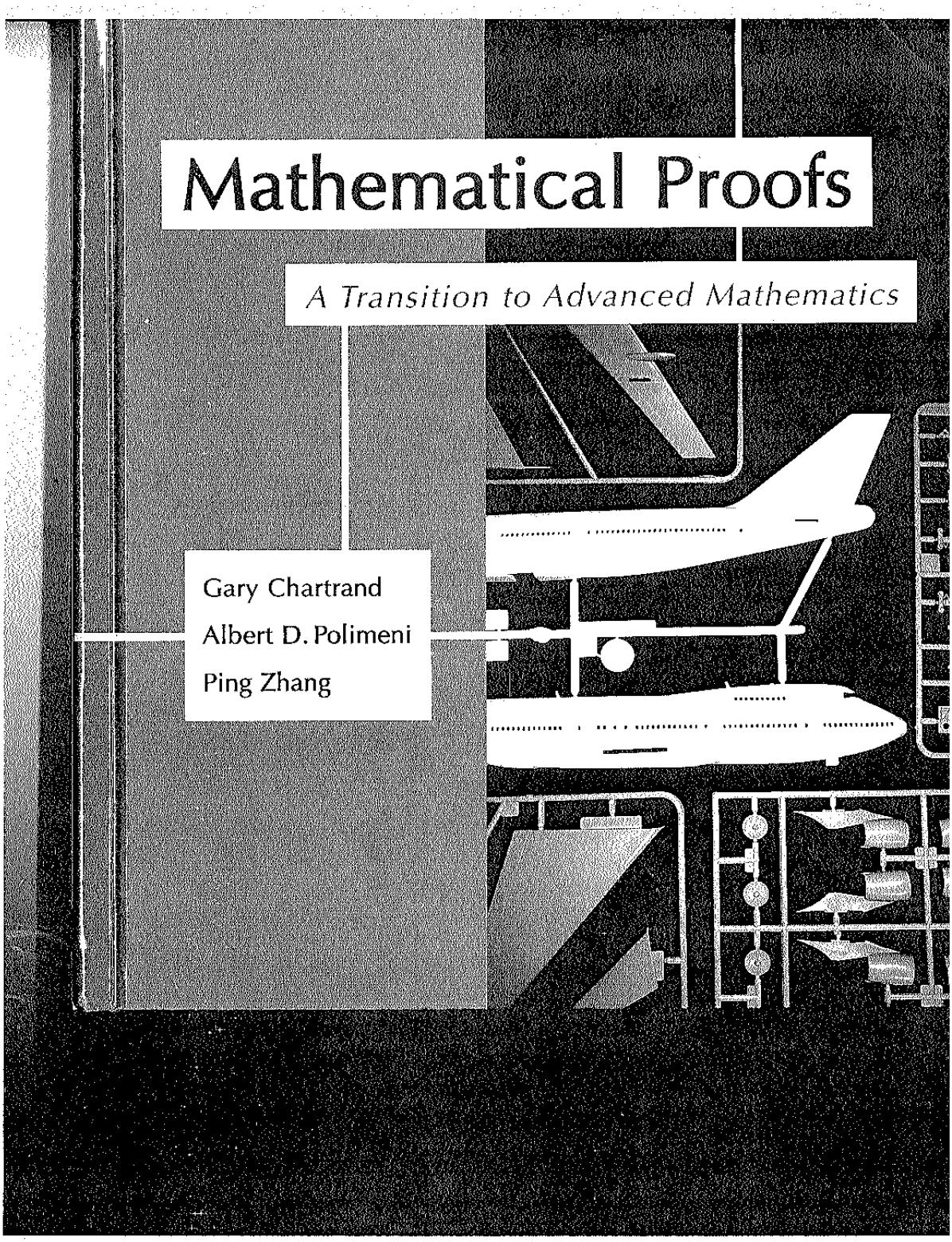
Mathematical Proofs

A Transition to Advanced Mathematics



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Mathematical Proofs

A Transition to Advanced Mathematics

SECOND EDITION

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To

the memory of my mother and father *G.C.*

the memory of my mother and father *A.D.P.*

my mother and the memory of my father *P.Z.*

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Preface

As we mentioned in the preface of the first edition, the theoretical gap between the material presented in calculus and the mathematical background expected (or at least hoped for) in more advanced courses has widened. In an attempt to narrow this gap and to better prepare students for the more abstract mathematics courses to follow, many colleges and universities have introduced courses that are now commonly called "transition courses". In these courses, students are introduced to proof techniques and writing their own proofs, as well as topics such as relations, functions, and cardinalities of sets, which are encountered throughout theoretical mathematics courses. In addition, transition courses often include theoretical aspects of number theory, abstract algebra, and calculus. This text has been written for such a course.

The idea for this text originated in the early 1980s. Long before transition courses became fashionable, we realized that even advanced undergraduates lack a sound understanding of proof techniques and have difficulty writing correct and clear proofs. The first edition of this book emanated from notes developed for these students, which, in turn, has led to this second edition.

Our Approach

Since this text originated from notes that were written exclusively for undergraduates to help them understand proof techniques and to write good proofs, this is the tone in which both editions of this book have been written: to be student-friendly. Numerous examples of proofs are presented in the text. Following common practice, we indicate the end of a proof with the symbol ■. Often we precede a proof by a discussion, referred to as a *proof strategy*, where we think through what is needed to present a proof of the result in question. Other times, we find it useful to reflect on a proof we have just presented to point out certain key details; we refer to a discussion of this type as a *proof analysis*. Periodically, problems are presented and solved, and we may find it convenient to discuss some features of the solution; we refer to this simply as an *analysis*. For clarity, we indicate the end of a discussion of a proof strategy, proof analysis, analysis, or solution of an example with the diamond symbol ♦.

A major goal of this text is to help students learn to construct proofs of their own that are not only mathematically correct but also clearly written. More advanced mathematics students should strive to present proofs that are convincing, readable, notationally consistent, and grammatically correct. A secondary goal is to have students gain sufficient knowledge of and confidence with proofs so that they will recognize, understand, and appreciate a proof that is properly written.

This book is intended as an introduction to mathematics in a rigorous setting. We envision students taking a course based on this book after they have had a year of calculus (and possibly another course, such as elementary linear algebra). It is likely that, prior to taking this course, a student's training in mathematics consisted primarily of doing patterned problems; that is, students have been taught methods for solving problems, likely including some explanation as to why these methods worked. Students may very well have had exposure to some proofs in earlier courses but, more than likely, were unaware of the logic involved and the method of proof being used. There may have even been times when the students were not certain what was being proved.

Outline of the Contents

Since writing good proofs requires a certain degree of competence in writing, we have devoted Chapter 0 to writing mathematics. The emphasis of this chapter is on effective and clear exposition, correct usage of symbols, writing and displaying mathematical expressions, and using key words and phrases. Although every instructor will emphasize writing in his or her own way, we feel that it is useful to read Chapter 0 periodically throughout the course. It will mean more as the student progresses through the course. Only minor changes and additions have been made to Chapter 0 in the second edition.

There have been significant changes made to Chapters 1–11, with relatively minor changes in Chapter 12 (Proofs in Calculus) and Chapter 13 (Proofs in Group Theory). A large number of new examples have been added to support the ideas presented, and many new results have been added with the goal of achieving a better understanding of the material. There has been more than a 50% increase in the number of exercises in the second edition over the first edition. Among the new exercises are:

(1) exercises that relate to new examples and results that have been added,

(2) more proof evaluation exercises,

(3) additional exercises in which the proof of an unknown result is given, with the goal of determining the result being proved, and

(4) exercises of a different type that are, at the same time, unique and interesting.

Each chapter is now divided more formally into sections and includes end-of-section exercises. There is also a final section of exercises for the entire chapter. In contrast to the first edition, examples are now formally identified by labeling them as Example X.Y to indicate that they occur in Chapter X.

While the proof techniques in the first edition were introduced over the first nine chapters of the book, the material on mathematical induction has been moved forward in the book so that all proof techniques now appear in the first six chapters.

Chapter 1 contains a gentle introduction to sets, so that everyone has the same background and is using the same notation as we prepare for what lies ahead. No proofs involving sets occur until Chapter 4. Much of Chapter 1 may very well be a review for many.

Chapter 2 deals exclusively with logic. The goal here is to present what is needed to get into proofs as quickly as possible. Much of the emphasis in Chapter 2 is on statements, implications, and quantified statements. Sets are introduced before logic so that students' first encounter with mathematics here is a familiar one and because sets are needed to discuss quantified statements properly in Chapter 2.

In the second edition, the distinction between statements and open sentences is expanded upon and clarified. One way this is accomplished is by considering statements that result from open sentences by substituting values from the domain for the variables appearing in the open sentence.

The section on quantifiers has been moved so that it is now Section 2.10. This section has been expanded greatly and now includes a discussion of double quantifiers. Quantifiers are revisited and dealt with in more detail in Chapter 7. Mixed quantifiers are also discussed in Chapter 7. Chapter 2 now ends with Section 2.10 (Quantified Statements) and Section 2.11 (Characterizations of Statements).

The two proof techniques of direct proof and proof by contrapositive are introduced in Chapter 3 in the familiar setting of even and odd integers. Proof by cases is discussed in this chapter as well as proofs of "if and only if" statements. Chapter 4 continues this discussion in other settings, namely divisibility of integers, congruence, real numbers, and sets.

Chapter 5 has been completely restructured. The technique of proof by contradiction is introduced here. The sections dealing with counterexamples, existence proofs, and disproving statements, which appeared in Chapter 6 (Prove or Disprove) of the first edition, now appear in Chapter 5 as well. Although an existence proof is not a proof technique, we felt that it was appropriate to include it within methods of proofs. Since existence proofs and counterexamples have a connection with proof by contradiction, we placed all of these in the same chapter. The topic of uniqueness (of an element with specified properties) is also addressed in Chapter 5. Having all of these important topics in Chapter 5 instead of in a later chapter (as in the first edition) increases the likelihood that they will be covered in a course.

The former Chapter 9 (Mathematical Induction) is now Chapter 6 in this edition. This change was made (1) to place all methods of proof together prior to applying them to various areas of mathematics, and (2) so that mathematical induction can be applied to ideas that follow Chapter 6. In addition to the Principle of Mathematical Induction and the Strong Principle of Mathematical Induction, this chapter includes proof by minimum counterexample.

The main goal of Chapter 7 (Prove or Disprove) concerns the testing of statements where statements of unknown truth value are provided and where it is to be determined, with justification, whether each statement is true or false. In addition to the challenge of determining whether given statements are true or false, such problems provide added practice with counterexamples and the various proof techniques. Testing statements is preceded in this chapter by a historical discussion of conjectures in mathematics and a review of quantifiers, together with a discussion of mixed quantifiers.

Chapter 8 deals with relations, especially equivalence relations. Many examples involving congruence are presented, and the set of integers modulo n is described.

Chapter 9 involves functions, with emphasis on the properties of one-to-one and onto. This gives rise to a discussion of bijective functions and inverses of functions. The well-defined property of functions is discussed in more detail in the second edition.

Chapter 10 deals with infinite sets and a discussion of cardinalities of sets. This chapter now includes a historical discussion of infinite sets, beginning with Cantor and his fascination and difficulties with the Schröder–Bernstein Theorem, then to Zermelo and the Axiom of Choice, and ending with a proof of the Schröder–Bernstein Theorem.

All of the proof techniques are used in Chapter 11, where numerous results in the area of number theory are introduced and proved.

Web Site for Mathematical Proofs

Three additional chapters, Chapters 14–16 (dealing with proofs in ring theory, linear algebra, and topology), can be found on the Web site: http://www.aw.com/info/chartrand.

Teaching a Course from This Text

Although a course using this text could be designed in many ways, here are our views on such a course. As we noted earlier, we think it is useful for students to reread (at least portions of) Chapter 0 throughout the course, and we feel that with each reading the chapter becomes more meaningful. The first part of Chapter 1 (Sets) will likely be familiar to most students, although the last part may not. Chapters 2–6 will probably be part of any course, although certain topics could receive varying degrees of emphasis (with proof by minimum counterexample in Chapter 6 possibly omitted). Little or much time could be spent on Chapter 7, depending on how much time is used to discuss the large number of "prove or disprove" exercises. We think that most of Chapters 8 and 9 would be covered in such a course and that it would be useful to cover some of the fundamental ideas addressed in Chapter 10 (Cardinalities of Sets). As time permits, portions of the later chapters could be covered, especially those of interest to the instructor, including the possibility of going to the Web site for even more variety.

Exercises

There are numerous exercises for Chapters 1–13 (as well as for Chapters 14–16 on the Web site). The degree of difficulty of the exercises ranges from routine to medium difficulty to moderately challenging. There are exercises that present students with statements, asking them to decide whether they are true or false (with justification). There are proposed proofs of statements, asking if the argument is valid. There are proofs without a statement given, asking students to supply a statement of what has been proved. Also, there are exercises that call upon students to ask questions of their own and to provide answers.

Chapter 3 is the first chapter in which students will be called upon to write proofs. At such an early stage, we feel that students need to (1) concentrate on constructing a valid proof and not be distracted by unfamiliarity with the mathematics, (2) develop some self-confidence with this process, and (3) learn how to write a proof properly. With this in mind, many of the exercises in Chapter 3 have been intentionally structured so as to be similar to the examples in that chapter.

In general, there are exercises for each section at the end of a chapter (section exercises) and additional exercises for the entire chapter (chapter exercises). Answers to the odd-numbered section exercises appear at the end of text. One should also keep in mind, however, that proofs of results are not unique.

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0 Communicating Mathematics

All serious students of mathematics eventually reach the stage when they realize that mathematics is not simply the manipulation of numbers or using the right formula. Although there are certainly numerous advantages of using sophisticated graphing calculators and computer software, this is not mathematics either. Mathematics is many things. Mathematics is understanding, observing, reasoning, explaining, thinking. Mathematics is also discovery. When we believe that we have made a mathematical discovery, how can we be certain that we are right? We must be able to verify this. Furthermore, we must be able to convince others of this. The following quote is due to the famous mathematician and physicist Blaise Pascal.

> *We are usually convinced more easily by reasons we have found ourselves than by those which occurred to others.*

Learning Mathematics

One of the major goals of this book is to assist you as you progress from an individual who uses mathematics to an individual who understands mathematics. Perhaps this will mark the beginning of you becoming someone who actually develops mathematics of your own. This is an attainable goal if you have the desire.

The fact that you've gone this far in your study of mathematics suggests that you have ability in mathematics. This is a real opportunity for you. Much of the mathematics that you will encounter in the future is based on what you are about to learn here. The better you learn the material and the mathematical thought process now, the more you will understand later. To be sure, any area of study is considerably more enjoyable when you understand it. But getting to that point will require effort on your part.

There are probably as many excuses for doing poorly in mathematics as there are strategies for doing well in mathematics. We have all heard students say (sometimes, remarkably, even with pride) that they are not good at mathematics. That's only an alibi. Mathematics can be learned like any other subject. Even some students who have done well in mathematics say that they are not good with proofs. This, too, is unacceptable. What is required is determination and effort. To have done well on an exam with little

or no studying is nothing to be proud of. Confidence based on being well-prepared is good, however.

Here is some advice that has worked for several students. First, it is important to understand what goes on in class each day. This means being present and being prepared for every class. After each class, recopy any lecture notes. When recopying the notes, express sentences in your own words and add details so that everything is as clear as possible. If you run into snags (and you will), talk them over with a classmate or your instructor. In fact, it's a good idea (at least in our opinion) to have someone with whom to discuss the material on a regular basis. Not only does it often clarify ideas, it gets you into the habit of using correct terminology and notation.

In addition to learning mathematics from your instructor, solidifying your understanding by careful note-taking, and by talking with classmates, your text is (or at least should be) an excellent source as well. Read your text carefully with pen (or pencil) and paper in hand. Make a serious effort to do every homework problem assigned and, eventually, be certain that you know how to solve them. If there are exercises in the text that have not been assigned, you might even try to solve these as well. Another good idea is to try to create your own problems. In fact, when studying for an exam, try creating your own exam. If you start doing this for all of your classes, you might be surprised at how good you become. Creativity is a major part of mathematics. Discovering mathematics not only contributes to your understanding of the subject but has the potential to contribute to mathematics itself. Creativity can come in all forms. The following quote is due to the well-known writer J. K. Rowling (author of the *Harry Potter* novels).

> *Sometimes ideas just come to me. Other times I have to sweat and almost bleed to make ideas come. It's a mysterious process, but I hope I never find out exactly how it works.*

The composer-lyricist Stephen Schwartz (who wrote the songs for the musicals *Godspell* and *Wicked*) discussed creativity in his song "The Spark of Creation" from the musical *Children of Eden:* (Copyright © 1991 Grey Dog Music, administered by Williamson Music. International Copyright Secured. All Rights Reserved.)

The spark of creation
Burning bright within me
The spark of creation
Won't let me rest at all
Until I discover
Or build or uncover
A thing that I can call
My celebration
Of the spark of creation.

In her book *Defying Gravity* on the life and work of Stephen Schwartz, the author Carol de Giere writes:

> *In many ways, this song expresses the theme of Stephen Schwartz's life—the naturalness and importance of the creative urge within us. At the same time he created an anthem for artists.*

In mathematics our goal is to seek the truth. Finding answers to mathematical questions is important, but we cannot be satisfied with this alone. We must be certain

that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here. As writers, it is our responsibility to give an accurate, clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved so that a well-presented argument will be understood. Consequently, in mathematics writing is important, *very* important. Is it *really* important to write mathematics well? After all, isn't mathematics mainly equations and symbols? Not at all. It is not only important to write mathematics well, it is important to write well. You will be writing the rest of your life, at least reports, letters, and e-mail. Many people who never meet you will know you only by what you write and how you write.

Mathematics is a sufficiently complicated subject that we don't need vague, hazy, and boring writing to add to it. A teacher has a very positive impression of a student who hands in well-written and well-organized assignments and examinations. You want people to enjoy reading what you've written. It is important to have a good reputation as a writer. It's part of being an educated person. Especially with the large number of e-mail letters that so many of us write, it has become commonplace for writing to be more casual. Although all people would probably subscribe to this (since it is more efficient), we should know how to write well formally and professionally when the situation requires it.

You might think that considering how long you've been writing and that you're set in your ways, it will be very difficult to improve your writing. Not really. If you want to improve, you can and will. Even if you are a good writer, your writing can always be improved. Ordinarily, people don't think much about their writing. Often just thinking about your writing is the first step to writing better.

What Others Have Said about Writing

Many people who are well known in their areas of expertise have expressed their thoughts about writing. Here are quotes by some of these individuals.

> *Anything that helps communication is good. Anything that hurts it is bad.*
>
> *I like words more than numbers, and I always did–conceptual more than computational.*
>
> Paul Halmos, mathematician

> *Writing is easy. All you have to do is cross out all the wrong words.*
>
> Mark Twain, author (*The Adventures of Huckleberry Finn*)

> *You don't write because you want to say something; you write because you've got something to say.*
>
> F. Scott Fitzgerald, author (*The Great Gatsby*)

> *Writing comes more easily if you have something to say.*
>
> Scholem Asch, author

Either write something worth reading or do something worth writing.

Benjamin Franklin, statesman, writer, inventor

What is written without effort is in general read without pleasure.

Samuel Johnson, writer

Easy reading is damned hard writing.

Nathaniel Hawthorne, novelist (*The Scarlet Letter*)

Everything that is written merely to please the author is worthless.

The last thing one knows when writing a book is what to put first.

I have made this letter longer because I lack the time to make it short.

Blaise Pascal, mathematician and physicist

The best way to become acquainted with a subject is to write a book about it.

Benjamin Disraeli, prime minister of England

In a very real sense, the writer writes in order to teach himself, to understand himself, to satisfy himself; the publishing of his ideas, though it brings gratification, is a curious anticlimax.

Alfred Kazin, literary critic

The skill of writing is to create a context in which other people can think.

Edwin Schlossberg, exhibit designer

A writer needs three things, experience, observation, and imagination, any two of which, at times any one of which, can supply the lack of the other.

William Faulkner, writer (*The Sound and the Fury*)

If confusion runs rampant in the passage just read,
It may very well be that too much has been said.

So that's what he meant! Then why didn't he say so?

Frank Harary, mathematician

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

David Hilbert, mathematician

Everything should be made as simple as possible, but not simpler.

Albert Einstein, physicist

Never let anything you write be published without having had others critique it.

Donald E. Knuth, computer scientist and writer

Some books are to be tasted, others to be swallowed, and some few to be chewed and digested.

Reading maketh a full man, conference a ready man, and writing an exact man.

Francis Bacon, writer and philosopher

Judge an article not by the quality of what is framed and hanging on the wall, but by the quality of what's in the wastebasket.

Anonymous (Quote by Leslie Lamport)

We are all apprentices in a craft where no-one ever becomes a master.

Ernest Hemingway, author (*For Whom the Bell Tolls*)

There are three rules for writing a novel. Unfortunately, no one knows what they are.

W. Somerset Maugham, author (*Of Human Bondage*)

Mathematical Writing

Most of the quotes given above pertain to writing in general, not to mathematical writing in particular. However these suggestions for writing apply as well to writing mathematics. For us, mathematical writing means writing assignments for a mathematics course (particularly a course with proofs). Such an assignment might consist of writing a single proof, writing solutions to a number of problems, or perhaps writing a term paper that, more than likely, includes definitions, examples, background, *and* proofs. We'll refer to any of these as an "assignment". Your goal should be to write correctly, clearly, and in an interesting manner.

Before you even begin to write, you should have already thought about a number of things. First, you should know what examples and proofs you plan to include if this is appropriate for your assignment. You should not be overly concerned about writing good proofs on your first attempt—but be certain that you do have *proofs*.

As you're writing your assignment, you must be aware of your audience. What is the target group for your assignment? Of course, it should be written for your instructor. But it should be written so that a classmate would understand it. As you grow mathematically, your audience will grow with you and you will adapt your writing to this new audience.

Give yourself enough time to write your assignment. Don't try to put it together just a few minutes before it's due. The disappointing result will be obvious to your instructor. And to you! Find a place to write that is comfortable for you: your room, an office, a study room, the library, and sitting at a desk, at a table, in a chair. Do what works best for you. Perhaps you write best when it's quiet or when there is background music.

Now that you're comfortably settled and have allowed enough time to do a good job, let's put a plan together. If the assignment is fairly lengthy, you may need an outline, which, most likely, would include one or more of the following:

1. Background and motivation
2. The definitions to be presented and possibly the notation to be used
3. The examples to include
4. The results to be presented (whose proofs have already been written, probably in rough form)
5. References to other results you intend to use
6. The order of everything mentioned above.

If the assignment is a term paper, it may include extensive background material and may need to be carefully motivated. The subject of the paper should be placed in perspective. Where does it fit in with what we already know?

Many writers write in "spirals". Even though you have a plan for your assignment that includes an ordered list of things you want to say, it is likely that you will reach some point (perhaps sooner than you think) when you realize that you should have included something earlier—perhaps a definition, a theorem, an example, some notation. (This happened to us many times while writing this text.) Insert the missing material, start over again, and write until once again you realize that something is missing. It is important, as you reread, that you start at the beginning each time. Then repeat the steps listed above.

We are about to give you some advice, some "pointers", about writing mathematics. Such advice is necessarily subjective. Not everyone subscribes to these suggestions on writing. Indeed, writing "experts" don't agree on all issues. For the present, your instructor will be your best guide. But writing does not follow a list of rules. As you mature mathematically, perhaps the best advice about your writing is the same advice given by Jiminy Cricket to Disney's Pinocchio: *Always let your conscience be your guide.* You must be yourself. And one additional piece of advice: Be careful about accepting advice on writing. Originality and creativity don't follow rules. Until you reach the stage of being comfortable and confident with your own writing, however, we believe that it is useful to consider a few writing tips.

Since a number of these writing tips may not make sense (because, after all, we don't even have anything to write yet), it will probably be most useful to return to this chapter periodically as you proceed through the chapters that follow.

Using Symbols

Since mathematics is a symbol-oriented subject, mathematical writing involves a mixture of words and symbols. Here are several guidelines to which a number of mathematicians adhere.

1. *Never start a sentence with a symbol.*
   Writing mathematics follows the same practice as writing all sentences, namely, that the first word should be capitalized. This is confusing if the sentence were to begin with a symbol since the sentence appears to be incomplete. Also, in general, a sentence sounds better if it starts with a word. Instead of writing:

   $$x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

   write:

   $$\text{The equation } x^2 - 6x + 8 = 0 \text{ has two distinct roots.}$$

2. *Separate symbols not in a list by words if possible.*
   Separating symbols by words makes the sentence easier to read and therefore easier to understand. The sentence:

   $$\text{Except for } a, b \text{ is the only root of } (x - a)(x - b) = 0.$$

   would be clearer if it were written as:

   $$\text{Except for } a, \text{ the number } b \text{ is the only root of } (x - a)(x - b) = 0.$$

3. *Except when discussing logic, avoid writing the following symbols in your assignment:*

   $$\Rightarrow,\ \forall,\ \exists,\ \ni,\ \textit{etc.}$$

   The first four symbols stand for "implies", "for every", "there exists", and "such that", respectively. You may have already seen these symbols and know what they mean. If so, this is good. It is useful when taking notes or writing early drafts of an assignment to use shorthand symbols, but many mathematicians avoid such symbols in their professional writing. (We will visit these symbols later.)

4. *Be careful about using i.e. and e.g.*
   These stand for *that is* and *for example*, respectively. There are situations when writing the words is preferable to writing the abbreviations as there may be confusion with nearby symbols. For example, $\sqrt{-1}$ and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ are not rational numbers, that is, i and e are not rational numbers.

5. *Write out integers as words when they are used as adjectives and when the numbers are relatively small or easy to describe in words. Write out numbers numerically when they specify the value of something.*

   There are exactly two groups of order 4.
   Fifty million Frenchmen can't be wrong.
   There are one million positive integers less than 1,000,001.

6. *Don't mix words and symbols improperly.*
   Instead of writing:

   $$\text{Every integer} \geq 2 \text{ is a prime or is composite.}$$

   it is preferable to write:

   $$\text{Every integer exceeding 1 is a prime or is composite.}$$

   or

   $$\text{If } n \geq 2 \text{ is an integer, then } n \text{ is prime or composite.}$$

   Although

   $$\text{Since } (x - 2)(x - 3) = 0, \text{ it follows that } x = 2 \text{ or } 3.$$

   sounds correct, it is not written correctly. It should be:

   $$\text{Since } (x - 2)(x - 3) = 0, \text{ it follows that } x = 2 \text{ or } x = 3.$$

7. *Avoid using a symbol in the statement of a theorem when it's not needed.*
   Don't write:

   $$\text{Theorem } \textit{Every bijective function } f \textit{ has an inverse.}$$

   Delete "f". It serves no useful purpose. The theorem does not depend on what the function is called. A symbol should not be used in the statement of a

theorem (or in its proof) exactly once. If it is useful to have a name for an arbitrary bijective function in the proof (as it probably will be), then "f" can be introduced there.

8. *Explain the meaning of every symbol that you introduce.*
Although what you intended may seem clear, don't assume this. For example, if you write $n = 2k + 1$ and k has never appeared before, then say that k is an integer (if indeed k *is* an integer).

9. *Use "frozen symbols" properly.*
If m and n are typically used for integers (as they probably are), then don't use them for real numbers. If A and B are used for sets, then don't use these as typical elements of a set. If f is used for a function, then don't use this as an integer. Write symbols that the reader would expect. To do otherwise could very well confuse the reader.

10. *Use consistent symbols.*
Unless there is some special reason to the contrary, use symbols that "fit" together. Otherwise, it is distracting to the reader.
Instead of writing:

> If x and y are even integers, then $x = 2a$ and $y = 2r$ for some integers a and r.

replace $2r$ by $2b$ (where then, of course, we write "for some integers a and b"). On the other hand, you might prefer to write $x = 2r$ and $y = 2s$.

Writing Mathematical Expressions

There will be numerous occasions when you will want to write mathematical expressions in your assignment, such as algebraic equations, inequalities, and formulas. If these expressions are relatively short, then they should probably be written within the text of the proof or discussion. (We'll explain this in a moment.) If the expressions are rather lengthy, then it is probably preferred for these expressions to be written as "displays".

For example, suppose that we are discussing the Binomial Theorem. (It's not important if you don't recall what this theorem is.) It's possible that what we are writing includes the following passage:

> For example, if we expand $(a + b)^4$, then we obtain $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

It would probably be better to write the expansion of $(a + b)^4$ as a **display**, where the mathematical expression is placed on a line or lines by itself and is centered. This is illustrated below.

> For example, if we expand $(a + b)^4$, then we obtain
>
> $$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

If several mathematical expressions are linked by equal signs and inequality symbols, then we would almost certainly write this as a display. For example, suppose that we wanted to write $n^3 + 3n^2 - n + 4$ in terms of k, where $n = 2k + 1$. A possible display is given next:

Since $n = 2k + 1$, it follows that

$$\begin{aligned} n^3 + 3n^2 - n + 4 &= (2k + 1)^3 + 3(2k + 1)^2 - (2k + 1) + 4 \\ &= (8k^3 + 12k^2 + 6k + 1) + 3(4k^2 + 4k + 1) - 2k - 1 + 4 \\ &= 8k^3 + 24k^2 + 16k + 7 = 8k^3 + 24k^2 + 16k + 6 + 1 \\ &= 2(4k^3 + 12k^2 + 8k + 3) + 1. \end{aligned}$$

Notice how the equal signs are lined up. (We wrote two equal signs on one line since that line would have contained very little material otherwise, as well as to balance the lengths of the lines better.)

Let's return to the expression $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ for the moment. If we were to write this expression in the text of a paragraph (as we are doing) and if we find it necessary to write portions of this expression on two separate lines, then this expression should be broken so that the first line ends with an operation or comparative symbol such as $+$, $-$, $<$, $\geq$, or $=$. In other words, the second line should *not* begin with one of these symbols. The reason for doing this is that ending the line with one of these symbols alerts the reader that more will follow; otherwise, the reader might conclude (incorrectly) that the portion of the expression appearing on the first line is the entire expression. Consequently, write

> For example, if we expand $(a + b)^4$, then we obtain $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

and not

> For example, if we expand $(a + b)^4$, then we obtain $(a + b)^4 = a^4 + 4a^3b$
> $+ 6a^2b^2 + 4ab^3 + b^4$.

If there is an occasion to refer to an expression that has already appeared, then this expression should have been written as a display and labeled as follows:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \tag{1}$$

Then we can simply refer to expression (1) rather than writing it out each time.

Common Words and Phrases in Mathematics

Some words and phrases appear so often in mathematical writing that it is useful to discuss them.

1. *I We One Let's*

> I will now show that n is even.
> We will now show that n is even.
> One now shows that n is even.
> Let's now show that n is even.

These are four ways that we might write a sentence in a proof. Which of these sounds the best to you? It is not considered good practice to use "I"

unless you are writing a personal account of something. Otherwise, "I" sounds egotistical and can be annoying. Using "one" is often awkward. Using "we" is standard practice in mathematics. This word also brings the reader into the discussion with the author and gives the impression of a team effort. The word "let's" accomplishes this as well but is much less formal. There is a danger of being *too* casual, however. In general, your writing should be balanced, maintaining a professional style. Of course, there is the possibility of avoiding all of these words:

The integer n is now shown to be even.

2. *Clearly Obviously Of course Certainly*

These and similar words can turn a reader off if what's written is not clear to the reader. It can give the impression that the author is putting the reader down. These words should be used sparingly and with caution. If they *are* used, then at least be certain that what you say is true. There is also the possibility that the writer (a student?) has a lack of understanding of the mathematics or is not being careful and is using these words as a cover-up. This gives us even more reasons to avoid these words.

3. *Any Each Every*

This statement is true for any integer n.

Does this mean that the statement is true for *some* integer n or *all* integers n? Since the word "any" can be vague, perhaps it is best to avoid it. If by "any", we mean "each" or "every", then use one of these two words instead. When the word "any" is encountered, most of the time the author means "each" or "every".

4. *Since $\cdots$, then $\cdots$*

A number of people connect these two words. You should use either "If $\cdots$, then $\cdots$" (should this be the intended meaning) or "Since $\cdots$, it follows that $\cdots$" or, possibly, "Since $\cdots$, we have $\cdots$." For example, it is correct to write

If n^2 is even, then n is even.

or

Since n^2 is even, it follows that n is even.

or perhaps

Since n^2 is even, n is even.

but avoid

Since n^2 is even, then n is even.

In this context, the word "since" can be replaced by "because".

5. *Therefore Thus Hence Consequently So It follows that This implies that*

This is tricky. Mathematicians cannot survive without these words. Often within a proof, we proceed from something we've just learned to something else that can be concluded from it. There are many (many!) openings to sentences that attempt to say this. Although each of the words or phrases

Therefore Thus Hence Consequently So It follows that This implies that

is suitable, it is good to introduce some variety into your writing and not use the same words or phrases any more often than necessary.

6. *That Which*

These words are often confused with each other. Sometimes they are interchangeable; more often they are not.

The solution to the equation is the number less than 5 that is positive. (2)

The solution to the equation is the number less than 5 which is positive. (3)

Which of these two sentences is correct? The simple answer is: Both are correct—or, at least, both might be correct.

For example, sentence (2) could be the response to the question: Which of the numbers -2, 3, and 5 is the solution of the equation? Sentence (3) could be the response to the question: Which of the numbers 4.9 and 5.0 is the solution of the equation?

The word "that" introduces a *restrictive clause* and, as such, the clause is essential to the meaning of the sentence. That is, if sentence (2) were written only as "The solution to the equation is the number less than 5.", then the entire meaning is changed. Indeed, we no longer know what the solution of the equation is.

On the other hand, the word "which" does *not* introduce a restrictive clause. It introduces a nonrestrictive (or parenthetical) clause. A *nonrestrictive clause* only provides additional information that is not essential to the meaning of the sentence. In sentence (3) the phrase "which is positive" simply provides more information about the solution. This clause may have been added because the solution to an earlier equation is negative. In fact, it would be more appropriate to add a comma:

The solution to the equation is the number less than 5, which is positive.

For another illustration, consider the following two statements:

I always keep the math text that I like with me. (4)

I always keep the math text which I like with me. (5)

What is the difference between these two sentences? In (4), the writer of the sentence clearly has more than one math text and is referring to the one that he/she likes. In (5), the writer has only one math text and is providing the added information that he/she likes it. The nonrestrictive clause in (5) should be set off by commas:

I always keep the math text, which I like, with me.

A possible guideline to follow as you seek to determine whether "that" or "which" is the proper word to use is to ask yourself: Does it sound right if it reads "which, by the

way,"? In general, "that" is normally used considerably more often than "which". Hence the advice here is: Beware of wicked which's!

While we are discussing the word "that", we mention that the words "assume" and "suppose" often precede restrictive clauses and, as such, the word "that" should immediately follow one of these words. Omitting "that" leaves us with an *implied* "that". Many mathematicians prefer to include it rather than omit it.

In other words, instead of writing:

Assume N is a normal subgroup.

many would write

Assume that N is a normal subgroup.

Some Closing Comments about Writing

1. Use good English. Write in complete sentences, ending each sentence with a period (or a question mark when appropriate) and capitalize the first word of each sentence. (Remember: No sentence begins with a symbol!)
2. Capitalize theorem and lemma as in Theorem 1 and Lemma 4.
3. Many mathematicians do not hyphenate words containing the prefix "non", such as

   nonempty, nonnegative, nondecreasing, nonzero.
4. Many words that occur often in mathematical writing are commonly misspelled. Among these are:

   commutative (independent of order)
   complement (supplement, balance, remainder)
   consistent (conforming, agreeing)
   feasible (suitable, attainable)
   its (possessive, not "it is")
   occurrence (incident)
   parallel (does not intersect)
   preceding (foregoing, former)
   principle (postulate, regulation, rule)
   proceed (continue, move on)

   and, of course,

   corollary, lemma, theorem.
5. There are many pairs of words that fit together in mathematics (while interchanging words among the pairs do not). For example,

   We ask questions.
   We pose problems.
   We present solutions.
   We prove theorems.
   We solve problems.
   and
   We conclude this chapter.

1 Sets

In this initial chapter, you will be introduced to, or more than likely be reminded of, a fundamental idea that occurs throughout mathematics: sets. Indeed, a set is an object from which every mathematical structure is constructed (as we will often see in the succeeding chapters). Although there is a formal subject called set theory in which the properties of sets follow from a number of axioms, this is neither our interest nor our need. It is our desire to keep the discussion of sets informal without sacrificing clarity. It is almost a certainty that portions of this chapter will be familiar to you. Nevertheless, it is important that we understand what is meant by a set, how mathematicians describe sets, the notation used with sets, and several concepts that involve sets.

You've been experiencing sets all your life. In fact, all of the following are examples of sets: the members of a sports team, the items on a shopping list, the integers. As a small child, you learned to say the alphabet. When you did this, you were actually listing the letters that make up the set we call the alphabet. A set is a collection of objects. The objects that make up a set are called its **elements** (or **members**). The elements of a softball team are the players, while the elements of the alphabet are letters.

It is customary to use capital (uppercase) letters (such as A, B, C, S, X, Y) to designate sets and lowercase letters (for example, a, b, c, s, x, y) to represent elements of sets. If a is an element of the set A, then we write $a \in A$; if a does not belong to A, then we write $a \notin A$.

1.1 Describing a Set

There will be many occasions when we (or you) will need to describe a set. The most important requirement when describing a set is that the description makes it clear precisely which elements belong to the set.

If a set consists of a small number of elements, then this set can be described by explicitly listing its elements between braces (curly brackets) where the elements are separated by commas. Thus $S = \{1, 2, 3\}$ is a set, consisting of the numbers 1, 2, and 3. The order in which the elements are listed doesn't matter. Thus the set S just mentioned could be written as $S = \{3, 2, 1\}$ or $S = \{2, 1, 3\}$, for example. They describe the same set. If a set T consists of the first five letters of the alphabet, then it is not essential that we

write $T = \{a, b, c, d, e\}$, that is, the elements of T need not be listed in alphabetical order. On the other hand, listing the elements of T in any other order may create unnecessary confusion.

The set A of all people who signed the Declaration of Independence and later became president of the United States is $A = \{\text{John Adams, Thomas Jefferson}\}$ and the set B of all positive even integers less than 20 is $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$. Some sets contain too many elements to be listed this way. Perhaps even the set B just given contains too many elements to describe in this manner. In such cases, the ellipsis or "three dot notation" is often helpful. For example, $X = \{1, 3, 5, \ldots, 49\}$ is the set of all positive odd integers less than 50, while $Y = \{2, 4, 6, \ldots\}$ is the set of all positive even integers. The three dots mean "and so on" for Y and "and so on up to" for X. A set need not contain any elements. Although it may seem peculiar to consider sets without elements, these kinds of sets occur surprisingly often and in a variety of settings. For example, if S is the set of real number solutions of the equation $x^2 + 1 = 0$, then S contains no elements. There is only one set that contains no elements, and it is called the **empty set** (or sometimes the **null set** or **void set**). The empty set is denoted by $\emptyset$. We also write $\emptyset = \{\ \}$. In addition to the example given above, the set of all real numbers x such that $x^2 < 0$ is also empty.

The elements of a set may in fact be sets themselves. The symbol ♦ below indicates the conclusion of an example.

Example 1.1 *The set $S = \{1, 2, \{1, 2\}, \emptyset\}$ consists of four elements, two of which are sets, namely, $\{1, 2\}$ and $\emptyset$. If we write $C = \{1, 2\}$, then we can also write $S = \{1, 2, C, \emptyset\}$.*

The set $T = \{0, \{1, 2, 3\}, 4, 5\}$ also has four elements, namely, the three integers 0, 4, and 5, and the set $\{1, 2, 3\}$. Even though $2 \in \{1, 2, 3\}$, the number 2 is not an element of T; that is, $2 \notin T$. ♦

Often sets consist of those elements satisfying some condition or possessing some specified property. In this case, we can define such a set as $S = \{x : p(x)\}$, where, by this, we mean that S consists of all those elements x satisfying some condition $p(x)$ concerning x. Some mathematicians write $S = \{x \mid p(x)\}$; that is, some prefer to write a vertical line rather than a colon (which, by itself here, is understood to mean "such that"). For example, if we are studying real number solutions of equations, then

$$S = \{x : (x - 1)(x + 2)(x + 3) = 0\}$$

is the set of all real numbers x such that $(x - 1)(x + 2)(x + 3) = 0$, that is, S is the solution set of the equation $(x - 1)(x + 2)(x + 3) = 0$. We could have written $S = \{1, -2, -3\}$; however, even though this way of expressing S is apparently simpler, it does not tell us that we are interested in the solutions of an equation. The **absolute value** $|x|$ of a real number x is x if $x \geq 0$; while $|x| = -x$ if $x < 0$. Therefore,

$$T = \{x : |x| = 2\}$$

is the set of all real numbers having absolute value 2; that is, $T = \{2, -2\}$. In the sets S and T that we have just described, we understand that "x" refers to a real number x. If

there is a possibility that this wouldn't be clear to the reader, then we should specifically say that x is a real number. We'll say more about this soon. The set

$$P = \{x : x \text{ has been a president of the United States}\}$$

describes, rather obviously, all those individuals who have been president of the United States. So Abraham Lincoln belongs to P, but Benjamin Franklin does not.

Example 1.2 *Let $A = \{3, 4, 5, \ldots, 20\}$. If B denotes the set consisting of those elements of A that are less than 8, then we can write*

$$B = \{x \in A : x < 8\} = \{3, 4, 5, 6, 7\}.$$ ♦

Some sets are encountered so often that they are given special notation. We use $\mathbf{N}$ to denote the set of all **positive integers** (or **natural numbers**); that is, $\mathbf{N} = \{1, 2, 3, \ldots\}$. The set of all integers (positive, negative, and zero) is denoted by $\mathbf{Z}$. So $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. With the aid of the notation we've just introduced, we can now describe the set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers by

$$E = \{y : y \text{ is an even integer}\} \text{ or } E = \{2x : x \text{ is an integer}\}, \text{ or as}$$

$$E = \{y : y = 2x \text{ for some } x \in \mathbf{Z}\} \text{ or } E = \{2x : x \in \mathbf{Z}\}.$$

Also,

$$S = \{x^2 : x \text{ is an integer}\} = \{x^2 : x \in \mathbf{Z}\} = \{0, 1, 4, 9, \ldots\}$$

describes the set of squares of integers.

