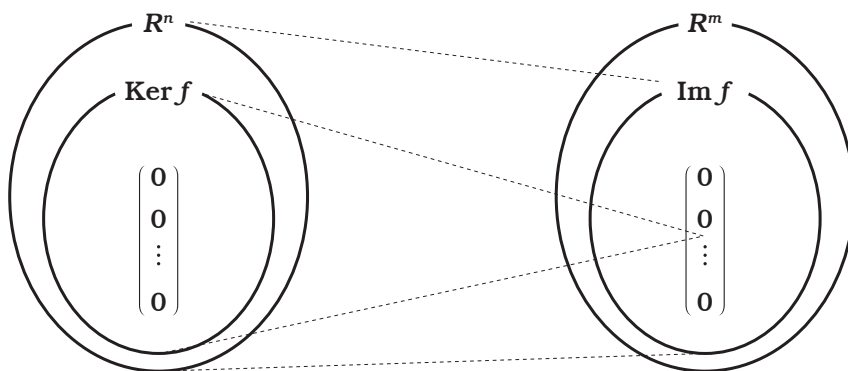
$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\}$$

(This is a more formal definition of image than what we saw in Chapter 2, but the concept is the same.)

An important observation is that $\text{Ker } f$ is a subspace of R^n and $\text{Im } f$ is a subspace of R^m . The *dimension theorem for linear transformations* further explores this observation by defining a relationship between the two:

$$\dim \text{Ker } f + \dim \text{Im } f = n$$

Note that the n above is equal to the first vector space's dimension ($\dim R^n$).*



* If you need a refresher on the concept of dimension, see “Basis and Dimension” on page 156.

EXAMPLE 1

Suppose that f is a linear transformation from R^2 to R^2 equal to the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$.
Then:

$$\left\{ \begin{array}{l} \text{Ker } f = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ \\ \text{Im } f = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = R^2 \end{array} \right.$$

$$\text{And: } \begin{cases} n &= 2 \\ \dim \text{Ker } f &= 0 \\ \dim \text{Im } f &= 2 \end{cases}$$

EXAMPLE 2

Suppose that f is a linear transformation from R^2 to R^2 equal to the matrix $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$.
Then:

$$\left\{ \begin{array}{l} \text{Ker } f = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \\ \\ \qquad \qquad \qquad = \left\{ c \begin{pmatrix} -2 \\ 1 \end{pmatrix} \mid c \text{ is an arbitrary number} \right\} \\ \\ \text{Im } f = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\} \\ \\ \qquad \qquad \qquad = \left\{ c \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mid c \text{ is an arbitrary number} \right\} \end{array} \right.$$

$$\text{And: } \begin{cases} n &= 2 \\ \dim \text{Ker } f &= 1 \\ \dim \text{Im } f &= 1 \end{cases}$$

EXAMPLE 3

Suppose f is a linear transformation from R^2 to R^3 equal to the 3×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Then:

$$\left\{ \begin{array}{l} \text{Ker } f = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ \\ \text{Im } f = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \\ \\ = \left\{ c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid c_1 \text{ and } c_2 \text{ are arbitrary numbers} \right\} \end{array} \right.$$

$$\text{And: } \left\{ \begin{array}{l} n = 2 \\ \dim \text{Ker } f = 0 \\ \dim \text{Im } f = 2 \end{array} \right.$$

EXAMPLE 4

Suppose that f is a linear transformation from R^4 to R^2 equal to the 2×4 matrix $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$. Then:

$$\left\{ \begin{aligned} \text{Ker } f &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mid x_1 + 3x_3 + x_4 = 0, \quad x_2 + x_3 + 2x_4 = 0 \right\} \\ &= \left\{ c_1 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \mid c_1 \text{ and } c_2 \text{ are arbitrary numbers} \right\} \\ \text{Im } f &= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mid \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} = R^2 \end{aligned} \right.$$

$$\text{And: } \begin{cases} n &= 4 \\ \dim \text{Ker } f &= 2 \\ \dim \text{Im } f &= 2 \end{cases}$$

RANK

The number of linearly independent vectors among the columns of the matrix M (which is also the dimension of the R^m subspace $\text{Im } f$) is called the *rank* of M , and it is written like this: $\text{rank } M$.

EXAMPLE 1

The linear system of equations $\begin{cases} 3x_1 + 1x_2 = y_1 \\ 1x_1 + 2x_2 = y_2 \end{cases}$, that is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 1x_2 \\ 1x_1 + 2x_2 \end{pmatrix}$,

can be rewritten as follows: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 1x_2 \\ 1x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The two vectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are linearly independent, as can be seen on pages 133 and 135, so the rank of $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ is 2.

Also note that $\det \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 1 \cdot 1 = 5 \neq 0$.

EXAMPLE 2

The linear system of equations $\begin{cases} 3x_1 + 6x_2 = y_1 \\ 1x_1 + 2x_2 = y_2 \end{cases}$, that is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 6x_2 \\ 1x_1 + 2x_2 \end{pmatrix}$,

can be rewritten as follows: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 3x_1 + 6x_2 \\ 1x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 6 \\ 2 \end{pmatrix}$
 $= x_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 2x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
 $= [x_1 + 2x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

So the rank of $\begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix}$ is 1.

Also note that $\det \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} = 3 \cdot 2 - 6 \cdot 1 = 0$.

EXAMPLE 3

The linear system of equations $\begin{cases} 1x_1 + 0x_2 = y_1 \\ 0x_1 + 1x_2 = y_2 \\ 0x_1 + 0x_2 = y_3 \end{cases}$, that is $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \\ 0x_1 + 0x_2 \end{pmatrix}$,

can be rewritten as: $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \\ 0x_1 + 0x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

The two vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent, as we discovered

on page 137, so the rank of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is 2.

The system could also be rewritten like this:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 \\ 0x_1 + 1x_2 \\ 0x_1 + 0x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Note that $\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$.

EXAMPLE 4

The linear system of equations $\begin{cases} 1x_1 + 0x_2 + 3x_3 + 1x_4 = y_1 \\ 0x_1 + 1x_2 + 1x_3 + 2x_4 = y_2 \end{cases}$, that is

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 3x_3 + 1x_4 \\ 0x_1 + 1x_2 + 1x_3 + 2x_4 \end{pmatrix}$, can be rewritten as follows:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 3x_3 + 1x_4 \\ 0x_1 + 1x_2 + 1x_3 + 2x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The rank of $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ is equal to 2, as we'll see on page 203.

The system could also be rewritten like this:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1x_1 + 0x_2 + 3x_3 + 1x_4 \\ 0x_1 + 1x_2 + 1x_3 + 2x_4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Note that $\det \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$.

The four examples seem to point to the fact that

$$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = 0 \text{ is the same as rank } \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \neq n.$$

This is indeed so, but no formal proof will be given in this book.

CALCULATING THE RANK OF A MATRIX

So far, we've only dealt with matrices where the rank was immediately apparent or where we had previously figured out how many linearly independent vectors made up the columns of that matrix. Though this might seem like "cheating" at first, these techniques can actually be very useful for calculating ranks in practice.

For example, take a look at the following matrix:

$$\begin{pmatrix} 1 & 4 & 4 \\ 2 & 5 & 8 \\ 3 & 6 & 12 \end{pmatrix}$$

It's immediately clear that the third column of this matrix is equal to the first column times 4. This leaves two linearly independent vectors (the first two columns), which means this matrix has a rank of 2.

Now look at this matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 5 \end{pmatrix}$$

It should be obvious right from the start that these vectors form a linearly independent set, so we know that the rank of this matrix is also 2.

Of course there are times when this method will fail you and you won't be able to tell the rank of a matrix just by eyeballing it. In those cases, you'll have to buckle down and actually calculate the rank. But don't worry, it's not too hard!

First we'll explain the **PROBLEM**, then we'll establish a good **WAY OF THINKING**, and then finally we'll tackle the **SOLUTION**.

PROBLEM

Calculate the rank of the following 2×4 matrix:

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

WAY OF THINKING

Before we can solve this problem, we need to learn a little bit about elementary matrices. An *elementary matrix* is created by starting with an identity matrix and performing exactly one of the elementary row operations used for Gaussian elimination (see Chapter 4). The resulting matrices can then be multiplied with any arbitrary matrix in such a way that the number of linearly independent columns becomes obvious.

With this information under our belts, we can state the following four useful facts about an arbitrary matrix A :

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

FACT 1

Multiplying the elementary matrix

$$\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

Row i
Row j

Column i Column j

to the left of an arbitrary matrix A will switch rows i and j in A .

If we multiply the matrix to the right of A , then the columns will switch places in A instead.

- Example 1 (Rows 1 and 4 are switched.)

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 1 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 1 \cdot a_{43} \\ 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} + 0 \cdot a_{43} \\ 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \end{pmatrix}$$

$$= \begin{pmatrix} a_{41} & a_{42} & a_{43} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$$

- **Example 2** (Columns 1 and 3 are switched.)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
 = \begin{pmatrix} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 & a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot 0 & a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 1 & a_{21} \cdot 0 + a_{22} \cdot 1 + a_{23} \cdot 0 & a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot 0 \\ a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 1 & a_{31} \cdot 0 + a_{32} \cdot 1 + a_{33} \cdot 0 & a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot 0 \\ a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 1 & a_{41} \cdot 0 + a_{42} \cdot 1 + a_{43} \cdot 0 & a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot 0 \end{pmatrix} \\
 = \begin{pmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \\ a_{43} & a_{42} & a_{41} \end{pmatrix}$$

FACT 2

Multiplying the elementary matrix

$$\begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & k & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

↑
Column i
↖
Row i

to the left of an arbitrary matrix A will multiply the i th row in A by k .

Multiplying the matrix to the right side of A will multiply the i th column in A by k instead.

- Example 1 (Row 3 is multiplied by k .)

$$\begin{aligned}
 & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \\
 &= \begin{pmatrix} 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\ 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + k \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + k \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + k \cdot a_{33} + 0 \cdot a_{43} \\ 0 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 1 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 1 \cdot a_{43} \end{pmatrix} \\
 &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ ka_{31} & ka_{32} & ka_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}
 \end{aligned}$$

- Example 2 (Column 2 is multiplied by k .)

$$\begin{aligned}
 & \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot k + a_{13} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot k + a_{23} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 1 \\ a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot 0 & a_{31} \cdot 0 + a_{32} \cdot k + a_{33} \cdot 0 & a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 1 \\ a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot 0 & a_{41} \cdot 0 + a_{42} \cdot k + a_{43} \cdot 0 & a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 1 \end{pmatrix} \\
 &= \begin{pmatrix} a_{11} & ka_{12} & a_{13} \\ a_{21} & ka_{22} & a_{23} \\ a_{31} & ka_{32} & a_{33} \\ a_{41} & ka_{42} & a_{43} \end{pmatrix}
 \end{aligned}$$

FACT 3

Multiplying the elementary matrix

$$\begin{pmatrix}
 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
 \vdots & \ddots & \vdots & & \vdots & & \vdots \\
 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\
 \vdots & & \vdots & \ddots & \vdots & & \vdots \\
 0 & \cdots & k & \cdots & 1 & \cdots & 0 \\
 \vdots & & \vdots & & \vdots & \ddots & \vdots \\
 0 & \cdots & 0 & \cdots & 0 & \cdots & 1
 \end{pmatrix}$$

Row i

Row j

Column i Column j

to the left of an arbitrary matrix A will add k times row i to row j in A .

Multiplying the matrix to the right side of A will add k times column j to column i instead.

- Example 1 (k times row 2 is added to row 4.)

$$\begin{pmatrix}
 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & k & 0 & 1
 \end{pmatrix}
 \begin{pmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \\
 a_{41} & a_{42} & a_{43}
 \end{pmatrix}$$

$$= \begin{pmatrix}
 1 \cdot a_{11} + 0 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 1 \cdot a_{12} + 0 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 1 \cdot a_{13} + 0 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\
 0 \cdot a_{11} + 1 \cdot a_{21} + 0 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 1 \cdot a_{22} + 0 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 1 \cdot a_{23} + 0 \cdot a_{33} + 0 \cdot a_{43} \\
 0 \cdot a_{11} + 0 \cdot a_{21} + 1 \cdot a_{31} + 0 \cdot a_{41} & 0 \cdot a_{12} + 0 \cdot a_{22} + 1 \cdot a_{32} + 0 \cdot a_{42} & 0 \cdot a_{13} + 0 \cdot a_{23} + 1 \cdot a_{33} + 0 \cdot a_{43} \\
 0 \cdot a_{11} + k \cdot a_{21} + 0 \cdot a_{31} + 1 \cdot a_{41} & 0 \cdot a_{12} + k \cdot a_{22} + 0 \cdot a_{32} + 1 \cdot a_{42} & 0 \cdot a_{13} + k \cdot a_{23} + 0 \cdot a_{33} + 1 \cdot a_{43}
 \end{pmatrix}$$

$$= \begin{pmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33} \\
 a_{41} + ka_{21} & a_{42} + ka_{22} & a_{43} + ka_{23}
 \end{pmatrix}$$

- Example 2 (k times column 3 is added to column 1.)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{pmatrix} \\
 = \begin{pmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot k & a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 \\ a_{21} \cdot 1 + a_{22} \cdot 0 + a_{23} \cdot k & a_{21} \cdot 0 + a_{22} \cdot 1 + a_{23} \cdot 0 & a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 1 \\ a_{31} \cdot 1 + a_{32} \cdot 0 + a_{33} \cdot k & a_{31} \cdot 0 + a_{32} \cdot 1 + a_{33} \cdot 0 & a_{31} \cdot 0 + a_{32} \cdot 0 + a_{33} \cdot 1 \\ a_{41} \cdot 1 + a_{42} \cdot 0 + a_{43} \cdot k & a_{41} \cdot 0 + a_{42} \cdot 1 + a_{43} \cdot 0 & a_{41} \cdot 0 + a_{42} \cdot 0 + a_{43} \cdot 1 \end{pmatrix} \\
 = \begin{pmatrix} a_{11} + ka_{13} & a_{12} & a_{13} \\ a_{21} + ka_{23} & a_{22} & a_{23} \\ a_{31} + ka_{33} & a_{32} & a_{33} \\ a_{41} + ka_{43} & a_{42} & a_{43} \end{pmatrix}$$

FACT 4

The following three $m \times n$ matrices all have the same rank:

1. The matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

2. The left product using an invertible $m \times m$ matrix:

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

3. The right product using an invertible $n \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

In other words, multiplying A by any elementary matrix—on either side—will not change A 's rank, since elementary matrices are invertible.