The set of **real numbers** is denoted by $\mathbf{R}$ and the set of positive real numbers is denoted by $\mathbf{R}^+$. A real number that can be expressed in the form $\frac{m}{n}$, where $m, n \in \mathbf{Z}$ and $n \neq 0$, is called a **rational number**. For example, $\frac{2}{3}$, $\frac{-5}{11}$, $17 = \frac{17}{1}$, and $\frac{4}{6}$ are rational numbers. The set of all rational numbers is denoted by $\mathbf{Q}$. Of course, $\frac{4}{6} = \frac{2}{3}$. A real number that is not rational is called **irrational.** The real numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{2}$, π, and e are known to be irrational; that is, none of these numbers can be expressed as the ratio of two integers. It is also known that the real numbers with infinite nonrepeating decimal expansions are precisely the irrational numbers. There is no common symbol to denote the set of irrational numbers. We will use $\mathbf{I}$ for the set of all irrational numbers. Thus, $\sqrt{2} \in \mathbf{R}$ and $\sqrt{2} \notin \mathbf{Q}$; so $\sqrt{2} \in \mathbf{I}$.

For a set S, we write $|S|$ to denote the number of elements in S. The number $|S|$ is also referred to as the **cardinal number** or **cardinality** of S. If $A = \{1, 2\}$ and $B = \{1, 2, \{1, 2\}, \emptyset\}$, then $|A| = 2$ and $|B| = 4$. Also, $|\emptyset| = 0$. Although the notation is identical for the cardinality of a set and the absolute value of a real number, we should have no trouble distinguishing between the two. A set S is **finite** if $|S| = n$ for some nonnegative integer n. A set S is **infinite** if it is not finite. For the present, we will use the notation $|S|$ only for finite sets S. In Chapter 10, we will discuss the cardinality of infinite sets.

Let's now consider a few examples of sets that are defined in terms of the special sets we have just described.

Example 1.3 *Let* $D = \{n \in \mathbf{N} : n \leq 9\}$, $E = \{x \in \mathbf{Q} : x \leq 9\}$, $H = \{x \in \mathbf{R} : x^2 - 2 = 0\}$, *and* $J = \{x \in \mathbf{Q} : x^2 - 2 = 0\}$.

(a) Describe the set D by listing its elements.
(b) Give an example of three elements that belong to E but do not belong to D.
(c) Describe the set H by listing its elements.
(d) Describe the set J in another manner.
(e) Determine the cardinality of each set D, H, and J.

Solution

(a) $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
(b) $\frac{7}{5}, 0, -3$.
(c) $H = \{\sqrt{2}, -\sqrt{2}\}$.
(d) $J = \emptyset$.
(e) $|D| = 9$, $|H| = 2$, and $|J| = 0$. ♦

A **complex number** is a number of the form $a + bi$, where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$. A complex number $a + bi$, where $b = 0$, can be expressed as $a + 0i$ or, more simply, as a. Hence $a + 0i = a$ is a real number. Thus every real number is a complex number. Let $\mathbf{C}$ denote the set of complex numbers. If $K = \{x \in \mathbf{C} : x^2 + 1 = 0\}$, then $K = \{i, -i\}$. Of course, if $L = \{x \in \mathbf{R} : x^2 + 1 = 0\}$, then $L = \emptyset$. You might recall that the sum of two complex numbers $a + bi$ and $c + di$ is $(a + c) + (b + d)i$, while their product is

$$(a + bi) \cdot (c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i.$$

The special sets that we've just described are now summarized as follows:

symbol	for the set of
$\mathbf{N}$	natural numbers (positive integers)
$\mathbf{Z}$	integers
$\mathbf{Q}$	rational numbers
$\mathbf{I}$	irrational numbers
$\mathbf{R}$	real numbers
$\mathbf{C}$	complex numbers

1.2 Subsets

A set A is called a **subset** of a set B if every element of A also belongs to B. If A is a subset of B, then we write $A \subseteq B$. If A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. This property of subsets might remind you of the property of real numbers where if $a, b, c \in \mathbf{R}$ such that if $a \leq b$ and $b \leq c$, then $a \leq c$. For the sets $X = \{1, 3, 6\}$ and $Y = \{1, 2, 3, 5, 6\}$, we have $X \subseteq Y$. Also, $\mathbf{N} \subseteq \mathbf{Z}$ and $\mathbf{Q} \subseteq \mathbf{R}$. In addition, $\mathbf{R} \subseteq \mathbf{C}$. Since $\mathbf{Q} \subseteq \mathbf{R}$ and $\mathbf{R} \subseteq \mathbf{C}$, it therefore follows that $\mathbf{Q} \subseteq \mathbf{C}$. Moreover, every set is a subset of itself.

Example 1.4 *Find two sets* A *and* B *such that* A *is both an element and a subset of* B.

Solution Suppose that we seek two sets A and B such that $A \in B$ and $A \subseteq B$. Let's start with a simple example for A, say $A = \{1\}$. Since we want $A \in B$, the set B must contain the set $\{1\}$ as one of its elements. On the other hand, we also require that $A \subseteq B$, so every element of A must belong to B. Since 1 is the only element of A, it follows that B must also contain the number 1. A possible choice for B is then $B = \{1, \{1\}\}$, although $B = \{1, 2, \{1\}\}$ would also satisfy the conditions. ♦

If a set C is *not* a subset of a set D, then we write $C \not\subseteq D$. In this case, there must be some element of C that is not an element of D. One consequence of this is that the empty set $\emptyset$ is a subset of every set. If this were not the case, then there must be some set A such that $\emptyset \not\subseteq A$. But this would mean there is some element, say x, in $\emptyset$ that is not in A. However, $\emptyset$ contains no elements. So $\emptyset \subseteq A$ for *every* set A.

Example 1.5 *Let* $S = \{1, \{2\}, \{1, 2\}\}$.

(a) Determine which of the following are elements of S:
$1, \{1\}, 2, \{2\}, \{1, 2\}, \{\{1, 2\}\}$.
(b) Determine which of the following are subsets of S:
$\{1\}, \{2\}, \{1, 2\}, \{\{1\}, 2\}, \{1, \{2\}\}, \{\{1\}, \{2\}\}, \{\{1, 2\}\}$.

Solution

(a) The following are elements of S: 1, $\{2\}$, $\{1, 2\}$.
(b) The following are subsets of S: $\{1\}$, $\{1, \{2\}\}$, $\{\{1, 2\}\}$. ♦

In a typical discussion of sets, we are ordinarily concerned with subsets of some specified set U, called **the universal set**. For example, we may be dealing only with integers, in which case the universal set is $\mathbf{Z}$, or we may be dealing only with real numbers, in which case the universal set is $\mathbf{R}$. On the other hand, the universal set being considered may be neither $\mathbf{Z}$ nor $\mathbf{R}$. Indeed, U may not even be a set of numbers.

Some frequently encountered subsets of $\mathbf{R}$ are the so-called "intervals". For $a, b \in \mathbf{R}$ and $a < b$, the **open interval** (a, b) is the set

$$(a, b) = \{x \in \mathbf{R} : a < x < b\}.$$

Therefore, all of the real numbers $\frac{5}{2}, \sqrt{5}, e, 3, \pi, 4.99$ belong to $(2, 5)$, but none of the real numbers $\sqrt{2}, 1.99, 2, 5$ belong to $(2, 5)$.

For $a, b \in \mathbf{R}$ and $a \leq b$, the **closed interval** $[a, b]$ is the set

$$[a, b] = \{x \in \mathbf{R} : a \leq x \leq b\}.$$

While $2, 5 \notin (2, 5)$, we do have $2, 5 \in [2, 5]$. The "interval" $[a, a]$ is therefore $\{a\}$. Thus, for $a < b$, we have $(a, b) \subseteq [a, b]$. For $a, b \in \mathbf{R}$ and $a < b$, the **half-open** or **half-closed intervals** $[a, b)$ and $(a, b]$ are defined as expected:

$$[a, b) = \{x \in \mathbf{R} : a \leq x < b\} \text{ and } (a, b] = \{x \in \mathbf{R} : a < x \leq b\}.$$

For $a \in \mathbf{R}$, the infinite intervals $(-\infty, a)$, $(-\infty, a]$, (a, ∞), and $[a, \infty)$ are defined as

$$(-\infty, a) = \{x \in \mathbf{R} : x < a\}, \ (-\infty, a] = \{x \in \mathbf{R} : x \leq a\},$$
$$(a, \infty) = \{x \in \mathbf{R} : x > a\}, \quad [a, \infty) = \{x \in \mathbf{R} : x \geq a\}.$$

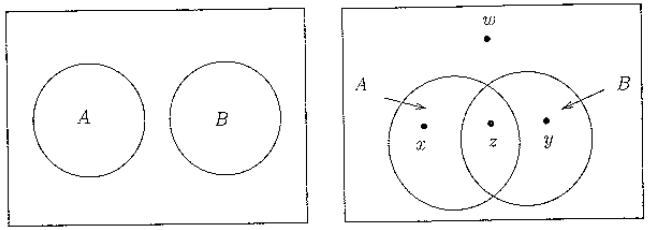

Figure 1.1 Venn diagrams for two sets A and B

The interval $(-\infty, \infty)$ is the set $\mathbf{R}$. Note that the infinity symbols ∞ and $-\infty$ are not real numbers; they are used only to help describe certain intervals. Therefore, $[1, \infty]$, for example, has no meaning.

Two sets A and B are **equal**, indicated by writing $A = B$, if they have exactly the same elements. Another way of saying $A = B$ is that every element of A is in B and every element of B is in A, that is, $A \subseteq B$ and $B \subseteq A$. This fact will be very useful to us in Chapter 4. If $A \neq B$, then there must be some element that belongs to one of A and B but does not belong to the other.

It is often convenient to represent sets by diagrams called **Venn diagrams.** For example, Figure 1.1 shows Venn diagrams for two sets A and B. The diagram on the left represents two sets A and B that have no elements in common, while the diagram on the right is more general. The element x belongs to A but not to B; the element y belongs to B but not to A; the element z belongs to both A and B; and w belongs to neither A nor B. In general, the elements of a set are understood to be those displayed within the region that describes the set. A rectangle in a Venn diagram represents the universal set in this case. Since every element under consideration belongs to the universal set, each element in a Venn diagram lies within the rectangle.

A set A is a **proper subset** of a set B if $A \subseteq B$ but $A \neq B$. If A is a proper subset of B, then we write $A \subset B$. For example, if $S = \{4, 5, 7\}$ and $T = \{3, 4, 5, 6, 7\}$, then $S \subset T$. (Although we write $A \subset B$ to indicate that A is a proper subset of B, it should be mentioned that some prefer to write $A \subsetneq B$ to indicate that A is a proper subset of B. Indeed, there are some who write $A \subset B$, rather than $A \subseteq B$, to indicate that A is a subset of B. We will follow the notation introduced above, however.)

The set consisting of all subsets of a given set A is called the **power set** of A and is denoted by $\mathcal{P}(A)$.

Example 1.6 *For each set A below, determine $\mathcal{P}(A)$. In each case, determine $|A|$ and $|\mathcal{P}(A)|$.*

(a) $A = \emptyset$, (b) $A = \{a, b\}$, (c) $A = \{1, 2, 3\}$.

Solution

(a) $\mathcal{P}(A) = \{\emptyset\}$. In this case, $|A| = 0$ and $|\mathcal{P}(A)| = 1$.

(b) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. In this case, $|A| = 2$ and $|\mathcal{P}(A)| = 4$.

(c) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
In this case, $|A| = 3$ and $|\mathcal{P}(A)| = 8$. ♦

Notice that for each set A in Example 1.6, we have $|\mathcal{P}(A)| = 2^{|A|}$. In fact, if A is any finite set, with n elements say, then $\mathcal{P}(A)$ has 2^n elements; that is,

$$|\mathcal{P}(A)| = 2^{|A|}$$

for every finite set A. (Later we will explain why this is true.)

Example 1.7 *If $C = \{\emptyset, \{\emptyset\}\}$, then*

$$\mathcal{P}(C) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}.$$

It is important to note that no two of the sets $\emptyset$, $\{\emptyset\}$, and $\{\{\emptyset\}\}$ are equal. (An empty box and a box containing an empty box are not the same.) For the set C above, it is therefore correct to write

$$\emptyset \subseteq C, \emptyset \subset C, \emptyset \in C, \{\emptyset\} \subseteq C, \{\emptyset\} \subset C, \{\emptyset\} \in C,$$

as well as

$$\{\{\emptyset\}\} \subseteq C, \{\{\emptyset\}\} \notin C, \{\{\emptyset\}\} \in \mathcal{P}(C).$$ ♦

1.3 Set Operations

Just as there are several ways of combining two integers to produce another integer (addition, subtraction, multiplication, and sometimes division), there are several ways to combine two sets to produce another set. The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements belonging to A or B; that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The use of the word "or" here, and in mathematics in general, allows an element of $A \cup B$ to belong to both A and B. That is, x is in $A \cup B$ if x is in A or x is in B or x is in both A and B. A Venn diagram for $A \cup B$ is shown in Figure 1.2.

Example 1.8 *For the sets $A_1 = \{2, 5, 7, 8\}$, $A_2 = \{1, 3, 5\}$, and $A_3 = \{2, 4, 6, 8\}$, we have*

$$A_1 \cup A_2 = \{1, 2, 3, 5, 7, 8\},$$
$$A_1 \cup A_3 = \{2, 4, 5, 6, 7, 8\},$$
$$A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 8\}.$$

Also, $\mathbf{N} \cup \mathbf{Z} = \mathbf{Z}$ and $\mathbf{Q} \cup \mathbf{I} = \mathbf{R}$. ♦

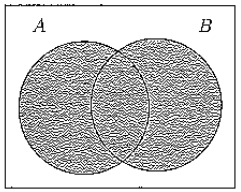

Figure 1.2 A Venn diagram for $A \cup B$

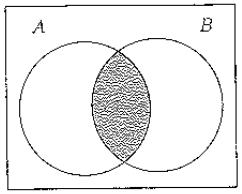

Figure 1.3 A Venn diagram for $A \cap B$

The **intersection** of two sets A and B is the set of all elements belonging to both A and B. The intersection of A and B is denoted by $A \cap B$. In symbols,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A Venn diagram for $A \cap B$ is shown in Figure 1.3.

Example 1.9 *For the sets A_1, A_2, and A_3 described in Example 1.8,*

$$A_1 \cap A_2 = \{5\},\ A_1 \cap A_3 = \{2, 8\}, \text{ and } A_2 \cap A_3 = \emptyset.$$

Also, $\mathbf{N} \cap \mathbf{Z} = \mathbf{N}$ *and* $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$. ♦

For every two sets A and B, it follows that

$$A \cap B \subseteq A \cup B.$$

If two sets A and B have no elements in common, then $A \cap B = \emptyset$ and A and B are said to be **disjoint**. Consequently, the sets A_2 and A_3 described in Example 1.8 are disjoint; however, A_1 and A_3 are not disjoint since 2 and 8 belong to both sets. Also, $\mathbf{Q}$ and $\mathbf{I}$ are disjoint.

The **difference** $A - B$ of two sets A and B (also written as $A \setminus B$ by some mathematicians) is defined as

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

A Venn diagram for $A - B$ is shown in Figure 1.4.

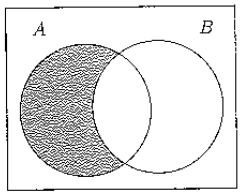

Figure 1.4 A Venn diagram for $A - B$

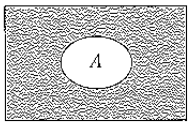

Figure 1.5 A Venn diagram for $\overline{A}$

Example 1.10 *For the sets $A_1 = \{2, 5, 7, 8\}$ and $A_2 = \{1, 3, 5\}$ in Examples 1.8 and 1.9, $A_1 - A_2 = \{2, 7, 8\}$ and $A_2 - A_1 = \{1, 3\}$. Furthermore,* $\mathbf{R} - \mathbf{Q} = \mathbf{I}$. ♦

Suppose that we are considering a certain universal set U; that is, all sets being discussed are subsets of U. For a set A, its **complement** is

$$\overline{A} = U - A = \{x : x \in U \text{ and } x \notin A\}.$$

If $U = \mathbf{Z}$, then $\overline{\mathbf{N}} = \{0, -1, -2, \ldots\}$; while if $U = \mathbf{R}$, then $\overline{\mathbf{Q}} = \mathbf{I}$. A Venn diagram for $\overline{A}$ is shown in Figure 1.5.

The set difference $A - B$ is sometimes called the **relative complement** of B in A. Indeed, from the definition, $A - B = \{x : x \in A \text{ and } x \notin B\}$. The set $A - B$ can also be expressed in terms of complements, namely, $A - B = A \cap \overline{B}$. This fact will be established later.

Example 1.11 *Let $U = \{1, 2, \ldots, 10\}$ be the universal set, $A = \{2, 3, 5, 7\}$, and $B = \{2, 4, 6, 8, 10\}$. Determine each of the following:*

(a) $\overline{B}$, (b) $A - B$, (c) $A \cap \overline{B}$, (d) $\overline{\overline{B}}$.

Solution

(a) $\overline{B} = \{1, 3, 5, 7, 9\}$.

(b) $A - B = \{3, 5, 7\}$.

(c) $A \cap \overline{B} = \{3, 5, 7\} = A - B$.

(d) $\overline{\overline{B}} = B = \{2, 4, 6, 8, 10\}$. ♦

Example 1.12 *Let $A = \{0, \{0\}, \{0, \{0\}\}\}$.*

(a) Determine which of the following are elements of A: 0, $\{0\}$, $\{\{0\}\}$.

(b) Determine $|A|$.

(c) Determine which of the following are subsets of A: 0, $\{0\}$, $\{\{0\}\}$.

For (d)–(i), determine the indicated sets.

(d) $\{0\} \cap A$

(e) $\{\{0\}\} \cap A$

(f) $\{\{\{0\}\}\} \cap A$

(g) $\{0\} \cup A$

(h) $\{\{0\}\} \cup A$

(i) $\{\{\{0\}\}\} \cup A$.

Solution

(a) While 0 and $\{0\}$ are elements of A, $\{\{0\}\}$ is not an element of A.

(b) The set A has three elements: 0, $\{0\}$, $\{0, \{0\}\}$. Therefore, $|A| = 3$.

(c) The integer 0 is not a set and so cannot be a subset of A (or a subset of any other set). Since $0 \in A$ and $\{0\} \in A$, it follows that $\{0\} \subseteq A$ and $\{\{0\}\} \subseteq A$.

(d) Since 0 is the only element that belongs to both $\{0\}$ and A, it follows that $\{0\} \cap A = \{0\}$.

(e) Since $\{0\}$ is the only element that belongs to both $\{\{0\}\}$ and A, it follows that $\{\{0\}\} \cap A = \{\{0\}\}$.

(f) Since $\{\{0\}\}$ is not an element of A, it follows that $\{\{\{0\}\}\}$ and A are disjoint sets and so $\{\{\{0\}\}\} \cap A = \emptyset$.

(g) Since $0 \in A$, it follows that $\{0\} \cup A = A$.

(h) Since $\{0\} \in A$, it follows that $\{\{0\}\} \cup A = A$.

(i) Since $\{\{0\}\} \notin A$, it follows that $\{\{\{0\}\}\} \cup A = \{0, \{0\}, \{\{0\}\}\}$. ♦

1.4 Indexed Collections of Sets

We will often encounter situations where more than two sets are combined using the set operations described above. In the case of three sets A, B, and C, the standard Venn diagram is shown in Figure 1.6.

The union $A \cup B \cup C$ is defined as

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$

Thus, in order for an element to belong to $A \cup B \cup C$, the element must belong to at least one of the sets A, B, and C. Because it is often useful to consider the union of several sets, additional notation is needed. The union of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is denoted by $A_1 \cup A_2 \cup \cdots \cup A_n$ or $\bigcup_{i=1}^{n} A_i$ and is defined as

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for some } i, 1 \leq i \leq n\}.$$

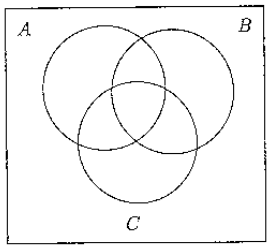

Figure 1.6 A Venn diagram for three sets

Thus, for an element a to belong to $\bigcup_{i=1}^{n} A_i$, it is necessary that a belongs to at least one of the sets $A_1, A_2, \ldots, A_n$.

Example 1.13 *Let $B_1 = \{1, 2\}$, $B_2 = \{2, 3\}, \ldots, B_{10} = \{10, 11\}$; that is, $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine each of the following:*

(a) $\bigcup_{i=1}^{5} B_i$. (b) $\bigcup_{i=1}^{10} B_i$. (c) $\bigcup_{i=3}^{7} B_i$. (d) $\bigcup_{i=j}^{k} B_i$, where $1 \leq j \leq k \leq 10$.

Solution

(a) $\bigcup_{i=1}^{5} B_i = \{1, 2, \ldots, 6\}$. (b) $\bigcup_{i=1}^{10} B_i = \{1, 2, \ldots, 11\}$.

(c) $\bigcup_{i=3}^{7} B_i = \{3, 4, \ldots, 8\}$. (d) $\bigcup_{i=j}^{k} B_i = \{j, j+1, \ldots, k+1\}$. ♦

We are often interested in the intersection of several sets as well. The intersection of the $n \geq 2$ sets $A_1, A_2, \ldots, A_n$ is expressed as $A_1 \cap A_2 \cap \cdots \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ and is defined by

$$\bigcap_{i=1}^{n} A_i = \{x : x \in A_i \text{ for every } i, 1 \leq i \leq n\}.$$

The next example concerns the sets mentioned in Example 1.13.

Example 1.14 *Let $B_i = \{i, i+1\}$ for $i = 1, 2, \ldots, 10$. Determine the following:*

(a) $\bigcap_{i=1}^{10} B_i$. (b) $B_i \cap B_{i+1}$. (c) $\bigcap_{i=j}^{j+1} B_i$, where $1 \leq j < 10$.

(d) $\bigcap_{i=j}^{k} B_i$ where $1 \leq j < k \leq 10$.

Solution

(a) $\bigcap_{i=1}^{10} B_i = \emptyset$. (b) $B_i \cap B_{i+1} = \{i+1\}$. (c) $\bigcap_{i=j}^{j+1} B_i = \{j+1\}$.

(d) $\bigcap_{i=j}^{k} B_i = \{j+1\}$ if $k = j+1$; while $\bigcap_{i=j}^{k} B_i = \emptyset$ if $k > j+1$. ♦

There are instances when the union or intersection of a collection of sets cannot be described conveniently (or perhaps at all) in the manner mentioned above. For this reason, we introduce a (nonempty) set I, called an **index set**, which is used as a mechanism for selecting those sets we want to consider. For example, for an index set I, suppose that there is a set S_α for each $\alpha \in I$. We write $\{S_\alpha\}_{\alpha \in I}$ to describe the collection of all sets S_α, where $\alpha \in I$. Such a collection is called an **indexed collection of sets**. We define the union of the sets in $\{S_\alpha\}_{\alpha \in I}$ by

$$\bigcup_{\alpha \in I} S_\alpha = \{x : x \in S_\alpha \text{ for some } \alpha \in I\},$$

and the intersection of these sets by

$$\bigcap_{\alpha\in I} S_\alpha = \{x : x \in S_\alpha \text{ for all } \alpha \in I\}.$$

Hence an element a belongs to $\bigcup_{\alpha\in I} S_\alpha$ if a belongs to at least one of the sets in the collection $\{S_\alpha\}_{\alpha\in I}$, while a belongs to $\bigcap_{\alpha\in I} S_\alpha$ if a belongs to every set in the collection $\{S_\alpha\}_{\alpha\in I}$. We refer to $\bigcup_{\alpha\in I} S_\alpha$ as the union of the collection $\{S_\alpha\}_{\alpha\in I}$ and $\bigcap_{\alpha\in I} S_\alpha$ as the intersection of the collection $\{S_\alpha\}_{\alpha\in I}$. Just as there is nothing special about our choice of i in $\bigcup_{i=1}^{n} A_i$ (that is, we could just as well describe this set by $\bigcup_{j=1}^{n} A_j$, say), there is nothing special about α in $\bigcup_{\alpha\in I} S_\alpha$. We could also describe this set by $\bigcup_{x\in I} S_x$. The variables i and α above are "dummy variables" and any appropriate symbol could be used. Indeed, we could write J or some other symbol for an index set.

Example 1.15 *For $n \in \mathbf{N}$, define $S_n = \{n, 2n\}$. For example, $S_1 = \{1, 2\}$, $S_2 = \{2, 4\}$, and $S_4 = \{4, 8\}$. Then $S_1 \cup S_2 \cup S_4 = \{1, 2, 4, 8\}$. We can also describe this set by means of an index set. If we let $I = \{1, 2, 4\}$, then*

$$\bigcup_{\alpha\in I} S_\alpha = S_1 \cup S_2 \cup S_4.$$ ♦

Example 1.16 *For each $n \in \mathbf{N}$, define A_n to be the closed interval $[-\frac{1}{n}, \frac{1}{n}]$ of real numbers; that is,*

$$A_n = \left\{x \in \mathbf{R} : -\frac{1}{n} \le x \le \frac{1}{n}\right\}.$$

So $A_1 = [-1, 1]$, $A_2 = [-\frac{1}{2}, \frac{1}{2}]$, $A_3 = [-\frac{1}{3}, \frac{1}{3}]$, and so on. We have now defined the sets $A_1, A_2, A_3, \ldots$. The union of these sets can be written as $A_1 \cup A_2 \cup A_3 \cup \cdots$ or $\bigcup_{i=1}^{\infty} A_i$. Using $\mathbf{N}$ as an index set, we can also write this union as $\bigcup_{n\in\mathbf{N}} A_n$. Since $A_n \subseteq A_1 = [-1, 1]$ for every $n \in \mathbf{N}$, it follows that $\bigcup_{n\in\mathbf{N}} A_n = [-1, 1]$. Certainly, $0 \in A_n$ for every $n \in \mathbf{N}$; in fact, $\bigcap_{n\in\mathbf{N}} A_n = \{0\}$. ♦

Example 1.17 *Let A denote the set of the letters of the alphabet, that is, $A = \{a, b, \ldots, z\}$. For $\alpha \in A$, let A_α consist of α and the two letters that follow α. So $A_a = \{a, b, c\}$ and $A_b = \{b, c, d\}$. By A_y, we will mean the set $\{y, z, a\}$ and $A_z = \{z, a, b\}$. Hence $|A_\alpha| = 3$ for every $\alpha \in A$. Therefore, $\bigcup_{\alpha\in A} A_\alpha = A$. Indeed, if*

$$B = \{a, d, g, j, m, p, s, v, y\},$$

then $\bigcup_{\alpha\in B} A_\alpha = A$ as well. On the other hand, if $I = \{p, q, r\}$, then $\bigcup_{\alpha\in I} A_\alpha = \{p, q, r, s, t\}$; while $\bigcap_{\alpha\in I} A_\alpha = \{r\}$. ♦

Example 1.18 *Let $S = \{1, 2, \ldots, 10\}$. Each of the sets*

$$S_1 = \{1, 2, 3, 4\},\ S_2 = \{4, 5, 6, 7, 8\},\ \text{and}\ S_3 = \{7, 8, 9, 10\}$$

is a subset of S. Also, $S_1 \cup S_2 \cup S_3 = S$. This union can be described in a number of ways. Define $I = \{1, 2, 3\}$ and $J = \{S_1, S_2, S_3\}$. Then the union of the three sets belonging to J is precisely $S_1 \cup S_2 \cup S_3$, which can also be written as

$$S = S_1 \cup S_2 \cup S_3 = \bigcup_{i=1}^{3} S_i = \bigcup_{\alpha\in I} S_\alpha = \bigcup_{X\in J} X.$$ ♦

1.5 Partitions of Sets

Recall that two sets are disjoint if their intersection is the empty set. A collection $\mathcal{S}$ of subsets of a set A is called **pairwise disjoint** if every two distinct subsets that belong to $\mathcal{S}$ are disjoint. For example, let $A = \{1, 2, \ldots, 7\}$, $B = \{1, 6\}$, $C = \{2, 5\}$, $D = \{4, 7\}$, and $S = \{B, C, D\}$. Then S is a pairwise disjoint collection of subsets of A since $B \cap C = B \cap D = C \cap D = \emptyset$. On the other hand, let $A' = \{1, 2, 3\}$, $B' = \{1, 2\}$, $C' = \{1, 3\}$, $D' = \{2, 3\}$, and $S' = \{B', C', D'\}$. Although S' is a collection of subsets of A' and $B' \cap C' \cap D' = \emptyset$, the set S' is *not* a pairwise disjoint collection of sets since $B' \cap C' \neq \emptyset$, for example. Indeed, $B' \cap D'$ and $C' \cap D'$ are also nonempty.

We will often have the occasion (especially in Chapter 8) to encounter, for a nonempty set A, a collection $\mathcal{S}$ of pairwise disjoint nonempty subsets of A with the added property that every element of A belongs to some subset in $\mathcal{S}$. Such a collection is called a **partition** of A. A **partition of A** can also be defined as a collection $\mathcal{S}$ of nonempty subsets of A such that every element of A belongs to exactly one subset in $\mathcal{S}$. Furthermore, a partition of A can be defined as a collection $\mathcal{S}$ of subsets of A satisfying the three properties:

(1) $X \neq \emptyset$ for every set $X \in \mathcal{S}$;
(2) for every two sets $X, Y \in \mathcal{S}$, either $X = Y$ or $X \cap Y = \emptyset$;
(3) $\bigcup_{X\in\mathcal{S}} X = A$.

Example 1.19 *Consider the following collections of subsets of the set $A = \{1, 2, 3, 4, 5, 6\}$:*

$$S_1 = \{\{1, 3, 6\}, \{2, 4\}, \{5\}\};$$
$$S_2 = \{\{1, 2, 3\}, \{4\}, \emptyset, \{5, 6\}\};$$
$$S_3 = \{\{1, 2\}, \{3, 4, 5\}, \{5, 6\}\};$$
$$S_4 = \{\{1, 4\}, \{3, 5\}, \{2\}\}.$$

Determine which of these sets are partitions of A.

Solution The set S_1 is a partition of A. The set S_2 is not a partition of A since $\emptyset$ is one of the elements of S_2. The set S_3 is not a partition of A either since the element 5 belongs to two distinct subsets in S_3, namely, $\{3, 4, 5\}$ and $\{5, 6\}$. Finally, S_4 is also not a partition of A because the element 6 belongs to no subset in S_4. ♦

As the word "partition" probably suggests, a partition of a nonempty set A is a division of A into nonempty subsets. The partition S_1 of the set A in Example 1.19 is illustrated in the diagram shown in Figure 1.7.

For example, the set $\mathbf{Z}$ of integers can be partitioned into the set of even integers and the set of odd integers. The set $\mathbf{R}$ of real numbers can be partitioned into the set $\mathbf{R}^+$ of positive real numbers, the set of negative real numbers, and the set $\{0\}$ consisting of

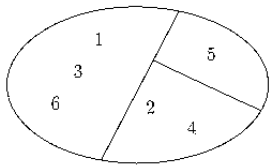

Figure 1.7 A partition of a set

the number 0. In addition, $\mathbf{R}$ can be partitioned into the set $\mathbf{Q}$ of rational numbers and the set $\mathbf{I}$ of irrational numbers.

Example 1.20 *Let* $A = \{1, 2, \ldots, 12\}$.

(a) Give an example of a partition S of A such that $|S| = 5$.

(b) Give an example of a subset T of the partition S in (a) such that $|T| = 3$.

(c) List all those elements B in the partition S in (a) such that $|B| = 2$.

Solution

(a) We are seeking a partition S of A consisting of five subsets. One such example is

$$S = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

(b) We are seeking a subset T of S (given in (a)) consisting of three elements. One such example is

$$T = \{\{1, 2\}, \{3, 4\}, \{7, 8, 9\}\}.$$

(c) We have been asked to list all those elements of S (given in (a)) consisting of two elements of A. These elements are: $\{1, 2\}, \{3, 4\}, \{5, 6\}$. ♦

1.6 Cartesian Products of Sets

We've already mentioned that when a set A is described by listing its elements, the order in which the elements of A are listed doesn't matter. That is, if the set A consists of two elements x and y, then $A = \{x, y\} = \{y, x\}$. When we speak of the **ordered pair** (x, y), however, this is another story. The ordered pair (x, y) is a single element consisting of a pair of elements in which x is the first element (or first coordinate) of the ordered pair (x, y) and y is the second element (or second coordinate). Moreover, for two ordered pairs (x, y) and (w, z) to be equal, that is, $(x, y) = (w, z)$, we must have $x = w$ and $y = z$. So, if $x \neq y$, then $(x, y) \neq (y, x)$.

The **Cartesian product** (or simply the product) $A \times B$ of two sets A and B is the set consisting of all ordered pairs whose first coordinate belongs to A and whose second coordinate belongs to B. In other words,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.21 *If* $A = \{x, y\}$ *and* $B = \{1, 2, 3\}$, *then*

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\},$$

while

$$B \times A = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}.$$

Since, for example, $(x, 1) \in A \times B$ *and* $(x, 1) \notin B \times A$, *these two sets do not contain the same elements; so* $A \times B \neq B \times A$. *Also,*

$$A \times A = \{(x, x), (x, y), (y, x), (y, y)\}$$

and

$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}.$$ ♦

We also note that if $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

The Cartesian product $\mathbf{R} \times \mathbf{R}$ is the set of all points in the Euclidean plane. For example, the graph of the straight line $y = 2x + 3$ is the set

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : \ y = 2x + 3\}.$$

For the sets $A = \{x, y\}$ and $B = \{1, 2, 3\}$ given in Example 1.21, $|A| = 2$ and $|B| = 3$; while $|A \times B| = 6$. Indeed, for all finite sets A and B,

$$|A \times B| = |A| \cdot |B|.$$

Cartesian products will be explored in more detail in Chapter 7.

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Which of the following are sets?

(a) 1, 2, 3

(b) {1, 2}, 3

(c) {{1}, 2}, 3

(d) {1, {2}, 3}

(e) {1, 2, a, b}

1.2. Let $S = \{-2, -1, 0, 1, 2, 3\}$. Describe each of the following sets as $\{x \in S : \ p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{1, 2, 3\}$

(b) $B = \{0, 1, 2, 3\}$

(c) $C = \{-2, -1\}$.

(d) $D = \{-2, 2, 3\}$.

1.3. Determine the cardinality of each of the following sets:

(a) $A = \{1, 2, 3, 4, 5\}$

(b) $B = \{0, 2, 4, \ldots, 20\}$

(c) $C = \{25, 26, 27, \ldots, 75\}$

(d) $D = \{\{1, 2\}, \{1, 2, 3, 4\}\}$
(e) $E = \{\emptyset\}$
(f) $F = \{2, \{2, 3, 4\}\}$

1.4. Write each of the following sets by listing its elements within braces.
(a) $A = \{n \in \mathbf{Z} : -4 < n \leq 4\}$
(b) $B = \{n \in \mathbf{Z} : n^2 < 5\}$
(c) $C = \{n \in \mathbf{N} : n^3 < 100\}$
(d) $D = \{x \in \mathbf{R} : x^2 - x = 0\}$
(e) $E = \{x \in \mathbf{R} : x^2 + 1 = 0\}$

1.5. Write each of the following sets in the form $\{x \in \mathbf{Z} : p(x)\}$, where $p(x)$ is a property concerning x.
(a) $A = \{-1, -2, -3, \ldots\}$
(b) $B = \{-3, -2, \ldots, 3\}$
(c) $C = \{-2, -1, 1, 2\}$

1.6. The set $E = \{2x : x \in \mathbf{Z}\}$ can be described by listing its elements, namely $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. List the elements of the following sets in a similar manner.
(a) $A = \{2x + 1 : x \in \mathbf{Z}\}$
(b) $B = \{4n : n \in \mathbf{Z}\}$
(c) $C = \{3q + 1 : q \in \mathbf{Z}\}$

1.7. The set $E = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers can be described by means of a defining condition by $E = \{y = 2x : x \in \mathbf{Z}\} = \{2x : x \in \mathbf{Z}\}$. Describe the following sets in a similar manner.
(a) $A = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$
(b) $B = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$
(c) $C = \{1, 8, 27, 64, 125, \ldots\}$

Section 1.2: Subsets

1.8. Give examples of three sets A, B, and C such that
(a) $A \subseteq B \subset C$.
(b) $A \in B$, $B \in C$, and $A \notin C$.
(c) $A \in B$ and $A \subset C$.

1.9. Let (a, b) be an open interval of real numbers and let $c \in (a, b)$. Describe an open interval I centered at c such that $I \subseteq (a, b)$.

1.10. Which of the following sets are equal?
$A = \{n \in \mathbf{Z} : |n| < 2\}$ $\quad D = \{n \in \mathbf{Z} : n^2 \leq 1\}$
$B = \{n \in \mathbf{Z} : n^3 = n\}$ $\quad E = \{-1, 0, 1\}$
$C = \{n \in \mathbf{Z} : n^2 \leq n\}$

1.11. For a universal set $U = \{1, 2, \ldots, 8\}$ and two sets $A = \{1, 3, 4, 7\}$ and $B = \{4, 5, 8\}$, draw a Venn diagram that represents these sets.

1.12. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for
(a) $A = \{1, 2\}$.
(b) $A = \{\emptyset, 1, \{a\}\}$.

1.13. Find $\mathcal{P}(A)$ for $A = \{0, \{0\}\}$.

1.14. Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

1.15. Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{0, \emptyset, \{\emptyset\}\}$.

1.16. Give an example of a set S such that
(a) $S \subseteq \mathcal{P}(\mathbf{N})$
(b) $S \in \mathcal{P}(\mathbf{N})$
(c) $S \subseteq \mathcal{P}(\mathbf{N})$ and $|S| = 5$.
(d) $S \in \mathcal{P}(\mathbf{N})$ and $|S| = 5$.

Section 1.3: Set Operations

1.17. Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$. Determine the following:
(a) $A \cup B$, (b) $A \cap B$, (c) $A - B$, (d) $B - A$, (e) $\overline{A}$, (f) $A \cap \overline{B}$.

1.18. Give examples of three sets A, B, and C such that
(a) $A \in B$, $A \subseteq C$, and $B \nsubseteq C$.
(b) $B \in A$, $B \subset C$, and $A \cap C \neq \emptyset$.
(c) $A \in B$, $B \subseteq C$, and $A \nsubseteq C$.

1.19. Give examples of three sets A, B, and C such that $B \neq C$ but $B - A = C - A$.

1.20. Give examples of two sets A and B such that $|A - B| = |A \cap B| = |B - A| = 3$. Draw the accompanying Venn diagram.

1.21. Let U be a universal set and let A and B be two subsets of U. Draw a Venn diagram for each of the following sets.
(a) $\overline{A \cup B}$ (b) $\overline{A} \cap \overline{B}$ (c) $\overline{A \cap B}$ (d) $\overline{A} \cup \overline{B}$
What can you say about parts (a) and (b)? parts (c) and (d)?

1.22. Give an example of a universal set U, two sets A and B, and an accompanying Venn diagram such that $|A \cap B| = |A - B| = |B - A| = |\overline{A \cup B}| = 2$.

1.23. Let A, B, and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.
(a) $(C - B) \cup A$
(b) $C \cap (A - B)$

1.24. Let $A = [\emptyset, \{\emptyset\}, \{\{\emptyset\}\}]$.
(a) Determine which of the following are elements of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
(b) Determine $|A|$.
(c) Determine which of the following are subsets of A: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$.
For (d)–(i), determine the indicated sets.
(d) $\emptyset \cap A$
(e) $\{\emptyset\} \cap A$
(f) $\{\emptyset, \{\emptyset\}\} \cap A$
(g) $\emptyset \cup A$
(h) $\{\emptyset\} \cup A$
(i) $\{\emptyset, \{\emptyset\}\} \cup A$.

Section 1.4: Indexed Collections of Sets

1.25. Give examples of a universal set U and sets A, B, and C such that each of the following sets contains exactly one element: $A \cap B \cap C$, $(A \cap B) - C$, $(A \cap C) - B$, $(B \cap C) - A$, $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$, $\overline{A \cup B \cup C}$. Draw the accompanying Venn diagram.

1.26. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1, r+1]$, and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine

(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$
(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$
(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

1.27. Let $A = \{1, 2, 5\}$, $B = \{0, 2, 4\}$, $C = \{2, 3, 4\}$, and $S = \{A, B, C\}$. Determine $\bigcup_{X \in S} X$ and $\bigcap_{X \in S} X$.

1.28. For a real number r, define S_r to be the interval $[r-1, r+2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

1.29. Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

1.30. For each of the following collections of sets, define a set A_n for each $n \in \mathbf{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbf{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

(a) $\{[1, 2+1), [1, 2+1/2), [1, 2+1/3), \ldots\}$
(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \ldots\}$

1.31. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbf{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.

(a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$.
(b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbf{Z}$.

Section 1.5: Partitions of Sets

1.32. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.

(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$
(b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$
(c) $S_3 = \{A\}$
(d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$
(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$

1.33. Which of the following sets are partitions of $A = \{1, 2, 3, 4, 5\}$?

(a) $S_1 = \{\{1, 3\}, \{2, 5\}\}$
(b) $S_2 = \{\{1, 2\}, \{3, 4, 5\}, \{2, 1\}\}$
(c) $S_3 = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$
(d) $S_4 = A$

1.34. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition S of A such that $|S| = 3$.

1.35. Give an example of a set A with $|A| = 4$ and two disjoint partitions S_1 and S_2 of A with $|S_1| = |S_2| = 3$.

1.36. Give an example of three sets A, S_1, and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1, and $|S_2| < |S_1| < |A|$.

1.37. Give an example of a partition of $\mathbf{Q}$ into three subsets.

1.38. Give an example of a partition of $\mathbf{N}$ into three subsets.

1.39. Give an example of a partition of $\mathbf{Z}$ into four subsets.

1.40. Let $A = \{1, 2, \ldots, 12\}$. Give an example of a partition S of A satisfying the following requirements: (i) $|S| = 5$, (ii) T is a subset of S such that $|T| = 4$ and $|\bigcup_{X \in T} X| = 10$, and (iii) there is no element $B \in S$ such that $|B| = 3$.