SOLUTION

The following table depicts calculating the rank of the 2×4 matrix:

$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

Begin with
$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$
Add $(-1 \cdot \text{column } 2)$ to column 3
$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$
Add $(-1 \cdot \text{column } 1)$ to column 4
$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$
Add $(-3 \cdot \text{column } 1)$ to column 3
$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$
Add $(-2 \cdot \text{column } 2)$ to column 4
$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Because of Fact 4, we know that both $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ have the same rank.

One look at the simplified matrix is enough to see that only $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent among its columns.

This means it has a rank of 2, and so does our initial matrix.

THE RELATIONSHIP BETWEEN LINEAR TRANSFORMATIONS AND MATRICES

We talked a bit about the relationship between linear transformations and matrices on page 168. We said that a linear transformation from R^n to R^m could be written as an $m \times n$ matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

As you probably noticed, this explanation is a bit vague. The more exact relationship is as follows:

THE RELATIONSHIP BETWEEN LINEAR TRANSFORMATIONS AND MATRICES

If $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is an arbitrary element in R^n and f is a function from R^n to R^m ,

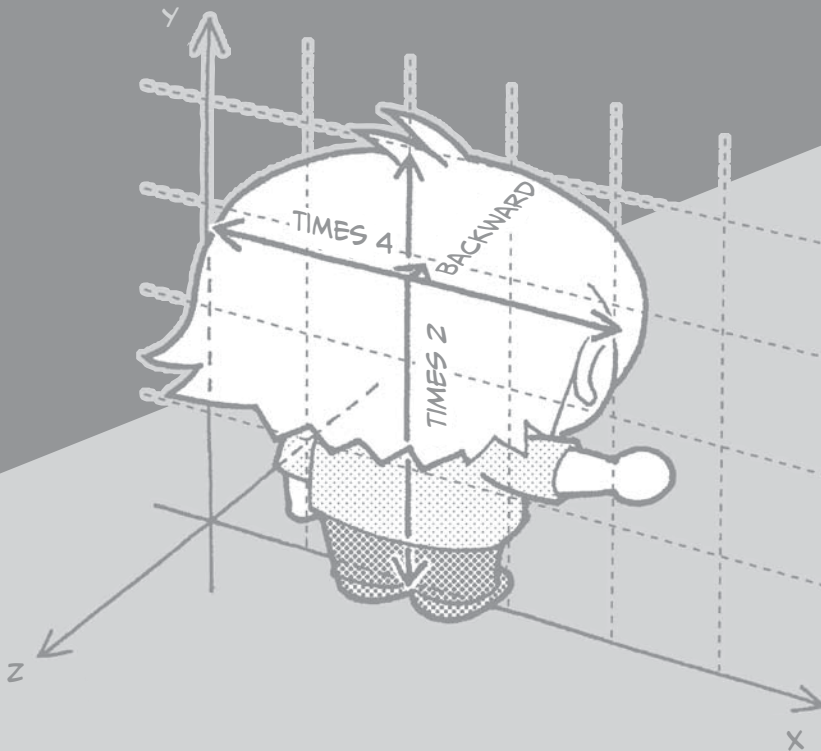
then f is a linear transformation from R^n to R^m if and only if

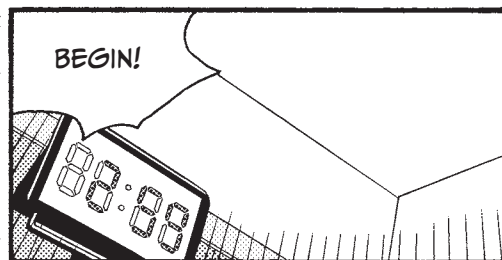
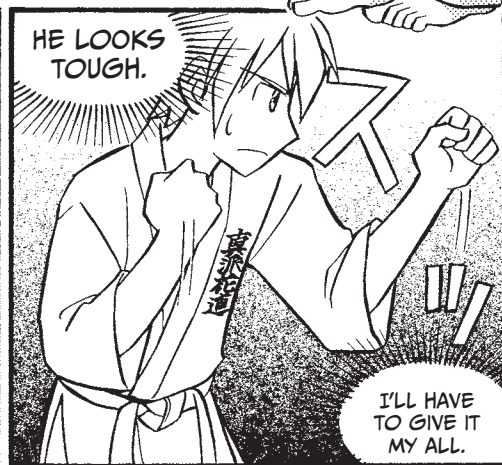
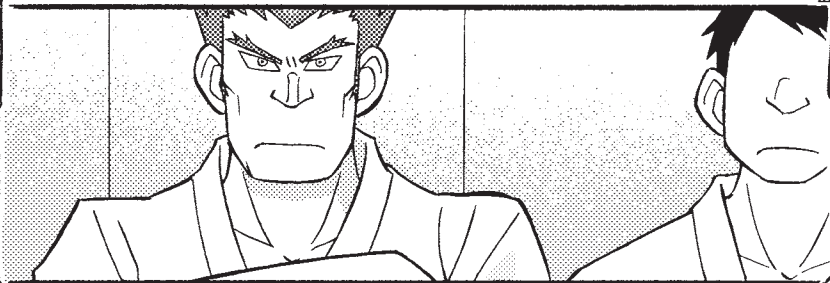
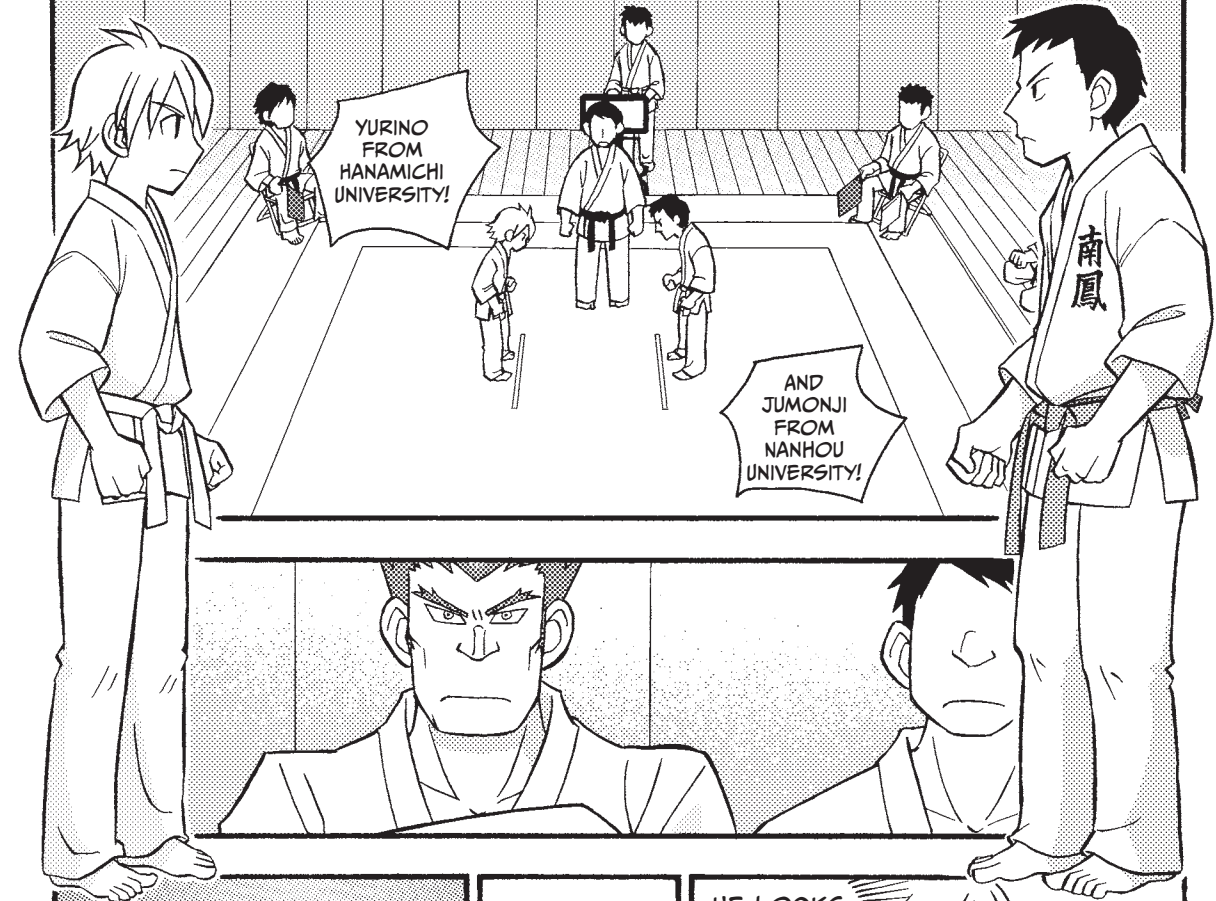
$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

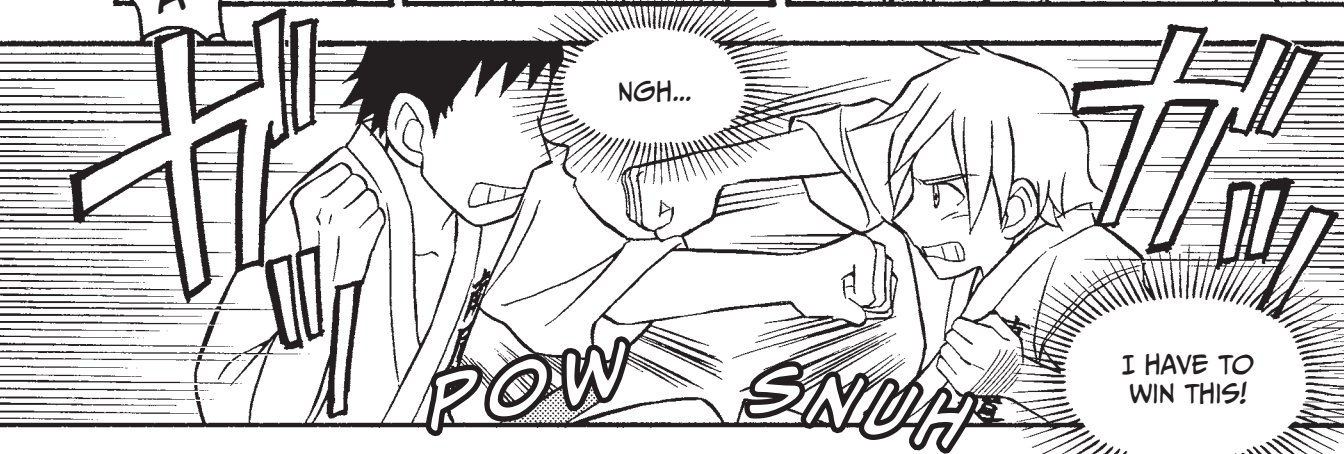
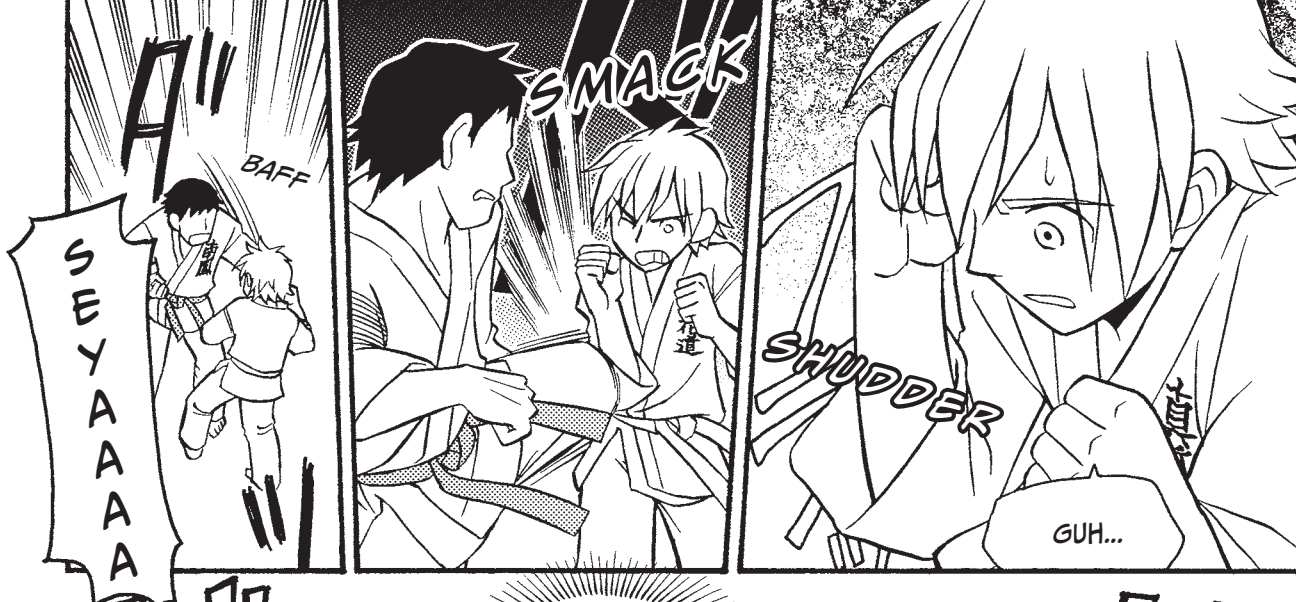
for some matrix A .