Section 1.6: Cartesian Products of Sets

1.41. Let $A = \{x, y, z\}$ and $B = \{x, y\}$. Determine $A \times B$.

1.42. Let $A = \{1, \{1\}, \{\{1\}\}\}$. Determine $A \times A$.

1.43. For $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

1.44. For $A = \{\emptyset, \{\emptyset\}\}$. Determine $A \times \mathcal{P}(A)$.

1.45. For $A = \{1, 2\}$ and $B = \{\emptyset\}$, determine $A \times B$ and $\mathcal{P}(A) \times \mathcal{P}(B)$.

1.46. Describe the graph of the circle whose equation is $x^2 + y^2 = 4$ as a subset of $\mathbf{R} \times \mathbf{R}$.

1.47. List the elements of the set $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z} : |x| + |y| = 3\}$. Plot the corresponding points in the Euclidean x-y plane.

ADDITIONAL EXERCISES FOR CHAPTER 1

1.48. Let $S = \{-10, -9, \ldots, 9, 10\}$. Describe each of the following sets as $\{x \in S : p(x)\}$, where $p(x)$ is some condition on x.

(a) $A = \{-10, -9, \ldots, -1, 1, \ldots, 9, 10\}$
(b) $B = \{-10, -9, \ldots, -1, 0\}$
(c) $C = \{-5, -4, \ldots, 7\}$
(d) $D = \{-10, -9, \ldots, 4, 6, 7, \ldots, 10\}$

1.49. Describe each of the following sets by listing its elements within braces.

(a) $\{x \in \mathbf{Z} : x^3 - 4x = 0\}$
(b) $\{x \in \mathbf{R} : |x| = -1\}$
(c) $\{m \in \mathbf{N} : 2 < m \leq 5\}$
(d) $\{n \in \mathbf{N} : 0 \leq n \leq 3\}$
(e) $\{k \in \mathbf{Q} : k^2 - 4 = 0\}$
(f) $\{k \in \mathbf{Z} : 9k^2 - 3 = 0\}$
(g) $\{k \in \mathbf{Z} : 1 \leq k^2 \leq 10\}$

1.50. Determine the cardinality of each of the following sets.

(a) $A = \{1, 2, 3, \{1, 2, 3\}, 4, \{4\}\}$
(b) $B = \{x \in \mathbf{R} : |x| = -1\}$
(c) $C = \{m \in \mathbf{N} : 2 < m \leq 5\}$
(d) $D = \{n \in \mathbf{N} : n < 0\}$
(e) $E = \{k \in \mathbf{N} : 1 \leq k^2 \leq 100\}$
(f) $F = \{k \in \mathbf{Z} : 1 \leq k^2 \leq 100\}$

1.51. For $A = \{-1, 0, 1\}$ and $B = \{x, y\}$, determine $A \times B$.

1.52. Let $U = \{1, 2, 3\}$ be the universal set, and let $A = \{1, 2\}$, $B = \{2, 3\}$, and $C = \{1, 3\}$. Determine the following.

(a) $(A \cup B) - (B \cap C)$

(b) $\overline{A}$

(c) $\overline{B \cup C}$

(d) $A \times B$

1.53. Let $A = \{1, 2, \ldots, 10\}$. Give an example of two sets S and B such that $S \subseteq \mathcal{P}(A)$, $|S| = 4$, $B \in S$, and $|B| = 2$.

1.54. For $A = \{1\}$ and $C = \{1, 2\}$, give an example of a set B such that $\mathcal{P}(A) \subset B \subset \mathcal{P}(C)$.

1.55. Give examples of two sets A and B such that

(a) $A \cap \mathcal{P}(A) \in B$

(b) $\mathcal{P}(A) \subseteq A \cup B$.

1.56. Which of the following sets are equal?

$A = \{n \in \mathbf{Z} : -4 \leq n \leq 4\}$ $\quad D = \{x \in \mathbf{Z} : x^3 = 4x\}$

$B = \{x \in \mathbf{N} : 2x + 2 = 0\}$ $\quad E = \{-2, 0, 2\}$

$C = \{x \in \mathbf{Z} : 3x - 2 = 0\}$

1.57. Let A and B be sets in some unknown universal set U. Suppose that $\overline{A} = \{3, 8, 9\}$, $A - B = \{1, 2\}$, $B - A = \{8\}$, and $A \cap B = \{5, 7\}$. Determine U, A, and B.

1.58. Let I denote the interval $[0, \infty)$. For each $r \in I$, define

$$A_r = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = r^2\},$$
$$B_r = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 \leq r^2\},$$
$$C_r = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 > r^2\}.$$

(a) Determine $\bigcup_{r \in I} A_r$ and $\bigcap_{r \in I} A_r$.

(b) Determine $\bigcup_{r \in I} B_r$ and $\bigcap_{r \in I} B_r$.

(c) Determine $\bigcup_{r \in I} C_r$ and $\bigcap_{r \in I} C_r$.

1.59. Give an example of four sets A_1, A_2, A_3, A_4 such that $|A_i \cap A_j| = |i - j|$ for every two integers i and j with $1 \leq i < j \leq 4$.

1.60. (a) Give an example of two problems suggested by Exercise 1.59 (above).

(b) Solve one of the problems in (a).

1.61. Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, and $C = \{1, 2, 3, 4, 5\}$. For the sets S and T described below, explain whether $|S| < |T|$, $|S| > |T|$, or $|S| = |T|$.

(a) Let B be the universal set and let S be the set of all subsets X of B for which $|X| \neq |\overline{X}|$. Let T be the set of 2-element subsets of C.

(b) Let S be the set of all partitions of the set A and let T be the set of 4-element subsets of C.

(c) Let $S = \{(b, a) : b \in B, a \in A, a + b \text{ is odd}\}$ and let T be the set of all nonempty proper subsets of A.

1.62. Give an example of a set $A = \{1, 2, \ldots, k\}$ for a smallest $k \in \mathbf{N}$ having subsets A_1, A_2, A_3 such that $|A_i - A_j| = |A_j - A_i| = |i - j|$ for every two integers i and j with $1 \leq i < j \leq 3$.

2

Logic

In mathematics our goal is to seek the truth. Are there connections between two given mathematical concepts? If so, what are they? Under what conditions does an object possess a particular property? Finding answers to questions such as these is important, but we cannot be satisfied only with this. We must be certain that we are right and that our explanation for why we believe we are correct is convincing to others. The reasoning we use as we proceed from what we know to what we wish to show must be logical. It must make sense to others, not just to ourselves.

There is joint responsibility here, however. It is the writer's responsibility to use the rules of logic to give a valid and clear argument with enough details provided to allow the reader to understand what we have written and to be convinced. It is the reader's responsibility to know the basics of logic and to study the concepts involved sufficiently well so that he or she will not only be able to understand a well-presented argument but can decide as well whether it is valid. Consequently, both writer and reader must have some familiarity with logic.

Although it is possible to spend a great deal of time studying logic, we will present only what we actually need and will instead use the majority of our time putting what we learn into practice.

2.1 Statements

In mathematics we are constantly dealing with statements. By a **statement** we mean a declarative sentence or assertion that is true or false (but not both). Statements therefore declare or assert the truth of something. Of course, the statements in which we will be primarily interested deal with mathematics. For example, the sentences

The integer 3 is odd.
The integer 57 is prime.

are statements (only the first of which is true).

Every statement has a **truth value**, namely **true** (denoted by T) or **false** (denoted by F). We often use P, Q, and R to denote statements, or perhaps $P_1, P_2, \ldots, P_n$ if

several statements are involved. We have seen that

$$P_1 : \text{The integer 3 is odd.}$$

and

$$P_2 : \text{The integer 57 is prime.}$$

are statements, where P_1 has truth value T and P_2 has truth value F.

Sentences that are imperative (commands) such as

Substitute the number 2 for x.
Find the derivative of $f(x) = e^{-x}\cos 2x$.

or are interrogative (questions) such as

Are these sets disjoint?
What is the derivative of $f(x) = e^{-x}\cos 2x$?

or are exclamatory such as

What an interesting question!
How difficult this problem is!

are not statements since these sentences are not declarative.

It may not be immediately clear whether a statement is true or false. For example, the sentence "The 100th digit in the decimal expansion of π is 7." is a statement, but it may be necessary to check some table to determine whether this statement is true. Indeed, for a sentence to be a statement, it is not a requirement that we be able to determine its truth value.

The sentence "The real number r is rational." is a statement *provided* we know what real number r is being referred to. Without this additional information, however, it is impossible to assign a truth value to it. This is an example of what is often referred to as an open sentence. In general, an **open sentence** is a declarative sentence that contains one or more variables, each variable representing a value in some prescribed set, called the **domain** of the variable, and which becomes a statement when values from their respective domains are substituted for these variables. For example, the open sentence "$3x = 12$" where the domain of x is the set of integers is a true statement only when $x = 4$.

An open sentence that contains a variable x is typically represented by $P(x)$, $Q(x)$, or $R(x)$. If $P(x)$ is an open sentence, where the domain of x is S, then we say $P(x)$ is an **open sentence over the domain** S. Also, $P(x)$ is a statement for each $x \in S$. For example, the open sentence

$$P(x) : \ (x-3)^2 \le 1$$

over the domain $\mathbf{Z}$ is a true statement when $x \in \{2, 3, 4\}$ and is a false statement otherwise.

Example 2.1 *For the open sentence*

$$P(x, y) : \ |x+1| + |y| = 1$$

P
T
F

Q
T
F

P	Q
T	T
T	F
F	T
F	F

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Figure 2.1 Truth tables for one, two, and three statements

in two variables, suppose that the domain of the variable x is $S = \{-2, -1, 0, 1\}$ and the domain of the variable y is $T = \{-1, 0, 1\}$. Then

$$P(-1, 1) : \ |(-1) + 1| + |1| = 1$$

is a true statement, while

$$P(1, -1) : \ |1 + 1| + |-1| = 1$$

is a false statement. In fact, $P(x, y)$ is a true statement when

$$(x, y) \in \{(-2, 0), (-1, -1), (-1, 1), (0, 0)\},$$

while $P(x, y)$ is a false statement for all other elements $(x, y) \in S \times T$. ♦

The possible truth values of a statement are often given in a table, called a **truth table.** The truth tables for two statements P and Q are given in Figure 2.1. Since there are two possible truth values for each of P and Q, there are four possible combinations of truth values for P and Q. The truth table showing all these combinations is also given in Figure 2.1. If a third statement R is involved, then there are eight possible combinations of truth values for P, Q, and R. This is displayed in Figure 2.1 as well. In general, a truth table involving n statements $P_1, P_2, \cdots, P_n$ contains 2^n possible combinations of truth values for these statements, and a truth table showing these combinations would have n columns and 2^n rows. Much of the time, we will be dealing with two statements, usually denoted by P and Q; so the associated truth table will have four rows with the first two columns headed by P and Q. In this case, it is customary to consider the four combinations of the truth values in the order TT, TF, FT, FF, from top to bottom.

2.2 The Negation of a Statement

Much of the interest in integers and other familiar sets of numbers comes not only from the numbers themselves but from properties of the numbers that result by performing

operations on them (such as taking their negatives, adding or multiplying them, or combinations of these). Similarly, much of our interest in statements comes from investigating the truth or falseness of new statements that can be produced from one or more given statements by performing certain operations on them. Our first example concerns producing a new statement from a single given statement.

The **negation** of a statement P is the statement:

not P.

and is denoted by $\sim P$. Although $\sim P$ could always be expressed as

It is not the case that P.

there are usually better ways to express the statement $\sim P$.

Example 2.2 *For the statement*

P_1 : *The integer* 3 *is odd.*

described above, we have

$\sim P_1$: *It is not the case that the integer* 3 *is odd.*

but it would be much preferred to write

$\sim P_1$: *The integer* 3 *is not odd.*

or better yet to write

$\sim P_1$: *The integer* 3 *is even.*

Similarly, the negation of the statement

P_2 : *The integer* 57 *is prime.*

considered above is

$\sim P_2$: *The integer* 57 *is not prime.*

Note that $\sim P_1$ *is false, while* $\sim P_2$ *is true.* ♦

Indeed, the negation of a true statement is always false, and the negation of a false statement is always true; that is, the truth value of $\sim P$ is opposite to that of P. This is summarized in Figure 2.2, which gives the truth table for $\sim P$ (in terms of the possible truth values of P).

P	$\sim P$
T	F
F	T

Figure 2.2 The truth table for negation

2.3 The Disjunction and Conjunction of Statements

For two given statements P and Q, a common way to produce a new statement from them is by inserting the word "or" or "and" between P and Q. The **disjunction** of the statements P and Q is the statement:

P **or** Q.

and is denoted by $P \vee Q$. The disjunction $P \vee Q$ is true if at least one of P and Q is true; otherwise, $P \vee Q$ is false. Therefore, $P \vee Q$ is true if exactly one of P and Q is true or if both P and Q are true.

Example 2.3 *For the statements*

P_1 : *The integer* 3 *is odd. and* P_2 : *The integer* 57 *is prime.*

described earlier, the disjunction is the new statement

$P_1 \vee P_2$: *Either* 3 *is odd or* 57 *is prime.*

which is true since at least one of P_1 *and* P_2 *is true (namely,* P_1 *is true). Of course, in this case exactly one of* P_1 *and* P_2 *is true.* ♦

For two statements P and Q, the truth table for $P \vee Q$ is shown in Figure 2.3. This truth table then describes precisely when $P \vee Q$ is true (or false).

Although the truth of "P or Q" allows for both P and Q to be true, there are instances when the use of "or" does not allow that possibility. For example, for an integer n, if we were to say "n is even or n is odd", then surely it is not possible for both "n is even" and "n is odd" to be true. When "or" is used in this manner, it is called the **exclusive or**. Suppose, for example, that $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, where $k \geq 2$, is a partition of a set S and x is some element of S. If

$x \in S_1$ or $x \in S_2$

is true, then it is impossible for both $x \in S_1$ and $x \in S_2$ to be true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 2.3 The truth table for disjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 2.4 The truth table for conjunction

The **conjunction** of the statements P and Q is the statement:

$$P \textbf{ and } Q.$$

and is denoted by $P \wedge Q$. The conjunction $P \wedge Q$ is true only when both P and Q are true; otherwise, $P \wedge Q$ is false.

Example 2.4 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime, the statement*

$$P_1 \wedge P_2 : 3 \textit{ is odd and } 57 \textit{ is prime.}$$

is false since P_2 is false and so not both P_1 and P_2 are true. ♦

The truth table for the conjunction of two statements is shown in Figure 2.4.

2.4 The Implication

The statement formed from two given statements in which we will be most interested is the implication (also called the conditional). For statements P and Q, the **implication** (or **conditional**) is the statement:

$$\textbf{If } P\textbf{, then } Q.$$

and is denoted by $P \Rightarrow Q$. In addition to the wording "If P, then Q.", we also express $P \Rightarrow Q$ in words as

$$P \textbf{ implies } Q.$$

The truth table for $P \Rightarrow Q$ is given in Figure 2.5.

Notice that $P \Rightarrow Q$ is false when P is true and Q is false, and is true otherwise.

Example 2.5 *For P_1 : The integer 3 is odd. and P_2 : The integer 57 is prime, the implication*

$$P_1 \Rightarrow P_2 : \textit{If 3 is an odd integer, then 57 is prime.}$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 2.5 The truth table for implication

is a false statement. The implication

$$P_2 \Rightarrow P_1 : \textit{If 57 is prime, then 3 is odd.}$$

is true, however. ♦

While the truth tables for the negation $\sim P$, the disjunction $P \vee Q$, and the conjunction $P \wedge Q$ are probably not unexpected, this may not be so for the implication $P \Rightarrow Q$. There is ample justification, however, for the truth values in the truth table of $P \Rightarrow Q$. We illustrate this with an example.

Example 2.6 *A student is taking a math class (let's say this one) and is currently receiving a B+. He visits his instructor a few days before the final examination and asks her, "Is there any chance that I can get an A in this course?" His instructor looks through her grade book and says, "If you earn an A on the final exam, then you will receive an A for your final grade." We now check the truth or falseness of this implication based on the various combinations of truth values of the statements*

$$P : \textit{You earn an A on the final exam.}$$

and

$$Q : \textit{You receive an A for your final grade.}$$

which make up the implication.

Analysis Suppose first that P and Q are both true. That is, the student receives an A on his final exam and later learns that he got an A for his final grade in the course. Did his instructor tell the truth? I think we would all agree that she did. So if P and Q are both true, then so too is $P \Rightarrow Q$, which agrees with the first row of the truth table of Figure 2.5.

Second, suppose that P is true and Q is false. So the student got an A on his final exam but did not receive an A as a final grade, say he received a B. Certainly, his instructor did not do as she promised (as she will soon be reminded by her student). What she said was false, which agrees with the second row of the table in Figure 2.5.

Third, suppose that P is false and Q is true. In this case, the student did not get an A on his final exam (say he earned a B), *but* when he received his final grades, he learned (and was pleasantly surprised) that his final grade was an A. How could this happen? Perhaps his instructor was lenient. Perhaps the final exam was unusually difficult, and a grade of B on it indicated an exceptionally good performance. Perhaps the instructor made a mistake. In any case, the instructor did not lie; so she told the truth. This agrees with the third row of the table in Figure 2.5.

Finally, suppose that P and Q are both false. That is, suppose the student did not get an A on his final exam, and he also did not get an A for a final grade. The instructor did not lie here either. She only promised the student an A *if* he got an A on the final exam. She promised nothing if the student did not get an A on the final exam. So the instructor told the truth, and this agrees with the fourth and final row of the table. ♦

In summary then, the only situation for which $P \Rightarrow Q$ is false is when P is true and Q is false (so $\sim Q$ is true). That is, the truth tables for

$$\sim(P \Rightarrow Q) \text{ and } P \wedge (\sim Q)$$

are the same. We'll revisit this observation again soon.

We have already mentioned that the implication $P \Rightarrow Q$ can be expressed as both "If P, then Q" and "P implies Q". In fact, there are several ways of expressing $P \Rightarrow Q$ in words, namely:

If P, then Q.
Q if P.
P implies Q.
P only if Q.
P is sufficient for Q.
Q is necessary for P.

It is probably not surprising that the first three of these say the same thing, but perhaps not at all obvious that the last three say the same thing as the first three. Consider the statement "P only if Q". This says that P is true only under the condition that Q is true; in other words, it cannot be the case that P is true and Q is false. Thus it says that if P is true, then necessarily Q must be true. We can also see from this that the statement "Q is necessary for P" has the same meaning as "P only if Q". The statement "P is sufficient for Q" states that the truth of P is sufficient for the truth of Q. In other words, the truth of P implies the truth of Q; that is, "P implies Q".

2.5 More on Implications

We have just discussed four ways to create new statements from one or two given statements. In mathematics, however, we are often interested in declarative sentences containing variables and whose truth or falseness is only known once we have assigned values to the variables. The values assigned to the variables come from their respective domains. These sentences are, of course, precisely the sentences we have referred to as open sentences. Just as new statements can be formed from statements P and Q by negation, disjunction, conjunction, or implication, new open sentences can be constructed from open sentences in the same manner.

Example 2.7 *Consider the open sentences*

$$P_1(x) : x = -3 \text{ and } P_2(x) : |x| = 3,$$

where $x \in \mathbf{R}$, that is, where the domain of x is $\mathbf{R}$ in each case. We can then form the following open sentences:

$$\sim P_1(x): \ x \neq -3.$$
$$P_1(x) \vee P_2(x): \ x = -3 \text{ or } |x| = 3.$$
$$P_1(x) \wedge P_2(x): \ x = -3 \text{ and } |x| = 3.$$
$$P_1(x) \Rightarrow P_2(x): \ \text{If } x = -3, \text{ then } |x| = 3.$$

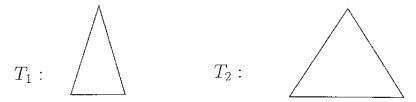

Figure 2.6 Isosceles and equilateral triangles

For a given real number the truth value of each resulting statement can be determined. For example, $\sim P_1(-3)$ is a false statement, while each of the remaining sentences above results in a true statement when $x = -3$. Both $P_1(2) \vee P_2(2)$ and $P_1(2) \wedge P_2(2)$ are false statements. On the other hand, both $\sim P_1(2)$ and $P_1(2) \Rightarrow P_2(2)$ are true statements. In fact, for each real number $x \neq -3$, the implication $P_1(x) \Rightarrow P_2(x)$ is a true statement since $P_1(x) : x = -3$ is a false statement. Thus $P_1(x) \Rightarrow P_2(x)$ is true for all $x \in \mathbf{R}$. We will see that open sentences that result in true statements for all values of the domain will be especially interesting to us.

Listed below are various ways of wording the implication $P_1(x) \Rightarrow P_2(x)$:

If $x = -3$, then $|x| = 3$.
$|x| = 3$ if $x = -3$.
$x = -3$ implies that $|x| = 3$.
$x = -3$ only if $|x| = 3$.
$x = -3$ is sufficient for $|x| = 3$.
$|x| = 3$ is necessary for $x = -3$. ♦

We now consider another example, this time from geometry. You may recall that a triangle is called **equilateral** if the lengths of its three sides are the same, while a triangle is **isosceles** if the lengths of any two of its three sides are the same. Figure 2.6 shows an isosceles triangle T_1 and an equilateral triangle T_2. Actually, since the lengths of any two of the three sides of T_2 are the same, T_2 is isosceles as well. Indeed, this is precisely the fact we wish to discuss.

Example 2.8 *For a triangle T, let*

$$P(T) : T \text{ is equilateral. and } Q(T) : T \text{ is isosceles.}$$

Thus, $P(T)$ and $Q(T)$ are open sentences over the domain S of all triangles. Consider the implication $P(T) \Rightarrow Q(T)$, where the domain then of the variable T is the set S. For an equilateral triangle T_1, both $P(T_1)$ and $Q(T_1)$ are true statements and so $P(T_1) \Rightarrow Q(T_1)$ is a true statement as well. If T_2 is not an equilateral triangle, then $P(T_2)$ is a false statement and so $P(T_2) \Rightarrow Q(T_2)$ is true. Therefore, $P(T) \Rightarrow Q(T)$ is a true statement for all $T \in S$. We now state $P(T) \Rightarrow Q(T)$ in a variety of ways:

If T is an equilateral triangle, then T is isosceles.
A triangle T is isosceles if T is equilateral.
A triangle T being equilateral implies that T is isosceles.
A triangle T is equilateral only if T is isosceles.
For a triangle T to be isosceles, it is sufficient that T be equilateral.
For a triangle T to be equilateral, it is necessary that T be isosceles. ♦

Notice that at times we change the wording to make the sentence sound better. In general, the sentence P in the implication $P \Rightarrow Q$ is commonly referred to as the **hypothesis** or **premise** of $P \Rightarrow Q$, while Q is called the **conclusion** of $P \Rightarrow Q$.

We now investigate the truth or falseness of implications involving open sentences for values of their variables.

Example 2.9 *Let* $S = \{2, 3, 5\}$ *and let*

$$P(n): n^2 - n + 1 \text{ is prime. and } Q(n): n^3 - n + 1 \text{ is prime.}$$

be open sentences over the domain S. *Determine the truth or falseness of the implication* $P(n) \Rightarrow Q(n)$ *for each* $n \in S$.

Solution In this case, we have the following:

$P(2)$: 3 is prime. $P(3)$: 7 is prime. $P(5)$: 21 is prime.
$Q(2)$: 7 is prime. $Q(3)$: 25 is prime. $Q(5)$: 121 is prime.

Consequently, $P(2) \Rightarrow Q(2)$ and $P(5) \Rightarrow Q(5)$ are true, while $P(3) \Rightarrow Q(3)$ is false. ♦

Example 2.10 *Let* $S = \{1, 2\}$ *and let* $T = \{-1, 4\}$. *Also, let*

$$P(x, y): ||x + y| - |x - y|| = 2 \text{ and } Q(x, y): x^{y+1} = y^x$$

be open sentences, where the domain of the variable x *is* S *and the domain of* y *is* T. *Determine the truth or falseness of the implication* $P(x, y) \Rightarrow Q(x, y)$ *for all* $(x, y) \in S \times T$.

Solution For $(x, y) = (1, -1)$, we have

$$P(1, -1) \Rightarrow Q(1, -1): \text{ If } 2 = 2, \text{ then } 1 = -1.$$

which is false. For $(x, y) = (1, 4)$, we have

$$P(1, 4) \Rightarrow Q(1, 4): \text{ If } 2 = 2, \text{ then } 1 = 4.$$

which is also false. For $(x, y) = (2, -1)$, we have

$$P(2, -1) \Rightarrow Q(2, -1): \text{ If } 2 = 2, \text{ then } 1 = 1.$$

which is true; while for $(x, y) = (2, 4)$, we have

$$P(2, 4) \Rightarrow Q(2, 4): \text{ If } 4 = 2, \text{ then } 32 = 16.$$

which is true. ♦

2.6 The Biconditional

For statements (or open sentences) P and Q, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$. The converse of an implication will often be of interest to us, either by itself or in conjunction with the original implication.

Example 2.11 *For the statements*

$$P_1: 3 \text{ is an odd integer.} \quad P_2: 57 \text{ is prime.}$$

the converse of the implication

$$P_1 \Rightarrow P_2: \text{If } 3 \text{ is an odd integer, then } 57 \text{ is prime.}$$

is the implication

$$P_2 \Rightarrow P_1: \text{If } 57 \text{ is prime, then } 3 \text{ is an odd integer.}$$ ♦

For statements (or open sentences) P and Q, the conjunction

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

of the implication $P \Rightarrow Q$ and its converse is called the **biconditional** of P and Q and is denoted by $P \Leftrightarrow Q$. For statements P and Q, the truth table for $P \Leftrightarrow Q$ can therefore be determined. This is given in Figure 2.7. From this table, we see that $P \Leftrightarrow Q$ is true whenever the statements P and Q are both true or are both false, while $P \Leftrightarrow Q$ is false otherwise. That is, $P \Leftrightarrow Q$ is true precisely when P and Q have the same truth values.

The biconditional $P \Leftrightarrow Q$ is often stated as

P is equivalent to Q.

or

P if and only if Q.

or as

P is a necessary and sufficient condition for Q.

For statements P and Q, it then follows that the biconditional "P if and only if Q" is true only when P and Q have the same truth values.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 2.7 The truth table for a biconditional

Example 2.12 *The biconditional*

$$3 \text{ is an odd integer if and only if } 57 \text{ is prime.}$$

is false, while the biconditional

$$100 \text{ is even if and only if } 101 \text{ is prime.}$$

is true. Furthermore, the biconditional

$$5 \text{ is even if and only if } 4 \text{ is odd.}$$

is also true. ♦

The phrase "if and only if" occurs often in mathematics, and we shall discuss this at greater length later. For the present, we consider two examples involving statements containing the phrase "if and only if".

Example 2.13 *We noted in Example 2.7 that for the open sentences*

$$P_1(x) : x = -3 \text{ and } P_2(x) : |x| = 3$$

over the domain $\mathbf{R}$*, the implication*

$$P_1(x) \Rightarrow P_2(x) : \text{If } x = -3, \text{ then } |x| = 3.$$

is a true statement for each $x \in \mathbf{R}$*. However, the converse*

$$P_2(x) \Rightarrow P_1(x) : \text{If } |x| = 3, \text{ then } x = -3.$$

is a false statement when $x = 3$ *since* $P_2(3)$ *is true and* $P_1(3)$ *is false. For all other real numbers* x*, the implication* $P_2(x) \Rightarrow P_1(x)$ *is true. Therefore, the biconditional*

$$P_1(x) \Leftrightarrow P_2(x) : \ x = -3 \text{ if and only if } |x| = 3.$$

is false when $x = 3$ *and is true for all other real numbers* x*.* ♦

Example 2.14 *For the open sentences*

$$P(T) : \ T \text{ is equilateral. and } Q(T) : \ T \text{ is isosceles.}$$

over the domain S *of all triangles, the converse of the implication*

$$P(T) \Rightarrow Q(T) : \text{If } T \text{ is equilateral, then } T \text{ is isosceles.}$$

is the implication

$$Q(T) \Rightarrow P(T) : \text{If } T \text{ is isosceles, then } T \text{ is equilateral.}$$

We noted that $P(T) \Rightarrow Q(T)$ *is a true statement for all triangles* T*, while* $Q(T) \Rightarrow P(T)$ *is a false statement when* T *is an isosceles triangle that is not equilateral. On the other hand, the second implication becomes a true statement for all other triangles* T*. Therefore, the biconditional*

$$P(T) \Leftrightarrow Q(T) : T \text{ is equilateral if and only if } T \text{ is isosceles.}$$

is false for all triangles that are isosceles and not equilateral, while it is true for all other triangles T*.* ♦

We now investigate the truth or falseness of biconditionals obtained by assigning to a variable each value in its domain.

Example 2.15 *Let* $S = \{0, 1, 4\}$*. Consider the following open sentences over the domain* S*:*

$$P(n) : \ \frac{n(n+1)(2n+1)}{6} \text{ is odd.}$$
$$Q(n) : \ (n+1)^3 = n^3 + 1.$$

Determine three distinct elements a, b, c *in* S *such that* $P(a) \Rightarrow Q(a)$ *is false,* $Q(b) \Rightarrow P(b)$ *is false, and* $P(c) \Leftrightarrow Q(c)$ *is true.*

Solution Observe that

$$P(0) : \ 0 \text{ is odd.} \quad P(1) : \ 1 \text{ is odd.} \quad P(4) : \ 30 \text{ is odd.}$$

$$Q(0) : \ 1 = 1. \quad Q(1) : \ 8 = 2. \quad Q(4) : \ 125 = 65.$$

Thus $P(0)$ and $P(4)$ are false, while $P(1)$ is true. Also, $Q(1)$ and $Q(4)$ are false, while $Q(0)$ is true. Thus $P(1) \Rightarrow Q(1)$ and $Q(0) \Rightarrow P(0)$ are false, while $P(4) \Leftrightarrow Q(4)$ is true. Hence we may take $a = 1$, $b = 0$, and $c = 4$. ♦

Analysis Notice in Example 2.15 that both $P(0) \Leftrightarrow Q(0)$ and $P(1) \Leftrightarrow Q(1)$ are false biconditionals. Hence the value 4 in S is the only choice for c. ♦

2.7 Tautologies and Contradictions

The symbols $\sim$, $\vee$, $\wedge$, $\Rightarrow$, and $\Leftrightarrow$ are sometimes referred to as **logical connectives**. From given statements, we can use these logical connectives to form more intricate statements. For example, the statement $(P \vee Q) \wedge (P \vee R)$ is a statement formed from the given statements P, Q, and R and the logical connectives $\vee$ and $\wedge$. We call $(P \vee Q) \wedge (P \vee R)$ a compound statement. More generally, a **compound statement** is a statement composed of one or more given statements (called **component statements** in this context) and at least one logical connective. For example, for a given component statement P, its negation $\sim P$ is a compound statement.

The compound statement $P \vee (\sim P)$, whose truth table is given in Figure 2.8, has the feature that it is true regardless of the truth value of P.

A compound statement S is called a **tautology** if it is true for all possible combinations of truth values of the component statements that comprise S. Hence $P \vee (\sim P)$ is

P	$\sim P$	$P \vee (\sim P)$
T	F	$\boldsymbol{T}$
F	T	$\boldsymbol{T}$

Figure 2.8 An example of a tautology

P	Q	$\sim Q$	$P \Rightarrow Q$	$(\sim Q) \vee (P \Rightarrow Q)$
T	T	F	T	$\boldsymbol{T}$
T	F	T	F	$\boldsymbol{T}$
F	T	F	T	$\boldsymbol{T}$
F	F	T	T	$\boldsymbol{T}$

Figure 2.9 Another tautology

a tautology, as is $(\sim Q) \vee (P \Rightarrow Q)$. This latter fact is verified in the truth table shown in Figure 2.9.

Letting

$$P_1 : 3 \text{ is odd. and } P_2 : 57 \text{ is prime.}$$

we see that not only is

57 is not prime, or 57 is prime if 3 is odd.

a true statement, but $(\sim P_2) \vee (P_1 \Rightarrow P_2)$ is true regardless of which statements P_1 and P_2 are being considered.

On the other hand, a compound statement S is called a **contradiction** if it is false for all possible combinations of truth values of the component statements that are used to form S. The statement $P \wedge (\sim P)$ is a contradiction, as is shown in Figure 2.10. Hence the statement

3 is odd and 3 is not odd.

is false.

Another example of a contradiction is $(P \wedge Q) \wedge (Q \Rightarrow (\sim P))$, which is verified in the truth table shown in Figure 2.11.

Indeed, if a compound statement S is a tautology, then its negation $\sim S$ is a contradiction.

P	$\sim P$	$P \wedge (\sim P)$
T	F	$\boldsymbol{F}$
F	T	$\boldsymbol{F}$

Figure 2.10 An example of a contradiction

P	Q	$\sim P$	$P \wedge Q$	$Q \Rightarrow \sim P$	$(P \wedge Q) \wedge (Q \Rightarrow \sim P)$
T	T	F	T	F	$\boldsymbol{F}$
T	F	F	F	T	$\boldsymbol{F}$
F	T	T	F	T	$\boldsymbol{F}$
F	F	T	F	T	$\boldsymbol{F}$

Figure 2.11 Another contradiction

2.8 Logical Equivalence

Figure 2.12 shows a truth table for the two statements $P \Rightarrow Q$ and $(\sim P) \vee Q$. The corresponding columns of these compound statements are identical; in other words, these two compound statements have exactly the same truth value for every combination of truth values of the statements P and Q. In general, whenever two (compound) statements R and S have the same truth values for all combinations of truth values of their component statements, then we say that R and S are **logically equivalent** and indicate this by writing $R \equiv S$. Hence $P \Rightarrow Q$ and $(\sim P) \vee Q$ are logically equivalent and so $P \Rightarrow Q \equiv (\sim P) \vee Q$.

Another, even simpler, example of logical equivalence concerns $P \wedge Q$ and $Q \wedge P$. That $P \wedge Q \equiv Q \wedge P$ is verified in the truth table shown in Figure 2.13.

What is the practical significance of logical equivalence? Suppose that R and S are logically equivalent compound statements. Then we know that R and S have the same truth values for all possible combinations of truth values of their component statements. But this means that the biconditional $R \Leftrightarrow S$ is true for all possible combinations of truth values of their component statements and hence $R \Leftrightarrow S$ is a tautology. Conversely, if $R \Leftrightarrow S$ is a tautology, then R and S are logically equivalent.

P	Q	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
F	F	T	$\boldsymbol{T}$	$\boldsymbol{T}$

Figure 2.12 Verification of $P \Rightarrow Q \equiv (\sim P) \vee Q$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	$\boldsymbol{T}$	$\boldsymbol{T}$
T	F	$\boldsymbol{F}$	$\boldsymbol{F}$
F	T	$\boldsymbol{F}$	$\boldsymbol{F}$
F	F	$\boldsymbol{F}$	$\boldsymbol{F}$

Figure 2.13 Verification of $P \wedge Q \equiv Q \wedge P$

Let R be a mathematical statement that we would like to show is true, and suppose that R and some statement S are logically equivalent. If we can show that S is true, then R is true as well. For example, suppose that we want to verify the truth of an implication $P \Rightarrow Q$. If we can establish the truth of the statement $(\sim P) \vee Q$, then the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ guarantees that $P \Rightarrow Q$ is true as well.

Example 2.16 *Returning to the mathematics instructor in Example 2.6 and whether she kept her promise that*

If you earn an A on the final exam, then you will receive an A for the final grade.

we need know only that the student did not receive an A on the final exam or the student received an A as a final grade to see that she kept her promise. ♦

Since the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$, verified in Figure 2.12, is especially important and we will have occasion to use this fact often, we state it as a theorem.

Theorem 2.17 *Let P and Q be two statements. Then*

$$P \Rightarrow Q \text{ and } (\sim P) \vee Q$$

are logically equivalent.

Let's return to the truth table in Figure 2.13, where we showed that $P \wedge Q$ and $Q \wedge P$ are logically equivalent for any two statements P and Q. In particular, this says that

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \text{ and } (Q \Rightarrow P) \wedge (P \Rightarrow Q)$$

are logically equivalent. Of course, $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is precisely what is called the biconditional of P and Q. Since $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ are logically equivalent, $(Q \Rightarrow P) \wedge (P \Rightarrow Q)$ represents the biconditional of P and Q as well. Since $Q \Rightarrow P$ can be written as "P if Q" and $P \Rightarrow Q$ can be expressed as "P only if Q", their conjunction can be written as "P if Q and P only if Q" or, more simply, as

$$P \text{ if and only if } Q.$$

Consequently, expressing $P \Leftrightarrow Q$ as "P if and only if Q" is justified. Furthermore, since $Q \Rightarrow P$ can be phrased as "P is necessary for Q" and $P \Rightarrow Q$ can be expressed as "P is sufficient for Q", writing $P \Leftrightarrow Q$ as "P is necessary and sufficient for Q" is likewise justified.

2.9 Some Fundamental Properties of Logical Equivalence

It probably comes as no surprise that the statements P and $\sim(\sim P)$ are logically equivalent. This fact is verified in Figure 2.14.

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

Figure 2.14 Verification of $P \equiv \sim(\sim P)$

We mentioned in Figure 2.13 that, for two statements P and Q, the statements $P \wedge Q$ and $Q \wedge P$ are logically equivalent. There are other fundamental logical equivalences that we often encounter as well.

Theorem 2.18 *For statements P, Q, and R,*

(1) *Commutative Laws*
   (*a*) $P \vee Q \equiv Q \vee P$
   (*b*) $P \wedge Q \equiv Q \wedge P$
(2) *Associative Laws*
   (*a*) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
   (*b*) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
(3) *Distributive Laws*
   (*a*) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
   (*b*) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
(4) *De Morgan's Laws*
   (*a*) $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
   (*b*) $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$.

Each part of Theorem 2.18 is verified by means of a truth table. We have already established the commutative law for conjunction (namely, that $P \wedge Q \equiv Q \wedge P$) in Figure 2.13. In Figure 2.15 $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ is verified by observing that the columns corresponding to the statements $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are identical.

The laws given in Theorem 2.18, together with other known logical equivalences, can be used to good advantage at times to prove other logical equivalences (without introducing a truth table).

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	***T***	T	T	***T***
T	T	F	F	***T***	T	T	***T***
T	F	T	F	***T***	T	T	***T***
T	F	F	F	***T***	T	T	***T***
F	T	T	T	***T***	T	T	***T***
F	T	F	F	***F***	T	F	***F***
F	F	T	F	***F***	F	T	***F***
F	F	F	F	***F***	F	F	***F***

Figure 2.15 Verification of the distributive law $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Example 2.19 *Suppose that we are asked to prove that*

$$\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$$

for every two statements P and Q. Using the logical equivalence of $P \Rightarrow Q$ and $(\sim P) \vee Q$ from Theorem 2.17 and Theorem 2.18(4a), we have the following:

$$\sim (P \Rightarrow Q) \equiv \sim((\sim P) \vee Q) \equiv (\sim(\sim P)) \wedge (\sim Q) \equiv P \wedge (\sim Q), \tag{2.1}$$

implying that the statements $\sim(P \Rightarrow Q)$ and $P \wedge (\sim Q)$ are logically equivalent, which we alluded to earlier. ♦

It is important to keep in mind what we have said about logical equivalence. For example, the logical equivalence of $P \wedge Q$ and $Q \wedge P$ allows us to replace a statement of the type $P \wedge Q$ by $Q \wedge P$ without changing its truth value. As an additional example, according to De Morgan's Laws in Theorem 2.18, if it is not the case that an integer a is even or an integer b is even, then it follows that a and b are both odd.

Example 2.20 *Using the second of De Morgan's Laws and (2.1), we can establish a useful logically equivalent form of the negation of $P \Leftrightarrow Q$ by the following string of logical equivalences:*

$$\begin{aligned} \sim (P \Leftrightarrow Q) &\equiv \sim((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \\ &\equiv (\sim(P \Rightarrow Q)) \vee (\sim(Q \Rightarrow P)) \\ &\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)). \end{aligned}$$ ♦

What we have observed about the negation of an implication and a biconditional is repeated in the following theorem.

Theorem 2.21 *For statements P and Q,*

(a) $\sim (P \Rightarrow Q) \equiv P \wedge (\sim Q)$
(b) $\sim (P \Leftrightarrow Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$.

2.10 Quantified Statements

We have mentioned that if $P(x)$ is an open sentence over a domain S, then $P(x)$ is a statement for each $x \in S$. We illustrate this again.

Example 2.22 *If $S = \{1, 2, \cdots, 7\}$, then*

$$P(n): \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is a statement for each $n \in S$. Therefore,

$P(1)$: 3 *is prime.*
$P(2)$: 7 *is prime.*
$P(3)$: 11 *is prime.*
$P(4)$: 19 *is prime.*

are true statements, while

$P(5)$: 27 *is prime.*
$P(6)$: 39 *is prime.*
$P(7)$: 51 *is prime.*

are false statements. ♦

There are other ways that an open sentence can be converted into a statement, namely by a method called **quantification**. Let $P(x)$ be an open sentence over a domain S. Adding the phrase "For every $x \in S$" to $P(x)$ produces a statement called a **quantified statement**. The phrase "for every" is referred to as the **universal quantifier** and is denoted by the symbol $\forall$. Other ways to express the universal quantifier are "for each" and "for all". This quantified statement is expressed in symbols by

$$\forall x \in S, P(x) \tag{2.2}$$

and is expressed in words by

$$\text{For every } x \in S, P(x). \tag{2.3}$$

The quantified statement (2.2) (or (2.3)) is true if $P(x)$ is true for every $x \in S$; while the quantified statement (2.2) is false if $P(x)$ is false for at least one element $x \in S$.

Another way to convert an open sentence $P(x)$ over a domain S into a statement through quantification is by the introduction of a quantifier called an existential quantifier. Each of the phrases "there exists", "there is", "for some", and "for at least one" is referred to as an **existential quantifier** and is denoted by the symbol $\exists$. The quantified statement

$$\exists x \in S, P(x) \tag{2.4}$$

can be expressed in words by

$$\text{There exists } x \in S \text{ such that } P(x). \tag{2.5}$$

The quantified statement (2.4) (or (2.5)) is true if $P(x)$ is true for at least one element $x \in S$, while the quantified statement (2.4) is false if $P(x)$ is false for all $x \in S$.

We now consider two quantified statements constructed from the open sentence we saw in Example 2.22.

Example 2.23 *For the open sentence*

$$P(n): \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

over the domain $S = \{1, 2, \cdots, 7\}$, the quantified statement

$$\forall n \in S, P(n): \text{For every } n \in S, \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is false since $P(5)$ is false, for example; while the quantified statement

$$\exists n \in S, P(n): \text{There exists } n \in S \text{ such that } \frac{2n^2 + 5 + (-1)^n}{2} \text{ is prime.}$$

is true since $P(1)$ is true, for example. ♦

The quantified statement $\forall x \in S, P(x)$ can also be expressed as

$$\text{If } x \in S, \text{ then } P(x).$$

Consider the open sentence $P(x): x^2 \geq 0$. over the set $\mathbf{R}$ of real numbers. Then

$$\forall x \in \mathbf{R}, P(x)$$

or, equivalently,

$$\forall x \in \mathbf{R}, x^2 \geq 0$$

can be expressed as

$$\text{For every real number } x, x^2 \geq 0.$$

or

$$\text{If } x \text{ is a real number, then } x^2 \geq 0.$$

as well as

$$\text{The square of every real number is nonnegative.}$$

In general, the universal quantifier is used to claim that the statement resulting from a given open sentence is true when each value of the domain of the variable is assigned to the variable. Consequently, the statement $\forall x \in \mathbf{R}, x^2 \geq 0$ is true since $x^2 \geq 0$ is true for every real number x.

Suppose now that we were to consider the open sentence $Q(x): x^2 \leq 0$. The statement $\forall x \in \mathbf{R}, Q(x)$ (that is, for every real number x, we have $x^2 \leq 0$) is false since, for example, $Q(1)$ is false. Of course, this means that its negation is true. If it were not the case that for every real number x, we have $x^2 \leq 0$, then there must exist some real number x such that $x^2 > 0$. This negation

$$\text{There exists a real number } x \text{ such that } x^2 > 0.$$

can be written in symbols as

$$\exists x \in \mathbf{R}, x^2 > 0 \text{ or } \exists x \in \mathbf{R}, \sim Q(x).$$

More generally, if we are considering an open sentence $P(x)$ over a domain S, then

$$\sim(\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x).$$

Example 2.24 *Suppose that we are considering the set $A = \{1, 2, 3\}$ and its power set $\mathcal{P}(A)$, the set of all subsets of A. Then the quantified statement*

$$\text{For every set } B \in \mathcal{P}(A), A - B \neq \emptyset. \tag{2.6}$$

is false since for the subset $B = A = \{1, 2, 3\}$, we have $A - B = \emptyset$. The negation of the statement (2.6) is

$$\text{There exists } B \in \mathcal{P}(A) \text{ such that } A - B = \emptyset. \tag{2.7}$$

The statement (2.7) is therefore true since for $B = A \in \mathcal{P}(A)$, we have $A - B = \emptyset$. The statement (2.6) can also be written as

$$\text{If } B \subseteq A, \text{ then } A - B \neq \emptyset. \tag{2.8}$$

Consequently, the negation of (2.8) can be expressed as

$$\text{There exists some subset } B \text{ of } A \text{ such that } A - B = \emptyset.$$ ♦

The existential quantifier is used to claim that at least one statement resulting from a given open sentence is true when the values of a variable are assigned from its domain. We know that for an open sentence $P(x)$ over a domain S, the quantified statement $\exists x \in S, P(x)$ is true provided $P(x)$ is a true statement for at least one element $x \in S$. Thus the statement $\exists x \in \mathbf{R}, x^2 > 0$ is true since, for example, $x^2 > 0$ is true for $x = 1$.

The quantified statement

$$\exists x \in \mathbf{R}, \ 3x = 12$$

is therefore true since there is some real number x for which $3x = 12$, namely $x = 4$ has this property. (Indeed, $x = 4$ is the *only* real number for which $3x = 12$.) On the other hand, the quantified statement

$$\exists n \in \mathbf{Z}, \ 4n - 1 = 0$$

is false as there is no integer n for which $4n - 1 = 0$. (Of course, $4n - 1 = 0$ when $n = 1/4$ but $1/4$ is not an integer.)

Suppose that $Q(x)$ is an open sentence over a domain S. If the statement $\exists x \in S, Q(x)$ is *not* true, then it must be the case that for every $x \in S, Q(x)$ is false. That is,

$$\sim(\exists x \in S, Q(x)) \equiv \forall x \in S, \sim Q(x).$$

We illustrate this with a specific example.

Example 2.25 *The following statement contains the existential quantifier:*

$$\text{There exists a real number } x \text{ such that } x^2 = 3. \tag{2.9}$$

If we let $P(x): x^2 = 3$, then (2.9) can be rewritten as $\exists x \in \mathbf{R}, P(x)$. The statement (2.9) is true since $P(x)$ is true when $x = \sqrt{3}$ (or when $x = -\sqrt{3}$). Hence the negation of (2.9) is:

$$\text{For every real number } x, x^2 \neq 3. \tag{2.10}$$

The statement (2.10) is therefore false. ♦

Let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. Then the quantified statement

$$\text{For all } x \in S \text{ and } y \in T, P(x, y).$$

can be expressed symbolically as

$$\forall x \in S, \forall y \in T, \ P(x, y). \tag{2.11}$$

The negation of the statement (2.11) is

$$\begin{aligned} \sim(\forall x \in S, \forall y \in T, \ P(x, y)) &\equiv \exists x \in S, \ \sim(\forall y \in T, \ P(x, y)) \\ &\equiv \exists x \in S, \exists y \in T, \ \sim P(x, y). \end{aligned} \tag{2.12}$$

We now consider examples of quantified statements involving two variables.

Example 2.26 *Consider the statement*

$$\text{For every two real numbers } x \text{ and } y,\ x^2 + y^2 \geq 0. \tag{2.13}$$

If we let

$$P(x, y):\ x^2 + y^2 \geq 0$$

where the domain of both x and y is $\mathbf{R}$, then statement (2.13) can be expressed as

$$\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y) \tag{2.14}$$

or as

$$\forall x, y \in \mathbf{R},\ P(x, y).$$

Since $x^2 \geq 0$ and $y^2 \geq 0$ for all real numbers x and y and so $x^2 + y^2 \geq 0$, $P(x, y)$ is true for all real numbers x and y and the quantified statement (2.14) is true.

The negation of statement (2.14) is therefore

$$\sim(\forall x \in \mathbf{R}, \forall y \in \mathbf{R},\ P(x, y)) \equiv \exists x \in \mathbf{R}, \exists y \in \mathbf{R},\ \sim P(x, y), \tag{2.15}$$

which, in words, is

$$\text{There exist real numbers } x \text{ and } y \text{ such that } x^2 + y^2 < 0. \tag{2.16}$$

The statement (2.16) is therefore false. ♦

For an open sentence containing two variables, the domains of the variables need not be the same.

Example 2.27 *Consider the statement*

$$\text{For every } s \in S \text{ and } t \in T,\ st + 2 \text{ is a prime.} \tag{2.17}$$

where the domain of the variable s is $S = \{1, 3, 5\}$ and the domain of the variable t is $T = \{3, 9\}$. If we let

$$Q(s, t):\ st + 2 \text{ is a prime.}$$

then the statement (2.17) can be expressed as

$$\forall s \in S, \forall t \in T,\ Q(s, t). \tag{2.18}$$

Since all of the statements

$Q(1, 3):\ 1 \cdot 3 + 2$ *is a prime.* $\quad Q(3, 3):\ 3 \cdot 3 + 2$ *is a prime.*
$Q(5, 3):\ 5 \cdot 3 + 2$ *is a prime.*

$Q(1, 9):\ 1 \cdot 9 + 2$ *is a prime.* $\quad Q(3, 9):\ 3 \cdot 9 + 2$ *is a prime.*
$Q(5, 9):\ 5 \cdot 9 + 2$ *is a prime.*

are true, the quantified statement (2.18) is true.

As we saw in (2.12), the negation of the quantified statement (2.18) is

$$\sim(\forall s \in S, \forall t \in T,\ Q(s, t)) \equiv \exists s \in S, \exists t \in T,\ \sim Q(s, t)$$

and so the negation of (2.17) is

$$\text{There exist } s \in S \text{ and } t \in T \text{ such that } st + 2 \text{ is not a prime.} \tag{2.19}$$

The statement (2.19) is therefore false. ♦

Again, let $P(x, y)$ be an open sentence, where the domain of the variable x is S and the domain of the variable y is T. The quantified statement

$$\text{There exist } x \in S \text{ and } y \in T \text{ such that } P(x, y).$$

can be expressed in symbols as

$$\exists x \in S, \exists y \in T,\ P(x, y). \tag{2.20}$$

The negation of the statement (2.20) is

$$\begin{aligned} \sim(\exists x \in S, \exists y \in T,\ P(x, y)) &\equiv \forall x \in S,\ \sim(\exists y \in T,\ P(x, y)) \\ &\equiv \forall x \in S, \forall y \in T,\ \sim P(x, y). \end{aligned} \tag{2.21}$$

We now illustrate this situation.

Example 2.28 *Consider the open sentence*

$$R(s, t):\ |s - 1| + |t - 2| \leq 2,$$

where the domain of the variable s is the set S of even integers and the domain of the variable t is the set T of odd integers. Then the quantified statement

$$\exists s \in S, \exists t \in T, R(s, t) \tag{2.22}$$

can be expressed in words as

$$\text{There exist an even integer } s \text{ and an odd integer } t \text{ such that } |s - 1| + |t - 2| \leq 2. \tag{2.23}$$

Since $R(2, 3):\ 1 + 1 \leq 2$ is true, the quantified statement (2.23) is true.

The negation of (2.22) is therefore

$$\sim(\exists s \in S, \exists t \in T,\ R(s, t)) \equiv \forall s \in S, \forall t \in T,\ \sim R(s, t) \tag{2.24}$$

and so the negation of (2.22), in words, is

$$\text{For every even integer } s \text{ and every odd integer } t,\ |s - 1| + |t - 2| > 2. \tag{2.25}$$

The quantified statement (2.25) is therefore false. ♦

Quantified statements may contain both universal and existential quantifiers. We will encounter this in Section 7.2.

Let's review some symbols that we have introduced in this chapter:

$\sim$	negation (not)
$\vee$	disjunction (or)
$\wedge$	conjunction (and)
$\Rightarrow$	implication
$\Leftrightarrow$	biconditional
$\forall$	universal quantifier (for every)
$\exists$	existential quantifier (there exists)

2.11 Characterizations of Statements

Let's return to the biconditional $P \Leftrightarrow Q$. Recall that $P \Leftrightarrow Q$ represents the compound statement $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. Earlier, we described how this compound statement can be expressed as

$$P \text{ if and only if } Q.$$

Many mathematicians abbreviate the phrase "if and only if" by writing "iff". Although "iff" is informal and, of course, is not a word, its use is common and you should be familiar with it.

Recall that whenever you see

$$P \text{ if and only if } Q.$$

or

$$P \text{ is necessary and sufficient for } Q.$$

this means

$$\text{If } P \text{ then } Q \textit{ and } \text{if } Q \text{ then } P.$$

Example 2.29 *Suppose that*

$$P(x) : x = -3 \textit{ and } Q(x) : |x| = 3,$$

where $x \in \mathbf{R}$. *Then the biconditional* $P(x) \Leftrightarrow Q(x)$ *can be expressed as*

$$x = -3 \textit{ if and only if } |x| = 3.$$

or

$$x = -3 \textit{ is necessary and sufficient for } |x| = 3.$$

or, perhaps better, as

$$x = -3 \textit{ is a necessary and sufficient condition for } |x| = 3.$$

Let's now consider the quantified statement $\forall x \in \mathbf{R}$, $P(x) \Leftrightarrow Q(x)$. *This statement is false because* $P(3) \Leftrightarrow Q(3)$ *is false.* ♦

Suppose that some concept (or object) is expressed in an open sentence $P(x)$ over a domain S and $Q(x)$ is another open sentence over the domain S concerning this concept. We say that this concept is **characterized** by $Q(x)$ if $\forall x \in S$, $P(x) \Leftrightarrow Q(x)$ is a true statement. The statement $\forall x \in S$, $P(x) \Leftrightarrow Q(x)$ is then called a **characterization** of this concept. For example, irrational numbers are defined as real numbers that are not rational and are characterized as real numbers whose decimal expansions are nonrepeating. This provides a characterization of irrational numbers:

A real number r is irrational if and only if r has a nonrepeating decimal expansion.

We saw that equilateral triangles are defined as triangles whose sides are equal. They are characterized however as triangles whose angles are equal. Therefore, we have the characterization:

A triangle T is equilateral if and only if T has three equal angles.

You might think that equilateral triangles are also characterized as those triangles having three equal sides, but the associated biconditional:

A triangle T is equilateral if and only if T has three equal sides.

is not a characterization of equilateral triangles. Indeed, this is the definition we gave of equilateral triangles. A characterization of a concept then gives an alternative, but equivalent, way of looking at this concept. Characterizations are often valuable in studying concepts or in proving other results. We will see examples of this in future chapters.

We mentioned that the following biconditional, though true, is not a characterization: A triangle T is equilateral if and only if T has three equal sides. Although this is the definition of equilateral triangles, mathematicians rarely use the phrase "if and only if" in a definition since this is what is meant in a definition. That is, a triangle is defined to be equilateral if it has three equal sides. Consequently, a triangle with three equal sides is equilateral, but a triangle that does not have three equal sides is not equilateral.

EXERCISES FOR CHAPTER 2

Section 2.1: Statements

2.1. Which of the following sentences are statements? For those that are, indicate the truth value.

(a) The integer 123 is prime.
(b) The integer 0 is even.
(c) Is $5 \times 2 = 10$?
(d) $x^2 - 4 = 0$.
(e) Multiply $5x + 2$ by 3.
(f) $5x + 3$ is an odd integer.
(g) What an impossible question!

2.2. Consider the sets A, B, C, and D below. Which of the following statements are true? Give an explanation for each false statement.

$$A = \{1, 4, 7, 10, 13, 16, \ldots\} \quad C = \{x \in \mathbf{Z} : x \text{ is prime and } x \neq 2\}$$
$$B = \{x \in \mathbf{Z} : x \text{ is odd}\} \quad D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$$

(a) $25 \in A$, (b) $33 \in D$, (c) $22 \notin A \cup D$, (d) $C \subseteq B$, (e) $\emptyset \in B \cap D$, (f) $53 \notin C$.

2.3. Which of the following statements are true? Give an explanation for each false statement.
(a) $\emptyset \in \emptyset$ (b) $\emptyset \in \{\emptyset\}$ (c) $\{1, 3\} = \{3, 1\}$
(d) $\emptyset = \{\emptyset\}$ (e) $\emptyset \subset \{\emptyset\}$ (f) $1 \subseteq \{1\}$.

2.4. The following is an open sentence over the domain $\mathbf{R}$:

$$P(x) : x(x - 1) = 6.$$

(a) For what values of x is $P(x)$ a true statement?
(b) For what values of x is $P(x)$ a false statement?

2.5. For the open sentence $P(x) : 3x - 2 > 4$ over the domain $\mathbf{Z}$, determine:

(a) the values of x for which $P(x)$ is true;
(b) the values of x for which $P(x)$ is false.

2.6. For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, determine:

(a) all $A \in S$ for which $P(A)$ is true;
(b) all $A \in S$ for which $P(A)$ is false;
(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$.

2.7. Let

$$P(n) : n \text{ and } n + 2 \text{ are primes.}$$

be an open sentence over the domain $\mathbf{N}$. Find six positive integers n for which $P(n)$ is true. If $n \in \mathbf{N}$ such that $P(n)$ is true, then the two integers n, $n + 2$ are called **twin primes**. It has been conjectured that there are infinitely many twin primes.

Section 2.2: The Negation of a Statement

2.8. State the negation of each of the following statements.

(a) $\sqrt{2}$ is a rational number.
(b) 0 is not a negative integer.
(c) 111 is a prime number.

2.9. Complete the truth table in Figure 2.16.

P	Q	$\sim P$	$\sim Q$
T	T		
T	F		
F	T		
F	F		

Figure 2.16 The truth table for Exercise 2.9.

P	Q	$\sim Q$	$P \wedge (\sim Q)$
T	T		
T	F		
F	T		
F	F		

Figure 2.17 The truth table for Exercise 2.12.

Section 2.3: The Disjunction and Conjunction of Statements

2.10. Let P: 15 is odd and Q : 21 is prime. State each of the following in words, and determine whether they are true or false. (a) $P \vee Q$ (b) $P \wedge Q$ (c) $(\sim P) \vee Q$ (d) $P \wedge (\sim Q)$.

2.11. For the sets $A = \{1, 2, \cdots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

$$P : A \subseteq B. \quad Q : |A - B| = 6.$$

Determine which of the following statements are true:
(a) $P \vee Q$ (b) $P \vee (\sim Q)$ (c) $P \wedge Q$
(d) $(\sim P) \wedge Q$ (e) $(\sim P) \vee (\sim Q)$.

2.12. Complete the truth table in Figure 2.17.

2.13. Let $S = \{1, 2, \ldots, 6\}$ and let

$$P(A) : A \cap \{2, 4, 6\} = \emptyset \text{ and } Q(A) : A \neq \emptyset.$$

be open sentences over the domain $\mathcal{P}(S)$.

(a) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \wedge Q(A)$ is true.
(b) Determine all $A \in \mathcal{P}(S)$ for which $P(A) \vee (\sim Q(A))$ is true.
(c) Determine all $A \in \mathcal{P}(S)$ for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Section 2.4: The Implication

2.14. Consider the statements P : 17 is even and Q : 19 is prime. Write each of the following statements in words, and indicate whether it is true or false.
(a) $\sim P$ (b) $P \vee Q$ (c) $P \wedge Q$ (d) $P \Rightarrow Q$.

2.15. For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\sim P)$.

2.16. Consider the statements P : $\sqrt{2}$ is rational and Q : 22/7 is rational. Write each of the following statements in words and indicate whether it is true or false.
(a) $P \Rightarrow Q$ (b) $Q \Rightarrow P$ (c) $(\sim P) \Rightarrow (\sim Q)$ (d) $(\sim Q) \Rightarrow (\sim P)$.

2.17. Consider the statements:

$$P : \sqrt{2} \text{ is rational}, \quad Q : \tfrac{2}{3} \text{ is rational}, \quad R : \sqrt{3} \text{ is rational}.$$

Write each of the following statements in words and indicate whether the statement is true or false.

(a) $(P \wedge Q) \Rightarrow R$
(b) $(P \wedge Q) \Rightarrow (\sim R)$
(c) $((\sim P) \wedge Q) \Rightarrow R$
(d) $(P \vee Q) \Rightarrow (\sim R)$.

Section 2.5: More on Implications

2.18. Consider the open sentences $P(n): 5n+3$ is prime and $Q(n): 7n+1$ is prime over the domain $\mathbf{N}$.

(a) State $P(n) \Rightarrow Q(n)$ in words.
(b) State $P(2) \Rightarrow Q(2)$ in words. Is this statement true or false?
(c) State $P(6) \Rightarrow Q(6)$ in words. Is this statement true or false?

2.19. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine the truth value of $P(x) \Rightarrow Q(x)$ for each $x \in S$.

(a) $P(x): |x| = 4$; $Q(x): x = 4$; $S = \{-4, -3, 1, 4, 5\}$.
(b) $P(x): x^2 = 16$; $Q(x): |x| = 4$; $S = \{-6, -4, 0, 3, 4, 8\}$.
(c) $P(x): x > 3$; $Q(x): 4x - 1 > 12$; $S = \{0, 2, 3, 4, 6\}$.

2.20. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.

(a) $P(x): x - 3 = 4$; $Q(x): x \geq 8$; $S = \mathbf{R}$.
(b) $P(x): x^2 \geq 1$; $Q(x): x \geq 1$; $S = \mathbf{R}$.
(c) $P(x): x^2 \geq 1$; $Q(x): x \geq 1$; $S = \mathbf{N}$.
(d) $P(x): x \in [-1, 2]$; $Q(x): x^2 \leq 2$; $S = [-1, 1]$.

2.21. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbf{Z}$. Determine the truth value of $P(x, y) \Rightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y): x^2 - y^2 = 0$ and $Q(x, y): x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.
(b) $P(x, y): |x| = |y|$ and $Q(x, y): x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.
(c) $P(x, y): x^2 + y^2 = 1$ and $Q(x, y): x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

Section 2.6: The Biconditional

2.22. Let P : 18 is odd and Q : 25 is even. State $P \Leftrightarrow Q$ in words. Is $P \Leftrightarrow Q$ true or false?

2.23. Consider the open sentences:

$$P(x): x = -2. \text{ and } Q(x): x^2 = 4.$$

over the domain $S = \{-2, 0, 2\}$. State each of the following in words and determine all values of $x \in S$ for which the resulting statements are true.

(a) $\sim P(x)$ (b) $P(x) \vee Q(x)$ (c) $P(x) \wedge Q(x)$ (d) $P(x) \Rightarrow Q(x)$
(e) $Q(x) \Rightarrow P(x)$ (f) $P(x) \Leftrightarrow Q(x)$.

2.24. For the following open sentences $P(x)$ and $Q(x)$ over a domain S, determine all values of $x \in S$ for which the biconditional $P(x) \Leftrightarrow Q(x)$ is true.

(a) $P(x): |x| = 4$; $Q(x): x = 4$; $S = \{-4, -3, 1, 4, 5\}$.
(b) $P(x): x \geq 3$; $Q(x): 4x - 1 > 12$; $S = \{0, 2, 3, 4, 6\}$.
(c) $P(x): x^2 = 16$; $Q(x): x^2 - 4x = 0$; $S = \{-6, -4, 0, 3, 4, 8\}$.

2.25. Let $P(x): x$ is odd. and $Q(x): x^2$ is odd. be open sentences over the domain $\mathbf{Z}$. State $P(x) \Leftrightarrow Q(x)$ in two ways: (1) using "if and only if" and (2) using "necessary and sufficient".

2.26. For the open sentences $P(x): |x - 3| < 1$ and $Q(x): x \in (2, 4)$. over the domain $\mathbf{R}$, state the biconditional. $P(x) \Leftrightarrow Q(x)$ in two different ways.

2.27. In each of the following, two open sentences $P(x, y)$ and $Q(x, y)$ are given, where the domain of both x and y is $\mathbf{Z}$. Determine the truth value of $P(x, y) \Leftrightarrow Q(x, y)$ for the given values of x and y.

(a) $P(x, y): x^2 - y^2 = 0$ and $Q(x, y): x = y$.
$(x, y) \in \{(1, -1), (3, 4), (5, 5)\}$.
(b) $P(x, y): |x| = |y|$ and $Q(x, y): x = y$.
$(x, y) \in \{(1, 2), (2, -2), (6, 6)\}$.
(c) $P(x, y): x^2 + y^2 = 1$ and $Q(x, y): x + y = 1$.
$(x, y) \in \{(1, -1), (-3, 4), (0, -1), (1, 0)\}$.

2.28. Let $S = \{1, 2, 3\}$. Consider the following open sentences over the domain S:

$$P(n): \frac{(n+4)(n+5)}{2} \text{ is odd.}$$
$$Q(n): 2^{n-2} + 3^{n-2} + 6^{n-2} > (2.5)^{n-1}.$$

Determine three distinct elements a, b, c in S such that $P(a) \Rightarrow Q(a)$ is false, $Q(b) \Rightarrow P(b)$ is false, and $P(c) \Leftrightarrow Q(c)$ is true.

2.29. Let $S = \{1, 2, 3, 4\}$. Consider the following open sentences over the domain S:

$$P(n): \frac{n(n-1)}{2} \text{ is even.}$$
$$Q(n): 2^{n-2} - (-2)^{n-2} \text{ is even.}$$
$$R(n): 5^{n-1} + 2^n \text{ is prime.}$$

Determine four distinct elements a, b, c, d in S such that

(i) $P(a) \Rightarrow Q(a)$ is false; (ii) $Q(b) \Rightarrow P(b)$ is true;
(iii) $P(c) \Leftrightarrow R(c)$ is true; (iv) $Q(d) \Leftrightarrow R(d)$ is false.

Section 2.7: Tautologies and Contradictions

2.30. For statements P and Q, show that $P \Rightarrow (P \vee Q)$ is a tautology.

2.31. For statements P and Q, show that $(P \wedge \sim Q) \wedge (P \wedge Q)$ is a contradiction.

2.32. For statements P and Q, show that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology. Then state $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ in words. (This is an important logical argument form, called **modus ponens**.)

2.33. For statements P, Q, and R, show that $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology. Then state this compound statement in words. (This is another important logical argument form, called **syllogism**.)

Section 2.8: Logical Equivalence

2.34. For statements P and Q, the implication $(\sim P) \Rightarrow (\sim Q)$ is called the **inverse** of the implication $P \Rightarrow Q$.

(a) Use a truth table to show that these statements are not logically equivalent.
(b) Find another implication that is logically equivalent to $\sim P \Rightarrow \sim Q$ and verify your answer.

2.35. Let P and Q be statements.

(a) Is $\sim(P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.
(b) What can you say about the biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?

2.36. For statements P, Q, and R, use a truth table to show that each of the following pairs of statements are logically equivalent.

(a) $(P \wedge Q) \Leftrightarrow P$ and $P \Rightarrow Q$.
(b) $P \Rightarrow (Q \vee R)$ and $(\sim Q) \Rightarrow ((\sim P) \vee R)$.

2.37. For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent.

2.38. For statements P, Q, and R, show that $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are logically equivalent.

Section 2.9: Some Fundamental Properties of Logical Equivalence

2.39. Verify the following laws stated in Theorem 2.18:

(a) Let P, Q, and R be statements. Then

$$P \vee (Q \wedge R) \text{ and } (P \vee Q) \wedge (P \vee R) \text{ are logically equivalent.}$$

(b) Let P and Q be statements. Then

$$\sim(P \vee Q) \text{ and } (\sim P) \wedge (\sim Q) \text{ are logically equivalent.}$$

2.40. Write negations of the following open sentences:

(a) Either $x = 0$ or $y = 0$.

(b) The integers a and b are both even.

2.41. Consider the implication: If x and y are even, then xy is even.

(a) State the implication using "only if ".

(b) State the converse of the implication.

(c) State the implication as a disjunction (see Theorem 2.17).

(d) State the negation of the implication as a conjunction (see Theorem 2.21(a)).

2.42. For a real number x, let $P(x) : x^2 = 2$ and $Q(x) : x = \sqrt{2}$. State the negation of the biconditional $P \Leftrightarrow Q$ in words (see Theorem 2.21(b)).

Section 2.10: Quantified Statements

2.43. Let S denote the set of odd integers, and let

$$P(x) : x^2 + 1 \text{ is even.} \quad \text{and} \quad Q(x) : x^2 \text{ is even.}$$

be open sentences over the domain S. State $\forall x \in S$, $P(x)$ and $\exists x \in S$, $Q(x)$ in words.

2.44. Define an open sentence $R(x)$ over some domain S and then state $\forall x \in S$, $R(x)$ and $\exists x \in S$, $R(x)$ in words.

2.45. State the negations of the following quantified statements, where all sets are subsets of some universal set U:

(a) For every set A, $A \cap \overline{A} = \emptyset$.

(b) There exists a set A such that $\overline{A} \subseteq A$.

2.46. State the negations of the following quantified statements:

(a) For every rational number r, the number $1/r$ is rational.

(b) There exists a rational number r such that $r^2 = 2$.

2.47. Let $P(n) : (5n - 6)/3$ is an integer. be an open sentence over the domain $\mathbf{Z}$. Determine, with explanations, whether the following statements are true:

(a) $\forall n \in \mathbf{Z}$, $P(n)$.

(b) $\exists n \in \mathbf{Z}$, $P(n)$.

2.48. Determine the truth value of each of the following statements.

(a) $\exists x \in \mathbf{R}$, $x^2 - x = 0$.

(b) $\forall n \in \mathbf{N}$, $n + 1 \geq 2$.

(c) $\forall x \in \mathbf{R}$, $\sqrt{x^2} = x$.

(d) $\exists x \in \mathbf{Q}$, $3x^2 - 27 = 0$.

(e) $\exists x \in \mathbf{R}, \exists y \in \mathbf{R}$, $x + y + 3 = 8$.

(f) $\forall x, y \in \mathbf{R}$, $x + y + 3 = 8$.

(g) $\exists x, y \in \mathbf{R}$, $x^2 + y^2 = 9$.

(h) $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}$, $x^2 + y^2 = 9$.

2.49. The statement

For every integer m, either $m \leq 1$ or $m^2 \geq 4$.

can be expressed using a quantifier as:

$$\forall m \in \mathbf{Z},\ m \leq 1 \text{ or } m^2 \geq 4.$$

Do this for the statements in parts (a) and (b).

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.

(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

(c) Express in words the negations of the statements in (a) and (b).

(d) Using quantifiers, express in symbols the negations of the statements in both (a) and (b).

2.50. Consider the open sentence

$$P(x, y, z) : (x - 1)^2 + (y - 2)^2 + (z - 2)^2 > 0.$$

where the domain of each of the variables x, y and z is $\mathbf{R}$.

(a) Express the quantified statement $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, \forall z \in \mathbf{R}$, $P(x, y, z)$ in words.

(b) Is the quantified statement in (a) true or false? Explain.

(c) Express the negation of the quantified statement in (a) in symbols.

(d) Express the negation of the quantified statement in (a) in words.

(e) Is the negation of the quantified statement in (a) true or false? Explain.

2.51. Consider the quantified statement

For every $s \in S$ and $t \in S$, $st - 2$ is prime.

where the domain of the variables s and t is $S = \{3, 5, 11\}$.

(a) Express this quantified statement in symbols.

(b) Is the quantified statement in (a) true or false? Explain.

(c) Express the negation of the quantified statement in (a) in symbols.

(d) Express the negation of the quantified statement in (a) in words.

(e) Is the negation of the quantified statement in (a) true or false? Explain.

Section 2.11: Characterizations of Statements

2.52. Give a definition of each of the following, and then state a characterization of each.

(a) two lines in the plane are perpendicular

(b) a rational number

2.53. Define an integer n to be odd if n is not even. State a characterization of odd integers.

2.54. Define a triangle to be isosceles if it has two equal sides. Which of the following statements are characterizations of isosceles triangles? If a statement is not a characterization of isosceles triangles, then explain why.

(a) If a triangle is equilateral, then it is isosceles.

(b) A triangle T is isosceles if and only if T has two equal sides.

(c) If a triangle has two equal sides, then it is isosceles.
(d) A triangle T is isosceles if and only if T is equilateral.
(e) If a triangle has two equal angles, then it is isosceles.
(f) A triangle T is isosceles if and only if T has two equal angles.

2.55. By definition, a right triangle is a triangle one of whose angles is a right angle. Also, two angles in a triangle are complementary if the sum of their degrees is 90°. Which of the following statements are characterizations of a right triangle? If a statement is not a characterization of a right triangle, then explain why.

(a) A triangle is a right triangle if and only if two of its sides are perpendicular.
(b) A triangle is a right triangle if and only if it has two complementary angles.
(c) A triangle is a right triangle if and only if its area is half of the product of the lengths of some pair of its sides.
(d) A triangle is a right triangle if and only if the square of the length of its longest side equals the sum of the squares of the lengths of the two smallest sides.
(e) A triangle is a right triangle if and only if twice the area of the triangle equals the area of some rectangle.

ADDITIONAL EXERCISES FOR CHAPTER 2

2.56. Construct a truth table for $P \wedge (Q \Rightarrow \sim P)$.

2.57. Given that the implication $(Q \vee R) \Rightarrow \sim P$ is false and Q is false, determine the truth values of R and P.

2.58. Find a compound statement involving the component statements P and Q that has the truth table given in Figure 2.18.

2.59. Determine the truth value of each of the following quantified statements:

(a) $\exists x \in \mathbf{R}, x^2 - x = 0$.
(b) $\forall n \in \mathbf{N}, n + 1 \geq 2$.
(c) $\forall x \in \mathbf{R}, \sqrt{x^2} = x$.
(d) $\exists x \in \mathbf{Q}, \frac{1}{x^2} = \frac{1}{2}$.
(e) $\exists x, y \in \mathbf{R}, x + y + 3 = 8$.
(f) $\forall x, y \in \mathbf{R}, x + y + 3 = 8$.

2.60. Rewrite each of the implications below using (1) only if and (2) sufficient.

(a) If a function f is differentiable, then f is continuous.
(b) If $x = -5$, then $x^2 = 25$.

P	Q	$\sim Q$	
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Figure 2.18 Truth table for Exercise 2.58.

2.61. Let

$$P(n) : n^2 - n + 5 \text{ is a prime.}$$

be an open sentence over a domain S.

(a) Determine the truth values of the quantified statements $\forall n \in S$, $P(n)$ and $\exists n \in S$, $\sim P(n)$ for $S = \{1, 2, 3, 4\}$.
(b) Determine the truth values of the quantified statements $\forall n \in S$, $P(n)$ and $\exists n \in S$, $\sim P(n)$ for $S = \{1, 2, 3, 4, 5\}$.
(c) How are the statements in (a) and (b) related?

2.62. (a) For statements P, Q, and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv ((P \wedge (\sim R)) \Rightarrow (\sim Q)).$$

(b) For statements P, Q, and R, show that

$$((P \wedge Q) \Rightarrow R) \equiv (Q \wedge (\sim R) \Rightarrow (\sim P)).$$

2.63. For a fixed integer n, use Exercise 2.62 to restate the following implication in two different ways:

If n is a prime and $n > 2$, then n is odd.

2.64. For fixed integers m and n, use Exercise 2.62 to restate the following implication in two different ways:

If m is even and n is odd, then $m + n$ is odd.

2.65. For a real valued function f and a real number x, use Exercise 2.62 to restate the following implication in two different ways:

If $f'(x) = 3x^2 - 2x$ and $f(0) = 4$, then $f(x) = x^3 - x^2 + 4$.

2.66. For the set $S = \{1, 2, 3\}$, give an example of three open sentences $P(n)$, $Q(n)$, and $R(n)$, each over the domain S, such that (1) each of $P(n)$, $Q(n)$, and $R(n)$ is a true statement for exactly two elements of S, (2) all of the implications $P(1) \Rightarrow Q(1)$, $Q(2) \Rightarrow R(2)$, and $R(3) \Rightarrow P(3)$ are true, and (3) the converse of each implication in (2) is false.

2.67. Do there exist a set S of cardinality 2 and a set $\{P(n), Q(n), R(n)\}$ of three open sentences over the domain S such that the implications $P(a) \Rightarrow Q(a)$, $Q(b) \Rightarrow R(b)$, and $R(c) \Rightarrow P(c)$ are true, where $a, b, c \in S$, and (2) the converses of the implications in (1) are false? Necessarily, at least two of these elements a, b, and c of S are equal.

2.68. Let $A = \{1, 2, \ldots, 6\}$ and $B = \{1, 2, \ldots, 7\}$. For $x \in A$, let $P(x) : 7x + 4$ is odd. For $y \in B$, let $Q(y) : 5y + 9$ is odd. Let

$$S = \{(P(x), Q(y)) : x \in A, y \in B, P(x) \Rightarrow Q(y) \text{ is false}\}.$$

What is $|S|$?

3

Direct Proof and Proof by Contrapositive

We are now prepared to begin discussing our main topic: mathematical proofs. Initially, we will be primarily concerned with one question: For a given true mathematical statement, how can we show that it is true? In this chapter you will be introduced to two important proof techniques.

A true mathematical statement whose truth is accepted without proof is referred to as an **axiom**. For example, an axiom of Euclid in geometry states that for every line ℓ and a point P not on ℓ, there is a unique line containing P that is parallel to ℓ. A true mathematical statement whose truth can be verified is often referred to as a **theorem**, although many mathematicians reserve the word "theorem" for such statements that are especially significant or interesting. For example, the mathematical statement "$2 + 3 = 5$" is true but few, if any, would consider this to be a theorem under this latter interpretation. In addition to the word "theorem", other common terms for this idea include proposition, result, observation, and fact, the choice often depending on the significance or degree of difficulty in its proof. We will use the word "theorem" sparingly, however, primarily reserving it for true mathematical statements that will be used later. Otherwise, we will simply use the word "result". For the most part then, our results are examples used to illustrate proof techniques, and our goal is to prove these results.

A **corollary** is a mathematical result that can be deduced from, and is thereby a consequence of, some earlier result. A **lemma** is a mathematical result that is useful in establishing the truth of some other result. Some people like to think of a lemma as a "helping result". Indeed, the German word for lemma is "hilfsatz", whose English translation is "helping theorem". Ordinarily then, a lemma is not of primary importance itself. Indeed, its very existence is due only to its usefulness in proving another (more interesting) result.

Most theorems (or results) are stated as implications. We now begin our study of proofs of such mathematical statements.

3.1 Trivial and Vacuous Proofs

In nearly all of the implications $P \Rightarrow Q$ that we will encounter, P and Q are open sentences; that is, we will actually be considering $P(x) \Rightarrow Q(x)$ or $P(n) \Rightarrow Q(n)$ or some related implication, depending on which variable is being used. The variables x or n (or some other symbols) are used to represent elements of some set S being discussed, that is, S is the domain of the variable. As we have seen, for each value of a variable from its domain, a statement results. (It is possible, of course, that P and Q are expressed in terms of two or more variables.) Whether $P(x)$ (or $Q(x)$) is true ordinarily depends on which element $x \in S$ we are considering; that is, it is rarely the case that $P(x)$ is true for all $x \in S$ (or that $P(x)$ is false for all $x \in S$). For example, for

$$P(n) : 3n^2 - 4n + 1 \text{ is even}$$

where $n \in \mathbf{Z}$, $P(1)$ is a true statement while $P(2)$ is a false statement. Likewise, it is seldom the case that $Q(x)$ is true for all $x \in S$ or that $Q(x)$ is false for all $x \in S$.

When the quantified statement $\forall x \in S$, $P(x) \Rightarrow Q(x)$ is expressed as a result or theorem, we often write such a statement as

$$\text{For } x \in S, \text{ if } P(x) \text{ then } Q(x).$$

or as

$$\text{Let } x \in S. \text{ If } P(x), \text{ then } Q(x). \tag{3.1}$$

Thus (3.1) is true if $P(x) \Rightarrow Q(x)$ is a true statement for each $x \in S$, while (3.1) is false if $P(x) \Rightarrow Q(x)$ is false for at least one element $x \in S$. In (3.1), if $Q(x)$ is true for all $x \in S$ or $P(x)$ is false for all $x \in S$, then determining the truth or falseness of (3.1) becomes considerably easier. Indeed, if it can be shown that $Q(x)$ is true for all $x \in S$ (regardless of the truth value of $P(x)$), then, according to the truth table for the implication (shown in Figure 2.5), (3.1) is true. This constitutes a proof of (3.1) and is called a **trivial proof**. Accordingly, the statement

$$\text{Let } n \in \mathbf{Z}. \text{ If } n^3 > 0, \text{ then 3 is odd.}$$

is true and a (trivial) proof consists only of observing that 3 is an odd integer. However, let's look at a more interesting example of a trivial proof.

Result 3.1 *Let $x \in \mathbf{R}$. If $x < 0$, then $x^2 + 1 > 0$.*

Proof Since $x^2 \geq 0$ for each real number x, it follows that

$$x^2 + 1 > x^2 \geq 0.$$

Hence $x^2 + 1 > 0$. ∎

Consider

$$P(x) : x < 0 \text{ and } Q(x) : x^2 + 1 > 0$$

where $x \in \mathbf{R}$. Then Result 3.1 asserts the truth of: For all $x \in \mathbf{R}$, $P(x) \Rightarrow Q(x)$. Since we verified that $Q(x)$ is true for every $x \in \mathbf{R}$, it follows that $P(x) \Rightarrow Q(x)$ is true for all $x \in \mathbf{R}$ and so Result 3.1 is true. In this case, when considered over the domain $\mathbf{R}$, $Q(x)$ is actually a true statement. It is this fact that allowed us to give a trivial proof of Result 3.1.

The proof of Result 3.1 does not depend on $x < 0$. Indeed, provided that $x \in \mathbf{R}$, we could have replaced "$x < 0$" by any hypothesis (including the more satisfying "$x \in \mathbf{R}$") and the result would still be true. In fact, this new result has the same proof.

The symbol ∎ that occurs at the end of the proof of Result 3.1 indicates that the proof is complete. There are definite advantages to using ∎ (or some other symbol) to indicate the conclusion of a proof. First, as you start reading a proof, you can look ahead for this symbol (to determine the length of the proof). Also, without this symbol, you may continue to read past the end of the proof, still thinking that you're reading a proof of the result. When you reach this symbol, you are *supposed* to be convinced that the result is true. If you are, this is good! Everything happened as planned. On the other hand, if you're not convinced, then, to you, the writer hasn't presented a proof. This may not be the writer's fault, however.

In the past, the most common way to indicate that a proof has concluded was to write Q.E.D., which stands for the Latin phrase "quod erat demonstrandum", whose English translation is "which was to be demonstrated". Some still use it.

Let $P(x)$ and $Q(x)$ be open sentences over a domain S. Then $\forall x \in S$, $P(x) \Rightarrow Q(x)$ is a true statement if it can be shown that $P(x)$ is false for all $x \in S$ (regardless of the truth value of $Q(x)$), according to the truth table for implication. Such a proof is called a **vacuous proof** of $\forall x \in S$, $P(x) \Rightarrow Q(x)$. Therefore,

$$\text{Let } n \in \mathbf{Z}. \text{ If 3 is even, then } n^3 > 0.$$

is a true statement. Let's take a look, however, at a more interesting example of a vacuous proof.

Result 3.2 *Let $x \in \mathbf{R}$. If $x^2 - 2x + 2 \leq 0$, then $x^3 \geq 8$.*

Proof First observe that

$$x^2 - 2x + 1 = (x - 1)^2 \geq 0.$$

Therefore, $x^2 - 2x + 2 = (x - 1)^2 + 1 \geq 1 > 0$. Thus $x^2 - 2x + 2 \leq 0$ is false for all $x \in \mathbf{R}$ and the implication is true. ∎

For

$$P(x) : x^2 - 2x + 2 \leq 0 \text{ and } Q(x) : x^3 \geq 8$$

over the domain $\mathbf{R}$, Result 3.2 asserts the truth of $\forall x \in \mathbf{R}$, $P(x) \Rightarrow Q(x)$. Since we verified that $P(x)$ is false for every $x \in \mathbf{R}$, it follows that $P(x) \Rightarrow Q(x)$ is true for each $x \in \mathbf{R}$. Hence Result 3.2 is true. In this case, $P(x)$ is a false statement for each $x \in \mathbf{R}$. This is what permitted us to give a vacuous proof of Result 3.2.