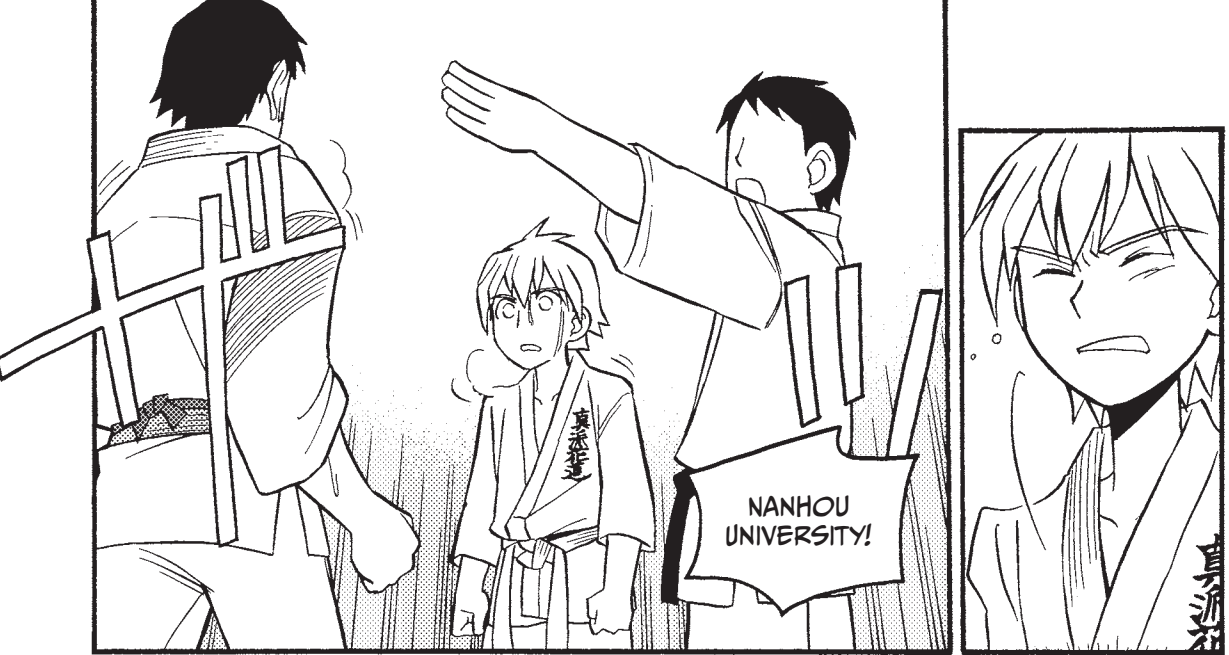
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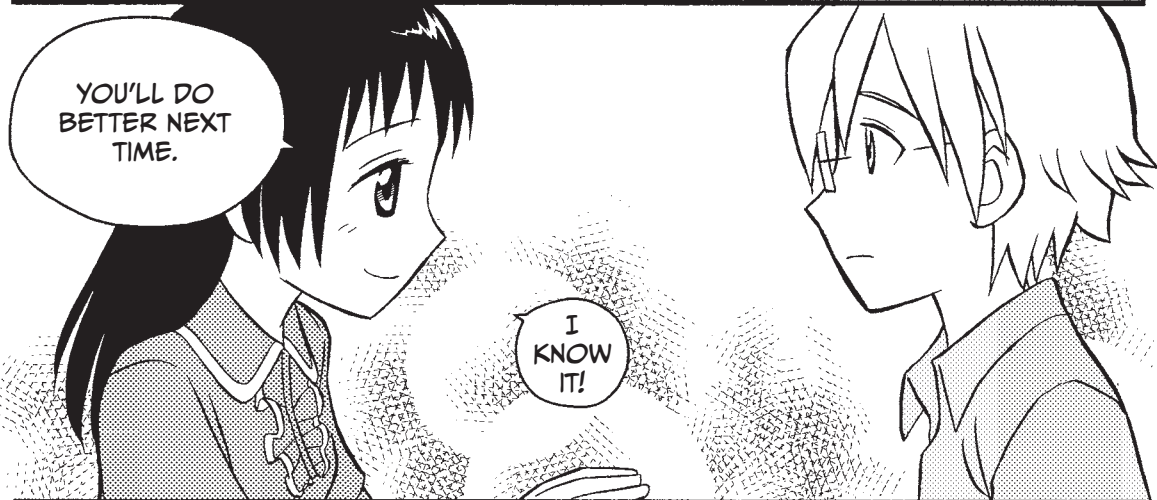
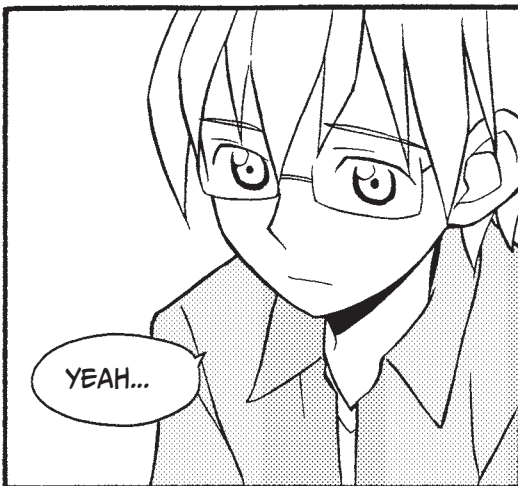
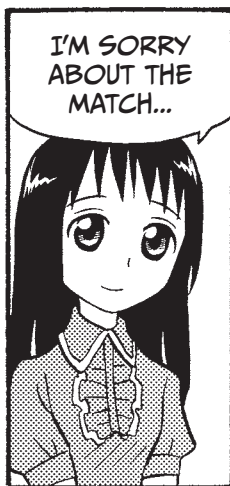
EIGENVALUES AND EIGENVECTORS

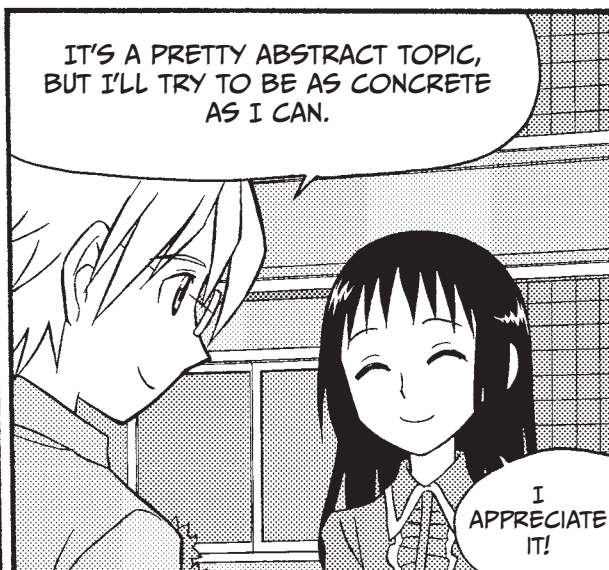
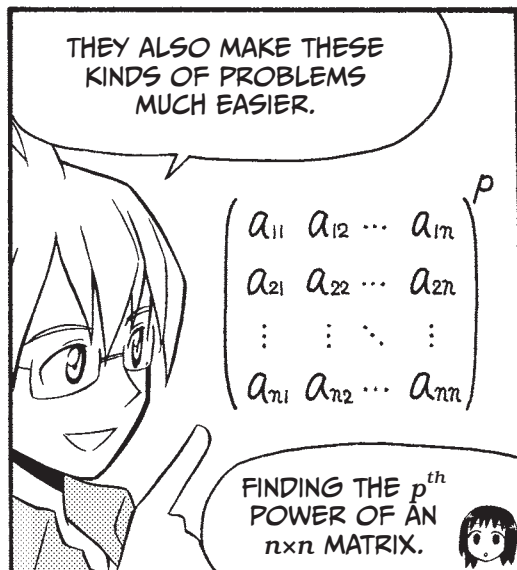
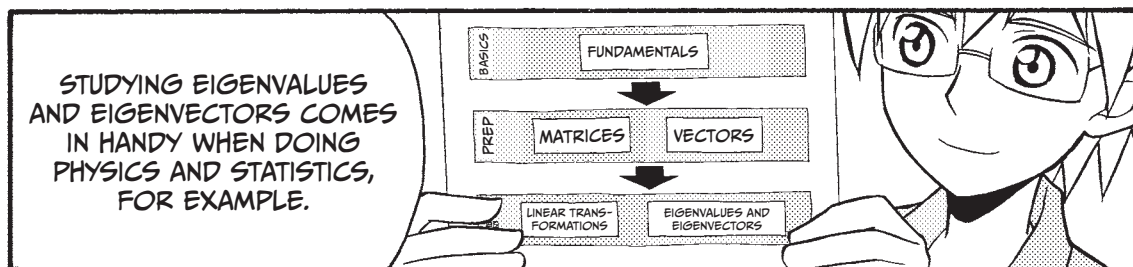
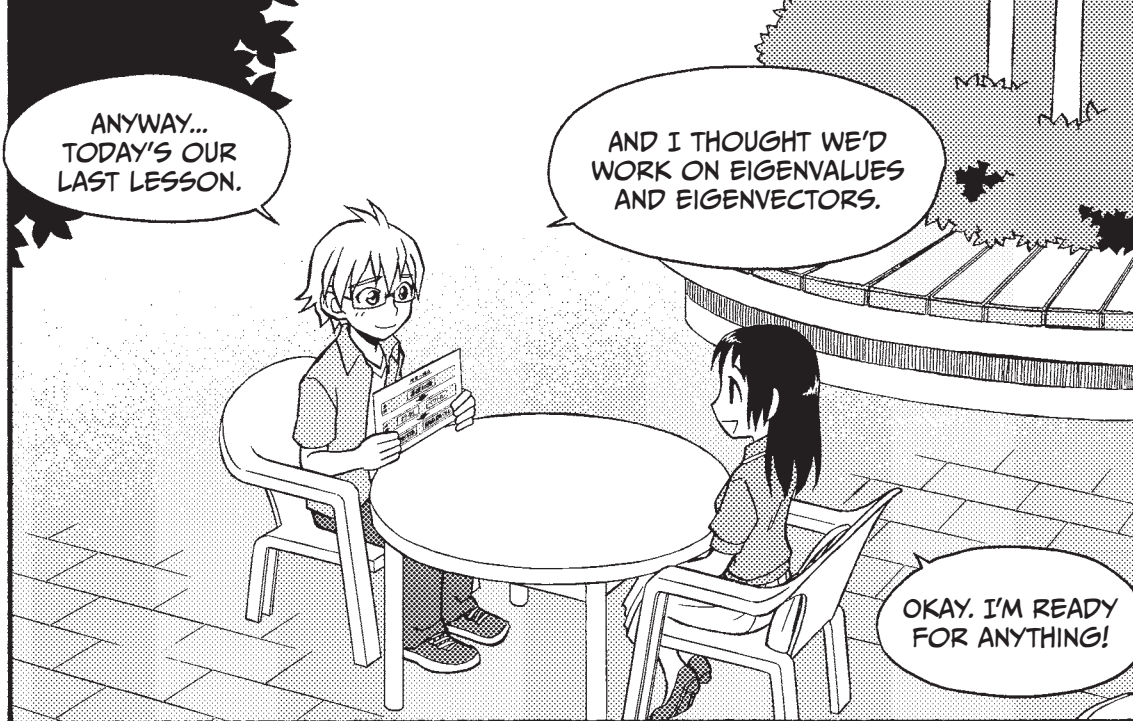












WHAT ARE EIGENVALUES AND EIGENVECTORS?

WHAT DO YOU SAY WE START OFF WITH A FEW PROBLEMS?

SURE.

OKAY, FIRST PROBLEM.
FIND THE IMAGE OF

$$c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

USING THE LINEAR
TRANSFORMATION
DETERMINED BY THE
2x2 MATRIX

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}$$

(WHERE c_1 AND c_2
ARE REAL NUMBERS).

HMM...

$$\begin{aligned} & \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \left[c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right] \\ &= c_1 \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 8 \cdot 3 + (-3) \cdot 1 \\ 2 \cdot 3 + 1 \cdot 1 \end{pmatrix} + c_2 \begin{pmatrix} 8 \cdot 1 + (-3) \cdot 2 \\ 2 \cdot 1 + 1 \cdot 2 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 21 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} \end{aligned}$$

LIKE THIS?

SO CLOSE!

OH, LIKE
THIS?

$$\begin{aligned} &= c_1 \begin{pmatrix} 21 \\ 7 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ &= c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

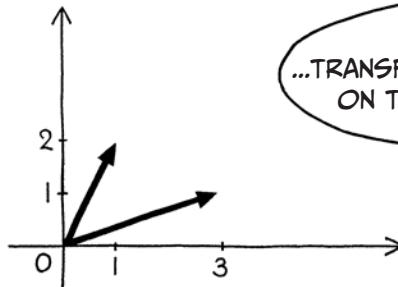
EXACTLY!

$$\mathbb{R}^2 \quad c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}} \mathbb{R}^2 \quad c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

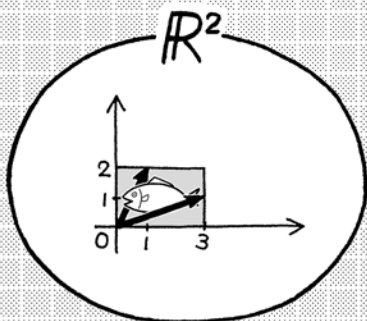
SO...THE ANSWER CAN
BE EXPRESSED USING
MULTIPLES OF THE
ORIGINAL TWO VECTORS?

THAT'S RIGHT! SO YOU
COULD SAY THAT THE
LINEAR TRANSFORMATION
EQUAL TO THE MATRIX

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}$$

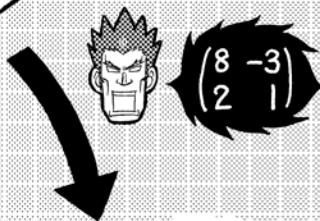


...TRANSFORMS ALL POINTS
ON THE $x_1 x_2$ PLANE...

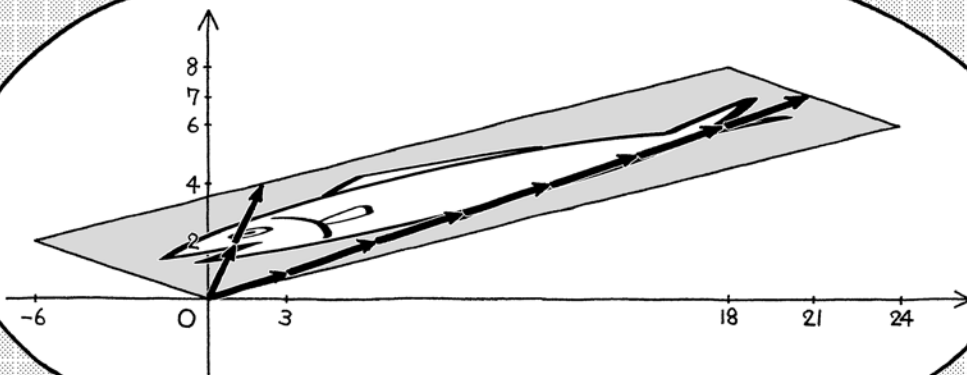


LIKE SO.

OH...



\mathbb{R}^2



LET'S MOVE ON TO ANOTHER PROBLEM.

FIND THE IMAGE OF $c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ USING

THE LINEAR TRANSFORMATION
DETERMINED BY THE 3×3 MATRIX $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

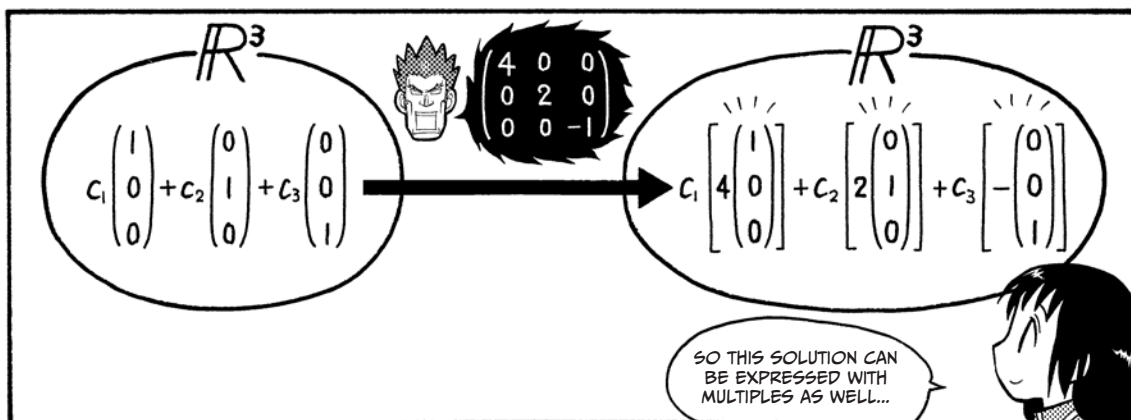
(WHERE c_1 , c_2 , AND c_3 ARE REAL NUMBERS).



$$\begin{aligned} & \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \left[c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \\ &= c_1 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= c_1 \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ &= c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

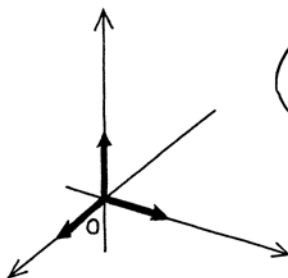
LIKE THIS?

CORRECT.

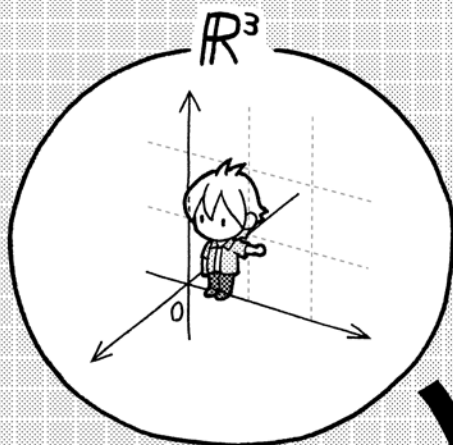


SO YOU COULD
SAY THAT THE
LINEAR TRANSFORMATION
EQUAL TO THE MATRIX

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



...TRANSFORMS EVERY
POINT IN THE
 $x_1 x_2 x_3$ SPACE...

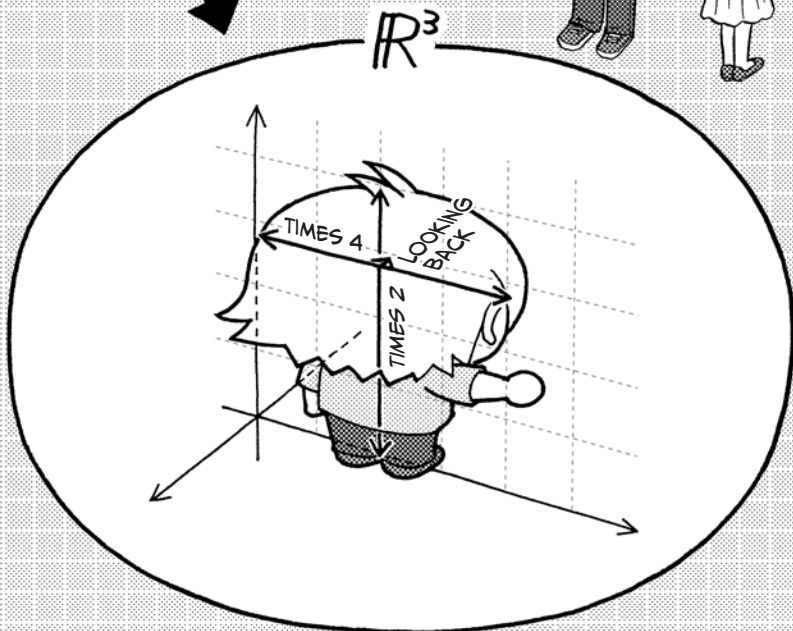


LIKE THIS.

I GET
IT!



$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



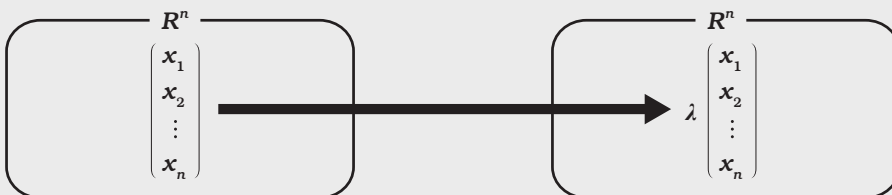


EIGENVALUES AND EIGENVECTORS

If the image of a vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ through the linear transformation determined by the matrix $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ is equal to $\lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, λ is said to be an *eigenvalue* to the matrix,

and $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is said to be an *eigenvector* corresponding to the eigenvalue λ .

The zero vector can never be an eigenvector.



SO THE TWO EXAMPLES COULD BE SUMMARIZED LIKE THIS?

EXACTLY!

MATRIX	$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
EIGENVALUE	$\lambda = 7, 2$	$\lambda = 4, 2, -1$
EIGENVECTOR	THE VECTOR CORRESPONDING TO $\lambda = 7$	THE VECTOR CORRESPONDING TO $\lambda = 4$
	$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	THE VECTOR CORRESPONDING TO $\lambda = 2$	THE VECTOR CORRESPONDING TO $\lambda = 2$
	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
		THE VECTOR CORRESPONDING TO $\lambda = -1$
		$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



CALCULATING EIGENVALUES AND EIGENVECTORS

LET'S HAVE A LOOK AT
CALCULATING THESE VECTORS
AND VALUES.

THE 2×2 MATRIX

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}$$

WILL DO FINE AS
AN EXAMPLE.

OKAY.

LET'S START
OFF WITH THE
RELATIONSHIP...

BETWEEN THE
DETERMINANT AND
EIGENVALUES OF
A MATRIX.

THE RELATIONSHIP BETWEEN THE DETERMINANT AND EIGENVALUES OF A MATRIX

λ is an eigenvalue of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ if and only if } \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} = 0$$

THIS MEANS THAT SOLVING THIS CHARACTERISTIC EQUATION GIVES US ALL EIGENVALUES CORRESPONDING TO THE UNDERLYING MATRIX.

$$\det \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix} = 0$$

IT'S PRETTY COOL.

GO AHEAD, GIVE IT A SHOT.

OKAY...

$$\begin{aligned} \det \begin{pmatrix} 8-\lambda & -3 \\ 2 & 1-\lambda \end{pmatrix} &= (8-\lambda) \cdot (1-\lambda) - (-3) \cdot 2 \\ &= (\lambda-8) \cdot (\lambda-1) - (-3) \cdot 2 \\ &= \lambda^2 - 9\lambda + 8 + 6 \\ &= \lambda^2 - 9\lambda + 14 \\ &= (\lambda-7)(\lambda-2) = 0 \\ \lambda &= 7, 2 \end{aligned}$$

SO...

THE VALUES ARE SEVEN AND TWO?

CORRECT!

FINDING EIGENVECTORS IS ALSO PRETTY EASY.

FOR EXAMPLE, WE CAN USE OUR PREVIOUS VALUES IN THIS FORMULA:

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ THAT IS } \begin{pmatrix} 8 - \lambda & -3 \\ 2 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



PROBLEM 1

Find an eigenvector corresponding to $\lambda = 7$.

Let's plug our value into the formula:

$$\begin{pmatrix} 8 - 7 & -3 \\ 2 & 1 - 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 - 6x_2 \end{pmatrix} = [x_1 - 3x_2] \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that $x_1 = 3x_2$, which leads us to our eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3c_1 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

where c_1 is an arbitrary nonzero real number.

PROBLEM 2

Find an eigenvector corresponding to $\lambda = 2$.

Let's plug our value into the formula:

$$\begin{pmatrix} 8 - 2 & -3 \\ 2 & 1 - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6x_1 - 3x_2 \\ 2x_1 - x_2 \end{pmatrix} = [2x_1 - x_2] \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This means that $x_2 = 2x_1$, which leads us to our eigenvector

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ 2c_2 \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where c_2 is an arbitrary nonzero real number.



CALCULATING THE PTH POWER OF AN $N \times N$ MATRIX

IT'S FINALLY TIME TO TACKLE TODAY'S REAL PROBLEM! FINDING THE p TH POWER OF AN $n \times n$ MATRIX.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^p$$

WE'VE ALREADY FOUND THE EIGENVALUES AND EIGENVECTORS OF THE MATRIX

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}$$

SO LET'S JUST BUILD ON THAT EXAMPLE.

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 7 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \cdot 7 \\ 1 \cdot 7 \end{pmatrix} \quad \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 2 \\ 2 \cdot 2 \end{pmatrix}$$

FOR SIMPLICITY'S SAKE, LET'S CHOOSE $c_1 = c_2 = 1$.

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 7 & 1 \cdot 2 \\ 1 \cdot 7 & 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}$$

USING THE TWO CALCULATIONS ABOVE...

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$

LET'S MULTIPLY $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$ TO THE RIGHT OF BOTH SIDES OF THE EQUATION. REFER TO PAGE 91 TO SEE WHY $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$ EXISTS.

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$

MAKES SENSE.



TRY USING
THE FORMULA
TO CALCULATE

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}^2$$

HMM...
OKAY.

$$\begin{aligned} & \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}^2 \\ &= \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}^2 \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7^2 & 0 \\ 0 & 2^2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \end{aligned}$$

IS...THIS IT?

YEP!

YAY!

LOOKING AT YOUR
CALCULATIONS,
WOULD YOU SAY THIS
RELATIONSHIP MIGHT
BE TRUE?

$$\begin{pmatrix} 8 & -3 \\ 2 & 1 \end{pmatrix}^p = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 7^p & 0 \\ 0 & 2^p \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$$

UHHH...

IT ACTUALLY IS!
THIS FORMULA IS VERY USEFUL
FOR CALCULATING ANY POWER
OF AN $n \times n$ MATRIX THAT CAN BE
WRITTEN IN THIS FORM.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^p = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1^p & 0 & \cdots & 0 \\ 0 & \lambda_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^p \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}^{-1}$$

THE EIGENVECTOR CORRESPONDING TO λ_1

THE EIGENVECTOR CORRESPONDING TO λ_2

THE EIGENVECTOR CORRESPONDING TO λ_n

GOT IT!

OH, AND BY
THE WAY...