In the proof of Result 3.2, the truth or falseness of $x^3 \geq 8$ played no role whatsoever. Indeed, had we replaced $x^3 \geq 8$ by $x^3 \leq 8$, for example, then neither the truth nor the proof of Result 3.2 would be affected. Whenever there is a vacuous proof of a result, we often say that the result follows **vacuously**. Although a trivial proof is almost never encountered in mathematics, the same thing cannot be said about vacuous proofs, as we will see later.

We consider one additional example.

Result 3.3 *Let* $S = \{n \in \mathbf{Z} : n \geq 2\}$ *and let* $n \in S$. *If* $2n + \frac{2}{n} < 5$, *then* $4n^2 + \frac{4}{n^2} < 25$.

Proof First, we observe that if $n = 2$, then $2n + \frac{2}{n} = 5$. Of course, $5 < 5$ is false. If $n \geq 3$, then $2n + \frac{2}{n} > 2n \geq 6$. So, when $n \geq 3$, $2n + \frac{2}{n} < 5$ is false as well. Thus $2n + \frac{2}{n} < 5$ is false for all $n \in S$. Hence the implication is true. ∎

In two of the examples that we presented to illustrate trivial and vacuous proofs, we used the fact (and assumed it was known) that 3 is odd. Also, in the proofs of Results 3.1 and 3.2, we used the fact that if r is any real number, then $r^2 \geq 0$. Although you are certainly familiar with this property of real numbers, it is essential that any facts used within a proof are known to and likely to be recalled by the reader. Facts used within a proof should not come as a surprise to the reader. This subject will be discussed in more detail shortly.

3.2 Direct Proofs

Typically, when we are discussing an implication $P(x) \Rightarrow Q(x)$ over some domain S, there is ordinarily some connection between $P(x)$ and $Q(x)$. That is, the truth value of $Q(x)$ for a particular $x \in S$ often depends on the truth value of $P(x)$ for that same element x, or the truth value of $P(x)$ depends on the truth value of $Q(x)$. These are the kinds of implications in which we are primarily interested, and it is the proofs of these types of results that will occupy much of our attention. We begin with the first major proof technique, which occurs more often in mathematics than any other technique.

Let $P(x)$ and $Q(x)$ be open sentences over a domain S. If $P(x)$ is false for some $x \in S$, then $P(x) \Rightarrow Q(x)$ is true for this element x. Hence we need only be concerned with showing that $P(x) \Rightarrow Q(x)$ is true for all $x \in S$ for which $P(x)$ is true. In a **direct proof** of $P(x) \Rightarrow Q(x)$ for all $x \in S$, we consider an arbitrary element $x \in S$ for which $P(x)$ is true and show that $Q(x)$ is true for this element x. To summarize then, to give a direct proof of $P(x) \Rightarrow Q(x)$ for all $x \in S$, we assume that $P(x)$ is true for some arbitrary element $x \in S$ and show that $Q(x)$ must be true as well for this element x.

In order to illustrate this type of proof (and others as well), we need to deal with mathematical topics with which we're all familiar. Let's first consider the integers and some of their elementary properties. We assume that you are familiar with the integers and the following properties of integers:

1. The negative of every integer is an integer.
2. The sum (and difference) of every two integers is an integer.
3. The product of every two integers is an integer.

We will agree that we can use any of these properties. No justification is required or expected. Initially, we will use even and odd integers to illustrate our proof techniques. In this case, however, any properties of even and odd integers must be verified before they can be used. For example, you probably know that the sum of every two even integers is even, but this must first be proved to be used. We need to lay some groundwork before any examples of direct proofs are given.

Since we will be working with even and odd integers, it is essential that we have precise definitions of these kinds of numbers. An integer n is defined to be **even** if $n = 2k$ for some integer k. For example, 10 is even since $10 = 2 \cdot 5$ (where, of course, 5 is an integer). Also, $-14 = 2(-7)$ is even, as is $0 = 2 \cdot 0$. The integer 17 is not even since there is no *integer* k for which $17 = 2k$. Thus we see that the set of all even integers is the set

$$S = \{2k : k \in \mathbf{Z}\} = \{\cdots, -4, -2, 0, 2, 4, \cdots\}.$$

We could define an integer n to be odd if it's not even, but it would be difficult to work with this definition. Instead, we define an integer n to be **odd** if $n = 2k + 1$ for some integer k. Now 17 is odd since $17 = 2 \cdot 8 + 1$. Also, -5 is odd because $-5 = 2(-3) + 1$. On the other hand, 26 is not odd since there is no *integer* k such that $26 = 2k + 1$. In fact, 26 is even. Hence, according to the definition of odd integers that we have just given, we see that the set of all odd integers is precisely the set

$$T = \{2k + 1 : k \in \mathbf{Z}\} = \{\cdots, -5, -3, -1, 1, 3, 5, \cdots\}.$$

Observe that S and T are disjoint sets and $S \cup T = \mathbf{Z}$; that is, $\mathbf{Z}$ is partitioned into S and T. Therefore, every integer is either even or odd.

From time to time, we will find ourselves in a position where we have a result to prove and it may not be entirely clear how to proceed. In such a case, we need to consider our options and develop a plan, which we refer to as a **proof strategy**. The idea is to discuss a proof strategy for the result and, from it, construct a proof. At other times, we may wish to reflect on a proof that we have just given in order to understand it better. Such a discussion will be referred to as a **proof analysis**. As with examples, we conclude both a proof strategy and a proof analysis with the symbol ♦.

We are now prepared to illustrate the direct proof technique. We follow the proof by a proof analysis.

Result 3.4 *If* n *is an odd integer, then* $3n + 7$ *is an even integer.*

Proof Assume that n is an odd integer. Since n is odd, we can write $n = 2k + 1$ for some integer k. Now

$$3n + 7 = 3(2k + 1) + 7 = 6k + 3 + 7 = 6k + 10 = 2(3k + 5).$$

Since $3k + 5$ is an integer, $3n + 7$ is even. ∎

PROOF ANALYSIS First, notice that Result 3.4 could have been stated as:

For every odd integer n, the integer $3n + 7$ is even.

Thus the domain of the variable n in Result 3.4 is the set of odd integers. In the proof of Result 3.4, the expression $2k + 1$ was substituted for n in $3n + 7$ and simplified as $6k + 10$. Since our goal was to show that $3n + 7$ is even, we needed to show that $3n + 7$ can be expressed as twice an integer. Consequently, we factored 2 from $6k + 10$ and wrote it as $2(3k + 5)$. Since 3 and k are integers, so is $3k$ (the product of two integers is an integer). Since $3k$ and 5 are integers, so is $3k + 5$ (the sum of two integers is an integer). Therefore, $3n + 7$ satisfies the definition of an even integer.

One other remark deserves mention here. In the second sentence, we wrote:

Since n is odd, we can write $n = 2k + 1$ for some integer k.

It would be incorrect to write: "*If* n is odd" rather than "Since n is odd" because we have already assumed that n is odd and therefore n is now known to be odd. ♦

We defined an integer n to be odd if we can write n as $2k + 1$ for some integer k. This means that whenever we want to show that an integer, say m, is odd, we must follow this definition; that is, we must show that $m = 2k + 1$ for some integer k. (Of course, the use of the symbol k is not important. For example, an odd integer n can be written as $n = 2\ell + 1$ for some integer ℓ.) We could have defined an integer n to be odd if it is possible to write $n = 2k - 1$ for some integer k, but we *didn't*. However, if we could prove that an integer n is odd if and only if n can be expressed as $2k - 1$ for some integer k, then we could use this characterization of odd integers to show that an integer is odd. This, however, would require additional work on our part, with no obvious benefit. Similarly, we could have defined an integer n to be even if we can write $n = 2k + 2$, or $n = 2k - 2$, or perhaps $n = 2k + 100$ for some integer k. The definitions of even and odd integers that we chose are probably the most commonly used. Any other definitions that could have been given provide no special advantage to us.

Result 3.4 is an example of a direct proof. Let

$$P(n) : n \text{ is an odd integer. and } Q(n) : 3n + 7 \text{ is an even integer.}$$

over the domain S of odd integers. Then we have verified Result 3.4 by assuming that $P(n)$ is true for an arbitrary element $n \in S$ and then showing that $Q(n)$ is true for this element. Showing that $Q(n)$ is true essentially required one step on our part. As we venture further into proofs, we will see that we can't always establish the truth of the desired conclusion so quickly. It may be necessary to establish the truth of some other mathematical statements along the way that can then be used to establish the truth of $Q(n)$. We will see examples of this later.

Let's consider another example. For variety, we use an alternative opening sentence and different symbols in the proof of the following result.

Result 3.5 *If n is an even integer, then $-5n - 3$ is an odd integer.*

Proof Let n be an even integer. Then $n = 2x$, where x is an integer. Therefore,

$$-5n - 3 = -5(2x) - 3 = -10x - 3 = -10x - 4 + 1 = 2(-5x - 2) + 1.$$

Since $-5x - 2$ is an integer, $-5n - 3$ is an odd integer. ■

We now consider another example, which may have a surprise ending.

Result 3.6 *If n is an odd integer, then $4n^3 + 2n - 1$ is odd.*

Proof Assume that n is odd. Then $n = 2y + 1$ for some integer y. Therefore,

$$\begin{aligned} 4n^3 + 2n - 1 &= 4(2y + 1)^3 + 2(2y + 1) - 1 \\ &= 4(8y^3 + 12y^2 + 6y + 1) + 4y + 2 - 1 \\ &= 32y^3 + 48y^2 + 28y + 5 \\ &= 2(16y^3 + 24y^2 + 14y + 2) + 1. \end{aligned}$$

Since $16y^3 + 24y^2 + 14y + 2$ is an integer, $4n^3 + 2n - 1$ is odd. ■

PROOF ANALYSIS Although the direct proof of Result 3.6 that we gave is correct, this is *not* the desired proof. Indeed, had we observed that

$$4n^3 + 2n - 1 = 4n^3 + 2n - 2 + 1 = 2(2n^3 + n - 1) + 1$$

and that $2n^3 + n - 1 \in \mathbf{Z}$, we could have concluded immediately that $4n^3 + 2n - 1$ is odd for *every* integer n. Hence a trivial proof of Result 3.6 could be given and, in fact, is preferred. The fact that $4n^3 + 2n - 1$ is odd does not depend on n being odd. Indeed, it would be far better to replace the statement of Result 3.6 by

If n is an integer, then $4n^3 + 2n - 1$ is odd. ♦

We give an additional example of a somewhat different type.

Result 3.7 *Let $S = \{1, 2, 3\}$ and let $n \in S$. If $\dfrac{n(n+3)}{2}$ is even, then $\dfrac{(n+2)(n-5)}{2}$ is even.*

Proof Let $n \in S$ such that $n(n + 3)/2$ is even. Since $n(n + 3)/2 = 2$ when $n = 1$, $n(n + 3)/2 = 5$ when $n = 2$, and $n(n + 3)/2 = 9$ when $n = 3$, it follows that $n = 1$. When $n = 1$, $(n + 2)(n - 5)/2 = -6$, which is even. Therefore, the implication is true. ■

PROOF ANALYSIS In the proof of Result 3.7, we were concerned only with those elements $n \in S$ for which $n(n + 3)/2$ is even. Furthermore, it is not initially clear for which elements n of S the integer $n(n + 3)/2$ is even. Since S consists only of three elements, this can be determined rather quickly, which is what we did. We saw that only $n = 1$ has the desired property and this is the only element we needed to consider. ♦

If our goal is to establish the truth of $P(x) \Rightarrow Q(x)$ for all x in a domain S by means of a direct proof, then the proof begins by assuming that $P(x)$ is true for an arbitrary element $x \in S$. It is often common in this situation, however, to omit the initial assumption that $P(x)$ is true for an arbitrary element $x \in S$. It is then understood that we are giving a direct proof. We illustrate this with a short example.

Result 3.8 *If n is an even integer, then $3n^5$ is an even integer.*

Proof Since n is an even integer, $n = 2x$ for some integer x. Therefore,

$$3n^5 = 3(2x)^5 = 3(32x^5) = 96x^5 = 2(48x^5).$$

Since $48x^5 \in \mathbf{Z}$, the integer $3n^5$ is even. ■

For the present, when giving a direct proof of $P(x) \Rightarrow Q(x)$ for all x in a domain S, we will often include the initial assumption that $P(x)$ is true for an arbitrary element $x \in S$ in order to solidify this technique in your mind.

3.3 Proof by Contrapositive

For statements P and Q, the **contrapositive** of the implication $P \Rightarrow Q$ is the implication $(\sim Q) \Rightarrow (\sim P)$. For example, for P_1 : 3 is odd and P_2 : 57 is prime, the contrapositive of the implication

$$P_1 \Rightarrow P_2 : \text{ If 3 is odd, then 57 is prime.}$$

is the implication

$$(\sim P_2) \Rightarrow (\sim P_1) : \text{ If 57 is not prime, then 3 is even.}$$

The most important feature of the contrapositive $(\sim Q) \Rightarrow (\sim P)$ is that it is logically equivalent to $P \Rightarrow Q$. This fact is stated formally as a theorem and is verified in the truth table shown in Figure 3.1.

Theorem 3.9 *For every two statements P and Q, the implication $P \Rightarrow Q$ and its contrapositive are logically equivalent; that is,*

$$P \Rightarrow Q \equiv (\sim Q) \Rightarrow (\sim P).$$

Let

$$P(x) : x = 2 \text{ and } Q(x) : x^2 = 4$$

where $x \in \mathbf{R}$. The contrapositive of the implication

$$P(x) \Rightarrow Q(x) : \text{If } x = 2, \text{ then } x^2 = 4.$$

is the implication

$$(\sim Q(x)) \Rightarrow (\sim P(x)) : \text{If } x^2 \neq 4, \text{ then } x \neq 2.$$

Suppose that we wish to prove a result (or theorem) which is expressed as

$$\text{Let } x \in S. \text{ If } P(x), \text{ then } Q(x). \tag{3.2}$$

or as

$$\text{For all } x \in S, \text{ if } P(x), \text{ then } Q(x). \tag{3.3}$$

We have seen that a proof of such a result consists of establishing the truth of the implication $P(x) \Rightarrow Q(x)$ for all $x \in S$. If it can be shown that $(\sim Q(x)) \Rightarrow (\sim P(x))$ is

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Figure 3.1 The logical equivalence of an implication and its contrapositive

true for all $x \in S$, then $P(x) \Rightarrow Q(x)$ is true for all $x \in S$. A **proof by contrapositive** of the result (3.2) (or of (3.3)) is a direct proof of its contrapositive:

$$\text{Let } x \in S. \text{ If } \sim Q(x), \text{ then } \sim P(x).$$

or

$$\text{For all } x \in S, \text{ if } \sim Q(x), \text{ then } \sim P(x).$$

Thus to give a proof by contrapositive of (3.2) (or of (3.3)), we assume that $\sim Q(x)$ is true for an arbitrary element $x \in S$ and show that $\sim P(x)$ is true for this element x.

There are certain types of results where a proof by contrapositive is preferable, or perhaps even essential. We now give some examples to illustrate this method of proof.

Result 3.10 *Let $x \in \mathbf{Z}$. If $5x - 7$ is even, then x is odd.*

Proof Assume that x is even. Then $x = 2a$ for some integer a. So

$$5x - 7 = 5(2a) - 7 = 10a - 7 = 10a - 8 + 1 = 2(5a - 4) + 1.$$

Since $5a - 4 \in \mathbf{Z}$, the integer $5x - 7$ is odd. ■

PROOF ANALYSIS Some comments are now in order. The goal of Result 3.10 was to prove $P(x) \Rightarrow Q(x)$ for all $x \in \mathbf{Z}$, where $P(x) : 5x - 7$ is even and $Q(x) : x$ is odd. Since we chose to give a proof by contrapositive, we gave a direct proof of $(\sim Q(x)) \Rightarrow (\sim P(x))$ for all $x \in \mathbf{Z}$. Hence the proof began by assuming that x is not odd, that is, x is even. The object then was to show that $5x - 7$ is odd.

If we had attempted to prove Result 3.10 with a direct proof, then we would have begun by assuming that $5x - 7$ is even. We could then write $5x - 7 = 2a$, where $a \in \mathbf{Z}$. So $x = (2a + 7)/5$. We then would want to show that x is odd. With the expression we have for x, it is not even clear that x is an integer, much less that x is an *odd* integer, although, of course, we were told in the statement of Result 3.10 that the domain of x is the set of integers. Therefore, it is not only that a proof by contrapositive provides us with a rather simple method of proving Result 3.10, it may not be immediately clear how or whether a direct proof can be used.

How did we know beforehand that it is a proof by contrapositive that we should use here? This is not as difficult as it may appear. If we use a direct proof, then we begin by assuming that $5x - 7$ is even for an arbitrary integer x; while if we use a proof by contrapositive, then we begin by assuming that x is even. Therefore, using a proof by contrapositive allows us to work with x initially rather than the more complicated expression $5x - 7$. ♦

In all of the examples that we have seen so far, we have considered only implications. Now we look at a biconditional.

Result 3.11 *Let $x \in \mathbf{Z}$. Then $11x - 7$ is even if and only if x is odd.*

Proof There are two implications to prove here, namely,

(1) if x is odd, then $11x - 7$ is even, and
(2) if $11x - 7$ is even, then x is odd.

We begin with (1). In this case, a direct proof is appropriate. Assume that x is odd. Then $x = 2r + 1$, where $r \in \mathbf{Z}$. So

$$11x - 7 = 11(2r + 1) - 7 = 22r + 11 - 7 = 22r + 4 = 2(11r + 2).$$

Since $11r + 2$ is an integer, $11x - 7$ is even.

We now prove (2), which is the converse of (1). We use a proof by contrapositive here. Assume that x is even. Then $x = 2s$, where $s \in \mathbf{Z}$. Therefore,

$$11x - 7 = 11(2s) - 7 = 22s - 7 = 22s - 8 + 1 = 2(11s - 4) + 1.$$

Since $11s - 4$ is an integer, $11x - 7$ is odd. ∎

A comment concerning the statements of Results 3.10 and 3.11 bears repeating here. These results begin with the sentence: Let $x \in \mathbf{Z}$. This, of course, is informing us that the domain in this case is $\mathbf{Z}$. That is, we are being told that x represents an integer. We need not state this assumption in the proof. The sentence "Let $x \in \mathbf{Z}$." is commonly called an "overriding" assumption or hypothesis, and so x is assumed to be an integer throughout the proofs of Results 3.10 and 3.11.

In the proof of Result 3.11, we discussed our plan of attack. Namely, we stated that there were two implications to prove and we specifically stated each. Ordinarily we don't include such information within the proof – unless the proof is quite long, in which case a roadmap indicating the steps we plan to take may be helpful. We give an additional example of this type, where this time a more conventional condensed proof is presented. The following example will be useful to us in the future; thus we refer to it as a theorem.

Theorem 3.12 *Let $x \in \mathbf{Z}$. Then x^2 is even if and only if x is even.*

Proof Assume that x is even. Then $x = 2a$ for some integer a. Therefore,

$$x^2 = (2a)^2 = 4a^2 = 2(2a^2).$$

Because $2a^2 \in \mathbf{Z}$, the integer x^2 is even.

For the converse, assume that x is odd. So $x = 2b + 1$, where $b \in \mathbf{Z}$. Then

$$x^2 = (2b + 1)^2 = 4b^2 + 4b + 1 = 2(2b^2 + 2b) + 1.$$

Since $2b^2 + 2b$ is an integer, x^2 is odd. ∎

Suppose now that you were asked to prove the following result:

Let $x \in \mathbf{Z}$. Then x^2 is odd if and only if x is odd. **(3.4)**

How would you do this? You might think of proving the implication "If x is odd, then x^2 is odd." by a direct proof and its converse "If x^2 is odd, then x is odd." by a proof by contrapositive, where, of course, the domain of x is $\mathbf{Z}$. If we look at what is happening here, we see that we are duplicating the proof of Theorem 3.12. This is no surprise whatsoever. Theorem 3.12 states that if x is even, then x^2 is even, and if x^2 is even, then x is even. The contrapositive of the first implication is "If x^2 is odd, then x is odd."; while the contrapositive of the second implication is "If x is odd, then x^2 is odd." In

other words, (3.4) simply restates Theorem 3.12 in terms of contrapositives. Thus, (3.4) requires no proof at all. It is essentially a restatement of Theorem 3.12. And speaking of restatements of Theorem 3.12, we need to recognize that this theorem can be restated in other ways. For example, we could restate

If x is an even integer, then x^2 is even.

as

The square of every even integer is even.

Hence Theorem 3.12 could be stated as:

An integer is even if and only if its square is even.

It is not only useful to sometimes restate results in different manners for variety, it is important to recognize what a result is saying regardless of the manner in which it may be stated.

At this point, it is convenient to pause and discuss how theorems (or results) can be used and why we may be interested in proving a particular theorem in the first place. Suppose that we have been successful in proving $P(x) \Rightarrow Q(x)$ for all x in some domain S (by whatever method). We therefore know that for every $x \in S$ for which the statement $P(x)$ is true, the statement $Q(x)$ is true. Also, for any $x \in S$ for which the statement $Q(x)$ is false, the statement $P(x)$ is false. For example, since we know that Result 3.10 is true, if we ever encounter an integer n for which $5n - 7$ is even, then we know that n is odd. Furthermore, if we should encounter an integer n for which n^2 is odd, then we can conclude by statement (3.4) or, better yet, by Theorem 3.12, that n itself must be odd.

It is not only knowing that a particular theorem might be useful to us in the future, it is perhaps that a theorem seems surprising, interesting, or even beautiful. (Yes – to mathematicians, and hopefully to you as well, a theorem can be beautiful.)

We next describe a type of result that we have not yet encountered. Consider the following result, which we would like to prove.

Result to Prove Let $x \in \mathbf{Z}$. If $5x - 7$ is odd, then $9x + 2$ is even.

PROOF STRATEGY This result doesn't seem to fit into the kinds of results we've been proving. (This is not unusual. After learning how to prove certain statements, we encounter new statements that require us to ... think.) If we attempt to give either a direct proof or a proof by contrapositive of this result, we may be headed for difficulties. There is, however, another approach. Even though we must be very careful about what we are assuming, from what we know about even and odd integers, it appears that if $5x - 7$ is odd, then x must be even. In fact, if we *knew* that whenever $5x - 7$ is odd then x is even, this fact would be extremely helpful. We illustrate this next. Don't forget that our goal is to prove the following result, which we will refer to as Result 3.14: Let $x \in \mathbf{Z}$. If $5x - 7$ is odd, then $9x + 2$ is even. The (unusual) numbering of this result is because we will first state and prove a lemma (Lemma 3.13) that will aid us in the proof of Result 3.14. ♦

In order to verify the truth of Result 3.14, we first prove the following lemma.

Lemma 3.13 *Let $x \in \mathbf{Z}$. If $5x - 7$ is odd, then x is even.*

Proof Assume that x is odd. Then $x = 2y + 1$, where $y \in \mathbf{Z}$. Therefore,

$$5x - 7 = 5(2y + 1) - 7 = 10y - 2 = 2(5y - 1).$$

Since $5y - 1$ is an integer, $5x - 7$ is even. ∎

We are now prepared to give a proof of Result 3.14.

Result 3.14 *Let $x \in \mathbf{Z}$. If $5x - 7$ is odd, then $9x + 2$ is even.*

Proof Let $5x - 7$ be an odd integer. By Lemma 3.13, the integer x is even. Since x is even, $x = 2z$ for some integer z. Thus

$$9x + 2 = 9(2z) + 2 = 18z + 2 = 2(9z + 1).$$

Because $9z + 1$ is an integer, $9x + 2$ is even. ∎

So, with the aid of Lemma 3.13, we have produced a very uncomplicated (and, hopefully, easy-to-follow) proof of Result 3.14.

The main reason for presenting Result 3.14 was to show how helpful a lemma can be in producing a proof of another result. However, having just said this, we now show how we can prove Result 3.14 without the aid of a lemma, by performing a bit of algebraic manipulation.

Alternative Proof of Result 3.14 Assume that $5x - 7$ is odd. Then $5x - 7 = 2n + 1$ for some integer n. Observe that

$$\begin{aligned} 9x + 2 &= (5x - 7) + (4x + 9) = 2n + 1 + 4x + 9 \\ &= 2n + 4x + 10 = 2(n + 2x + 5). \end{aligned}$$

Because $n + 2x + 5$ is an integer, $9x + 2$ is even. ∎

You may prefer one proof of Result 3.14 over the other. Whether you do or not, it is important to know that two different methods can be used. These methods might prove to be useful for future results you encounter. Also, you might think we used a trick to give the second proof of Result 3.14, but, as we will see, if the same "trick" can be used often, then it becomes a technique.

3.4 Proof by Cases

While attempting to give a proof of a mathematical statement concerning an element x in some set S, it is sometimes useful to observe that x possesses one of two or more properties. A common property which x may possess is that of belonging to a particular subset of S. If we can verify the truth of the statement regardless of which of these properties that x may have, then we have a proof of the statement. Such a proof is then divided into parts called **cases**, one case for each property that x may possess or for each

subset to which x may belong. This method is called **proof by cases**. Indeed, it may be useful in a proof by cases to further divide a case into other cases, called **subcases**.

For example, in a proof of $\forall n \in \mathbf{Z}$, $R(n)$, it might be convenient to use a proof by cases whose proof is divided into the two cases

Case 1. *n is even* and *Case* 2. *n is odd.*

Other possible proofs by cases might involve proving $\forall x \in \mathbf{R}$, $P(x)$ using the cases

Case 1. $x = 0$, *Case* 2. $x < 0$, and *Case* 3. $x > 0$.

Also, we might attempt to prove $\forall n \in \mathbf{N}$, $P(n)$ using the cases

Case 1. $n = 1$ and *Case* 2. $n \geq 2$.

Furthermore, for $S = \mathbf{Z} - \{0\}$, we might try to prove $\forall x, y \in S$, $P(x, y)$ by using the cases

Case 1. $xy > 0$ and *Case* 2. $xy < 0$.

Case 1 could, in fact, be divided into two subcases:

Subcase 1.1. $x > 0$ *and* $y > 0$. and *Subcase* 1.2. $x < 0$ *and* $y < 0$;

while Case 2 could be divided into the two subcases:

Subcase 2.1. $x > 0$ *and* $y < 0$. and *Subcase* 2.2. $x < 0$ *and* $y > 0$.

Let's look at an example of a proof by cases.

Result 3.15 *If $n \in \mathbf{Z}$, then $n^2 + 3n + 5$ is an odd integer.*

Proof We proceed by cases, according to whether n is even or odd.

Case 1. *n is even.* Then $n = 2x$ for some $x \in \mathbf{Z}$. So

$$n^2 + 3n + 5 = (2x)^2 + 3(2x) + 5 = 4x^2 + 6x + 5 = 2(2x^2 + 3x + 2) + 1.$$

Since $2x^2 + 3x + 2 \in \mathbf{Z}$, the integer $n^2 + 3n + 5$ is odd.

Case 2. *n is odd.* Then $n = 2y + 1$, where $y \in \mathbf{Z}$. Thus

$$\begin{aligned} n^2 + 3n + 5 &= (2y + 1)^2 + 3(2y + 1) + 5 = 4y^2 + 10y + 9 \\ &= 2(2y^2 + 5y + 4) + 1. \end{aligned}$$

Because $2y^2 + 5y + 4 \in \mathbf{Z}$, the integer $n^2 + 3n + 5$ is odd. ∎

Two integers x and y are said to be **of the same parity** if x and y are both even or are both odd. The integers x and y are **of opposite parity** if one of x and y is even and the other is odd. For example, 5 and 13 are of the same parity, while 8 and 11 are of opposite parity. Because the definition of two integers having the same (or opposite) parity requires the two integers to satisfy one of two properties, any result containing these terms is likely to be proved by cases. The following theorem presents a characterization of two integers that are of the same parity.

Theorem 3.16 *Let $x, y \in \mathbf{Z}$. Then x and y are of the same parity if and only if $x + y$ is even.*

Proof First, assume that x and y are of the same parity. We consider two cases.

Case 1. *x and y are even.* Then $x = 2a$ and $y = 2b$ for some integers a and b. So $x + y = 2a + 2b = 2(a + b)$. Since $a + b \in \mathbf{Z}$, the integer $x + y$ is even.

Case 2. *x and y are odd.* Then $x = 2a + 1$ and $y = 2b + 1$, where $a, b \in \mathbf{Z}$. Therefore,

$$x + y = (2a + 1) + (2b + 1) = 2a + 2b + 2 = 2(a + b + 1).$$

Since $a + b + 1$ is an integer, $x + y$ is even.

For the converse, assume that x and y are of opposite parity. Again, we consider two cases.

Case 1. *x is even and y is odd.* Then $x = 2a$ and $y = 2b + 1$, where $a, b \in \mathbf{Z}$. Then

$$x + y = 2a + (2b + 1) = 2(a + b) + 1.$$

Since $a + b \in \mathbf{Z}$, the integer $x + y$ is odd.

Case 2. *x is odd and y is even.* The proof is similar to the proof of the preceding case and is therefore omitted. ■

PROOF ANALYSIS There is another comment regarding the proof of Theorem 3.16 we want to make. Although there is always some concern when omitting steps or proofs, it should be clear that it is truly a waste of effort by writer and reader alike to give a proof of the case when x is odd and y is even in Theorem 3.16. Indeed, there is an alternative when the converse is considered:

For the converse, assume that x and y are of opposite parity. Without loss of generality, assume that x is even and y is odd. Then $x = 2a$ and $y = 2b + 1$, where $a, b \in \mathbf{Z}$. Then

$$x + y = 2a + (2b + 1) = 2(a + b) + 1.$$

Since $a + b \in \mathbf{Z}$, the integer $x + y$ is odd. ♦

We used the phrase **without loss of generality** (some abbreviate this as *WOLOG* or *WLOG*) to indicate that the proofs of the two situations are similar, so the proof of only one of these is needed. Sometimes it is rather subjective to say that two situations are similar. We present one additional example to illustrate this.

Theorem to Prove Let a and b be integers. Then ab is even if and only if a is even or b is even.

PROOF STRATEGY Before we begin a proof of this result (Theorem 3.17 below), let's see what we will be required to show. We need to prove two implications, namely, (1) If a is even or b is even, then ab is even and (2) if ab is even, then a is even or b is even. We consider (1) first. A direct proof seems appropriate. Here, we will assume that a is even or b is even. We could give a proof by cases: (i) a is even, (ii) b is even. On the other hand, since the proofs of these cases will certainly be similar, we could say, without loss of generality, that a is even. We will see that it is unnecessary to make any assumption about b.

If we were to give a direct proof of (2), then we would begin by assuming that ab is even, say $ab = 2k$ for some integer k. But how could we deduce any information about a and b individually? Let's try another approach. If we use a proof by contrapositive, then we would begin by assuming that it is not the case that a is even or b is even. This is exactly the situation covered by one of De Morgan's laws:

$$\sim(P \vee Q) \text{ is logically equivalent to } (\sim P) \wedge (\sim Q).$$

It is important not to forget this. In this case, we have P : a is even. and Q : b is even. So the negation of "a is even or b is even" is "a is odd *and* b is odd". ♦

Let's now prove this result.

Theorem 3.17 *Let a and b be integers. Then ab is even if and only if a is even or b is even.*

Proof First, assume that a is even or b is even. Without loss of generality, let a be even. Then $a = 2x$ for some integer x. Thus $ab = (2x)b = 2(xb)$. Since xb is an integer, ab is even.

For the converse, assume that a is odd and b is odd. Then $a = 2x + 1$ and $b = 2y + 1$, where $x, y \in \mathbf{Z}$. Hence

$$ab = (2x + 1)(2y + 1) = 4xy + 2x + 2y + 1 = 2(2xy + x + y) + 1.$$

Since $2xy + x + y$ is an integer, ab is odd. ■

3.5 Proof Evaluations

We have now stated several results and have given a proof of each result (sometimes preceding a proof by a proof strategy or following the proof with a proof analysis). Let's reverse this process by giving an example of a proof of a result but not stating the result being proved. We will follow the proof with several options for the statements of the result being proved.

Example 3.18 *Given below is a proof of a result.*

Proof Assume that n is an odd integer. Then $n = 2k + 1$ for some integer k. Then

$$3n - 5 = 3(2k + 1) - 5 = 6k + 3 - 5 = 6k - 2 = 2(3k - 1).$$

Since $3k - 1$ is an integer, $3n - 5$ is even. ■

Which of the following is proved above?

(1) $3n - 5$ is an even integer.
(2) If n is an odd integer, then $3n - 5$ is an even integer.
(3) Let n be an integer. If $3n - 5$ is an even integer, then n is an odd integer.
(4) Let n be an integer. If $3n - 5$ is an odd integer, then n is an even integer.

The correct answers are (2) and (4). The proof given is a direct proof of (2) and a proof by contrapositive of (4). The sentence (1) is an open sentence, not a statement, and is only the conclusion of (2). Statement (3) is the converse of (2). ♦

When learning any mathematical subject, it is not the least bit unusual to make mistakes along the way. In fact, part of learning mathematics is to learn from your mistakes and those of others. For this reason, you will see a few exercises at the end of most chapters (beginning with this chapter) where you are asked to evaluate the proof of a result. That is, a result and a proposed proof of this result will be given. You are then asked to read this proposed proof and determine whether, in your opinion, it is, in fact, a proof. If you don't believe that the given argument provides a proof of the result, then you should point out the (or *a*) mistake. We give two examples of this.

Problem 3.19 *Evaluate the proposed proof of the following result.*

Result *If x and y are integers of the same parity, then $x - y$ is even.*

Proof Let x and y be two integers of the same parity. We consider two cases, according to whether x and y are both even or are both odd.

Case 1. *x and y are both even.* Let $x = 6$ and $y = 2$, which are both even. Then $x - y = 4$, which is even.

Case 2. *x and y are both odd.* Let $x = 7$ and $y = 1$, which are both odd. Then $x - y = 6$, which is even. ■

Proof Evaluation Although the proof started correctly, assuming that x and y are two integers of the same parity and dividing the proof into these two cases, the proof of each case is incorrect. When we assume that x and y are both even, for example, x and y must represent arbitrary even integers, not specific even integers. ♦

Problem 3.20 *Evaluate the proposed proof of the following result.*

Result *If m is an even integer and n is an odd integer, then $3m + 5n$ is odd.*

Proof Let m be an even integer and n an odd integer. Then $m = 2k$ and $n = 2k + 1$, where $k \in \mathbf{Z}$. Therefore,

$$\begin{aligned} 3m + 5n &= 3(2k) + 5(2k + 1) = 6k + 10k + 5 \\ &= 16k + 5 = 2(8k + 2) + 1. \end{aligned}$$

Since $8k + 2$ is an integer, $3m + 5n$ is odd. ■

Proof Evaluation There is a mistake in the second sentence of the proposed proof, where it is written that $m = 2k$ and $n = 2k + 1$, where $k \in \mathbf{Z}$. Since the same symbol k is used for both m and n, we have inadvertently added the assumption that $n = m + 1$. This is incorrect, however, as it was never stated that m and n must be consecutive integers. In other words, we should write $m = 2k$ and $n = 2\ell + 1$, say, where $k, \ell \in \mathbf{Z}$. ♦

EXERCISES FOR CHAPTER 3

Section 3.1: Trivial and Vacuous Proofs

3.1. Let $x \in \mathbf{R}$. Prove that if $0 < x < 1$, then $x^2 - 2x + 2 \neq 0$.

3.2. Let $n \in \mathbf{N}$. Prove that if $|n - 1| + |n + 1| \leq 1$, then $|n^2 - 1| \leq 4$.

3.3. Let $r \in \mathbf{Q}^+$. Prove that if $\frac{r^2+1}{r} \leq 1$, then $\frac{r^2+2}{r} \leq 2$.

3.4. Let $x \in \mathbf{R}$. Prove that if $x^3 - 5x - 1 \geq 0$, then $(x - 1)(x - 3) \geq -2$.

3.5. Let $n \in \mathbf{N}$. Prove that if $n + \frac{1}{n} < 2$, then $n^2 + \frac{1}{n^2} < 4$.

Section 3.2: Direct Proofs

3.6. Prove that if x is an odd integer, then $9x + 5$ is even.

3.7. Prove that if x is an even integer, then $5x - 3$ is an odd integer.

3.8. Prove that if a and c are odd integers, then $ab + bc$ is even.

3.9. Let $n \in \mathbf{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

3.10. Let $x \in \mathbf{Z}$. Prove that if 2^{2x} is an odd integer, then 4^x is an odd integer.

3.11. Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n + 1)^2(n + 2)^2/4$ is even, then $(n + 2)^2(n + 3)^2/4$ is even.

Section 3.3: Proof by Contrapositive

3.12. Let $x \in \mathbf{Z}$. Prove that if $7x + 5$ is odd, then x is even.

3.13. Let $n \in \mathbf{Z}$. Prove that if $15n$ is even, then $9n$ is even.

3.14. Let $x \in \mathbf{Z}$. Prove that $5x - 11$ is even if and only if x is odd.

3.15. Let $x \in \mathbf{Z}$. Use a lemma to prove that if $7x + 4$ is even, then $3x - 11$ is odd.

3.16. Let $x \in \mathbf{Z}$. Prove that $3x + 1$ is even if and only if $5x - 2$ is odd.

3.17. Let $S = \{2, 3, 4\}$ and let $n \in S$. Use a proof by contrapositive to prove that if $n^2(n - 1)^2/4$ is even, then $n^2(n + 1)^2/4$ is even.

3.18. Let $n \in \mathbf{Z}$. Prove that $(n + 1)^2 - 1$ is even if and only if n is even.

Section 3.4: Proof by Cases

3.19. Prove that if $n \in \mathbf{Z}$, then $n^2 - 3n + 9$ is odd.

3.20. Prove that if $n \in \mathbf{Z}$, then $n^3 - n$ is even.

3.21. Let $x, y \in \mathbf{Z}$. Prove that if xy is odd, then x and y are odd.

3.22. Let $a, b \in \mathbf{Z}$. Prove that if ab is odd, then $a^2 + b^2$ is even.

3.23. Let $x, y \in \mathbf{Z}$. Prove that $x - y$ is even if and only if x and y are of the same parity.

3.24. Let $a, b \in \mathbf{Z}$. Prove that if $a + b$ and ab are of the same parity, then a and b are even.

3.25. (a) Let x and y be integers. Prove that $(x + y)^2$ is even if and only if x and y are of the same parity.
(b) Restate the result in (a) in terms of odd integers.

3.26. A collection of nonempty subsets of a nonempty set S is called a **cover** of S if every element of S belongs to at least one of the subsets. (A cover is a partition of S if every element of S belongs to exactly one of the subsets.) Consider the following.

Result Let $a, b \in \mathbf{Z}$. If a is even or b is even, then ab is even.

Proof Assume that a is even or b is even. We consider the following cases.

Case 1. *a is even.* Then $a = 2k$, where $k \in \mathbf{Z}$. Thus $ab = (2k)b = 2(kb)$. Since $kb \in \mathbf{Z}$, it follows that ab is even.

Case 2. *b is even.* Then $b = 2\ell$, where $\ell \in \mathbf{Z}$. Thus $ab = a(2\ell) = 2(a\ell)$. Since $a\ell \in \mathbf{Z}$, it follows that ab is even. ∎

Since the domain is $\mathbf{Z}$ for both a and b, we might think of $\mathbf{Z} \times \mathbf{Z}$ being the domain of (a, b). Consider the following subsets of $\mathbf{Z} \times \mathbf{Z}$:

$S_1 = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : a \text{ and } b \text{ are odd}\}$

$S_2 = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : a \text{ is even}\}$

$S_3 = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} : b \text{ is even}\}$.

(a) Why is $\{S_1, S_2, S_3\}$ a cover of $\mathbf{Z} \times \mathbf{Z}$ and not a partition of $\mathbf{Z} \times \mathbf{Z}$?

(b) Why does the set S_1 not appear in the proof above?

(c) Give a proof by cases of the result above where the cases are determined by a partition and not a cover.

Section 3.5: Proof Evaluations

3.27. Below is a proof of a result.

Proof We consider two cases.

Case 1. *a and b are even.* Then $a = 2r$ and $b = 2s$ for integers r and s. Thus

$$a^2 - b^2 = (2r)^2 - (2s)^2 = 4r^2 - 4s^2 = 2(2r^2 - 2s^2).$$

Since $2r^2 - 2s^2$ is an integer, $a^2 - b^2$ is even.

Case 2. *a and b are odd.* Then $a = 2r + 1$ and $b = 2s + 1$ for integers r and s. Thus

$$\begin{aligned} a^2 - b^2 &= (2r + 1)^2 - (2s + 1)^2 = (4r^2 + 4r + 1) - (4s^2 + 4s + 1) \\ &= 4r^2 + 4r - 4s^2 - 4s = 2(2r^2 + 2r - 2s^2 - 2s). \end{aligned}$$

Since $2r^2 + 2r - 2s^2 - 2s$ is an integer, $a^2 - b^2$ is even. ∎

Which of the following is being proved?

(1) Let $a, b \in \mathbf{Z}$. Then a and b are of the same parity if and only if $a^2 - b^2$ is even.

(2) Let $a, b \in \mathbf{Z}$. Then $a^2 - b^2$ is even.

(3) Let $a, b \in \mathbf{Z}$. If a and b are of the same parity, then $a^2 - b^2$ is even.

(4) Let $a, b \in \mathbf{Z}$. If $a^2 - b^2$ is even, then a and b are of the same parity.

3.28. Below is given a proof of a result. What result is being proved?

Proof Assume that x is even. Then $x = 2a$ for some integer a. So

$$3x^2 - 4x - 5 = 3(2a)^2 - 4(2a) - 5 = 12a^2 - 8a - 5 = 2(6a^2 - 4a - 3) + 1.$$

Since $6a^2 - 4a - 3$ is an integer, $3x^2 - 4x - 5$ is odd.