WHEN $p = 1$, WE SAY THAT THE FORMULA
DIAGONALIZES THE $n \times n$ MATRIX

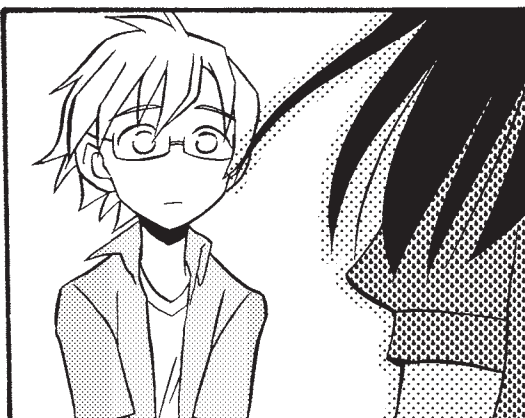
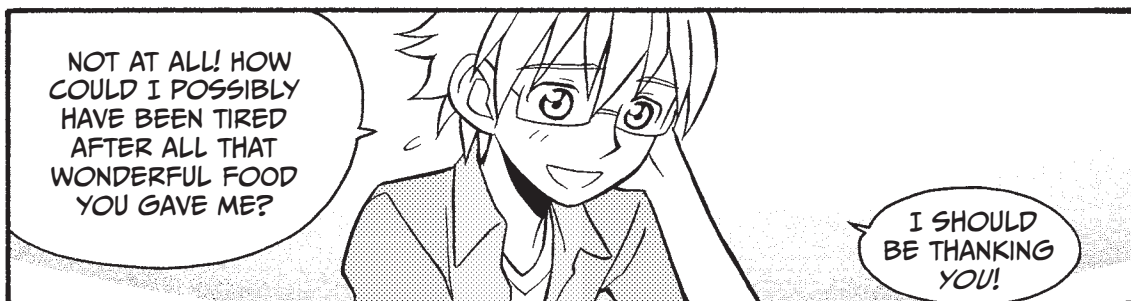
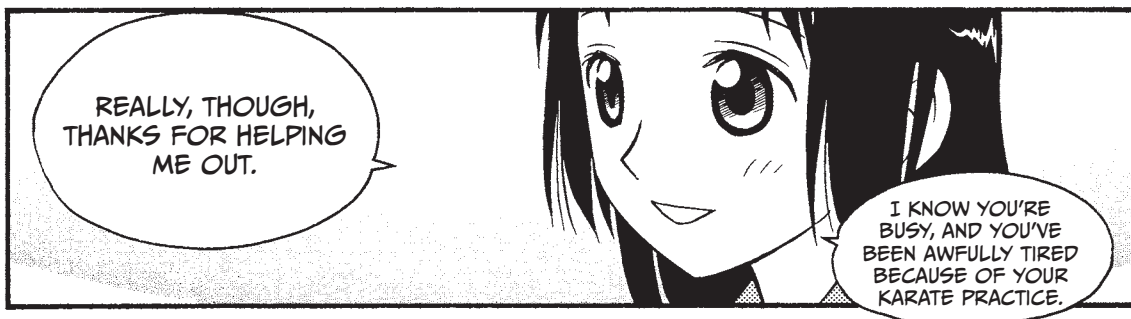
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

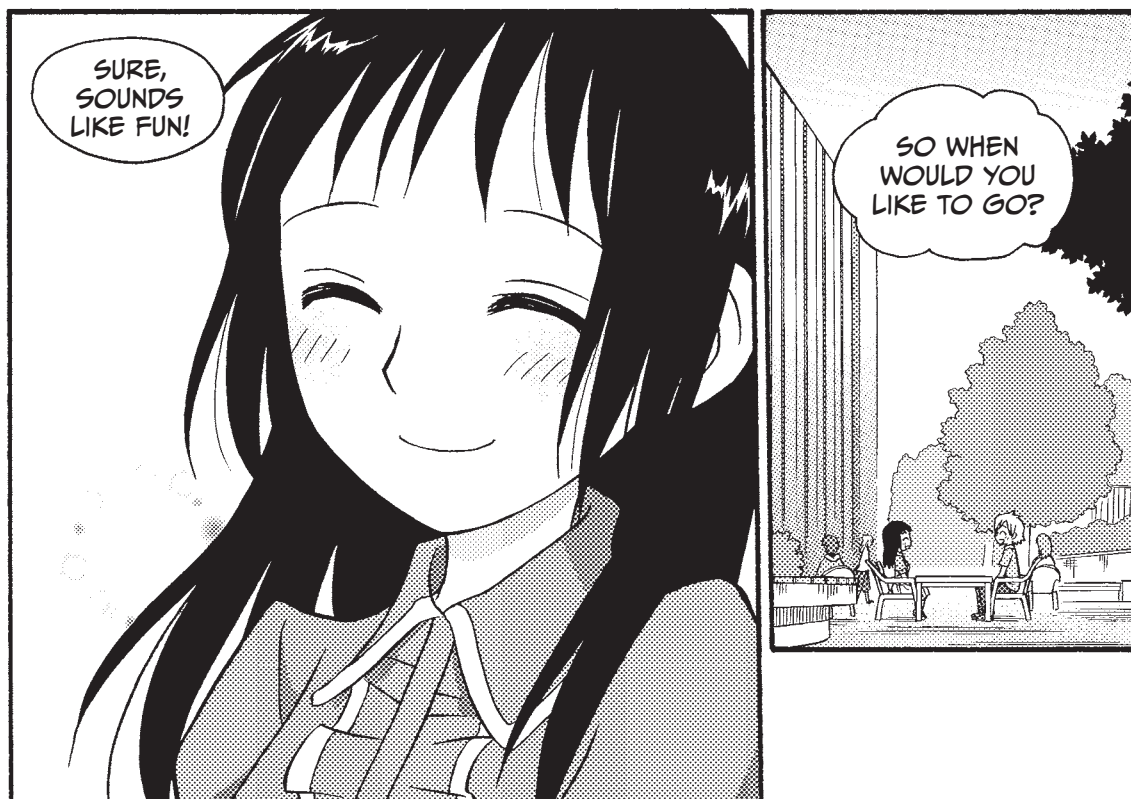
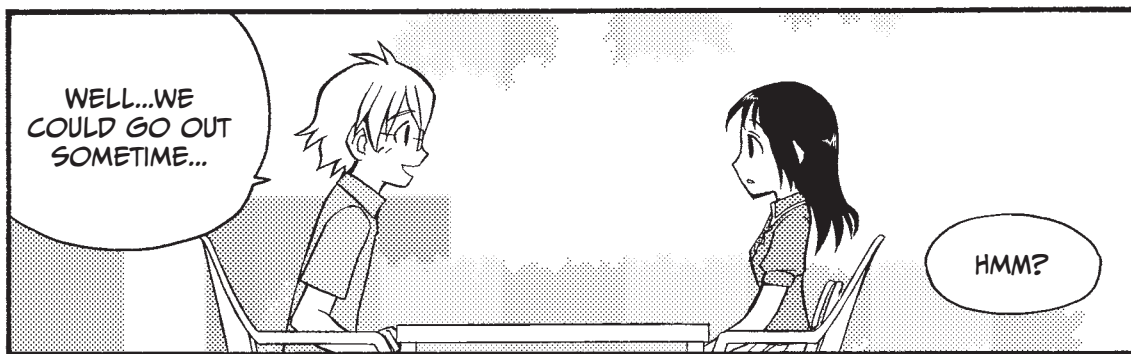
$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

THE RIGHT SIDE OF THE EQUATION IS THE DIAGONALIZED
FORM OF THE MIDDLE MATRIX ON THE LEFT SIDE.

AND
THAT'S IT!

NICE!





MULTIPLICITY AND DIAGONALIZATION

We said on page 221 that any $n \times n$ matrix could be expressed in this form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}^{-1}$$

The eigenvector corresponding to λ_1
 The eigenvector corresponding to λ_2
 The eigenvector corresponding to λ_n

This isn't totally true, as the concept of *multiplicity*¹ plays a large role in whether a matrix can be diagonalized or not. For example, if all n solutions of the following equation

$$\det \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix} = 0$$

are real and have multiplicity 1, then diagonalization is possible. The situation becomes more complicated when we have to deal with eigenvalues that have multiplicity greater than 1. We will therefore look at a few examples involving:

- Matrices with eigenvalues having multiplicity greater than 1 that can be diagonalized
- Matrices with eigenvalues having multiplicity greater than 1 that cannot be diagonalized

1. The multiplicity of any polynomial root reveals how many identical copies of that same root exist in the polynomial. For instance, in the polynomial $f(x) = (x - 1)^4(x + 2)^2x$, the factor $(x - 1)$ has multiplicity 4, $(x + 2)$ has 2, and x has 1.

A DIAGONALIZABLE MATRIX WITH AN EIGENVALUE HAVING MULTIPLICITY 2

PROBLEM

Use the following matrix in both problems:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$$

1. Find all eigenvalues and eigenvectors of the matrix.
2. Express the matrix in the following form:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}^{-1}$$

SOLUTION

1. The eigenvalues λ of the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix}$$

are the roots of the characteristic equation: $\det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & -1 \\ -2 & 0 & 3 - \lambda \end{pmatrix} = 0$.

$$\det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & -1 \\ -2 & 0 & 3 - \lambda \end{pmatrix}$$

$$\begin{aligned} &= (1 - \lambda)(1 - \lambda)(3 - \lambda) + 0 \cdot (-1) \cdot (-2) + 0 \cdot 1 \cdot 0 \\ &\quad - 0 \cdot (1 - \lambda) \cdot (-2) - 0 \cdot 1 \cdot (3 - \lambda) - (1 - \lambda) \cdot (-1) \cdot 0 \\ &= (1 - \lambda)^2(3 - \lambda) = 0 \end{aligned}$$

$$\lambda = 3, 1$$

Note that the eigenvalue 1 has multiplicity 2.

A. The eigenvectors corresponding to $\lambda = 3$

Let's insert our eigenvalue into the following formula:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ that is } \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 1 & 1 - \lambda & -1 \\ -2 & 0 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us:

$$\begin{pmatrix} 1 - 3 & 0 & 0 \\ 1 & 1 - 3 & -1 \\ -2 & 0 & 3 - 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & -1 \\ -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ x_1 - 2x_2 - x_3 \\ -2x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions are as follows:

$$\begin{cases} x_1 = 0 \\ x_3 = -2x_2 \end{cases} \text{ and the eigenvector } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ -2c_1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

where c_1 is a real nonzero number.

B. The eigenvectors corresponding to $\lambda = 1$

Repeating the steps above, we get

$$\begin{pmatrix} 1 - 1 & 0 & 0 \\ 1 & 1 - 1 & -1 \\ -2 & 0 & 3 - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 - x_3 \\ -2x_1 + 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and see that $x_3 = x_1$ and x_2 can be any real number. The eigenvector consequently becomes

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where c_1 and c_2 are arbitrary real numbers that cannot both be zero.

3. We then apply the formula from page 221:

The eigenvector corresponding to 3

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}^{-1}$$

The linearly independent eigenvectors corresponding to 1

A NON-DIAGONALIZABLE MATRIX WITH A REAL EIGENVALUE HAVING MULTIPLICITY 2

? PROBLEM

Use the following matrix in both problems:

$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix}$$

1. Find all eigenvalues and eigenvectors of the matrix.
2. Express the matrix in the following form:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}^{-1}$$

! SOLUTION

1. The eigenvalues λ of the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix}$$

are the roots of the characteristic equation: $\det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ -7 & 1 - \lambda & -1 \\ 4 & 0 & 3 - \lambda \end{pmatrix} = 0.$

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ -7 & 1-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (1-\lambda)(1-\lambda)(3-\lambda) + 0 \cdot (-1) \cdot 4 + 0 \cdot (-7) \cdot 0 \\ &\quad - 0 \cdot (1-\lambda) \cdot 4 - 0 \cdot (-7) \cdot (3-\lambda) - (1-\lambda) \cdot (-1) \cdot 0 \\ &= (1-\lambda)^2(3-\lambda) = 0 \end{aligned}$$

$$\lambda = 3, 1$$

Again, note that the eigenvalue 1 has multiplicity 2.

A. The eigenvectors corresponding to $\lambda = 3$

Let's insert our eigenvalue into the following formula:

$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ that is } \begin{pmatrix} 1-\lambda & 0 & 0 \\ -7 & 1-\lambda & -1 \\ 4 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us

$$\begin{pmatrix} 1-3 & 0 & 0 \\ -7 & 1-3 & -1 \\ 4 & 0 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ -7 & -2 & -1 \\ 4 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -7x_1 - 2x_2 - x_3 \\ 4x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions are as follows:

$$\begin{cases} x_1 = 0 \\ x_3 = -2x_2 \end{cases} \text{ and the eigenvector } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \\ -2c_1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

where c_1 is a real nonzero number.

B. The eigenvectors corresponding to $\lambda = 1$

We get

$$\begin{pmatrix} 1-1 & 0 & 0 \\ -7 & 1-1 & -1 \\ 4 & 0 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -7 & 0 & -1 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -7x_1 - x_3 \\ 4x_1 + 2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and see that $\begin{cases} x_3 = -7x_1 \\ x_3 = -2x_1 \end{cases}$

But this could only be true if $x_1 = x_3 = 0$. So the eigenvector has to be

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \\ 0 \end{pmatrix} = c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where c_2 is an arbitrary real nonzero number.