For the converse, assume that x is odd. So $x = 2b + 1$, where $b \in \mathbf{Z}$. Therefore,

$$\begin{aligned} 3x^2 - 4x - 5 &= 3(2b + 1)^2 - 4(2b + 1) - 5 = 3(4b^2 + 4b + 1) - 8b - 4 - 5 \\ &= 12b^2 + 4b - 6 = 2(6b^2 + 2b - 3). \end{aligned}$$

Since $6b^2 + 2b - 3$ is an integer, $3x^2 - 4x - 5$ is even. ∎

3.29. Evaluate the proof of the following result.

Result Let $n \in \mathbf{Z}$. If $3n - 8$ is odd, then n is odd.

Proof Assume that n is odd. Then $n = 2k + 1$ for some integer k. Then $3n - 8 = 3(2k + 1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k - 3) + 1$. Since $3k - 3$ is an integer, $3n - 8$ is odd. ∎

3.30. Evaluate the proof of the following result.

Result Let $a, b \in \mathbf{Z}$. Then $a - b$ is even if and only if a and b are of the same parity.

Proof We consider two cases.

Case 1. *a and b are of the same parity.* We now consider two subcases.

Subcase 1.1. *a and b are both even.* Then $a = 2x$ and $b = 2y$, where $x, y \in \mathbf{Z}$. Then $a - b = 2x - 2y = 2(x - y)$. Since $x - y$ is an integer, $a - b$ is even.

Subcase 1.2. *a and b are both odd.* Then $a = 2x + 1$ and $b = 2y + 1$, where $x, y \in \mathbf{Z}$. Then $a - b = (2x + 1) - (2y + 1) = 2(x - y)$. Since $x - y$ is an integer, $a - b$ is even.

Case 2. *a and b are of opposite parity.* We again have two subcases.

Subcase 2.1. *a is odd and b is even.* Then $a = 2x + 1$ and $b = 2y$, where $x, y \in \mathbf{Z}$. Then $a - b = (2x + 1) - 2y = 2(x - y) + 1$. Since $x - y$ is an integer, $a - b$ is odd.

Subcase 2.2. *a is even and b is odd.* Then $a = 2x$ and $b = 2y + 1$, where $x, y \in \mathbf{Z}$. Then $a - b = 2x - (2y + 1) = 2x - 2y - 1 = 2(x - y - 1) + 1$. Since $x - y - 1$ is an integer, $a - b$ is odd. ∎

ADDITIONAL EXERCISES FOR CHAPTER 3

3.31. Let $x \in \mathbf{Z}$. Prove that if $7x - 8$ is even, then x is even.

3.32. Let $x \in \mathbf{Z}$. Prove that x^3 is even if and only if x is even.

3.33. Let $x \in \mathbf{Z}$. Use one or two lemmas to prove that $3x^3$ is even if and only if $5x^2$ is even.

3.34. Give a direct proof of the following: Let $x \in \mathbf{Z}$. If $11x - 5$ is odd, then x is even.

3.35. Let $x, y \in \mathbf{Z}$. Prove that if $x + y$ is odd, then x and y are of opposite parity.

3.36. Let $x, y \in \mathbf{Z}$. Prove that if $3x + 5y$ is even, then x and y are of the same parity.

3.37. Let $x, y \in \mathbf{Z}$. Prove that $(x + 1)y^2$ is even if and only if x is odd or y is even.

3.38. Let $x, y \in \mathbf{Z}$. Prove that if xy and $x + y$ are even, then both x and y are even.

3.39. Prove, for every integer x, that the integers $3x + 1$ and $5x + 2$ are of opposite parity.

3.40. Let $S = \{a, b, c, d\}$ be a set of four distinct integers. Prove that if either (1) for each $x \in S$, the integer x and the sum of any two of the remaining three integers of S are of the same parity or (2) for each $x \in S$, the

integer x and the sum of any two of the remaining three integers of S are of opposite parity, then every pair of integers of S are of the same parity.

3.41. Evaluate the proof of the following result.

Result Let $x, y \in \mathbf{Z}$ and let a and b be odd integers. If $ax + by$ is even, then x and y are of the same parity.

Proof Assume that x and y are of opposite parity. Then $x = 2p$ and $y = 2q + 1$ for some integers p and q. Since a and b are odd integers, $a = 2r + 1$ and $b = 2s + 1$ for integers r and s. Hence

$$\begin{aligned} ax + by &= (2r + 1)(2p) + (2s + 1)(2q + 1) \\ &= 4pr + 2p + 4qs + 2s + 2q + 1 \\ &= 2(2pr + p + 2qs + s + q) + 1. \end{aligned}$$

Since $2pr + p + 2qs + s + q$ is an integer, $ax + by$ is odd. ∎

3.42. Let $x, y \in \mathbf{Z}$. Prove that if a and b are even integers, then $ax + by$ is even.

3.43. Prove that for every two distinct real numbers a and b, either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$.

3.44. Let $a, b \in \mathbf{Z}$. Prove that if $ab = 4$, then $(a - b)^3 - 9(a - b) = 0$.

3.45. Prove that if a and b are two positive integers, then $a^2(b + 1) + b^2(a + 1) \geq 4ab$.

3.46. Prove the following two results:

(a) Result A: Let $n \in \mathbf{Z}$. If n^3 is even, then n is even.

(b) Result B: If n is an odd integer, then $5n^9 + 13$ is even.

3.47. Let a, b, and c be the lengths of the sides of a triangle T, where $a \leq b \leq c$. Prove that if T is a right triangle, then

$$(abc)^2 = \frac{c^6 - a^6 - b^6}{3}.$$

4

More on Direct Proof and Proof by Contrapositive

All of the examples illustrating direct proof and proof by contrapositive that we have seen involve properties of even and odd integers. In this chapter, we will give additional examples of direct proofs and proofs by contrapositive in new surroundings. First, we will see how even and odd integers can be studied in a more general setting, through divisibility of integers. We will then explore some of the properties of real numbers and, finally, look at properties of set operations.

4.1 Proofs Involving Divisibility of Integers

We have now seen many examples of integers that can be written as $2x$ for some integer x. These are precisely the even integers, of course. However, some integers can also be expressed as $3x$ or $4x$, or as $-5x$ for some integer x. In general, for integers a and b with $a \neq 0$, we say that a **divides** b if there is an integer c such that $b = ac$. In this case, we write $a \mid b$. Hence if n is an even integer, then $2 \mid n$; moreover, if 2 divides some integer n, then n is even. That is, an integer n is even if and only if $2 \mid n$. Theorem 3.17 can therefore be restated for integers a and b as: $2 \mid ab$ if and only if $2 \mid a$ or $2 \mid b$.

If $a \mid b$, then we also say that b is a **multiple** of a and that a is a **divisor** of b. Thus every even integer is a multiple of 2. If a does not divide b, then we write $a \nmid b$. For example, $4 \mid 48$ since $48 = 4 \cdot 12$ and $-3 \mid 57$ since $57 = (-3) \cdot (-19)$. On the other hand, $4 \nmid 66$ as there is no integer c such that $66 = 4c$.

We now apply the techniques we've learned to prove some results concerning divisibility properties of integers.

Result to Prove Let a, b, and c be integers with $a \neq 0$ and $b \neq 0$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

PROOF STRATEGY It seems reasonable here to use a direct proof and to begin by assuming that $a \mid b$ and $b \mid c$. This means that $b = ax$ and $c = by$ for some integers x and y. Since our goal is to show that $a \mid c$, we need to show that c can be written as the product of a and some other integer. Hence it is logical to consider c and determine how we can express it. ♦

sum of any two of the remaining three integers of S are of opposite parity, then every pair
e of the same parity.

f of the following result.

$\in \mathbf{Z}$ and let a and b be odd integers. If $ax + by$ is even, then x and y are of the same parity.

hat x and y are of opposite parity. Then $x = 2p$ and $y = 2q + 1$ for some integers p and
re odd integers, $a = 2r + 1$ and $b = 2s + 1$ for integers r and s. Hence

$$\begin{aligned} ax + by &= (2r + 1)(2p) + (2s + 1)(2q + 1) \\ &= 4pr + 2p + 4qs + 2s + 2q + 1 \\ &= 2(2pr + p + 2qs + s + q) + 1. \end{aligned}$$

$2qs + s + q$ is an integer, $ax + by$ is odd. ∎

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4

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4.1 Proofs Involving Divisibility of Integers

We have now seen many examples of integers that can be written as $2x$ for some integer x. These are precisely the even integers, of course. However, some integers can also be expressed as $3x$ or $4x$, or as $-5x$ for some integer x. In general, for integers a and b with $a \neq 0$, we say that a **divides** b if there is an integer c such that $b = ac$. In this case, we write $a \mid b$. Hence if n is an even integer, then $2 \mid n$; moreover, if 2 divides some integer n, then n is even. That is, an integer n is even if and only if $2 \mid n$. Theorem 3.17 can therefore be restated for integers a and b as: $2 \mid ab$ if and only if $2 \mid a$ or $2 \mid b$.

If $a \mid b$, then we also say that b is a **multiple** of a and that a is a **divisor** of b. Thus every even integer is a multiple of 2. If a does not divide b, then we write $a \nmid b$. For example, $4 \mid 48$ since $48 = 4 \cdot 12$ and $-3 \mid 57$ since $57 = (-3) \cdot (-19)$. On the other hand, $4 \nmid 66$ as there is no integer c such that $66 = 4c$.

We now apply the techniques we've learned to prove some results concerning divisibility properties of integers.

Result to Prove Let a, b, and c be integers with $a \neq 0$ and $b \neq 0$. If $a \mid b$ and $b \mid c$, then $a \mid c$.

PROOF STRATEGY It seems reasonable here to use a direct proof and to begin by assuming that $a \mid b$ and $b \mid c$. This means that $b = ax$ and $c = by$ for some integers x and y. Since our goal is to show that $a \mid c$, we need to show that c can be written as the product of a and some other integer. Hence it is logical to consider c and determine how we can express it. ♦

Result 4.1 *Let a, b, and c be integers with $a \neq 0$ and $b \neq 0$. If $a \mid b$ and $b \mid c$, then $a \mid c$.*

Proof Assume that $a \mid b$ and $b \mid c$. Then $b = ax$ and $c = by$, where $x, y \in \mathbf{Z}$. Therefore, $c = by = (ax)y = a(xy)$. Since xy is an integer, $a \mid c$. ∎

We now verify two other divisibility properties of integers.

Result 4.2 *Let a, b, c, and d be integers with $a \neq 0$ and $b \neq 0$. If $a \mid c$ and $b \mid d$, then $ab \mid cd$.*

Proof Let $a \mid c$ and $b \mid d$. Then $c = ax$ and $d = by$, where $x, y \in \mathbf{Z}$. Then

$$cd = (ax)(by) = (ab)(xy).$$

Since xy is an integer, $ab \mid cd$. ∎

Result 4.3 *Let $a, b, c, x, y \in \mathbf{Z}$, where $a \neq 0$. If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$.*

Proof Assume that $a \mid b$ and $a \mid c$. Then $b = ar$ and $c = as$, where $r, s \in \mathbf{Z}$. Then

$$bx + cy = (ar)x + (as)y = a(rx + sy).$$

Since $rx + sy$ is an integer, $a \mid (bx + cy)$. ∎

The examples that we have presented thus far concern general properties of divisibility of integers. We now look at some specialized properties of divisibility.

Result 4.4 *Let $x \in \mathbf{Z}$. If $2 \mid (x^2 - 1)$, then $4 \mid (x^2 - 1)$.*

Proof Assume that $2 \mid (x^2 - 1)$. So $x^2 - 1 = 2y$ for some integer y. Thus $x^2 = 2y + 1$ is an odd integer. It then follows by Theorem 3.12 that x too is odd. Hence $x = 2z + 1$ for some integer z. Then

$$x^2 - 1 = (2z + 1)^2 - 1 = (4z^2 + 4z + 1) - 1 = 4z^2 + 4z = 4(z^2 + z).$$

Since $z^2 + z$ is an integer, $4 \mid (x^2 - 1)$. ∎

For each of the Results 4.1–4.4, a direct proof worked very well. For the following result, however, the situation is quite different.

Result to Prove Let $x, y \in \mathbf{Z}$. If $3 \nmid xy$, then $3 \nmid x$ and $3 \nmid y$.

PROOF STRATEGY If we let

$$P\colon\ 3 \nmid xy, \quad Q\colon 3 \nmid x, \quad \text{and} \quad R\colon\ 3 \nmid y,$$

then we wish to prove that $P \Rightarrow Q \wedge R$. (It should be clear that P, Q, and R are open sentences in this case, but we omit the variables here for simplicity.) If we use a direct proof, then we would assume that $3 \nmid xy$ and attempt to show that $3 \nmid x$ and $3 \nmid y$. Thus we would know that xy *cannot* be expressed as 3 times an integer. On the other hand, if we use a proof by contrapositive, then we are considering the implication $(\sim(Q \wedge R)) \Rightarrow (\sim P)$, which, by De Morgan's Law, is logically equivalent to $((\sim Q) \vee (\sim R)) \Rightarrow (\sim P)$

and which, in words, is: If $3 \mid x$ or $3 \mid y$, then $3 \mid xy$. This method looks more promising. ♦

Result 4.5 *Let $x, y \in \mathbf{Z}$. If $3 \nmid xy$, then $3 \nmid x$ and $3 \nmid y$.*

Proof Assume that $3 \mid x$ or $3 \mid y$. Without loss of generality, assume that 3 divides x. Then $x = 3z$ for some integer z. Hence $xy = (3z)y = 3(zy)$. Since zy is an integer, $3 \mid xy$. ∎

We have already mentioned that if an integer n is not a multiple of 2, then we can write $n = 2q + 1$ for some integer q (that is, if an integer n is not even, then it is odd). This is a consequence of knowing that 0 and 1 are the only possible remainders when an integer is divided by 2. Along the same lines, if an integer n is not a multiple of 3, then we can write $n = 3q + 1$ or $n = 3q + 2$ for some integer q, that is, every integer can be expressed as $3q$, $3q + 1$, or $3q + 2$ for some integer q since 0, 1, and 2 are the only remainders that can result when an integer is divided by 3. Similarly, if an integer n is not a multiple of 4, then n can be expressed as $4q + 1$, $4q + 2$, or $4q + 3$ for some integer q. This topic is explored in more detail in Chapter 11.

Result to Prove Let $x \in \mathbf{Z}$. If $3 \nmid (x^2 - 1)$, then $3 \mid x$.

PROOF STRATEGY We have two options here, namely, (1) use a direct proof and begin a proof by assuming that $3 \nmid (x^2 - 1)$, or (2) use a proof by contrapositive and begin a proof by assuming that $3 \nmid x$. Certainly, we cannot avoid assuming that 3 does *not* divide *some* integer. However, it appears far easier to know that $3 \nmid x$ and attempt to show that $3 \mid (x^2 - 1)$ than to know that $3 \nmid (x^2 - 1)$ and show that $3 \mid x$. Also, if $3 \nmid x$, then we now know that $x = 3q + 1$ or $x = 3q + 2$ for some integer q, which suggests a proof by cases. ♦

Result 4.6 *Let $x \in \mathbf{Z}$. If $3 \nmid (x^2 - 1)$, then $3 \mid x$.*

Proof Assume that $3 \nmid x$. Then either $x = 3q + 1$ for some integer q, or $x = 3q + 2$ for some integer q. We consider these two cases.

Case 1. $x = 3q + 1$ *for some integer* q. Then

$$\begin{aligned} x^2 - 1 &= (3q + 1)^2 - 1 = (9q^2 + 6q + 1) - 1 \\ &= 9q^2 + 6q = 3(3q^2 + 2q). \end{aligned}$$

Since $3q^2 + 2q$ is an integer, $3 \mid (x^2 - 1)$.

Case 2. $x = 3q + 2$ *for some integer* q. Then

$$\begin{aligned} x^2 - 1 &= (3q + 2)^2 - 1 = (9q^2 + 12q + 4) - 1 \\ &= 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1). \end{aligned}$$

Since $3q^2 + 4q + 1$ is an integer, $3 \mid (x^2 - 1)$. ∎

We now consider a biconditional involving divisibility.

Result 4.7 *Let $x, y \in \mathbf{Z}$. Then $4 \mid (x^2 - y^2)$ if and only if x and y are of the same parity.*

Proof Assume first that x and y are of the same parity. We show that $4 \mid (x^2 - y^2)$. There are two cases.

Case 1. *x and y are both even.* Thus $x = 2a$ and $y = 2b$ for some integers a and b. Then

$$x^2 - y^2 = (2a)^2 - (2b)^2 = 4a^2 - 4b^2 = 4(a^2 - b^2).$$

Since $a^2 - b^2$ is an integer, $4 \mid (x^2 - y^2)$.

Case 2. *x and y are both odd.* So $x = 2c + 1$ and $y = 2d + 1$ for some integers c and d. Then

$$\begin{aligned} x^2 - y^2 &= (2c + 1)^2 - (2d + 1)^2 = (4c^2 + 4c + 1) - (4d^2 + 4d + 1) \\ &= 4c^2 + 4c - 4d^2 - 4d = 4(c^2 + c - d^2 - d). \end{aligned}$$

Since $c^2 + c - d^2 - d$ is an integer, $4 \mid (x^2 - y^2)$.

For the converse, assume that x and y are of opposite parity. We show that $4 \nmid (x^2 - y^2)$. We consider two cases.

Case 1. *x is even and y is odd.* Thus $x = 2a$ and $y = 2b + 1$ for some integers a and b. Then

$$\begin{aligned} x^2 - y^2 &= (2a)^2 - (2b + 1)^2 = 4a^2 - [4b^2 + 4b + 1] \\ &= 4a^2 - 4b^2 - 4b - 1 = 4a^2 - 4b^2 - 4b - 4 + 3 \\ &= 4(a^2 - b^2 - b - 1) + 3. \end{aligned}$$

Since $a^2 - b^2 - b - 1$ is an integer, it follows that there is a remainder of 3 when $x^2 - y^2$ is divided by 4, and so $4 \nmid (x^2 - y^2)$.

Case 2. *x is odd and y is even.* The proof of this case is similar to that of Case 1 and is therefore omitted. ■

We consider a result of a somewhat different nature.

Result to Prove For every integer $n \geq 7$, there exist positive integers a and b such that $n = 2a + 3b$.

PROOF STRATEGY First, notice that we can write $7 = 2 \cdot 2 + 3 \cdot 1$, $8 = 2 \cdot 1 + 3 \cdot 2$, and $9 = 2 \cdot 3 + 3 \cdot 1$. So the result is certainly true for $n = 7, 8, 9$. On the other hand, there is no pair a, b of positive integers such that $6 = 2a + 3b$. Of course, this observation shows only that we cannot replace $n \geq 7$ by $n \geq 6$.

Suppose that n is an integer such that $n \geq 7$. We could bring the integer 2 into the discussion by observing that we can write $n = 2q$ or $n = 2q + 1$, where $q \in \mathbf{Z}$. Actually, if $n = 2q$, then $q \geq 4$ since $n \geq 7$; while if $n = 2q + 1$, then $q \geq 3$ since $n \geq 7$. This is a useful observation. ♦

Result 4.8 *For every integer $n \geq 7$, there exist positive integers a and b such that $n = 2a + 3b$.*

Proof Let n be an integer such that $n \geq 7$. Then $n = 2q$ or $n = 2q + 1$ for some integer q. We consider these two cases.

Case 1. $n = 2q$. Since $n \geq 7$, it follows that $q \geq 4$. Thus

$$n = 2q = 2(q - 3) + 6 = 2(q - 3) + 3 \cdot 2.$$

Since $q \geq 4$, it follows that $q - 3 \in \mathbf{N}$.

Case 2. $n = 2q + 1$. Since $n \geq 7$, it follows that $q \geq 3$. Thus

$$n = 2q + 1 = 2(q - 1) + 2 + 1 = 2(q - 1) + 3 \cdot 1.$$

Since $q \geq 3$, it follows that $q - 1 \in \mathbf{N}$. ■

4.2 Proofs Involving Congruence of Integers

We know that an integer x is even if $x = 2q$ for some integer q, while x is odd if $x = 2q + 1$ for some integer q. Furthermore, two integers x and y are of the same parity if they are both even or are both odd. From this, it follows that x and y are of the same parity if and only if $2 \mid (x - y)$. Consequently, $2 \mid (x - y)$ if and only if x and y have the same remainder when divided by 2. We also know that an integer x can be expressed as $3q$, $3q + 1$, or $3q + 2$ for some integer q, according to whether the remainder is 0, 1, or 2 when x is divided by 3. If integers x and y are both of the form $3q + 1$, then $x = 3s + 1$ and $y = 3t + 1$, where $s, t \in \mathbf{Z}$, and so $x - y = 3(s - t)$. Since $s - t$ is an integer, $3 \mid (x - y)$. Similarly, if x and y are both of the form $3q$ or are both of the form $3q + 2$, then $3 \mid (x - y)$ as well. Hence if x and y have the same remainder when divided by 3, then $3 \mid (x - y)$. The converse of this implication is true as well. This suggests a special interest in pairs x, y of integers such that $2 \mid (x - y)$ or $3 \mid (x - y)$, or, in fact, in pairs x, y of integers such that $n \mid (x - y)$ for some integer $n \geq 2$.

For integers a, b, and $n \geq 2$, we say that a is **congruent to b modulo n,** written $a \equiv b \pmod{n}$, if $n \mid (a - b)$. For example, $15 \equiv 7 \pmod{4}$ since $4 \mid (15 - 7)$, and $3 \equiv -15 \pmod{9}$ since $9 \mid (3 - (-15))$. On the other hand, 14 is not congruent to 4 modulo 6, written $14 \not\equiv 4 \pmod{6}$, since $6 \nmid (14 - 4)$.

Since we know that every integer x can be expressed as $x = 2q$ or as $x = 2q + 1$ for some integer q, it follows that either $2 \mid (x - 0)$ or $2 \mid (x - 1)$; that is, $x \equiv 0 \pmod{2}$ or $x \equiv 1 \pmod{2}$. Also, since each integer x can be expressed as $x = 3q$, $x = 3q + 1$, or $x = 3q + 2$ for some integer q, it follows that $3 \mid (x - 0)$, $3 \mid (x - 1)$, or $3 \mid (x - 2)$. Hence

$$x \equiv 0 \pmod{3}, \quad x \equiv 1 \pmod{3}, \quad \text{or} \quad x \equiv 2 \pmod{3}.$$

Moreover, for each integer x, exactly one of

$$x \equiv 0 \pmod{4}, \quad x \equiv 1 \pmod{4}, \quad x \equiv 2 \pmod{4}, \quad x \equiv 3 \pmod{4}$$

holds, according to whether the remainder is 0, 1, 2, or 3, respectively, when x is divided by 4. Similar statements can also be made when x is divided by n for each integer $n \geq 5$.

We now consider some properties of congruence of integers.

Result to Prove Let a, b, k, and n be integers, where $n \geq 2$. If $a \equiv b \pmod{n}$, then $ka \equiv kb \pmod{n}$.

PROOF STRATEGY A direct proof seems reasonable here. So, we begin by assuming that $a \equiv b \pmod{n}$. Our goal is to show that $ka \equiv kb \pmod{n}$. Because we know that $a \equiv b \pmod{n}$, it

follows, from the definition, that $n \mid (a - b)$, which implies that $a - b = nx$ for some integer x. We need to show that $ka \equiv kb \pmod{n}$, which means that we need to show that $n \mid (ka - kb)$. Thus, we must show that $ka - kb = nt$ for some integer t. This suggests considering the expression $ka - kb$. ♦

Result 4.9 *Let a, b, k, and n be integers, where $n \geq 2$. If $a \equiv b \pmod{n}$, then $ka \equiv kb \pmod{n}$.*

Proof Assume that $a \equiv b \pmod{n}$. Then $n \mid (a - b)$. Hence $a - b = nx$ for some integer x. Therefore,

$$ka - kb = k(a - b) = k(nx) = n(kx).$$

Since kx is an integer, $n \mid (ka - kb)$ and so $ka \equiv kb \pmod{n}$. ■

Result 4.10 *Let $a, b, c, d, n \in \mathbf{Z}$, where $n \geq 2$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv b + d \pmod{n}$.*

Proof Assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then $a - b = nx$ and $c - d = ny$ for some integers x and y. Adding these two equations, we obtain

$$(a - b) + (c - d) = nx + ny$$

and so

$$(a + c) - (b + d) = n(x + y).$$

Since $x + y$ is an integer, $n \mid [(a + c) - (b + d)]$. Hence $a + c \equiv b + d \pmod{n}$. ■

The next result parallels that of Result 4.10 in terms of multiplication.

Result to Prove Let $a, b, c, d, n \in \mathbf{Z}$, where $n \geq 2$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.

PROOF STRATEGY This result and Result 4.10 have the same hypothesis. In the proof of Result 4.10, we arrived at the equations $a - b = nx$ and $c - d = ny$ and needed only to add them to complete the proof. This suggests that in the current result, it would be reasonable to multiply these two equations. However, if we multiply them, we obtain $(a - b)(c - d) = (nx)(ny)$, which does not give us the desired conclusion that $ac - bd$ is a multiple of n. It is essential, though, that we work $ac - bd$ into the proof. By rewriting $a - b = nx$ and $c - d = ny$ as $a = b + nx$ and $c = d + ny$, respectively, and *then* multiplying, we can accomplish this, however. ♦

Result 4.11 *Let $a, b, c, d, n \in \mathbf{Z}$, where $n \geq 2$. If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$.*

Proof Assume that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Then $a - b = nx$ and $c - d = ny$, where $x, y \in \mathbf{Z}$. Thus $a = b + nx$ and $c = d + ny$. Multiplying these two equations, we obtain

$$\begin{aligned} ac &= (b + nx)(d + ny) = bd + dnx + bny + n^2xy \\ &= bd + n(dx + by + nxy) \end{aligned}$$

and so $ac - bd = n(dx + by + nxy)$. Since $dx + by + nxy$ is an integer, $ac \equiv bd \pmod{n}$. ■

The proofs of the preceding three results use a direct proof. This is not a convenient proof technique for the next result, however.

Result to Prove Let $n \in \mathbf{Z}$. If $n^2 \not\equiv n \pmod{3}$, then $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$.

PROOF STRATEGY Let

$$P(n) : n^2 \not\equiv n \pmod{3},\ Q(n) : n \not\equiv 0 \pmod{3}, \text{ and } R(n) : n \not\equiv 1 \pmod{3}.$$

Our goal is then to show that $P(n) \Rightarrow (Q(n) \wedge R(n))$ is true for every integer n. A direct proof does not appear to be a good choice. However, a proof by contrapositive would lead us to the implication $\sim(Q(n) \wedge R(n)) \Rightarrow (\sim P(n))$, which, by De Morgan's Law, is logically equivalent to

$$((\sim Q(n)) \vee (\sim R(n))) \Rightarrow (\sim P(n)).$$

In words, we then have: If $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{3}$, then $n^2 \equiv n \pmod{3}$. ♦

Result 4.12 *Let $n \in \mathbf{Z}$. If $n^2 \not\equiv n \pmod{3}$, then $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$.*

Proof Let n be an integer such that $n \equiv 0 \pmod{3}$ or $n \equiv 1 \pmod{3}$. We consider these two cases.

Case 1. $n \equiv 0 \pmod{3}$. Then $n = 3k$ for some integer k. Hence

$$n^2 - n = (3k)^2 - (3k) = 9k^2 - 3k = 3(3k^2 - k).$$

Since $3k^2 - k$ is an integer, $3 \mid (n^2 - n)$. Thus $n^2 \equiv n \pmod{3}$.

Case 2. $n \equiv 1 \pmod{3}$. So $n = 3\ell + 1$ for some integer ℓ, and

$$\begin{aligned} n^2 - n &= (3\ell + 1)^2 - (3\ell + 1) = (9\ell^2 + 6\ell + 1) - (3\ell + 1) \\ &= 9\ell^2 + 3\ell = 3(3\ell^2 + \ell). \end{aligned}$$

Since $3\ell^2 + \ell$ is an integer, $3 \mid (n^2 - n)$ and so $n^2 \equiv n \pmod{3}$. ■

As a consequence of Result 4.12, if an integer n and its square n^2 have different remainders when divided by 3, then the remainder for n (when divided by 3) is 2.

4.3 Proofs Involving Real Numbers

We now apply the proof techniques we have introduced to verify some mathematical statements involving real numbers. To be certain that we are working under the same set of rules, let us recall some facts about real numbers that can be used without justification. We have already mentioned that $a^2 \geq 0$ for every real number a. Indeed, $a^n \geq 0$ for every real number a if n is a positive even integer. If $a < 0$ and n is a positive odd integer, then $a^n < 0$. Of course, the product of two real numbers is positive if and only if both numbers are positive or both are negative.

Now let $a, b, c \in \mathbf{R}$. If $a \geq b$ and $c \geq 0$, then the inequality $ac \geq bc$ holds. Indeed, if $c > 0$, then $a/c \geq b/c$.

$$\text{If } a > b \text{ and } c > 0, \text{ then } ac > bc \text{ and } a/c > b/c. \tag{4.1}$$

If $c < 0$, then the inequalities in (4.1) are reversed; namely:

$$\text{If } a > b \text{ and } c < 0, \text{ then } ac < bc \text{ and } a/c < b/c. \tag{4.2}$$

Another important and well-known property of real numbers is that if the product of two real numbers is 0, then at least one of these numbers is 0.

Theorem to Prove If x and y are real numbers such that $xy = 0$, then $x = 0$ or $y = 0$.

PROOF STRATEGY If we use a direct proof, then we begin by assuming that $xy = 0$. If $x = 0$, then we already have the desired result. On the other hand, if $x \neq 0$, then we are required to show that $y = 0$. However, if $x \neq 0$, then $1/x$ is a real number. This suggests multiplying $xy = 0$ by $1/x$. ♦

Theorem 4.13 *Let $x, y \in \mathbf{R}$. If $xy = 0$, then $x = 0$ or $y = 0$.*

Proof Assume that $xy = 0$. We consider two cases, according to whether $x = 0$ or $x \neq 0$.

Case 1. $x = 0$. Then we have the desired result.

Case 2. $x \neq 0$. Multiplying $xy = 0$ by the number $1/x$, we obtain $\frac{1}{x}(xy) = \frac{1}{x} \cdot 0 = 0$.

Since

$$\frac{1}{x}(xy) = \left(\frac{1}{x}x\right) y = 1 \cdot y = y,$$

it follows that $y = 0$. ■

We now use Theorem 4.13 to prove the next result.

Result 4.14 *Let $x \in \mathbf{R}$. If $x^3 - 5x^2 + 3x = 15$, then $x = 5$.*

Proof Assume that $x^3 - 5x^2 + 3x = 15$. Thus $x^3 - 5x^2 + 3x - 15 = 0$. Observe that

$$x^3 - 5x^2 + 3x - 15 = x^2(x - 5) + 3(x - 5) = (x^2 + 3)(x - 5).$$

Since $x^3 - 5x^2 + 3x - 15 = 0$, it follows that $(x^2 + 3)(x - 5) = 0$. By Theorem 4.13, $x^2 + 3 = 0$ or $x - 5 = 0$. However, $x^2 + 3 > 0$, so $x - 5 = 0$, implying that $x = 5$. ■

Next we consider an example of a proof by contrapositive involving an inequality.

Result 4.15 *Let $x \in \mathbf{R}$. If $x^5 - 3x^4 + 2x^3 - x^2 + 4x - 1 \geq 0$, then $x \geq 0$.*

Proof Assume that $x < 0$. Then $x^5 < 0$, $2x^3 < 0$, and $4x < 0$. In addition, $-3x^4 < 0$ and $-x^2 < 0$. Thus

$$x^5 - 3x^4 + 2x^3 - x^2 + 4x - 1 < 0 - 1 < 0,$$

as desired. ■

On occasion we may encounter problems that involve the verification of a certain equality or inequality and where it is convenient to find an equivalent formulation of the equality or inequality whose truth is clear. This then becomes the starting point of a proof. We now verify an inequality whose proof uses this common approach.

Result to Prove If $x, y \in \mathbf{R}$, then

$$\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy.$$

PROOF STRATEGY Let's first eliminate fractions from the expression. Showing that $\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy$ is equivalent to showing that

$$12\left(\frac{1}{3}x^2 + \frac{3}{4}y^2\right) \geq 12xy,$$

that is,

$$4x^2 + 9y^2 \geq 12xy,$$

which, in turn, is equivalent to

$$4x^2 - 12xy + 9y^2 \geq 0.$$

Making a simple observation about $4x^2 - 12xy + 9y^2$ leads to a proof. ♦

Result 4.16 *If $x, y \in \mathbf{R}$, then*

$$\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy.$$

Proof Since $(2x - 3y)^2 \geq 0$, it follows that $4x^2 - 12xy + 9y^2 \geq 0$ and so $4x^2 + 9y^2 \geq 12xy$. Dividing this inequality by 12, we obtain

$$\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy$$

producing the desired inequality. ■

Recall that for a real number x, its **absolute value** $|x|$ is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The following theorem gives a familiar property of absolute values of real numbers (called the **triangle inequality**) that has numerous applications. Since the definition of $|x|$ is essentially a definition by cases, proofs involving $|x|$ are often by cases.

Theorem 4.17 *For every two real numbers x and y,*

$$|x + y| \leq |x| + |y|.$$

Proof Since $|x + y| = |x| + |y|$ if either x or y is 0, we can assume that x and y are nonzero. We proceed by cases.

Case 1. $x > 0$ *and* $y > 0$. Then $x + y > 0$ and

$$|x + y| = x + y = |x| + |y|.$$

Case 2. $x < 0$ *and* $y < 0$. Since $x + y < 0$,

$$|x + y| = -(x + y) = (-x) + (-y) = |x| + |y|.$$

Case 3. *One of* x *and* y *is positive and the other is negative, say* $x > 0$ *and* $y < 0$. We consider two subcases.

Subcase 3.1. $x + y \geq 0$. Then

$$|x| + |y| = x + (-y) = x - y > x + y = |x + y|.$$

Subcase 3.2. $x + y < 0$. Here

$$|x| + |y| = x + (-y) = x - y > -x - y = -(x + y) = |x + y|.$$

Therefore, $|x + y| \leq |x| + |y|$ for every two real numbers x and y. ■

4.4 Proofs Involving Sets

We now turn our attention to proofs concerning properties of sets. Recall, for sets A and B contained in some universal set U, that the **intersection** of A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$

the **union** of A and B is

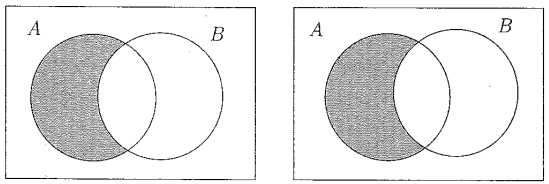
$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and the **difference** of A and B is

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

The set $A - B$ is also called the **relative complement** of B in A, and the relative complement of A in U is called simply the **complement** of A and is denoted by $\overline{A}$. Thus, $\overline{A} = U - A$. In what follows, we will always assume that the sets under discussion are subsets of some universal set U.

Figure 4.1 shows Venn diagrams of $A - B$ and $A \cap \overline{B}$ for arbitrary sets A and B. The diagrams suggest that these two sets are equal. This is, in fact, the case. Recall that to show the equality of two sets C and D, we can verify the two set inclusions $C \subseteq D$



$A - B$ $\qquad$ $A \cap \overline{B}$

Figure 4.1 Venn diagrams for $A - B$ and $A \cap \overline{B}$

and $D \subseteq C$. To establish the inclusion $C \subseteq D$, we show that every element of C is also an element of D; that is, if $x \in C$ then $x \in D$. This is accomplished with a direct proof, by letting x be an (arbitrary) element of C and showing that x must belong to D as well. Recall that we need not be concerned if C contains no elements; for in this case $x \in C$ is false for every element x and so the implication "If $x \in C$, then $x \in D$." is true for all $x \in U$. As a consequence of this observation, if $C = \emptyset$, then C contains no elements and it follows that $C \subseteq D$.

Result 4.18 *For every two sets* A *and* B,

$$A - B = A \cap \overline{B}.$$

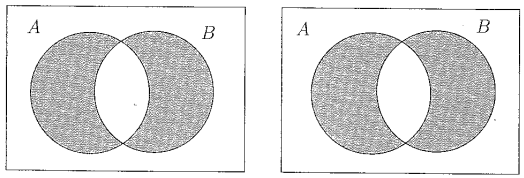
Proof First we show that $A - B \subseteq A \cap \overline{B}$. Let $x \in A - B$. Then $x \in A$ and $x \notin B$. Since $x \notin B$, it follows that $x \in \overline{B}$. Therefore, $x \in A$ and $x \in \overline{B}$; so $x \in A \cap \overline{B}$. Hence $A - B \subseteq A \cap \overline{B}$.

Next we show that $A \cap \overline{B} \subseteq A - B$. Let $y \in A \cap \overline{B}$. Then $y \in A$ and $y \in \overline{B}$. Since $y \in \overline{B}$, we see that $y \notin B$. Now because $y \in A$ and $y \notin B$, we conclude that $y \in A - B$. Thus, $A \cap \overline{B} \subseteq A - B$. ■

PROOF ANALYSIS In the second paragraph of the proof of Result 4.18, we used y (rather than x) to denote an arbitrary element of $A \cap \overline{B}$. We did this only for variety. We could have used x twice. Once we decided to use distinct symbols, y was the logical choice since x was used in the first paragraph of the proof. This keeps our use of symbols consistent. Another possibility would have been to use a in the first paragraph and b in the second. This has some disadvantages, however. Since the sets are being called A and B, we might have a tendency to think that $a \in A$ and $b \in B$, which may confuse the reader. For this reason, we chose x and y over a and b.

Before leaving the proof of Result 4.18, we have one other remark. At one point in the second paragraph, we learned that $y \in A$ and $y \notin B$. From this we could have concluded (correctly) that $y \notin A \cap B$, but this is not what we wanted. Instead, we wrote that $y \in A - B$. It is always a good idea to keep our goal in sight. We wanted to show that $y \in A - B$; so it was important to keep in mind that it was the set $A - B$ in which we were interested, not $A \cap B$. ♦

Next, let's consider the Venn diagrams for $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$, which are shown in Figure 4.2. From these two diagrams, we might



$(A \cup B) - (A \cap B)$ $\qquad$ $(A - B) \cup (B - A)$

Figure 4.2 Venn diagrams for $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$

conclude (correctly) that the two sets $(A \cup B) - (A \cap B)$ and $(A - B) \cup (B - A)$ are equal. Indeed, all that is lacking is a *proof* that these two sets are equal. That is, Venn diagrams can be useful in suggesting certain results concerning sets, but they are only drawings and do not constitute a proof.

Result 4.19 *For every two sets A and B,*

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

Proof First we show that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$. Let $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Since $x \in A \cup B$, it follows that $x \in A$ or $x \in B$. Without loss of generality, let $x \in A$. Since $x \notin A \cap B$, the element $x \notin B$. Therefore, $x \in A - B$ and so $x \in (A - B) \cup (B - A)$. Hence

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A).$$

Next we show that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. Let $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$, say the former. So $x \in A$ and $x \notin B$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Consequently, $x \in (A \cup B) - (A \cap B)$. Therefore,

$$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B),$$

as desired. ■

PROOF ANALYSIS In the proof of Result 4.19, when we were verifying the set inclusion

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A),$$

we concluded that $x \in A$ or $x \in B$. At that point, we could have divided the proof into two cases (*Case* 1. $x \in A$ and *Case* 2. $x \in B$); however, the proofs of the two cases would be identical, except that A and B would be interchanged. Therefore, we decided to consider only one of these. Since it really didn't matter which case we handled, we simply chose the case where $x \in A$. This was accomplished by writing:

Without loss of generality, assume that $x \in A$.

In the proof of the reverse set containment, we found ourselves in a similar situation, namely, $x \in A - B$ or $x \in B - A$. Again, these two situations were basically identical, and we simply chose to work with the first (former) situation. (Had we decided to assume that $x \in B - A$, we would have considered the *latter* case.) ♦

We now look at an example of a biconditional concerning sets.

Result 4.20 *Let A and B be sets. Then $A \cup B = A$ if and only if $B \subseteq A$.*

Proof First we prove that if $A \cup B = A$, then $B \subseteq A$. We use a proof by contrapositive. Assume that B is not a subset of A. Then there must be some element $x \in B$ such that $x \notin A$. Since $x \in B$, it follows that $x \in A \cup B$. However, since $x \notin A$, we have $A \cup B \neq A$.

Next we prove the converse, namely, if $B \subseteq A$, then $A \cup B = A$. We use a direct proof here. Assume that $B \subseteq A$. To verify that $A \cup B = A$, we show that $A \subseteq A \cup B$ and $A \cup B \subseteq A$. The set inclusion $A \subseteq A \cup B$ is immediate (if $x \in A$, then $x \in A \cup B$). It remains only to show then that $A \cup B \subseteq A$. Let $y \in A \cup B$. Thus $y \in A$ or $y \in B$. If $y \in A$, then we already have the desired result. If $y \in B$, then since $B \subseteq A$, it follows that $y \in A$. Thus $A \cup B \subseteq A$. ■

PROOF ANALYSIS In the first paragraph of the proof of Result 4.20 we indicated that we were using a proof by contrapositive, while in the second paragraph we mentioned that we were using a direct proof. This really wasn't necessary as the assumptions we made would inform the reader what technique we were applying. Also, in the proof of Result 4.20, we used a proof by contrapositive for one implication and a direct proof for its converse. This wasn't necessary either. Indeed, it is quite possible to interchange the techniques we used (see Exercise 4.28). ♦

4.5 Fundamental Properties of Set Operations

Many results concerning sets follow from some very fundamental properties of sets, which, in turn, follow from corresponding results about logical statements that were described in Chapter 2. For example, we know that if P and Q are two statements, then $P \vee Q$ and $Q \vee P$ are logically equivalent. Similarly, if A and B are two sets, then $A \cup B = B \cup A$. We list some of the fundamental properties of set operations in the following theorem.

Theorem 4.21 *For sets A, B, and C,*

(1) *Commutative Laws*
   (*a*) $A \cup B = B \cup A$
   (*b*) $A \cap B = B \cap A$

(2) *Associative Laws*
   (*a*) $A \cup (B \cup C) = (A \cup B) \cup C$
   (*b*) $A \cap (B \cap C) = (A \cap B) \cap C$

(3) *Distributive Laws*
   (*a*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (*b*) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) *De Morgan's Laws*
   (*a*) $\overline{A \cup B} = \overline{A} \cap \overline{B}$
   (*b*) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We present proofs of only three parts of Theorem 4.21, beginning with the commutative law of the union of two sets.