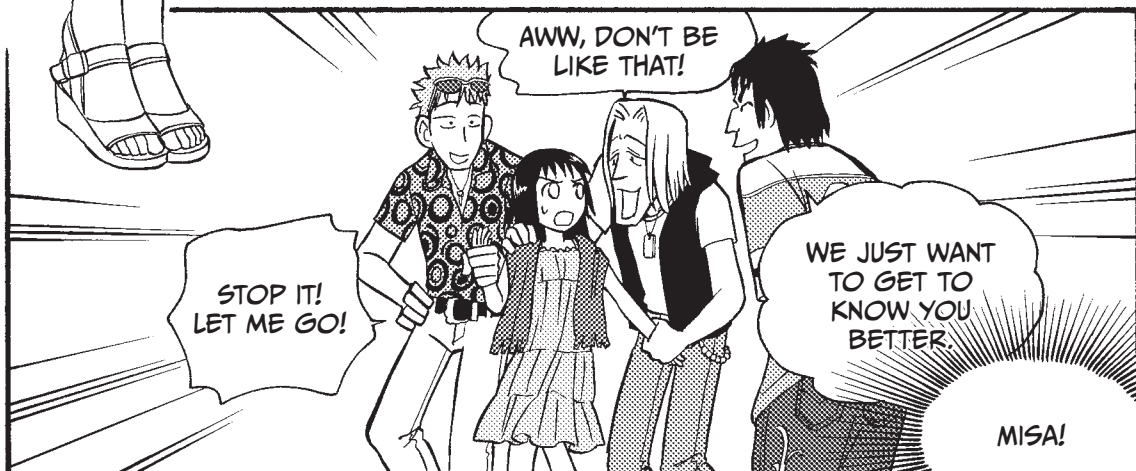
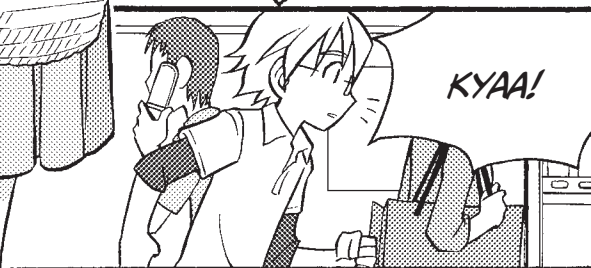
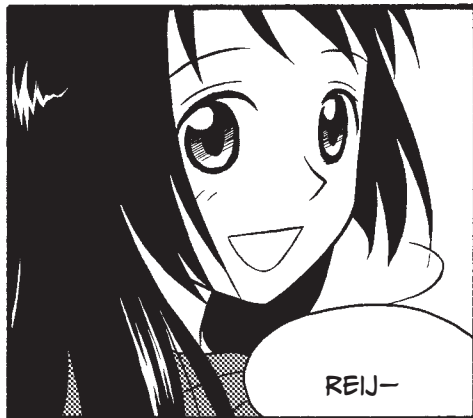
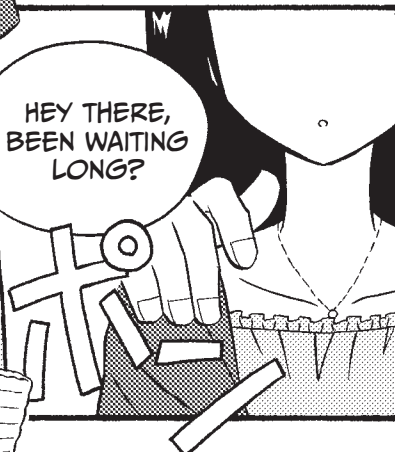
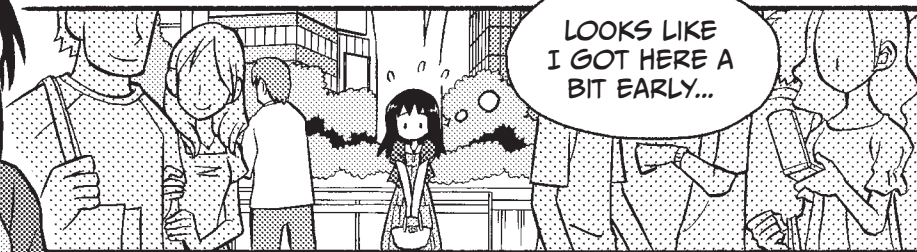
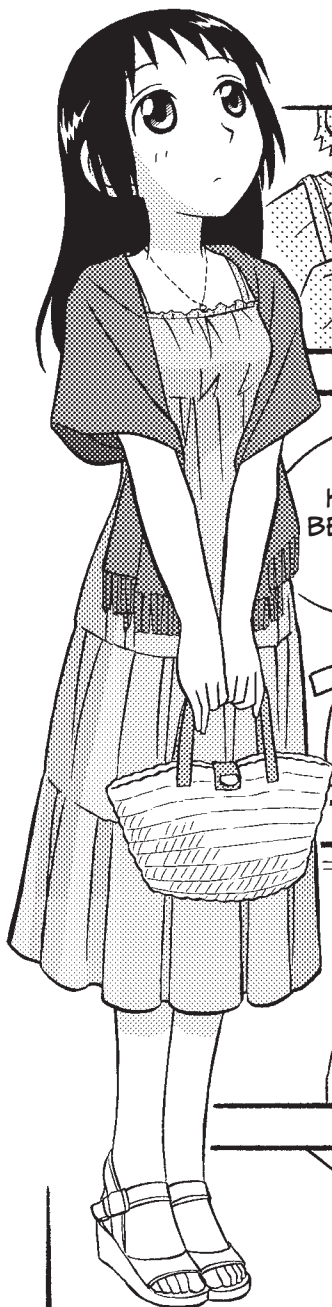
3. Since there were no eigenvectors in the form

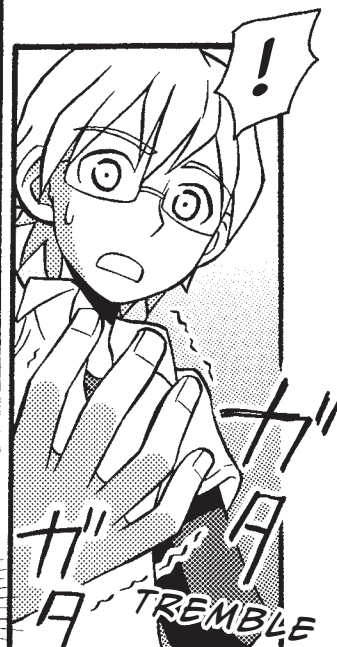
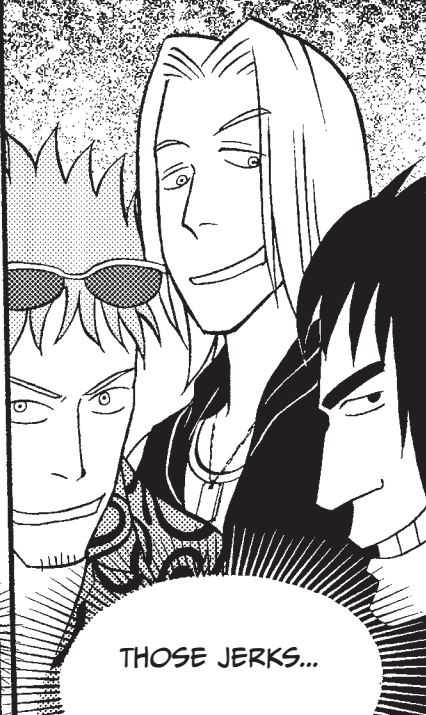
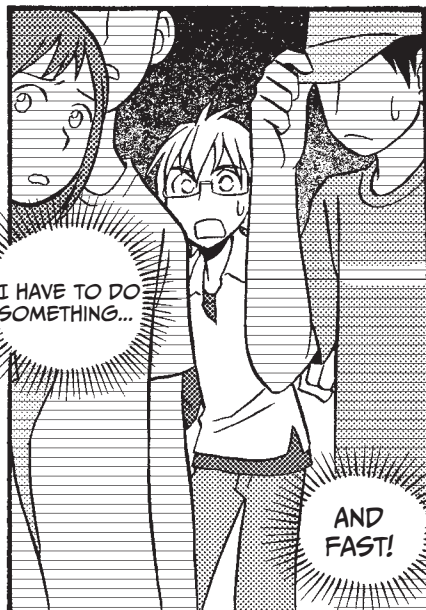
$$c_2 \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \end{pmatrix} + c_3 \begin{pmatrix} x_{13} \\ x_{23} \\ x_{33} \end{pmatrix}$$

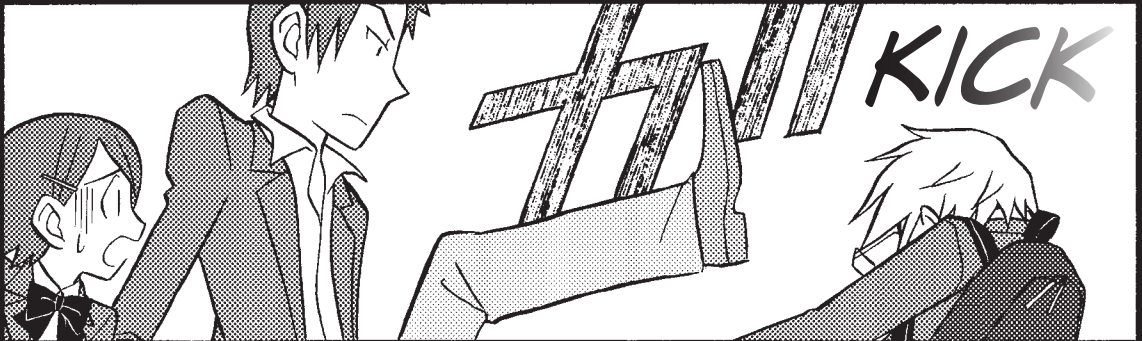
for $\lambda = 1$, there are not enough linearly independent eigenvectors to express

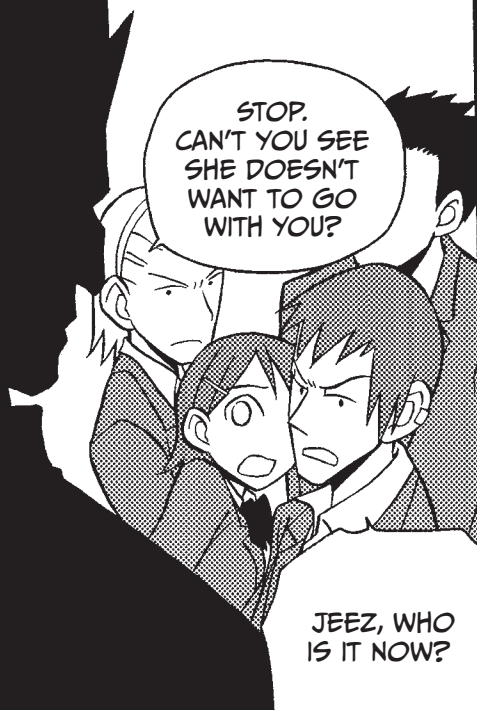
$$\begin{pmatrix} 1 & 0 & 0 \\ -7 & 1 & -1 \\ 4 & 0 & 3 \end{pmatrix} \text{ in the form } \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}^{-1}$$

It is important to note that all diagonalizable $n \times n$ matrices *always* have n linearly independent eigenvectors. In other words, there is always a basis in R^n consisting solely of eigenvectors, called an *eigenbasis*.







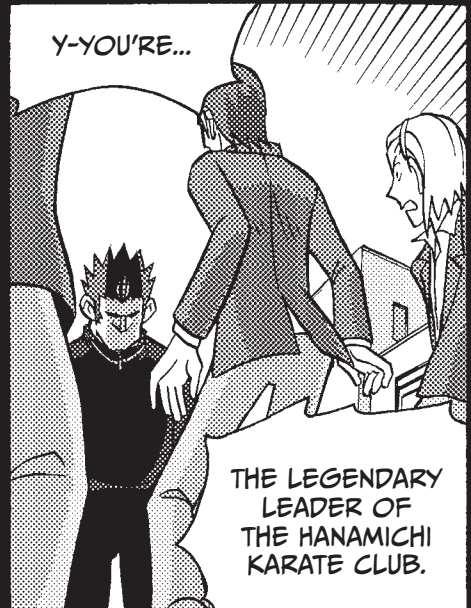


STOP.
CAN'T YOU SEE
SHE DOESN'T
WANT TO GO
WITH YOU?

JEEZ, WHO
IS IT NOW?



OH NO!



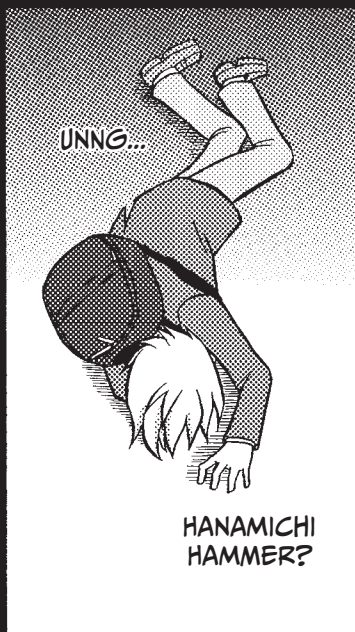
Y-YOU'RE...

THE LEGENDARY
LEADER OF
THE HANAMICHI
KARATE CLUB.



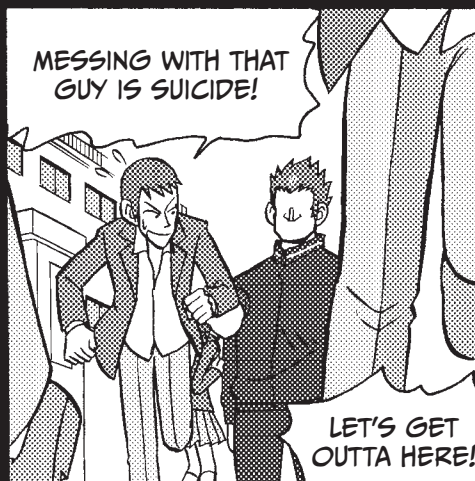
"THE HANAMICHI
HAMMER!"