Proof of Theorem 4.21(1a) We show that $A \cup B \subseteq B \cup A$. Assume that $x \in A \cup B$. Then $x \in A$ or $x \in B$. Applying the commutative law for disjunction of statements, we conclude that $x \in B$ or $x \in A$; so $x \in B \cup A$. Thus, $A \cup B \subseteq B \cup A$. The proof of the reverse set inclusion $B \cup A \subseteq A \cup B$ is similar and is therefore omitted. ■

Next we verify one of the distributive laws.

Proof of Theorem 4.21(3a) First we show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$. Thus $x \in (A \cup B) \cap (A \cup C)$, as desired. On the other hand, if $x \in B \cap C$, then $x \in B$ and $x \in C$; and again, $x \in A \cup B$ and $x \in A \cup C$. So $x \in (A \cup B) \cap (A \cup C)$. Therefore, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To verify the reverse set inclusion, let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. So we may assume that $x \notin A$. Then the fact that $x \in A \cup B$ and $x \notin A$ implies that $x \in B$. By the same reasoning, $x \in C$. Therefore, $x \in B \cap C$, and so $x \in A \cup (B \cap C)$. Therefore, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. ■

As a final example, we prove one of De Morgan's laws.

Proof of Theorem 4.21(4a) First, we show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$. Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. Hence $x \notin A$ and $x \notin B$. Therefore, $x \in \overline{A}$ and $x \in \overline{B}$, so $x \in \overline{A} \cap \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Next we show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. Let $x \in \overline{A} \cap \overline{B}$. Then $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \notin A$ and $x \notin B$, so $x \notin A \cup B$. Therefore, $x \in \overline{A \cup B}$. Hence $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. ■

PROOF ANALYSIS In the proof of the De Morgan law that we just presented, we arrived at the step $x \notin A \cup B$ at one point and then next wrote $x \notin A$ *and* $x \notin B$. Since $x \in A \cup B$ implies that $x \in A$ *or* $x \in B$, you might have expected us to write that $x \notin A$ *or* $x \notin B$ after writing $x \notin A \cup B$; but this would not be the correct conclusion. When we say that $x \notin A \cup B$, this is equivalent to writing $\sim(x \in A \cup B)$, which is logically equivalent to $\sim((x \in A)$ or $(x \in B))$. By the De Morgan law for the negation of the disjunction of two statements (or two open sentences), we have that $\sim((x \in A)$ or $(x \in B))$ is logically equivalent to $\sim(x \in A)$ and $\sim(x \in B)$; that is, $x \notin A$ *and* $x \notin B$. ♦

Proofs of some other parts of Theorem 4.21 are left as exercises.

4.6 Proofs Involving Cartesian Products of Sets

Recall that the **Cartesian product** (or simply the **product**) $A \times B$ of two sets A and B is defined as

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

Before looking at several examples of proofs concerning Cartesian products of sets, it is important to keep in mind that an arbitrary element of the Cartesian product $A \times B$ of two sets A and B is of the form (a, b), where $a \in A$ and $b \in B$.

Result 4.22 *Let A, B, C, and D be sets. If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.*

Proof Let $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Since $A \subseteq C$ and $B \subseteq D$, it follows that $x \in C$ and $y \in D$. Hence $(x, y) \in C \times D$. ■

Result 4.23 *For sets A, B, and C,*

$$A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof We first show that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Let $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. Thus $y \in B$ or $y \in C$, say the former. Then $(x, y) \in A \times B$, and so $(x, y) \in (A \times B) \cup (A \times C)$. Consequently, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Next we show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Let $(x, y) \in (A \times B) \cup (A \times C)$. Then $(x, y) \in A \times B$ or $(x, y) \in A \times C$, say the former. Then $x \in A$ and $y \in B \subseteq B \cup C$. Hence $(x, y) \in A \times (B \cup C)$, implying that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. ■

We give one additional example of a proof involving the Cartesian products of sets.

Result 4.24 *For sets A, B, and C,*

$$A \times (B - C) = (A \times B) - (A \times C).$$

Proof First we show that $A \times (B - C) \subseteq (A \times B) - (A \times C)$. Let $(x, y) \in A \times (B - C)$. Then $x \in A$ and $y \in B - C$. Since $y \in B - C$, it follows that $y \in B$ and $y \notin C$. Because $x \in A$ and $y \in B$, we have $(x, y) \in A \times B$. Since $y \notin C$, however, $(x, y) \notin A \times C$. Therefore, $(x, y) \in (A \times B) - (A \times C)$. Hence $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

We now show that $(A \times B) - (A \times C) \subseteq A \times (B - C)$. Let $(x, y) \in (A \times B) - (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \notin A \times C$. Since $(x, y) \in A \times B$, it follows that $x \in A$ and $y \in B$. Also, since $x \in A$ and $(x, y) \notin A \times C$, it follows that $y \notin C$. So $y \in B - C$. Thus $(x, y) \in A \times (B - C)$ and $(A \times B) - (A \times C) \subseteq A \times (B - C)$. ■

PROOF ANALYSIS We add one comment concerning the preceding proof. During the proof of $(A \times B) - (A \times C) \subseteq A \times (B - C)$, we needed to show that $y \notin C$. We learned that $(x, y) \notin A \times C$. However, this information alone did not allow us to conclude that $y \notin C$. Indeed, if $(x, y) \notin A \times C$, then $x \notin A$ *or* $y \notin C$. Since we knew, however, that $x \in A$ *and* $(x, y) \notin A \times C$, we were able to conclude that $y \notin C$. ♦

EXERCISES FOR CHAPTER 4

Section 4.1: Proofs Involving Divisibility of Integers

4.1. Let a and b be integers, where $a \neq 0$. Prove that if $a \mid b$, then $a^2 \mid b^2$.

4.2. Let $a, b \in \mathbf{Z}$, where $a \neq 0$ and $b \neq 0$. Prove that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

4.3. Let $m \in \mathbf{Z}$.

(a) Give a direct proof of the following: If $3 \mid m$, then $3 \mid m^2$.

(b) State the contrapositive of the implication in (a).

(c) Give a direct proof of the following: If $3 \nmid m$, then $3 \nmid m^2$.

(d) State the contrapositive of the implication in (c).

(e) State the conjunction of the implications in (a) and (c) using "if and only if".

4.4. Let $x, y \in \mathbf{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3 \mid (x^2 - y^2)$.

4.5. Let $a, b, c \in \mathbf{Z}$, where $a \neq 0$. Prove that if $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.

4.6. Let $a \in \mathbf{Z}$. Prove that if $3 \mid 2a$, then $3 \mid a$.

4.7. Let $n \in \mathbf{Z}$. Prove that $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.

4.8. Let $n \in \mathbf{Z}$. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

4.9. Prove that for every integer $n \geq 8$, there exist nonnegative integers a and b such that $n = 3a + 5b$.

Section 4.2: Proofs Involving Congruence of Integers

4.10. Let $a, b, n \in \mathbf{Z}$, where $n \geq 2$. Prove that if $a \equiv b \pmod{n}$, then $a^2 \equiv b^2 \pmod{n}$.

4.11. Let $a, b, c, n \in \mathbf{Z}$, where $n \geq 2$. Prove that if $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$, then $b \equiv c \pmod{n}$.

4.12. Let $a, b \in \mathbf{Z}$. Prove that if $a^2 + 2b^2 \equiv 0 \pmod{3}$, then either a and b are both congruent to 0 modulo 3 or neither is congruent to 0 modulo 3.

4.13. (a) Prove that if a is an integer such that $a \equiv 1 \pmod{5}$, then $a^2 \equiv 1 \pmod{5}$.
(b) Given that b is an integer such that $b \equiv 1 \pmod{5}$, what can we conclude from (a)?

4.14. (a) Result 4.12 states: Let $n \in \mathbf{Z}$. If $n^2 \not\equiv n \pmod{3}$, then $n \not\equiv 0 \pmod{3}$ and $n \not\equiv 1 \pmod{3}$. State and prove the converse of this result.
(b) State the conjunction of Result 4.12 and its converse using "if and only if".

4.15. Let $a, b \in \mathbf{Z}$. Show that if $a \equiv 5 \pmod{6}$ and $b \equiv 3 \pmod{4}$, then $4a + 6b \equiv 6 \pmod{8}$.

4.16. Let $n \in \mathbf{Z}$. Prove each of the statements (a)–(f).

(a) If $n \equiv 0 \pmod{7}$, then $n^2 \equiv 0 \pmod{7}$.
(b) If $n \equiv 1 \pmod{7}$, then $n^2 \equiv 1 \pmod{7}$.
(c) If $n \equiv 2 \pmod{7}$, then $n^2 \equiv 4 \pmod{7}$.
(d) If $n \equiv 3 \pmod{7}$, then $n^2 \equiv 2 \pmod{7}$.
(e) For each integer n, $n^2 \equiv (7 - n)^2 \pmod{7}$.
(f) For every integer n, n^2 is congruent to exactly one of 0, 1, 2, or 4 modulo 7.

4.17. Let $a \in \mathbf{Z}$. Prove that $a^3 \equiv a \pmod{3}$.

Section 4.3: Proofs Involving Real Numbers

4.18. Let $x, y \in \mathbf{R}$. Prove that if $x^2 - 4x = y^2 - 4y$ and $x \neq y$, then $x + y = 4$.

4.19. Let a, b, and m be integers. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.

4.20. Let $x \in \mathbf{R}$. Prove that if $3x^4 + 1 \leq x^7 + x^3$, then $x > 0$.

4.21. (a) Recall that $\sqrt{r} > 0$ for every positive real number r. Prove that if a and b are positive real numbers, then

$$0 < \sqrt{ab} \leq \frac{a+b}{2}.$$

(The number $\sqrt{ab}$ is called the **geometric mean** of a and b, while $(a + b)/2$ is called the **arithmetic mean** or **average** of a and b.)
(b) Under what conditions does $\sqrt{ab} = (a + b)/2$ for positive real numbers a and b? Justify your answer.

4.22. (a) Prove that if r is a real number such that $0 < r < 1$, then

$$\frac{1}{r(1-r)} \geq 4.$$

(b) If the real number r in part (a) is an integer, is the implication true in this case? Explain.

4.23. Let $x, y \in \mathbf{R}$. Prove that $|xy| = |x| \cdot |y|$.

4.24. Prove that for every two real numbers x and y,

$$|x + y| \geq |x| - |y|.$$

[Hint: Observe that $|x| = |(x + y) + (-y)|$.]

4.25. Prove that for every three real numbers x, y, and z,

$$|x - z| \leq |x - y| + |y - z|.$$

4.26. Prove that if r is a real number such that $|r - 1| < 1$, then $\frac{4}{r(4-r)} \geq 1$.

Section 4.4: Proofs Involving Sets

4.27. Let A and B be sets. Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

4.28. In Result 4.20, it was proved for sets A and B that $A \cup B = A$ if and only if $B \subseteq A$. Provide another proof of this result by giving a direct proof of the implication "If $A \cup B = A$, then $B \subseteq A$" and a proof by contrapositive of its converse.

4.29. Let A and B be sets. Prove that $A \cap B = A$ if and only if $A \subseteq B$.

4.30. (a) Give an example of three sets A, B, and C such that $A \cap B = A \cap C$ but $B \neq C$.
(b) Give an example of three sets A, B, and C such that $A \cup B = A \cup C$ but $B \neq C$.
(c) Let A, B, and C be sets. Prove that if $A \cap B = A \cap C$ *and* $A \cup B = A \cup C$, then $B = C$.

4.31. Prove that if A and B are sets such that $A \cup B \neq \emptyset$, then $A \neq \emptyset$ or $B \neq \emptyset$.

4.32. Let $A = \{n \in \mathbf{Z} : n \equiv 1 \pmod{2}\}$ and $B = \{n \in \mathbf{Z} : n \equiv 3 \pmod{4}\}$. Prove that $B \subseteq A$.

4.33. Let A and B be sets. Prove that $A \cup B = A \cap B$ if and only if $A = B$.

Section 4.5: Fundamental Properties of Set Operations

4.34. Prove that $A \cap B = B \cap A$ for every two sets A and B (Theorem 4.21(1b)).

4.35. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for every three sets A, B, and C (Theorem 4.21(3b)).

4.36. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ for every two sets A and B (Theorem 4.21(4b)).

4.37. Let A, B, and C be sets. Prove that $(A - B) \cap (A - C) = A - (B \cup C)$.

4.38. Let A, B, and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

4.39. Let A, B, and C be sets. Use Theorem 4.21 to prove that $\overline{\overline{A} \cup (\overline{B} \cap C)} = (A \cap B) \cup (A - C)$.

Section 4.6: Proofs Involving Cartesian Products of Sets

4.40. Let A and B be sets. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

4.41. For sets A and B, find a necessary and sufficient condition for $A \times B = B \times A$.

4.42. For sets A and B, find a necessary and sufficient condition for $(A \times B) \cap (B \times A) = \emptyset$. Verify that this condition *is* necessary and sufficient.

4.43. Let A, B, and C be nonempty sets. Prove that $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

4.44. Result 4.22 states that if A, B, C, and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.
(a) Show that the converse of Result 4.22 is false.
(b) Under what added hypothesis is the converse true? Prove your assertion.

4.45. Let A, B, and C be sets. Prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

4.46. Let A, B, C, and D be sets. Prove that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

4.47. Let A, B, C, and D be sets. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

4.48. Let A and B be sets. Show, in general, that $\overline{A \times B} \neq \overline{A} \times \overline{B}$.

ADDITIONAL EXERCISES FOR CHAPTER 4

4.49. Let $n \in Z$. Prove that $5 \mid n^2$ if and only if $5 \mid n$.

4.50. Prove for integers a and b that $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$.

4.51. Prove that if n is an odd integer, then $8 \mid [n^2 + (n+6)^2 + 6]$.

4.52. Prove that if n is an odd integer, then $8 \mid (n^4 + 4n^2 + 11)$.

4.53. Let $n, m \in \mathbf{Z}$. Prove that if $n \equiv 1 \pmod 2$ and $m \equiv 3 \pmod 4$, then $n^2 + m \equiv 0 \pmod 4$.

4.54. Find two distinct positive integer values of a for which the following is true and give a proof in each case:

For every integer n, $a \nmid (n^2 + 1)$.

4.55. Prove for every two real numbers a and b that $ab \leq \sqrt{a^2}\sqrt{b^2}$.

4.56. Prove for every four real numbers a, b, c, and d that $ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$. [Hint: Observe that $(ad - bc)^2 \geq 0$.]

4.57. Prove the following: Let $x \in \mathbf{R}$. If $x(x-5) = -4$, then $\sqrt{5x^2 - 4} = 1$ implies that $x + \frac{1}{x} = 2$.

4.58. Evaluate the proposed proof of the following result.

Result Let $x, y \in \mathbf{Z}$. If $x \equiv 2 \pmod 3$ and $y \equiv 2 \pmod 3$, then $xy \equiv 1 \pmod 3$.

Proof Let $x \equiv 2 \pmod 3$ and $y \equiv 2 \pmod 3$. Then $x = 3k + 2$ and $y = 3k + 2$ for some integer k. Hence

$$\begin{aligned} xy &= (3k+2)(3k+2) = 9k^2 + 12k + 4 = 9k^2 + 12k + 3 + 1 \\ &= 3(3k^2 + 4k + 1) + 1. \end{aligned}$$

Since $3k^2 + 4k + 1$ is an integer, $xy \equiv 1 \pmod 3$. ∎

4.59. Below is given a proof of a result. What result is proved?

Proof Assume that $x \equiv 1 \pmod 5$ and $y \equiv 2 \pmod 5$. Then $5 \mid (x-1)$ and $5 \mid (y-2)$. Hence $x - 1 = 5a$ and $y - 2 = 5b$ for some integers a and b. So $x = 5a + 1$ and $y = 5b + 2$. Therefore,

$$\begin{aligned} x^2 + y^2 &= (5a+1)^2 + (5b+2)^2 = (25a^2 + 10a + 1) + (25b^2 + 20b + 4) \\ &= 25a^2 + 10a + 25b^2 + 20b + 5 = 5(5a^2 + 2a + 5b^2 + 4b + 1). \end{aligned}$$

Since $5a^2 + 2a + 5b^2 + 4b + 1$ is an integer, $5 \mid (x^2 + y^2)$ and so $x^2 + y^2 \equiv 0 \pmod 5$. ∎

4.60. A proof of the following result is given.

Result Let $n \in \mathbf{Z}$. If n^4 is even, then $3n + 1$ is odd.

Proof Assume that $n^4 = (n^2)^2$ is even. Since n^4 is even, n^2 is even. Furthermore, since n^2 is even, n is even. Because n is even, $n = 2k$ for some integer k. Then

$$3n + 1 = 3(2k) + 1 = 6k + 1 = 2(3k) + 1.$$

Since $3k$ is an integer, $3n + 1$ is odd. ∎

Answer the following questions.

(1) Which proof technique is being used?
(2) What is the starting assumption?
(3) What must be shown to give a complete proof?
(4) Give a reason for each of the following steps in the proof.
   (a) Since n^4 is even, n^2 is even.
   (b) Furthermore, since n^2 is even, n is even.
   (c) Because n is even, $n = 2k$ for some integer k.
   (d) Then $3n + 1 = 3(2k) + 1 = 6k + 1 = 2(3k) + 1$.
   (e) Since $3k$ is an integer, $3n + 1$ is odd.

4.61. Given below is an attempted proof of a result.

Proof First, we show that $A \subseteq (A \cup B) - B$. Let $x \in A$. Since $A \cap B = \emptyset$, it follows that $x \notin B$. Therefore, $x \in A \cup B$ and $x \notin B$, so $x \in (A \cup B) - B$. Thus $A \subseteq (A \cup B) - B$.

Next, we show that $(A \cup B) - B \subseteq A$. Let $x \in (A \cup B) - B$. Then $x \in A \cup B$ and $x \notin B$. From this, it follows that $x \in A$. Hence $(A \cup B) - B \subseteq A$. ∎

(a) What result is being proved above?
(b) What change (or changes) in this proof would make it better (from your point of view)?

4.62. Evaluate the proposed proof of the following result.

Result Let $x, y \in \mathbf{Z}$ such that $3 \mid x$. If $3 \mid (x + y)$, then $3 \mid y$.

Proof Since $3 \mid x$, it follows that $x = 3a$, where $a \in \mathbf{Z}$. Assume that $3 \mid (x + y)$. Then $x + y = 3b$ for some integer b. Hence $y = 3b - x = 3b - 3a = 3(b - a)$. Since $b - a$ is an integer, $3 \mid y$.

For the converse, assume that $3 \mid y$. Therefore, $y = 3c$, where $c \in \mathbf{Z}$. Thus $x + y = 3a + 3c = 3(a + c)$. Since $a + c$ is an integer, $3 \mid (x + y)$. ∎

4.63. Evaluate the proposed proof of the following result.

Result Let $x, y \in \mathbf{Z}$. If $x \equiv 1 \pmod 3$ and $y \equiv 1 \pmod 3$, then $xy \equiv 1 \pmod 3$.

Proof Assume that $x \equiv 1 \pmod 3$ and $y \equiv 1 \pmod 3$. Then $3 \mid (x-1)$ and $3 \mid (y-1)$. Hence $x - 1 = 3q$ and $y - 1 = 3q$ for some integer q and so $x = 3q + 1$ and $y = 3q + 1$. Thus

$$xy = (3q+1)(3q+1) = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$$

and so $xy - 1 = 3(3q^2 + 2q)$. Since $3q^2 + 2q$ is an integer, $3 \mid (xy - 1)$. Hence $xy \equiv 1 \pmod 3$. ∎

4.64. Evaluate the proposed proof of the following result.

Result For every three sets A, B, and C,

$$(A \times C) - (B \times C) \subseteq (A - B) \times C.$$

Proof Let $(x, y) \in (A \times C) - (B \times C)$. Then $(x, y) \in A \times C$ and $(x, y) \notin B \times C$. Since $(x, y) \in A \times C$, it follows that $x \in A$ and $y \in C$. Since $(x, y) \notin B \times C$, we have $x \notin B$. Thus $x \in A - B$. Hence $(x, y) \in (A - B) \times C$. ∎

4.65. Prove that for every three integers a, b, and c, the sum

$$|a - b| + |a - c| + |b - c|$$

is an even integer.

4.66. Prove that for every two positive real numbers a and b,

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

4.67. Prove that for every real number x,

$$\sin^6 x + 3 \sin^2 x \cos^2 x + \cos^6 x = 1.$$

[Hint: Consider $(\sin^2 x + \cos^2 x)^3$.]

4.68. Let $x, y \in \mathbf{R}$. Prove that if $x < 0$, then

$$x^3 - x^2 y \leq x^2 y - x y^2.$$

5

Existence and Proof by Contradiction

Thus far, we have considered quantified statements involving universal quantifiers, namely statements of the type $\forall x \in S$, $R(x)$. We now consider problems that involve, either directly or indirectly, quantified statements involving existential quantifiers, that is, statements of the type $\exists x \in S$, $R(x)$.

5.1 Counterexamples

It must certainly come as no surprise that some quantified statements of the type $\forall x \in S$, $R(x)$ are false. We have seen that

$$\sim(\forall x \in S, R(x)) \equiv \exists x \in S, \sim R(x),$$

that is, if the statement $\forall x \in S$, $R(x)$ is false, then there exists some element $x \in S$ for which $R(x)$ is false. Such an element x is called a **counterexample** of the (false) statement $\forall x \in S$, $R(x)$. Finding a counterexample verifies that $\forall x \in S$, $R(x)$ is false.

Example 5.1 *Consider the statement:*

$$\textit{If } x \in \mathbf{R}, \textit{ then } (x^2 - 1)^2 > 0. \tag{5.1}$$

or, equivalently,

$$\textit{For every real number } x, (x^2 - 1)^2 > 0.$$

Show that the statement (5.1) *is false by exhibiting a counterexample.*

Solution For $x = 1$, $(x^2 - 1)^2 = (1^2 - 1)^2 = 0$. Thus $x = 1$ is a counterexample. ♦

It might be noticed that the number $x = -1$ is also a counterexample. In fact, $x = 1$ and $x = -1$ are the only two counterexamples of the statement (5.1). That is, the statement

$$\text{If } x \in \mathbf{R} - \{1, -1\}, \text{ then } (x^2 - 1)^2 > 0. \tag{5.2}$$

is true.

4.64. Evaluate the proposed proof of the following result.

Result For every three sets A, B, and C,

$$(A \times C) - (B \times C) \subseteq (A - B) \times C.$$

Proof Let $(x, y) \in (A \times C) - (B \times C)$. Then $(x, y) \in A \times C$ and $(x, y) \notin B \times C$. Since $(x, y) \in A \times C$, it follows that $x \in A$ and $y \in C$. Since $(x, y) \notin B \times C$, we have $x \notin B$. Thus $x \in A - B$. Hence $(x, y) \in (A - B) \times C$. ∎

4.65. Prove that for every three integers a, b, and c, the sum

$$|a - b| + |a - c| + |b - c|$$

is an even integer.

4.66. Prove that for every two positive real numbers a and b,

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

4.67. Prove that for every real number x,

$$\sin^6 x + 3 \sin^2 x \cos^2 x + \cos^6 x = 1.$$

[Hint: Consider $(\sin^2 x + \cos^2 x)^3$.]

4.68. Let $x, y \in \mathbf{R}$. Prove that if $x < 0$, then

$$x^3 - x^2 y \leq x^2 y - x y^2.$$

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$$\text{If } x \in \mathbf{R} - \{1, -1\}, \text{ then } (x^2 - 1)^2 > 0. \tag{5.2}$$

is true.

If a statement P is shown to be false in some manner, then P is said to be **disproved**. The counterexample $x = 1$ therefore disproves the statement (5.1).

Example 5.2 *Disprove the statement:*

$$\textit{If } x \textit{ is a real number, then } \tan^2 x + 1 = \sec^2 x. \tag{5.3}$$

Solution Since $\tan x$ and $\sec x$ are not defined when $x = \pi/2$, it follows that $\tan^2 x + 1$ and $\sec^2 x$ have no numerical value when $x = \pi/2$ and, consequently, $\tan^2 x + 1$ and $\sec^2 x$ are not equal when $x = \pi/2$. That is, $x = \pi/2$ is a counterexample to the statement (5.3). ♦

Although $\tan^2 x + 1 = \sec^2 x$ is a well-known identity from trigonometry, statement (5.3), as presented, is false. The following is true, however:

$$\text{If } x \text{ is a real number for which } \tan x \text{ and } \sec x \text{ are defined,}$$
$$\text{then } \tan^2 x + 1 = \sec^2 x. \tag{5.4}$$

Indeed, it is probably statement (5.4) that was intended in Example 5.2, rather than statement (5.3). Since $\tan x$ and $\sec x$ are defined for precisely the same real numbers x (namely, those numbers x such that $\cos x \neq 0$), we can restate (5.4) as

$$\text{If } x \in \mathbf{R} - \left\{n\pi + \tfrac{\pi}{2} : n \in \mathbf{Z}\right\}, \text{ then } \tan^2 x + 1 = \sec^2 x.$$

Example 5.3 *Disprove the statement:*

$$\textit{If } x \in \mathbf{Z}, \textit{ then } \frac{x^2 + x}{x^2 - x} = \frac{x + 1}{x - 1}. \tag{5.5}$$

Solution If $x = 0$, then $x^2 - x = 0$ and so $\frac{x^2+x}{x^2-x}$ is undefined. On the other hand, if $x = 0$, then $\frac{x+1}{x-1} = -1$; so the expressions $\frac{x^2+x}{x^2-x}$ and $\frac{x+1}{x-1}$ are certainly not equal when $x = 0$. Thus $x = 0$ is a counterexample to the statement (5.5). ♦

Since neither $\frac{x^2+x}{x^2-x}$ nor $\frac{x+1}{x-1}$ is defined when $x = 1$, it follows that $x = 1$ is also a counterexample of statement (5.5). Indeed, $x = 0$ and $x = 1$ are the only counterexamples of statement (5.5) and so the statement

$$\text{If } x \in \mathbf{Z} - \{0, 1\}, \text{ then } \frac{x^2 + x}{x^2 - x} = \frac{x + 1}{x - 1}.$$

is true.

The three preceding examples illustrate the fact that an open sentence $R(x)$ that is false over some domain S may very well be true over a subset of S. Therefore, the truth (or falseness) of a statement $\forall x \in S$, $R(x)$ depends not only on the open sentence $R(x)$ but on its domain as well.

Example 5.4 *Disprove the statement:*

$$\textit{For every odd positive integer } n, \ 3 \mid (n^2 - 1). \tag{5.6}$$

Solution Since $3 \nmid (3^2 - 1)$, it follows that $n = 3$ is a counterexample. ♦

You might have noticed that even though $3 \nmid (3^2 - 1)$, it is the case that $3 \mid (n^2 - 1)$ for *some* odd positive integers. For example, $3 \mid (n^2 - 1)$ if $n = 1, 5, 7, 11, 13, 17$, while $3 \nmid (n^2 - 1)$ if $n = 3, 9, 15, 21$. This should make you wonder for which odd positive integers n, the open sentence $3 \mid (n^2 - 1)$ is true. (See Result 4.6.)

We have seen that a quantified statement of the type

$$\forall x \in S, \ R(x)$$

is false if

$$\exists x \in S, \ \sim R(x)$$

is true, that is, if there exists some element $x \in S$ for which $R(x)$ is false. There will be many instances when $R(x)$ is an implication $P(x) \Rightarrow Q(x)$. Therefore, the quantified statement

$$\forall x \in S, \ P(x) \Rightarrow Q(x) \tag{5.7}$$

is false if

$$\exists x \in S, \ \sim (P(x) \Rightarrow Q(x)) \tag{5.8}$$

is true. By Theorem 2.21(a), the statement (5.8) can be expressed as

$$\exists x \in S, \ (P(x) \wedge (\sim Q(x))).$$

That is, to show that the statement (5.7) is false, we need to exhibit a counterexample, which is then an element $x \in S$ for which $P(x)$ is true and $Q(x)$ is false.

Example 5.5 *Disprove the statement:*

$$\textit{Let } n \in \mathbf{Z}. \textit{ If } n^2 + 3n \textit{ is even, then } n \textit{ is odd.}$$

Solution If $n = 2$, then $n^2 + 3n = 2^2 + 3 \cdot 2 = 10$ is even and 2 is even. Thus $n = 2$ is a counterexample. ♦

In the preceding example, not only is 2 a counterexample, *every* even integer is a counterexample.

Example 5.6 *Disprove the statement:*

$$\textit{If } n \textit{ is an odd integer, then } n^2 - n \textit{ is odd.} \tag{5.9}$$

Solution For the odd integer $n = 1$, the integer $n^2 - n = 1^2 - 1 = 0$ is even. Thus $n = 1$ is a counterexample. ♦

Actually, it is not difficult to prove that the statement

$$\text{If } n \text{ is an odd integer, then } n^2 - n \text{ is even.}$$

is true. Although it may very well be of interest to know this, to show that statement (5.9) is false requires exhibiting only a single counterexample. It does not require proving some other result. One should know the difference between these two.

Example 5.7 *Show that the statement:*

$$\text{Let } n \in \mathbf{Z}. \text{ If } 4 \mid (n^2 - 1), \text{ then } 4 \mid (n - 1).$$

is false.

Solution Since $4 \mid (3^2 - 1)$ but $4 \nmid (3 - 1)$, it follows that $n = 3$ is a counterexample. ♦

Example 5.8 *Show that the statement*

$$\text{For positive integers } a, b, c, a^{b^c} = (a^b)^c.$$

is false.

Solution Let $a = 2$, $b = 2$, and $c = 3$. Then $a^{b^c} = 2^{2^3} = 2^8 = 256$, while $(a^b)^c = (2^2)^3 = 4^3 = 64$. Since $256 \neq 64$, the positive integers $a = 2$, $b = 2$, and $c = 3$ constitute a counterexample. ♦

Example 5.9 *Show that the statement:*

$$\text{Let } a \text{ and } b \text{ be nonzero real numbers. If } x, y \in \mathbf{R}^+, \text{ then}$$

$$\frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 > xy. \tag{5.10}$$

is false.

Solution Let $x = b^2$ and $y = a^2$. Then

$$\frac{a^2}{2b^2}x^2 + \frac{b^2}{2a^2}y^2 = \frac{a^2b^2}{2} + \frac{a^2b^2}{2} = a^2b^2 = xy.$$

Thus $x = b^2$ and $y = a^2$ is a counterexample and so the inequality is false. ♦

Analysis After reading the solution of Example 5.9, the only question that may occur to you is where the counterexample $x = b^2$ and $y = a^2$ came from. Multiplying the inequality (5.10) by $2a^2b^2$ (which eliminates all fractions) produces the equivalent inequality

$$a^4x^2 + b^4y^2 > 2a^2b^2xy$$

and so

$$a^4x^2 - 2a^2b^2xy + b^4y^2 > 0,$$

which can be expressed as

$$(a^2x - b^2y)^2 > 0.$$

Of course, $(a^2x - b^2y)^2 \geq 0$. Thus any values of x and y for which $a^2x - b^2y = 0$ produce a counterexample. Although there are many choices for x and y, one such choice is $x = b^2$ and $y = a^2$. ♦

5.2 Proof by Contradiction

Suppose, as usual, that we would like to show that a certain mathematical statement R is true. If R is expressed as the quantified statement $\forall\, x \in S,\ P(x) \Rightarrow Q(x)$, then we have already introduced two proof techniques, namely direct proof and proof by contrapositive, that could be used to establish the truth of R. We now introduce a third method that can be used to establish the truth of R, regardless of whether R is expressed in terms of an implication.

Suppose that we assume R is a false statement and, from this assumption, we are able to arrive at or deduce a statement that contradicts some assumption we made in the proof or some known fact. (The known fact might be a definition, an axiom, or a theorem.) If we denote this assumption or known fact by P, then what we have deduced is $\sim P$ and have thus produced the contradiction $C : P \wedge (\sim P)$. We have therefore established the truth of the implication

$$(\sim R) \Rightarrow C.$$

However, because $(\sim R) \Rightarrow C$ is true and C is false, it follows that $\sim R$ is false and so R is true, as desired. This technique is called **proof by contradiction**.

If R is the quantified statement $\forall\, x \in S,\ P(x) \Rightarrow Q(x)$, then a proof by contradiction of this statement consists of verifying the implication

$$\sim (\forall\, x \in S,\ P(x) \Rightarrow Q(x)) \Rightarrow C$$

for some contradiction C. However, since

$$\begin{aligned} \sim (\forall\, x \in S,\ P(x) \Rightarrow Q(x)) &\equiv \exists x \in S,\ \sim (P(x) \Rightarrow Q(x)) \\ &\equiv \exists x \in S,\ (P(x) \wedge (\sim Q(x))), \end{aligned}$$

it follows that a proof by contradiction of $\forall\, x \in S,\ P(x) \Rightarrow Q(x)$ would begin by assuming the existence of some element $x \in S$ such that $P(x)$ is true and $Q(x)$ is false. That is, a proof by contradiction of $\forall\, x \in S,\ P(x) \Rightarrow Q(x)$ begins by assuming the existence of a counterexample of this quantified statement. Often the reader is alerted that a proof by contradiction is being used by saying (or writing)

Suppose that R is false.

or

Assume, to the contrary, that R is false.

Therefore, if R is the quantified statement $\forall\, x \in S,\ P(x) \Rightarrow Q(x)$, then a proof by contradiction might begin with:

> *Assume, to the contrary, that there exists some element $x \in S$ for which $P(x)$ is true and $Q(x)$ is false.*

(or something along these lines). The remainder of the proof then consists of showing that this assumption leads to a contradiction.

Let's now look at some examples of proof by contradiction. We begin by establishing a fact about positive real numbers.

Result to Prove There is no smallest positive real number.

PROOF STRATEGY In a proof by contradiction, we begin by assuming that the statement is false and attempt to show that this leads us to a contradiction. Hence we begin by assuming that there *is* a smallest positive real number. It is useful to represent this number by a symbol, say r. Our goal is to produce a contradiction. How do we go about doing this? Of course, if we could think of a positive real number that is less than r, then this would give us a contradiction. ♦

Result 5.10 *There is no smallest positive real number.*

Proof Assume, to the contrary, that there is a smallest positive real number, say r. Since $0 < r/2 < r$, it follows that $r/2$ is a positive real number that is smaller than r. This, however, is a contradiction. ■

PROOF ANALYSIS The contradiction referred to in the proof of Result 5.10 is the statement: r is the smallest positive real number *and* $r/2$ is a positive real number that is less than r. This statement is certainly false. We have assumed that the reader understands what contradiction has been obtained. If we think that the reader may not see this, then, of course, we should specifically state (in the proof) what the contradiction is.

There is another point concerning Result 5.10 that should be made. This result states that "there is *no* smallest positive real number". This is a negative-sounding result. In the vast majority of cases, proofs of negative-sounding results are given by contradiction. Thus the proof technique used in Result 5.10 is not unexpected. ♦

Let's consider two additional examples.

Result 5.11 *No odd integer can be expressed as the sum of three even integers.*

Proof Assume, to the contrary, that there exists an odd integer n which can be expressed as the sum of three even integers x, y, and z. Then $x = 2a$, $y = 2b$, and $z = 2c$ with $a, b, c \in \mathbf{Z}$. Therefore,

$$n = x + y + z = 2a + 2b + 2c = 2(a + b + c).$$

Since $a + b + c$ is an integer, n is even. This is a contradiction. ■

PROOF ANALYSIS Consider the statement:

R: No odd integer can be expressed as the sum of three even integers.

Obviously, Result 5.11 states that R is a true statement. In order to give a proof by contradiction of Result 5.11, we attempted to prove an implication of the type $(\sim R) \Rightarrow C$ for some contradiction C. The negation $\sim R$ is

$\sim R$: There exists an odd integer that can be expressed as the sum of three even integers.

The proof we gave of Result 5.11 began by assuming the truth of $\sim R$. We introduced symbols for the four integers involved to make it easier to explain the proof. Eventually,

we were able to show that n is an even integer. On the other hand, we knew that n is odd. Hence n was both even and odd. This was our contradiction C. ♦

In the two examples of proof by contradiction that we have given, neither statement to be proved is expressed as an implication. For our next example, we consider an implication.

Result 5.12 *If a is an even integer and b is an odd integer, then $4 \nmid (a^2 + 2b^2)$.*

Proof Assume, to the contrary, that there exist an even integer a and an odd integer b such that $4 \mid (a^2 + 2b^2)$. Thus $a = 2x$, $b = 2y + 1$, and $a^2 + 2b^2 = 4z$ for some integers x, y, and z. Hence $(2x)^2 + 2(2y + 1)^2 = 4z$. Simplifying, we obtain $4x^2 + 8y^2 + 8y + 2 = 4z$ or, equivalently,

$$2 = 4z - 4x^2 - 8y^2 - 8y = 4(z - x^2 - 2y^2 - 2y).$$

Since $z - x^2 - 2y^2 - 2y$ is an integer, $4 \mid 2$, which is impossible. ■

PROOF ANALYSIS Let S be the set of even integers and T the set of odd integers. In Result 5.12, our goal was to prove that

$$\forall a \in S, \forall b \in T,\ P(a, b). \tag{5.11}$$

is true, where

$$P(a, b):\ 4 \nmid (a^2 + 2b^2).$$

Since we were attempting to prove (5.11) by contradiction, we wanted to establish the truth of

$$\sim (\forall a \in S, \forall b \in T,\ P(a, b)) \Rightarrow C$$

for some contradiction C or, equivalently, the truth of

$$\exists a \in S, \exists b \in T,\ (\sim P(a, b)) \Rightarrow C.$$

Hence we began by assuming that there exist an even integer a and an odd integer b such that $4 \mid (a^2 + 2b^2)$. We eventually deduced that $4 \mid 2$, which is a false statement and thereby produced a desired contradiction.

Using some facts we discussed earlier, we could have given a direct proof of Result 5.12. Once we wrote $a = 2x$ and $b = 2y + 1$, we have

$$\begin{aligned} a^2 + 2b^2 &= (2x)^2 + 2(2y + 1)^2 = 4x^2 + 8y^2 + 8y + 2 \\ &= 4(x^2 + 2y^2 + 2y) + 2. \end{aligned}$$

Hence we have expressed $a^2 + 2b^2$ as $4q + 2$, where $q = x^2 + 2y^2 + 2y$. That is, dividing $a^2 + 2b^2$ by 4 results in a remainder of 2, and so $4 \nmid (a^2 + 2b^2)$. At this stage, however, a proof by contradiction of Result 5.12 is probably preferred, in order to both practice and understand this proof technique. ♦

Let's consider two other negative–sounding results.

Result 5.13 *The integer* 100 *cannot be written as the sum of three integers, an odd number of which are odd.*

Proof Assume, to the contrary, that 100 *can* be written as the sum of three integers a, b, and c, an odd number of which are odd. We consider two cases.

Case 1. *Exactly one of* a, b, *and* c *is odd, say* a. Then $a = 2x + 1$, $b = 2y$, and $c = 2z$, where $x, y, z \in \mathbf{Z}$. So

$$100 = a + b + c = (2x + 1) + 2y + 2z = 2(x + y + z) + 1.$$

Since $x + y + z \in \mathbf{Z}$, the integer 100 is odd, producing a contradiction.

Case 2. *All of* a, b, *and* c *are odd.* Then $a = 2x + 1$, $b = 2y + 1$, and $c = 2z + 1$, where $x, y, z \in \mathbf{Z}$. So

$$100 = a + b + c = (2x + 1) + (2y + 1) + (2z + 1) = 2(x + y + z + 1) + 1.$$

Since $x + y + z + 1 \in \mathbf{Z}$, the integer 100 is odd, again a contradiction. ■

PROOF ANALYSIS Observe that the proof of Result 5.13 begins by assuming that 100 can be written as the sum of three integers, an odd number of which are odd (as expected). However, by introducing symbols for these integers, namely a, b, and c, this made for an easier and clearer proof. ♦

Result 5.14 *For every integer* m *such that* $2 \mid m$ *and* $4 \nmid m$, *there exist no integers* x *and* y *for which* $x^2 + 3y^2 = m$.

Proof Assume, to the contrary, that there exist an integer m such that $2 \mid m$ and $4 \nmid m$ and integers x and y for which $x^2 + 3y^2 = m$. Since $2 \mid m$, it follows that m is even. By Theorem 3.16, x^2 and $3y^2$ are of the same parity. We consider two cases.

Case 1. x^2 *and* $3y^2$ *are even.* Since $3y^2$ is even and 3 is odd, it follows by Theorem 3.17 that y^2 is even. Because x^2 and y^2 are both even, we have by Theorem 3.12 that x and y are even. Thus $x = 2a$ and $y = 2b$, where $a, b \in \mathbf{Z}$. Therefore,

$$\begin{aligned} x^2 + 3y^2 &= (2a)^2 + 3(2b)^2 = 4a^2 + 12b^2 \\ &= 4(a^2 + 3b^2) = m. \end{aligned}$$

Since $a^2 + 3b^2 \in \mathbf{Z}$, it follows that $4 \mid m$, producing a contradiction.

Case 2. x^2 *and* $3y^2$ *are odd.* Since $3y^2$ is odd and 3 is odd, it follows by (the contrapositive of) Theorem 3.17 that y^2 is odd. By (the contrapositive of) Theorem 3.12, x and y are both odd. Then $x = 2a + 1$ and $y = 2b + 1$, where $a, b \in \mathbf{Z}$. Thus

$$\begin{aligned} x^2 + 3y^2 &= (2a + 1)^2 + 3(2b + 1)^2 = (4a^2 + 4a + 1) + 3(4b^2 + 4b + 1) \\ &= 4a^2 + 4a + 12b^2 + 12b + 4 = 4(a^2 + a + 3b^2 + 3b + 1) = m. \end{aligned}$$

Since $a^2 + a + 3b^2 + 3b + 1 \in \mathbf{Z}$, it follows that $4 \mid m$, producing a contradiction. ■

The next result concerns irrational numbers. Recall that a real number is rational if it can be expressed as m/n for some $m, n \in \mathbf{Z}$, where $n \neq 0$. Since "irrational" means "not rational", it is not surprising that proof by contradiction is the proof technique we will use.