IN THE
FLESH.



UNNG...

HANAMICHI
HAMMER?



MESSING WITH THAT
GUY IS SUICIDE!

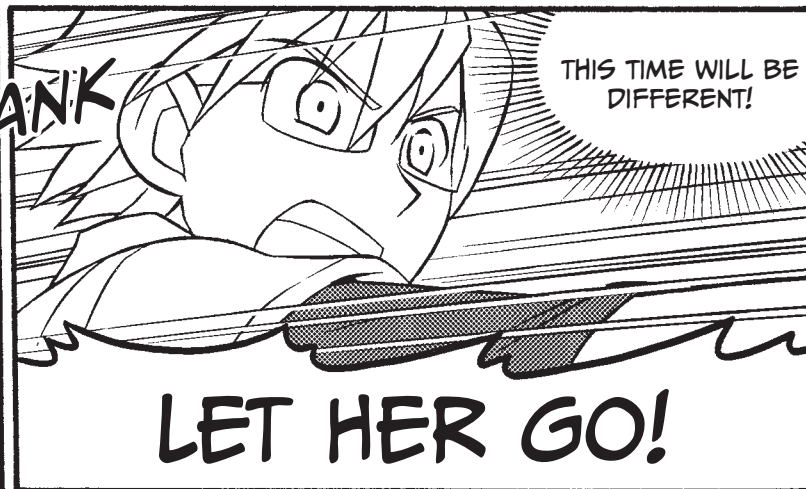
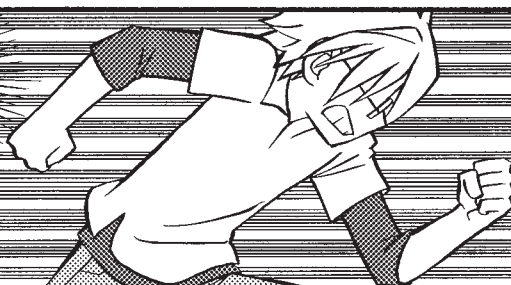
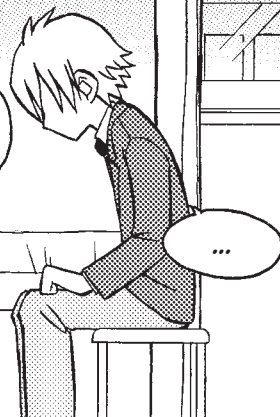
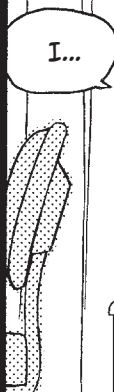
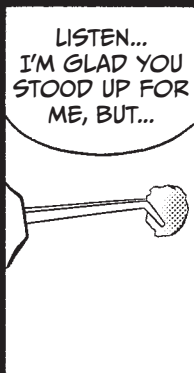
LET'S GET
OUTTA HERE!

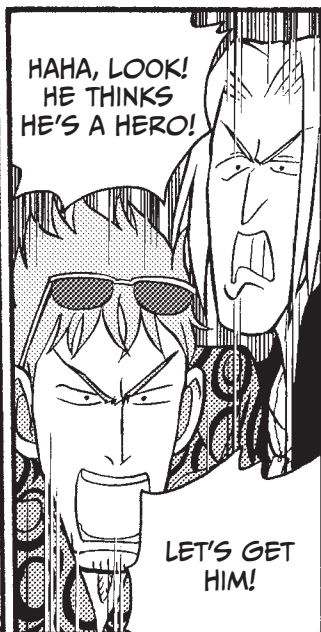
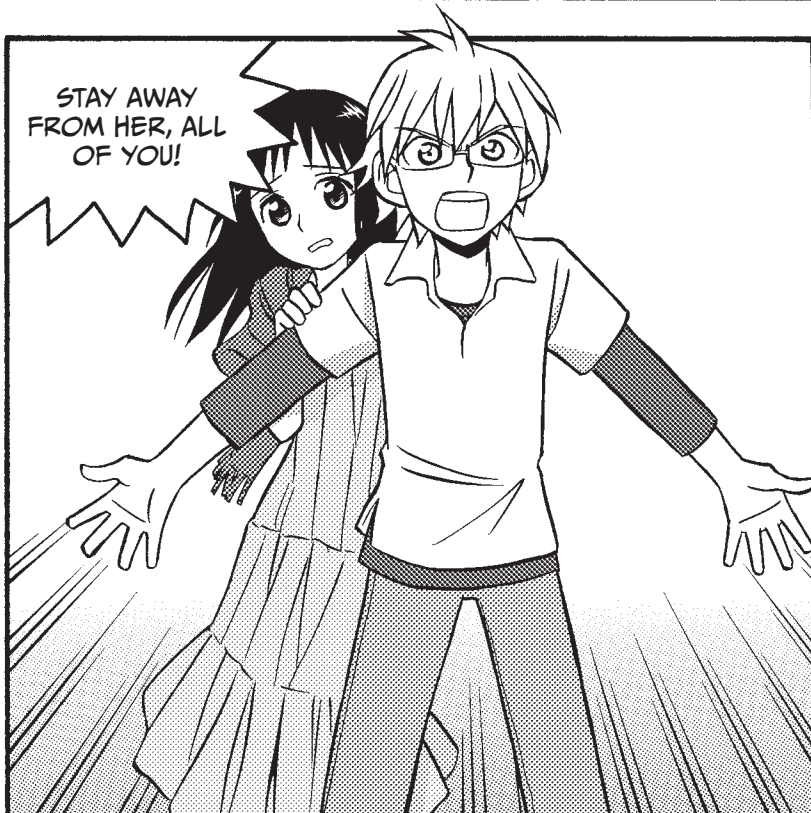
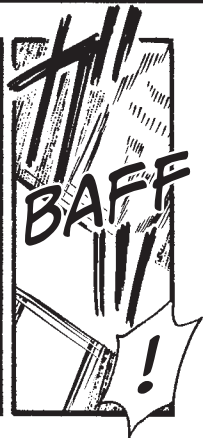
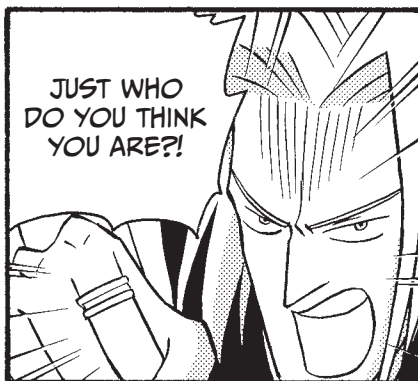
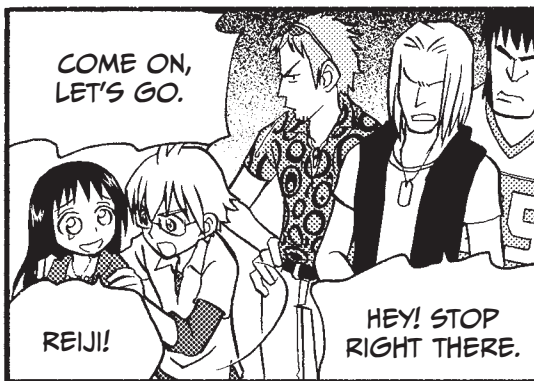


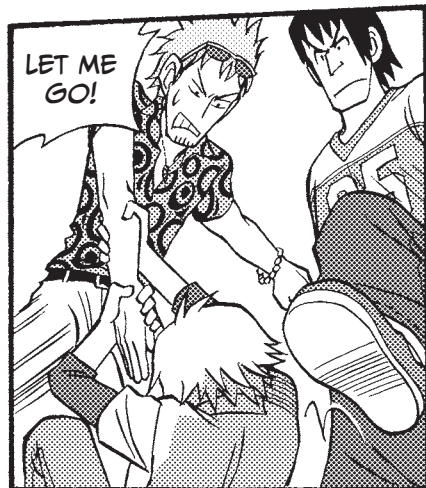
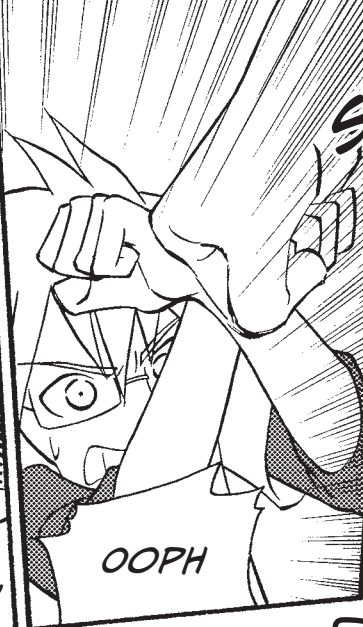
T-THANK YOU!

DON'T WORRY
ABOUT IT...BUT
I THINK YOUR
BOYFRIEND
NEEDS MEDICAL
ATTENTION...

MOAN...









ATTACKING MY
LITTLE SISTER,
ARE WE?

TETSUO!



I DON'T LIKE
EXCESSIVE
VIOLENCE...BUT IN
ATTACKING MISA,
YOU HAVE GIVEN ME
LITTLE CHOICE...

CRACK



IT'S
ICHINOSE!



THE HANAMICHI
HAMMER!



MOMMY!

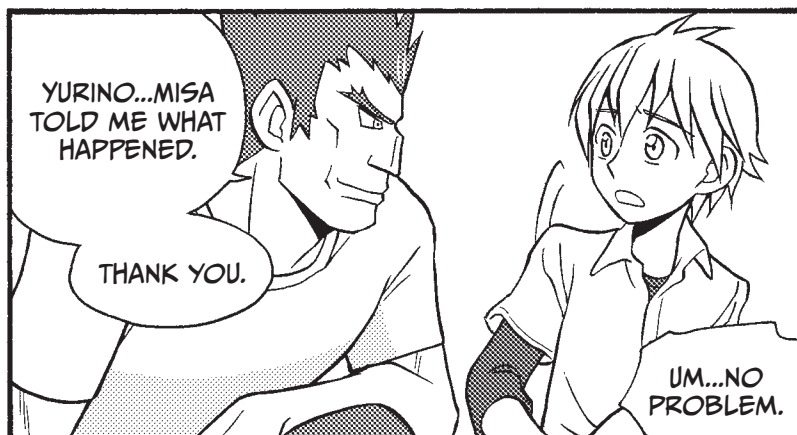
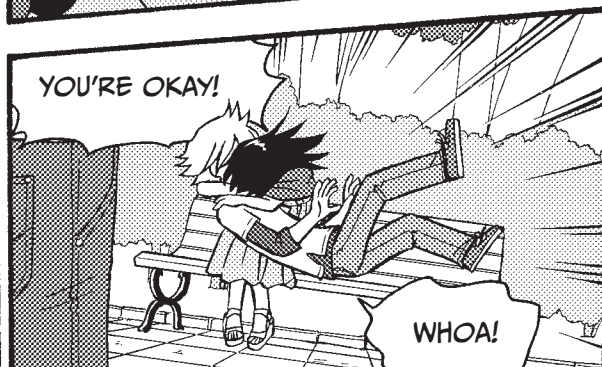
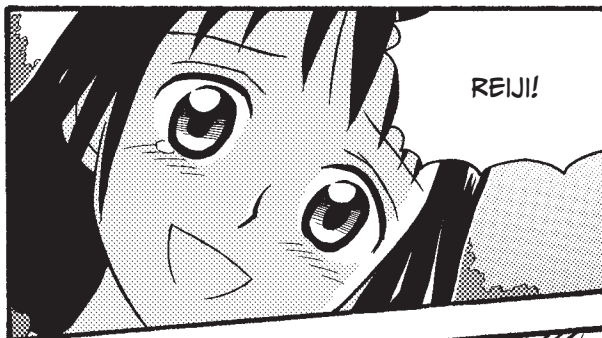
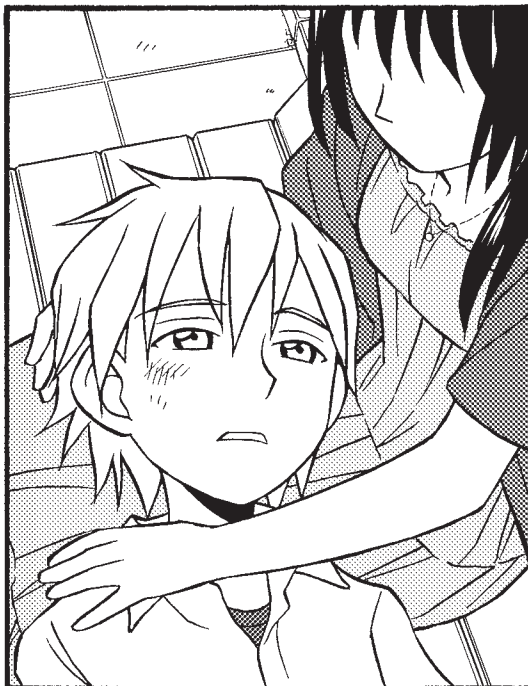
RUN!

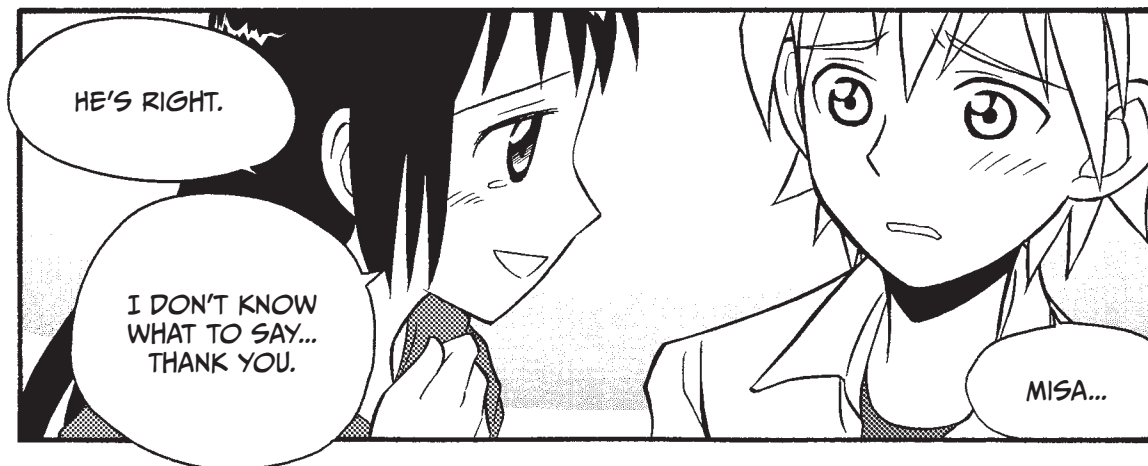
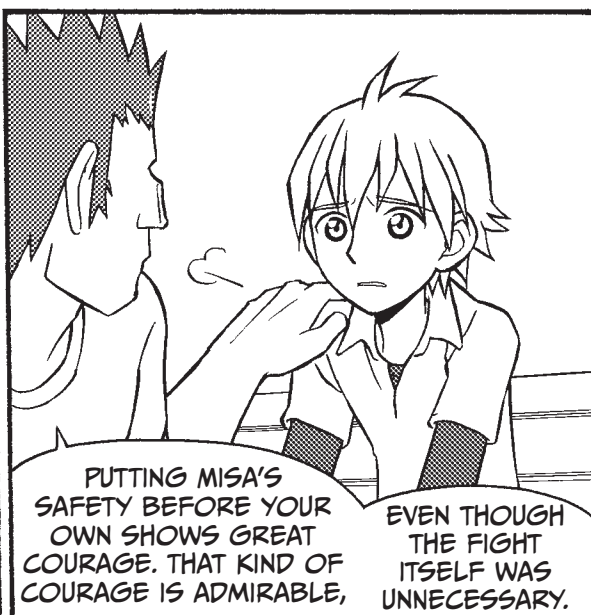


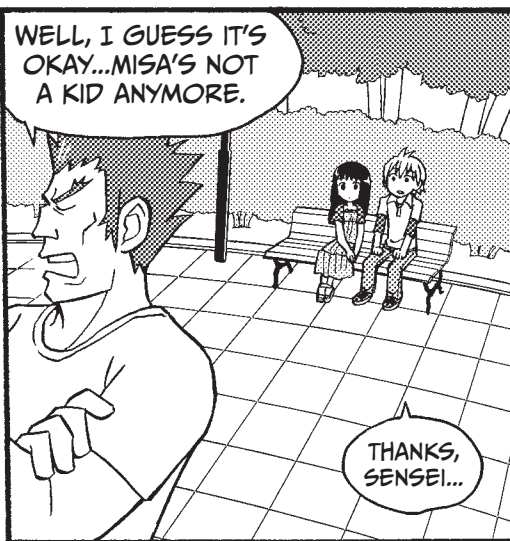
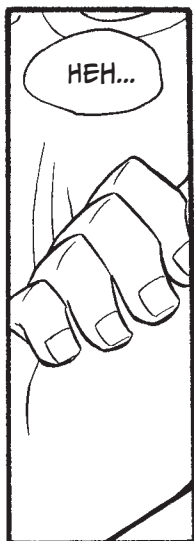
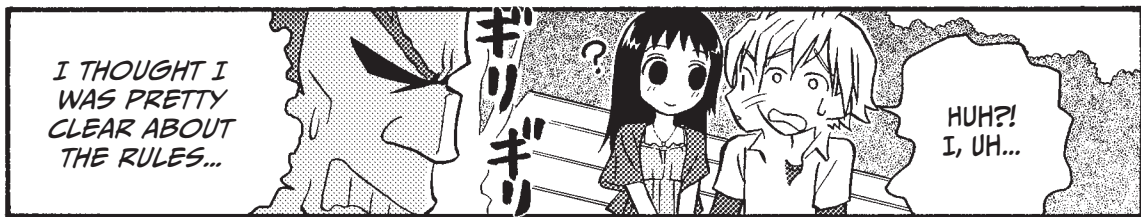
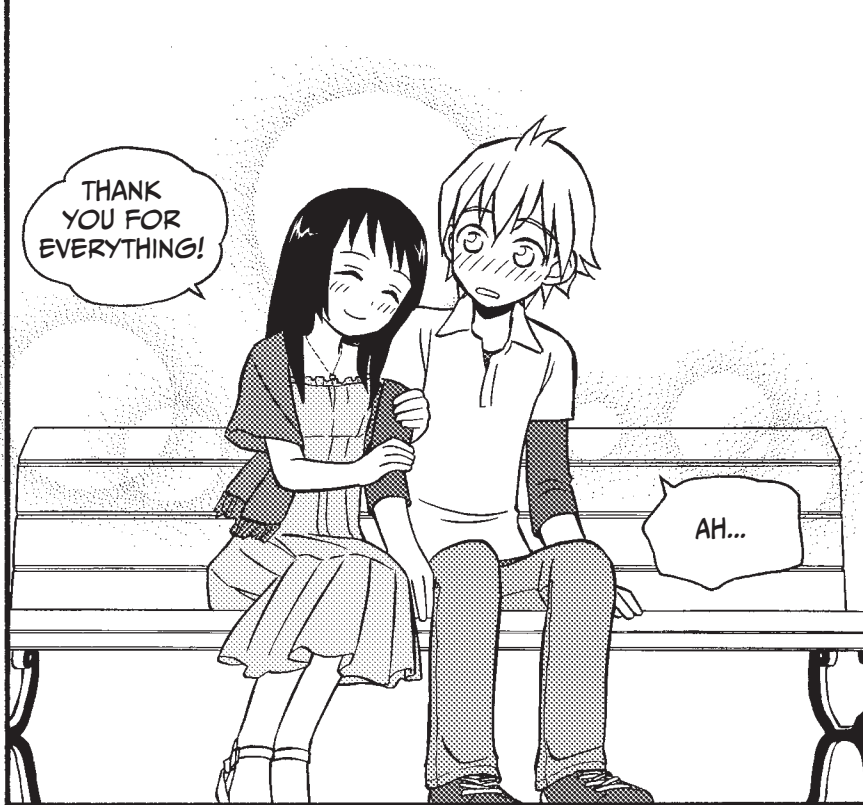
SENSEI?

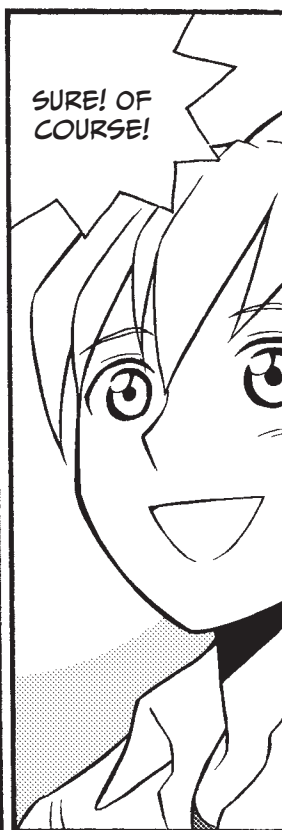
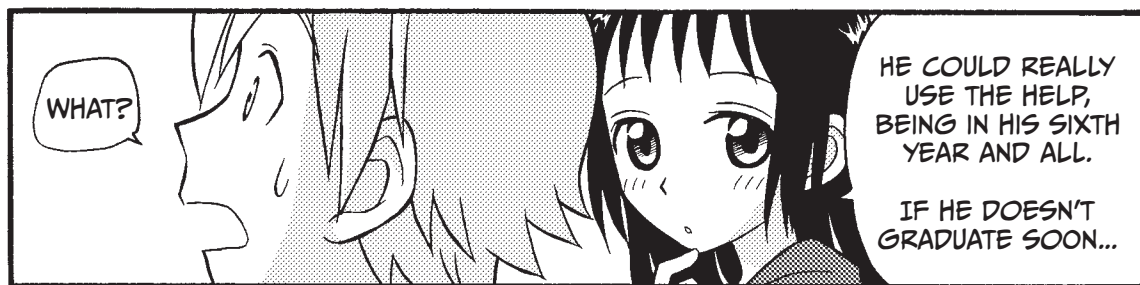
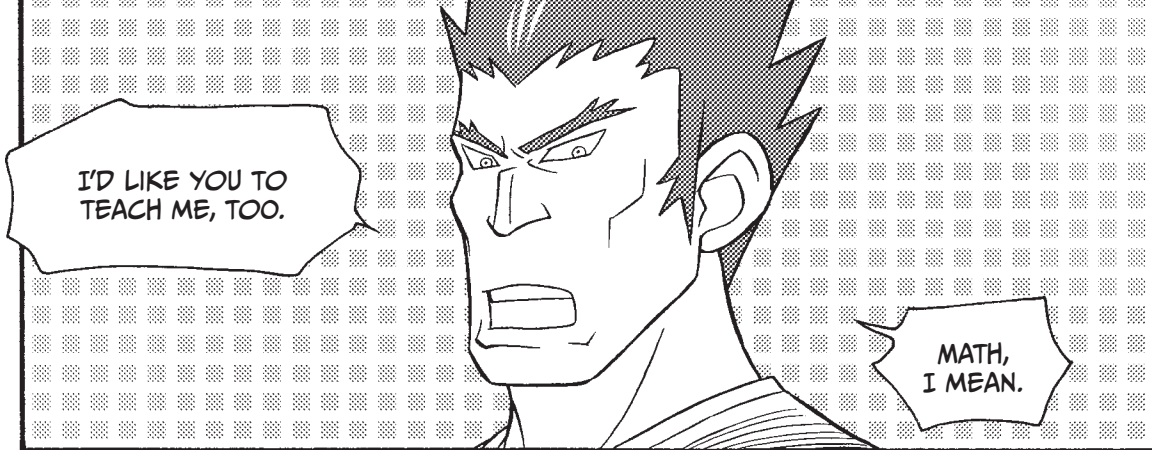


HE'S OUT
COLD.









ONLINE RESOURCES

THE APPENDIXES

The appendixes for *The Manga Guide to Linear Algebra* can be found online at <http://www.nostarch.com/linearalgebra>. They include:

- Appendix A: Workbook
- Appendix B: Vector Spaces
- Appendix C: Dot Product
- Appendix D: Cross Product
- Appendix E: Useful Properties of Determinants

UPDATES

Visit <http://www.nostarch.com/linearalgebra> for updates, errata, and other information.





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NOTES



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Shin Takahashi was born 1972 in Niigata. He received a master's degree from Kyushu Institute of Design (known as Kyushu University today). Having previously worked both as an analyst and as a seminar leader, he is now an author specializing in technical literature.

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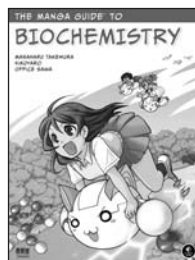
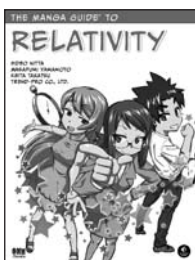
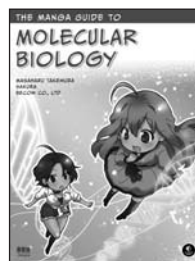
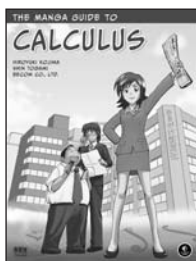
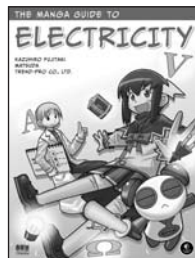
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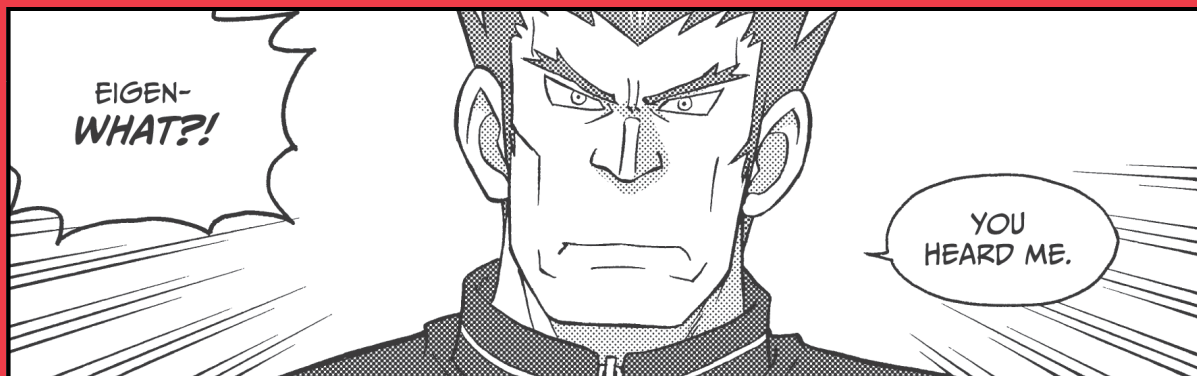


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