Result 5.15 *The sum of a rational number and an irrational number is irrational.*

Proof Assume, to the contrary, that there exist a rational number x and an irrational number y whose sum is a rational number z. Thus, $x + y = z$, where $x = a/b$ and $z = c/d$ for some integers $a, b, c, d \in \mathbf{Z}$ and $b, d \neq 0$. This implies that

$$y = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$

Since $bc - ad$ and bd are integers and $bd \neq 0$, it follows that y is rational, which is a contradiction. ■

Result 5.15 concerns the irrationality of numbers. One of the best known irrational numbers is $\sqrt{2}$. Although we have never verified that this number is irrational, we establish this fact now.

Theorem to Prove The real number $\sqrt{2}$ is irrational.

PROOF STRATEGY In the proof of this result, we will use Theorem 3.12 which states that an integer x is even if and only if x^2 is even. Also, in the proof, it will be useful to express a rational number m/n, where $m, n \in \mathbf{Z}$ and $n \neq 0$, in lowest terms, which means that m and n contain no common divisor greater than 1. ♦

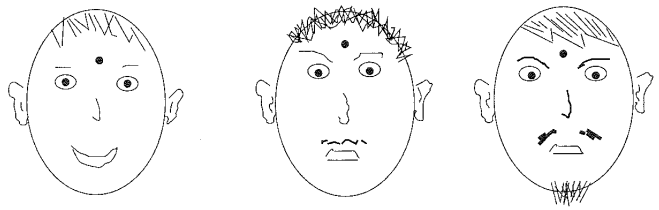
Theorem 5.16 *The real number* $\sqrt{2}$ *is irrational.*

Proof Assume, to the contrary, that $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$, where $a, b \in \mathbf{Z}$ and $b \neq 0$. We may further assume that a/b has been expressed in (or reduced to) lowest terms. Then $2 = a^2/b^2$; so $a^2 = 2b^2$. Since b^2 is an integer, a^2 is even. By Theorem 3.12, a is even. So $a = 2c$, where $c \in \mathbf{Z}$. Thus, $(2c)^2 = 2b^2$, and so $4c^2 = 2b^2$. Therefore, $b^2 = 2c^2$. Because c^2 is an integer, b^2 is even, which implies by Theorem 3.12 that b is even. Since a and b are even, each has 2 as a divisor, which is a contradiction since a/b has been reduced to lowest terms. ■

The Three Prisoners Problem We now take a brief diversion from our discussion of proof by contradiction to present a "story" problem. Three prisoners (see Figure 5.1) have been sentenced to long terms in prison, but due to overcrowded conditions, one prisoner must be released.

The warden devises a scheme to determine which prisoner is to be released. He tells the prisoners that he will blindfold them and then paint a red dot or a blue dot on each forehead. After he paints the dots, he will remove the blindfolds, and a prisoner should raise his hand if he sees a red dot on at least one of the other two prisoners. The first prisoner to identify the color of the dot on his own forehead will be released. Of course, the prisoners agree to this. (What do they have to lose?)

The warden blindfolds the prisoners, as promised, and then paints a red dot on the foreheads of *all three prisoners*. He removes the blindfolds and, since each prisoner sees



#1 #2 #3

Figure 5.1 The three prisoners

a red dot (in fact two red dots), each prisoner raises his hand. Some time passes when one of the prisoners exclaims, "I know what color my dot is! It's red!" This prisoner is then released. Although the story of the three prisoners is over, there is a lingering question: How did this prisoner correctly identify the color of the dot painted on his forehead?

The solution is given next, but try to determine the answer for yourself before reading on.

Solution of the Three Prisoners Problem

Let's assume (without loss of generality) that it's prisoner #1 (see Figure 5.1) who determined that he had a red dot painted on his forehead. How did he come to this conclusion? Perhaps you think he just guessed since he had nothing to lose anyway. But this is not the answer we were looking for.

Prisoner #1 knows that the color of his dot is either red or blue. He thinks, "Assume, to the contrary, that my dot is blue. Then, of course, #2 knows this and he knows that #3 has a red dot. (That's why #2 raised his hand.) But #2 also knows that #3 raised his hand. So if my dot is blue, #2 knows his dot is red. Similarly, if my dot is blue, then #3 knows his dot is red. In other words, if my dot is blue, then both #2 and #3 should be able to identify the colors of their dots quite quickly. But time has passed, and they haven't determined the colors of their dots. So my dot can't be blue." Therefore, #1 exclaims, "I know what color my dot is! It's red!"

What you probably noticed is that the reasoning #1 used to conclude that his dot is red is proof by contradiction. It seems as if there is more to know about prisoner #1. But that's another story. ♦

5.3 A Review of Three Proof Techniques

We have seen that we're often in the situation where we want to prove the truth of a statement $\forall x \in S,\ P(x) \Rightarrow Q(x)$. You have now been introduced to three proof techniques: direct proof, proof by contrapositive, and proof by contradiction. For each of these three techniques, you should be aware of how to start a proof and what your goal should be. You should also know what *not* to do. Figure 5.2 gives several ways that we *might* start a proof. Only some of these are legitimate, however.

Let's now compare the three proof techniques with two examples.

	First Step of "Proof"	Remarks/Goal
1.	**Assume that P is true.**	**A direct proof is being used. Show that Q is true.**
2.	Assume that P is false.	A mistake has been made.
3.	Assume that Q is true.	A mistake has been made.
4.	**Assume that Q is false.**	**A proof by contrapositive is being used. Show that P is false.**
5.	Assume that P is true and Q is true.	A mistake has been made.
6.	**Assume that P is true and Q is false.**	**A proof by contradiction is being used. Obtain a contradiction.**
7.	Assume that P is false and Q is true.	A mistake has been made.
8.	Assume that P is false and Q is false.	A mistake has been made.
9.	Assume that $P \Rightarrow Q$ is true.	A mistake has been made.
10.	**Assume that $P \Rightarrow Q$ is false.**	**A proof by contradiction is being used. Obtain a contradiction.**

Figure 5.2 How to prove (and not to prove) that $\forall x \in S,\ P(x) \Rightarrow Q(x)$ is true

Result 5.17 *If n is an even integer, then $3n + 7$ is odd.*

Direct Proof Let n be an even integer. Then $n = 2x$ for some integer x. Therefore,

$$3n + 7 = 3(2x) + 7 = 6x + 7 = 2(3x + 3) + 1.$$

Since $3x + 3$ is an integer, $3n + 7$ is odd. ■

Proof by Contrapositive Assume that $3n + 7$ is even. Then $3n + 7 = 2y$ for some integer y. Hence

$$n = (3n + 7) + (-2n - 7) = 2y - 2n - 7 = 2(y - n - 4) + 1.$$

Since $y - n - 4$ is an integer, n is odd. ■

Proof by Contradiction Assume, to the contrary, that there exists an even integer n such that $3n + 7$ is even. Since n is even, $n = 2x$ for some integer x. Hence

$$3n + 7 = 3(2x) + 7 = 6x + 7 = 2(3x + 3) + 1.$$

Since $3x + 3$ is an integer, $3n + 7$ is odd, which is a contradiction. ■

Although a direct proof of Result 5.17 is certainly the preferred proof technique in this case, it is useful to compare all three techniques. The following example is more intricate.

Result 5.18 *Let x be a nonzero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.*

Direct Proof Assume that $x + \frac{1}{x} < 2$. Since $x \neq 0$, we know that $x^2 > 0$. Multiplying both sides of the inequality $x + \frac{1}{x} < 2$ by x^2, we obtain $x^2\left(x + \frac{1}{x}\right) < 2x^2$. Simplifying this inequality, we have $x^3 + x - 2x^2 < 0$; so

$$x(x^2 - 2x + 1) = x(x - 1)^2 < 0.$$

Since $(x-1)^2 \geq 0$ and $x(x-1)^2 \neq 0$, we must have $(x-1)^2 > 0$. Since $x(x-1)^2 < 0$ and $(x-1)^2 > 0$, it follows that $x < 0$, as desired. ■

Proof Strategy for Proof by Contrapositive For a proof by contrapositive, we will begin by assuming that $x \geq 0$ and attempt to show that $x + \frac{1}{x} \geq 2$. This inequality can be simplified by multiplying through by x, obtaining $x^2 + 1 \geq 2x$. Subtracting $2x$ from both sides, we have $x^2 - 2x + 1 = (x-1)^2 \geq 0$, which, of course, we know to be true. A proof is suggested then by reversing the order of these steps:

$$x + \frac{1}{x} \geq 2$$

$$x^2 + 1 \geq 2x$$

$$x^2 - 2x + 1 = (x-1)^2 \geq 0.$$

This method is common when dealing with inequalities. ♦

Proof by Contrapositive Assume that $x \geq 0$. Since $x \neq 0$, it follows that $x > 0$. Since $(x-1)^2 \geq 0$, we have $(x-1)^2 = x^2 - 2x + 1 \geq 0$. Adding $2x$ to both sides of this inequality, we obtain $x^2 + 1 \geq 2x$. Dividing both sides of the inequality $x^2 + 1 \geq 2x$ by the positive number x, we obtain $x + \frac{1}{x} \geq 2$, as desired. ■

Proof by Contradiction Assume, to the contrary, that there exists a nonzero real number x such that $x + \frac{1}{x} < 2$ and $x \geq 0$. Since $x \neq 0$, it follows that $x > 0$. Multiplying both sides of the inequality $x + \frac{1}{x} < 2$ by x, we obtain $x^2 + 1 < 2x$. Subtracting $2x$ from both sides, we have $x^2 - 2x + 1 < 0$. It then follows that $(x-1)^2 < 0$, which is a contradiction. ■

Many mathematicians feel that if a result can be verified by a direct proof, then this is the proof technique that should be used, as it is normally easier to understand. This is only a general guideline, however; it is *not* a hard and fast rule.

5.4 Existence Proofs

In an **existence theorem** the existence of an object (or objects) possessing some specified property or properties is asserted. Typically then, an existence theorem concerning an open sentence $R(x)$ over a domain S can be expressed as a quantified statement

$$\exists x \in S,\ R(x):\ \text{There exists } x \in S \text{ such that } R(x). \qquad \textbf{(5.12)}$$

We have seen that such a statement (5.12) is true provided that $R(x)$ is true for some $x \in S$. A proof of an existence theorem is called an **existence proof**. An existence proof may then consist of displaying or constructing an example of such an object or perhaps, with the aid of known results, verifying that such objects must exist without ever producing a single example of the desired type. For example, there are theorems in mathematics that tell us that every polynomial of odd degree with real coefficients has at least one real number as a solution, but we don't know how to find a real number solution for every such polynomial. Indeed, we quote the great mathematician David Hilbert, who used the following example in his lectures to illustrate the idea of an existence proof:

There is at least one student in this class . . . let us name him 'X' . . . for whom the following statement is true: No other student in the class has more hairs on his head than X. Which student is it? That we shall never know; but of his existence we can be absolutely certain.

Let's now see some examples of existence proofs.

Result to Prove There exists an integer whose cube equals its square.

PROOF STRATEGY Since this result is only asserting the existence of an integer whose cube equals its square, we have a proof once we can think of an example. The integer 1 has this property. ♦

Result 5.19 *There exists an integer whose cube equals its square.*

Proof Since $1^3 = 1^2 = 1$, the integer 1 has the desired property. ■

Suppose that we didn't notice that the integer 1 satisfied the required condition in the preceding theorem. Then an alternate proof may go something like this: Let $x \in \mathbf{Z}$ such that $x^3 = x^2$. Then $x^3 - x^2 = 0$ or $x^2(x-1) = 0$. Thus, there are only two possible integers with this property, namely 1 and 0, and, in fact, both integers have the desired property.

A common error in elementary algebra is to write $(a+b)^2 = a^2 + b^2$. Can this *ever* be true?

Result 5.20 *There exist real numbers a and b such that* $(a+b)^2 = a^2 + b^2$.

Proof Let $a, b \in \mathbf{Z}$ such that $(a+b)^2 = a^2 + b^2$. Then $a^2 + 2ab + b^2 = a^2 + b^2$, so $2ab = 0$. Since $a = 1, b = 0$ is a solution to this equation, we have

$$(a+b)^2 = (1+0)^2 = 1^2 = 1^2 + 0^2 = a^2 + b^2.$$ ■

The proof presented of Result 5.20 is longer than necessary. We could have written the following proof:

Proof Let $a = 1$ and $b = 0$. Then

$$(a+b)^2 = (1+0)^2 = 1^2 = 1^2 + 0^2 = a^2 + b^2.$$ ■

In the first proof, we actually presented an argument for how we thought of $a = 1$ and $b = 0$. In a proof, we are not required to explain where we got the idea for the proof, although it may very well be interesting to know this. If we feel that such information might be interesting or valuable, it may be worthwhile to include this in a discussion preceding or following the proof. The first proof we gave of Result 5.20 actually informs us of all real numbers a and b for which $(a+b)^2 = a^2 + b^2$, namely, $(a+b)^2 = a^2 + b^2$ if and only if at least one of a and b is 0. This is more than what was requested of us, but, nevertheless, it seems interesting.

We saw in Section 5.2 that $\sqrt{2}$ is irrational. Since $\sqrt{2} = 2^{1/2}$, it follows that there exist rational numbers a and b such that a^b is irrational; namely, $a = 2$ and $b = 1/2$ have

this property. Let's reverse this question. That is, do there exist *irrational numbers* a and b such that a^b is rational? Although there are many irrational numbers (in fact, an infinite number), we have verified only that $\sqrt{2}$ is irrational. (On the other hand, we know from the exercises for this section that $r + \sqrt{2}$ is irrational for every rational number r and that both $r\sqrt{2}$ and $r/\sqrt{2}$ are irrational for every nonzero rational number r.)

Result to Prove There exist irrational numbers a and b such that a^b is rational.

PROOF STRATEGY As we mentioned, there are only certain numbers that we know to be irrational, the simplest being $\sqrt{2}$. This might suggest considering the (real) number $\sqrt{2}^{\sqrt{2}}$. If this number is rational, then this answers our question. But perhaps $\sqrt{2}^{\sqrt{2}}$ is irrational. Then what do we do? This discussion suggests two cases. ♦

Result 5.21 *There exist irrational numbers a and b such that a^b is rational.*

Proof Consider the number $\sqrt{2}^{\sqrt{2}}$. Of course, this number is either rational or irrational. We consider these possibilities separately.

Case 1. $\sqrt{2}^{\sqrt{2}}$ *is rational.* Then we can take $a = b = \sqrt{2}$, and we have the desired result.

Case 2. $\sqrt{2}^{\sqrt{2}}$ *is irrational.* In this case, consider the number obtained by raising the (irrational) number $\sqrt{2}^{\sqrt{2}}$ to the (irrational) power $\sqrt{2}$; that is, consider a^b, where $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$. Observe that

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

which is rational. ■

The proof of Result 5.21 may seem unsatisfactory to you since we still don't know two specific irrational numbers a and b such that a^b is rational. We know only that two such numbers exist. We actually do know a bit more; namely, either (1) $\sqrt{2}^{\sqrt{2}}$ is rational, or (2) $\sqrt{2}^{\sqrt{2}}$ is irrational and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ is rational. (Actually it has been proved that $\sqrt{2}^{\sqrt{2}}$ is an irrational number. Hence there are also irrational numbers of the form of a^b, where a and b are both irrational.)

In the next result, we want to show that the equation $x^5 + 2x - 5 = 0$ has a real number solution between $x = 1$ and $x = 2$. It is not easy to find a number that satisfies this equation. Instead, we use a well-known theorem from calculus to show that such a solution exists. You may not remember all the terms used in the following theorem, but this is not crucial.

The Intermediate Value Theorem of Calculus If f is a function that is continuous on the closed interval $[a, b]$ and k is a number between $f(a)$ and $f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = k$.

We now give an example to show how this theorem can be used.

Result 5.22 *The equation $x^5 + 2x - 5 = 0$ has a real number solution between $x = 1$ and $x = 2$.*

Proof Let $f(x) = x^5 + 2x - 5$. Since f is a polynomial function, it is continuous on the set of all real numbers and so f is continuous on the interval $[1, 2]$. Now $f(1) = -2$ and $f(2) = 31$. Since 0 is between $f(1)$ and $f(2)$, it follows by the Intermediate Value Theorem of Calculus that there is a number c between 1 and 2 such that $f(c) = c^5 + 2c - 5 = 0$. Hence c is a solution. ■

As we just saw, the equation $x^5 + 2x - 5 = 0$ has a real number solution between $x = 1$ and $x = 2$. Actually, the equation $x^5 + 2x - 5 = 0$ has exactly one real number solution between $x = 1$ and $x = 2$. This brings up the topic of uniqueness. An element belonging to some prescribed set A and possessing a certain property P is **unique** if it is the only element of A having property P. Typically, to prove that only one element of A has property P, we proceed in one of two ways:

(1) We assume that a and b are elements of A possessing property P and show that $a = b$.
(2) We assume that a and b are distinct elements of A possessing property P and show that $a = b$.

Although (1) results in a direct proof and (2) results in a proof by contradiction, it is often the case that either proof technique can be used.

As an illustration, we return to Result 5.22 and show, in fact, that the equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.

Result 5.23 *The equation $x^5 + 2x - 5 = 0$ has a unique real number solution between $x = 1$ and $x = 2$.*

Proof Assume, to the contrary, that the equation $x^5 + 2x - 5 = 0$ has two distinct real number solutions a and b between $x = 1$ and $x = 2$. We may assume that $a < b$. Since $1 < a < b < 2$, it follows that $a^5 + 2a - 5 < b^5 + 2b - 5$. On the other hand, $a^5 + 2a - 5 = 0$ and $b^5 + 2b - 5 = 0$. Thus

$$0 = a^5 + 2a - 5 < b^5 + 2b - 5 = 0,$$

which produces a contradiction. ■

Actually, we could have omitted Result 5.22 altogether and replaced it by Result 5.23 only (renumbering this result by Result 5.22), including the proofs of both Results 5.22 and 5.23.

We now present another result concerning uniqueness.

Result to Prove For an irrational number r, let

$$S = \{sr + t : s, t \in \mathbf{Q}\}.$$

For every $x \in S$, there exist unique rational numbers a and b such that $x = ar + b$.

PROOF STRATEGY To verify that a and b are unique, we assume that x can be expressed in two ways, say as $ar + b$ and $cr + d$, where $a, b, c, d \in \mathbf{Q}$, and then show that $a = c$ and $b = d$. Hence $ar + b = cr + d$. If $a \neq c$, then we can show that r is a rational number, producing a contradiction. Thus $a = c$. Subtracting ar from both sides of $ar + b = cr + d$, we obtain $b = d$ as well. ♦

We now give a complete proof.

Result 5.24 *For an irrational number r, let*

$$S = \{sr + t : s, t \in \mathbf{Q}\}.$$

For every $x \in S$, there exist unique rational numbers a and b such that $x = ar + b$.

Proof Let $x \in S$ and suppose that $x = ar + b$ and $x = cr + d$, where $a, b, c, d \in \mathbf{Q}$. Then $ar + b = cr + d$. If $a \neq c$, then $(a - c)r = d - b$ and so

$$r = \frac{d - b}{a - c}.$$

Since $\frac{d-b}{a-c}$ is a rational number, this is impossible. So $a = c$. Subtracting $ar = cr$ from both sides of $ar + b = cr + d$, we obtain $b = d$. ■

Example 5.25 (a) *Show that the equation $6x^3 + x^2 - 2x = 0$ has a root in the interval $[-1, 1]$.*
(b) *Does this equation have a unique root in the interval $[-1, 1]$?*

Solution (a) By inspection, we can see that $x = 0$ is a root of the equation.
(b) Observe that

$$6x^3 + x^2 - 2x = x(6x^2 + x - 2) = x(3x + 2)(2x - 1).$$

Thus $x = -2/3$ and $x = 1/2$ are also roots of the equation $6x^3 + x^2 - 2x = 0$ and so this equation does not have a unique root in the interval $[-1, 1]$. ♦

5.5 Disproving Existence Statements

Let $R(x)$ be a statement for each element x in a domain S. We have already seen that to disprove a quantified statement of the type $\forall\, x \in S, R(x)$, it suffices to produce a counterexample (that is, an element x in S for which $R(x)$ is false). However, disproving a quantified statement of the type $\exists x \in S, R(x)$ requires a totally different approach. Since

$$\sim(\exists x \in S, R(x)) \equiv \forall x \in S, \sim R(x),$$

it follows that the statement $\exists x \in S, R(x)$ is false if $R(x)$ is false for *every* $x \in S$. Let's look at some examples of disproving existence statements.

Example 5.26 *Disprove the statement: There exists an odd integer n such that $n^2 + 2n + 3$ is odd.*

Solution We show that if n is an odd integer, then $n^2 + 2n + 3$ is even. Let n be an odd integer. Then $n = 2k + 1$ for some integer k. Thus

$$\begin{aligned} n^2 + 2n + 3 &= (2k + 1)^2 + 2(2k + 1) + 3 = 4k^2 + 4k + 1 + 4k + 2 + 3 \\ &= 4k^2 + 8k + 6 = 2(2k^2 + 4k + 3). \end{aligned}$$

Since $2k^2 + 4k + 3$ is an integer, $n^2 + 2n + 3$ is even. ♦

Example 5.27 *Disprove the statement: There is a real number x such that $x^6 + 2x^4 + x^2 + 2 = 0$.*

Solution Let $x \in \mathbf{R}$. Since x^6, x^4, and x^2 are all even powers of the real number x, it follows that $x^6 \geq 0$, $x^4 \geq 0$, and $x^2 \geq 0$. Therefore, $x^6 + 2x^4 + x^2 + 2 \geq 0 + 0 + 0 + 2 = 2$ and so $x^6 + 2x^4 + x^2 + 2 \neq 0$. Hence the equation $x^6 + 2x^4 + x^2 + 2 = 0$ has no real number solution. ♦

Example 5.28 *Disprove the statement: There exists an integer n such that $n^3 - n + 1$ is even.*

Solution Let $n \in \mathbf{Z}$. We consider two cases.

Case 1. *n is even.* Then $n = 2a$, where $a \in \mathbf{Z}$. So

$$n^3 - n + 1 = (2a)^3 - (2a) + 1 = 8a^3 - 2a + 1 = 2(4a^3 - a) + 1.$$

Since $4a^3 - a$ is an integer, $n^3 - n + 1$ is odd and so it is not even.
Case 2. *n is odd.* Then $n = 2b + 1$, where $b \in \mathbf{Z}$. Hence

$$\begin{aligned} n^3 - n + 1 &= (2b + 1)^3 - (2b + 1) + 1 \\ &= 8b^3 + 12b^2 + 6b + 1 - 2b - 1 + 1 \\ &= 8b^3 + 12b^2 + 4b + 1 = 2(4b^3 + 6b^2 + 2b) + 1. \end{aligned}$$

Since $4b^3 + 6b^2 + 2b$ is an integer, $n^3 - n + 1$ is odd and so it is not even. ♦

If we had replaced Example 5.26 by

For every odd integer n, $n^2 + 2n + 3$ is even.

replaced Example 5.27 by

For every real number x, $x^6 + 2x^4 + x^2 + 2 \neq 0$.

and replaced Example 5.28 by

For every integer n, $n^3 - n + 1$ is odd.

then we would have a true statement in each case, and the solutions of Examples 5.26 – 5.28 would become proofs.

EXERCISES FOR CHAPTER 5

Section 5.1: Counterexamples

5.1. Disprove the statement: If a and b are any two real numbers, then $\log(ab) = \log(a) + \log(b)$.

5.2. Disprove the statement: If $n \in \{0, 1, 2, 3, 4\}$, then $2^n + 3^n + n(n-1)(n-2)$ is prime.

5.3. Disprove the statement: If $n \in \{1, 2, 3, 4, 5\}$, then $3 \mid (2n^2 + 1)$.

5.4. Disprove the statement: Let $n \in \mathbf{N}$. If $\frac{n(n+1)}{2}$ is odd, then $\frac{(n+1)(n+2)}{2}$ is odd.

5.5. Disprove the statement: For every two positive integers a and b, $(a+b)^3 = a^3 + 2a^2b + 2ab + 2ab^2 + b^3$.

5.6. Let $a, b \in \mathbf{Z}$. Disprove the statement: If ab and $(a+b)^2$ are of opposite parity, then a^2b^2 and $a + ab + b$ are of opposite parity.

Section 5.2: Proof by Contradiction

5.7. Prove that there is no largest negative rational number.

5.8. Prove that there is no smallest positive irrational number.

5.9. Prove that 200 cannot be written as the sum of an odd integer and two even integers.

5.10. Use proof by contradiction to prove that if a and b are odd integers, then $4 \nmid (a^2 + b^2)$.

5.11. Prove that if $a \geq 2$ and b are integers, then $a \nmid b$ or $a \nmid (b+1)$.

5.12. Prove that 1000 cannot be written as the sum of three integers, an even number of which are even.

5.13. Prove that the product of an irrational number and a nonzero rational number is irrational.

5.14. Prove that when an irrational number is divided by a (nonzero) rational number, the resulting number is irrational.

5.15. Let a be an irrational number and r a nonzero rational number. Prove that if s is a real number, then either $ar + s$ or $ar - s$ is irrational.

5.16. Prove that $\sqrt{3}$ is irrational. [Hint: First prove for an integer a that $3 \mid a^2$ if and only if $3 \mid a$. Recall that every integer can be written as $3q$, $3q+1$, or $3q+2$ for some integer q.]

5.17. Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

5.18. (a) Prove that $\sqrt{6}$ is an irrational number.
(b) Prove that there are infinitely many positive integers n such that $\sqrt{n}$ is irrational.

5.19. Let $S = \{p + q\sqrt{2} : p, q \in \mathbf{Q}\}$ and $T = \{r + s\sqrt{3} : r, s \in \mathbf{Q}\}$. Prove that $S \cap T = \mathbf{Q}$.

5.20. Prove that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

5.21. Prove that there exists no positive integer x such that $2x < x^2 < 3x$.

5.22. Let m be a positive integer of the form $m = 2s$, where s is an odd integer. Prove that there do not exist positive integers x and y such that $x^2 - y^2 = m$.

5.23. Prove that the sum of the squares of two odd integers cannot be a perfect square.

5.24. Use a proof by contradiction to prove the following. Let $m \in \mathbf{Z}$. If $3 \nmid (m^2 - 1)$, then $3 \mid m$.

Section 5.3: A Review of Three Proof Techniques

5.25. Prove that if n is an odd integer, then $7n - 5$ is even by (a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.

5.26. Let x be a positive real number. Prove that if $x - \frac{2}{x} > 1$, then $x > 2$ by
(a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.

5.27. Let $a, b \in \mathbf{R}$. Prove that if $ab \neq 0$, then $a \neq 0$ by using as many of the three proof techniques as possible.

5.28. Let $x, y \in \mathbf{R}^+$. Prove that if $x \leq y$, then $x^2 \leq y^2$ by
(a) a direct proof, (b) a proof by contrapositive, and (c) a proof by contradiction.

Section 5.4: Existence Proofs

5.29. Show that there exist a rational number a and an irrational number b such that a^b is rational.

5.30. Show that there exist a rational number a and an irrational number b such that a^b is irrational.

5.31. Show that there exist two distinct irrational numbers a and b such that a^b is rational.

5.32. Show that there exist no nonzero real numbers a and b such that $\sqrt{a^2 + b^2} = \sqrt[3]{a^3 + b^3}$.

5.33. Prove that there exists a unique real number solution to the equation $x^3 + x^2 - 1 = 0$ between $x = 2/3$ and $x = 1$.

5.34. Let $R(x)$ be an open sentence over a domain S. Suppose that $\forall x \in S$, $R(x)$ is a false statement and that the set T of counterexamples is a proper subset of S. Show that there exists a nonempty subset W of S such that $\forall x \in W$, $R(x)$ is true.

Section 5.5: Disproving Existence Statements

5.35. Disprove the statement: There exist odd integers a and b such that $4 \mid (3a^2 + 7b^2)$.

5.36. Disprove the statement: There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

5.37. Disprove the statement: There is an integer n such that $n^4 + n^3 + n^2 + n$ is odd.

ADDITIONAL EXERCISES FOR CHAPTER 5

5.38. (a) Prove that if $a \geq 2$ and $n \geq 1$ are integers such that $a^2 + 1 = 2^n$, then a is odd.
(b) Prove that there are no integers $a \geq 2$ and $n \geq 1$ such that $a^2 + 1 = 2^n$.

5.39. The king's daughter had three suitors and couldn't decide which one to marry. So the king said, "I have three gold crowns and two silver ones. I will put either a gold or silver crown on each of your heads. The suitor who can tell me which crown he has will marry my daughter." The first suitor looked around and said he could not tell. The second did the same. The third suitor said: "I have a gold crown." He is correct, but the daughter was puzzled: This suitor was blind. How did he know? (*Source:*)

5.40. Let $x, y \in \mathbf{R}^+$. Use a proof by contradiction to prove that if $x < y$, then $\sqrt{x} < \sqrt{y}$.

5.41. Prove that there do not exist positive integers a and n such that $a^2 + 3 = 3^n$.

5.42. (a) Let n be a positive integer. Show that every integer m with $1 \leq m \leq 2n$ can be expressed as $2^\ell k$, where ℓ is a nonnegative integer and k is an odd integer with $1 \leq k < 2n$.

(b) Prove for every positive integer n and every subset S of $\{1, 2, \ldots, 2n\}$ with $|S| = n + 1$ that there exist integers $a, b \in S$ such that $a \mid b$.

5.43. A proof of a result is given below. What result is proved?

Proof Let $a, b, c \in \mathbf{Z}$ such that $a^2 + b^2 = c^2$. Assume, to the contrary, that a, b, and c are all odd. Then $a = 2r + 1$, $b = 2s + 1$, and $c = 2t + 1$, where $r, s, t \in \mathbf{Z}$. Thus,

$$\begin{aligned} a^2 + b^2 &= (4r^2 + 4r + 1) + (4s^2 + 4s + 1) \\ &= 2(2r^2 + 2r + 2s^2 + 2s + 1). \end{aligned}$$

Since $2r^2 + 2r + 2s^2 + 2s + 1$ is an integer, it follows that $a^2 + b^2$ is even. On the other hand,

$$c^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 2(2t^2 + 2t) + 1.$$

Since $2t^2 + 2t$ is an integer, it follows that c^2 is odd. Therefore, $a^2 + b^2$ is even and c^2 is odd, contradicting that $a^2 + b^2 = c^2$. ∎

5.44. A proof of a result is given below. What result is proved?

Proof Let $a \equiv 2 \pmod 4$ and $b \equiv 1 \pmod 4$ and assume, to the contrary, that $4 \mid (a^2 + 2b)$. Since $a \equiv 2 \pmod 4$ and $b \equiv 1 \pmod 4$, it follows that $a = 4r + 2$ and $b = 2s + 1$, where $r, s \in \mathbf{Z}$. Therefore,

$$\begin{aligned} a^2 + 2b &= (4r + 2)^2 + 2(2s + 1) = (16r^2 + 16r + 4) + (4s + 2) \\ &= 16r^2 + 16r + 4s + 6. \end{aligned}$$

Since $4 \mid (a^2 + 2b)$, we have $a^2 + 2b = 4t$, where $t \in \mathbf{Z}$. So $16r^2 + 16r + 4s + 6 = 4t$ and

$$6 = 4t - 16r^2 - 16r - 4s = 4(t - 4r^2 - 4r - s).$$

Since $t - 4r^2 - 4r - s$ is an integer, $4 \mid 6$, which is a contradiction. ∎

5.45. Evaluate the proposed proof of the following result.

Result The number 25 cannot be written as the sum of three integers, an even number of which are odd.

Proof Assume, to the contrary, that 25 can be written as the sum of three integers, an even number of which are odd. Then $25 = x + y + z$, where $x, y, z \in \mathbf{Z}$. We consider two cases.

Case 1. *x and y are odd.* Then $x = 2a + 1$, $y = 2b + 1$, and $z = 2c$, where $a, b, c \in \mathbf{Z}$. Therefore,

$$\begin{aligned} 25 = x + y + z &= (2a + 1) + (2b + 1) + 2c \\ &= 2a + 2b + 2c + 2 = 2(a + b + c + 1). \end{aligned}$$

Since $a + b + c + 1$ is an integer, 25 is even, a contradiction.

Case 2. *x, y, and z are even.* Then $x = 2a$, $y = 2b$, and $z = 2c$, where $a, b, c \in \mathbf{Z}$. Hence

$$25 = x + y + z = 2a + 2b + 2c = 2(a + b + c).$$

Since $a + b + c$ is an integer, 25 is even, again a contradiction. ∎

5.46. Evaluate the proposed proof of the following result.

Result If x is an irrational number and y is a rational number, then $z = x - y$ is irrational.

Proof Assume, to the contrary, that $z = x - y$ is rational. Then $z = a/b$, where $a, b \in \mathbf{Z}$ and $b \neq 0$. Since $\sqrt{2}$ is irrational, we let $x = \sqrt{2}$. Since y is rational, $y = c/d$, where $c, d \in \mathbf{Z}$ and $d \neq 0$. Therefore,

$$\sqrt{2} = x = y + z = \frac{c}{d} + \frac{a}{b} = \frac{ad + bc}{bd}.$$

Since $ad + bc$ and bd are integers, where $bd \neq 0$, it follows that $\sqrt{2}$ is rational, producing a contradiction. ∎

5.47. Prove that the sum of the irrational numbers $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ is also irrational.

5.48. Let $a_1, a_2, \ldots, a_r$ be odd integers where $a_i > 1$ for $i = 1, 2, \ldots, r$. Prove that if $n = a_1 a_2 \cdots a_r + 2$, then $a_i \nmid n$ for each integer i $(1 \leq i \leq r)$.

6

Mathematical Induction

We have seen three proof techniques which could be used to prove that a quantified statement $\forall x \in S$, $P(x)$ is true: direct proof, proof by contrapositive, and proof by contradiction. For certain sets S, however, there is another possible method of proof: mathematical induction.

6.1 The Principle of Mathematical Induction

Let A be a nonempty set of real numbers. A number $m \in A$ is called a **least element** (or a **minimum** or **smallest element**) of A if $x \geq m$ for every $x \in A$. Some nonempty sets of real numbers have a least element; others do not. The set $\mathbf{N}$ has a smallest element, namely 1, while $\mathbf{Z}$ has no least element. The closed interval $[2, 5]$ has the minimum element 2, but the open interval $(2, 5)$ has no minimum element. The set

$$A = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\}$$

also has no least element.

If a nonempty set A of real numbers has a least element, then this element is necessarily unique. We will verify this fact. Recall that when attempting to prove that an element possessing a certain property is unique, it is customary to assume that there are two elements with this property. We then show that these elements are equal, implying that there is exactly one such element.

Theorem 6.1 *If a set A of real numbers has a least element, then A has a unique least element.*

Proof Let m_1 and m_2 be least elements of A. Since m_1 is a least element, $m_2 \geq m_1$. Also, since m_2 is a least element, $m_1 \geq m_2$. Therefore, $m_1 = m_2$. ∎

The proof we gave of Theorem 6.1 is a direct proof. Suppose that we had replaced the first sentence of this proof by

Assume, to the contrary, that A contains distinct least elements m_1 and m_2.

If the remainder of the proof of Theorem 6.1 were the same except for adding a concluding sentence that we have a contradiction, then this too would be a proof of

Theorem 6.1. That is, with a small change, the proof technique used to verify Theorem 6.1 can be transformed from a direct proof to a proof by contradiction.

There is a property possessed by some sets of real numbers that will be of great interest to us here. A nonempty set S of real numbers is said to be **well-ordered** if every nonempty subset of S has a least element. Let $S = \{-7, -1, 2\}$. The nonempty subsets of S are

$$\{-7, -1, 2\}, \{-7, -1\}, \{-7, 2\}, \{-1, 2\}, \{-7\}, \{-1\}, \text{and } \{2\}.$$

Since each of these subsets has a least element, S is well-ordered. Indeed, it should be clear that *every* nonempty finite set of real numbers is well-ordered. (See Exercise 6.25.) The open interval $(0, 1)$ is *not* well-ordered, since, for example, $(0, 1)$ itself has no least element. The closed interval $[0, 1]$ has the least element 0; however, $[0, 1]$ is *not* well-ordered since the open interval $(0, 1)$ is a (nonempty) subset of $[0, 1]$ without a least element. Because none of the sets $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$ has a least element, none of these sets is well-ordered. Hence, having a least element is a necessary condition for a nonempty set to be well-ordered, but it is not sufficient.

Although it may appear evident that the set $\mathbf{N}$ of positive integers is well-ordered, this statement cannot be proved from the properties of positive integers that we have used and derived thus far. Consequently, this statement is accepted as an axiom, which we state below.

The Well-Ordering Principle

The set $\mathbf{N}$ of positive integers is well-ordered.

A consequence of the Well-Ordering Principle is another principle, which serves as the foundation of another and important proof technique.

Theorem 6.2 **(The Principle of Mathematical Induction)** *For each positive integer n, let $P(n)$ be a statement. If*

(1) *$P(1)$ is true and*
(2) *the implication*

$$\text{If } P(k), \text{ then } P(k+1).$$

is true for every positive integer k,

then $P(n)$ is true for every positive integer n.

Proof Assume, to the contrary, that the theorem is false. Then conditions (1) and (2) are satisfied but there exist some positive integers n for which $P(n)$ is a false statement. Let

$$S = \{n \in \mathbf{N} : P(n) \text{ is false}\}.$$

Since S is a nonempty subset of $\mathbf{N}$, it follows by the Well-Ordering Principle that S contains a least element s. Since $P(1)$ is true, $1 \notin S$. Thus $s \geq 2$ and $s - 1 \in \mathbf{N}$. Therefore, $s - 1 \notin S$ and so $P(s-1)$ is a true statement. By condition (2), $P(s)$ is also true and so $s \notin S$. This, however, contradicts our assumption that $s \in S$. ∎

The Principle of Mathematical Induction is stated more symbolically next.

The Principle of Mathematical Induction

For each positive integer n, let $P(n)$ be a statement. If

(1) *$P(1)$ is true and*
(2) *$\forall k \in \mathbf{N},\ P(k) \Rightarrow P(k+1)$ is true,*

then $\forall n \in \mathbf{N},\ P(n)$ is true.

As a consequence of the Principle of Mathematical Induction, the quantified statement $\forall n \in \mathbf{N},\ P(n)$ can be proved to be true if

(1) we can show that the statement $P(1)$ is true and
(2) we can establish the truth of the implication

$$\text{If } P(k), \text{ then } P(k+1).$$

for every positive integer k.

A proof using the Principle of Mathematical Induction is called an **induction proof** or a **proof by induction**. The verification of the truth of $P(1)$ in an induction proof is called the **base step**, **basis step**, or the **anchor** of the induction. In the implication

$$\text{If } P(k), \text{ then } P(k+1).$$

for an arbitrary positive integer k, the statement $P(k)$ is called the **inductive** (or **induction**) **hypothesis**. Often we use a direct proof to verify

$$\forall k \in \mathbf{N},\ P(k) \Rightarrow P(k+1), \tag{6.1}$$

although any proof technique is acceptable. That is, we typically assume that the inductive hypothesis $P(k)$ is true for an arbitrary positive integer k and attempt to show that $P(k+1)$ is true. Establishing the truth of (6.1) is called the **inductive step** in the induction proof.

We illustrate this proof technique by showing that the sum of the first n positive integers is given by $n(n+1)/2$ for every positive integer n, that is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Result 6.3 *Let*

$$P(n):\ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

where $n \in \mathbf{N}$. Then $P(n)$ is true for every positive integer n.

Proof We employ induction. Since $1 = (1 \cdot 2)/2$, the statement $P(1)$ is true. Assume that $P(k)$ is true for an arbitrary positive integer k, that is, assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

We show that $P(k+1)$ is true, that is, we show that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.$$

Thus

$$\begin{aligned}1+2+3+\cdots+(k+1)&=(1+2+3+\cdots+k)+(k+1)\\&=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2},\end{aligned}$$

as desired.

By the Principle of Mathematical Induction, $P(n)$ is true for every positive integer n. ■

Typically, a statement to be proved by induction is not presented in terms of $P(n)$ or some other open sentence. In order to illustrate this, we give an alternative statement and proof of Result 6.3, as it is to be understood what $P(n)$ would represent.

Result 6.4 *For every positive integer n,*

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Proof We employ induction. Since $1=(1\cdot 2)/2$, the statement is true for $n=1$. Assume that

$$1+2+3+\cdots+k=\frac{k(k+1)}{2},$$

where k is a positive integer. We show that

$$1+2+3+\cdots+(k+1)=\frac{(k+1)(k+2)}{2}.$$

Thus

$$\begin{aligned}1+2+3+\cdots+(k+1)&=(1+2+3+\cdots+k)+(k+1)\\&=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}\\&=\frac{(k+1)(k+2)}{2}.\end{aligned}$$

By the Principle of Mathematical Induction,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}$$

for every positive integer n. ■

PROOF ANALYSIS The proof of Result 6.3 (or of Result 6.4) began by stating that induction was being used. This alerts the reader of what to expect in the proof. Also, in the proof of the inductive

step, it is assumed that

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

for a positive integer k, that is, for an *arbitrary* positive integer k. We do *not* assume that

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

for *every* positive integer k as this would be assuming what we are attempting to prove in Result 6.3 (and in Result 6.4). ♦

Carl Friedrich Gauss (1777–1855) is considered to be one of the most brilliant mathematicians of all time. The story goes that when he was very young (in grade school in Germany) his teacher gave him and his classmates the supposedly unpleasant task of adding the integers from 1 to 100. He obtained the correct result of 5050 quickly. It is believed that he considered both the sum $1+2+\cdots+100$ and its reverse sum $100+99+\cdots+1$ and added these to obtain the sum $101+101+\cdots+101$, which has 100 terms and so equals 10,100. Since this is twice the required sum, $1+2+\cdots+100=10100/2=5050$. This, of course, can be quite easily generalized to find a formula for $1+2+3+\cdots+n$, where $n\in\mathbf{N}$. Let

$$S=1+2+3+\cdots+n. \tag{6.2}$$

If we reverse the order of the terms on the right side of (6.2), then we obtain

$$S=n+(n-1)+(n-2)+\cdots+1. \tag{6.3}$$

Adding (6.2) and (6.3), we have

$$2S=(n+1)+(n+1)+(n+1)+\cdots+(n+1). \tag{6.4}$$

Since there are n terms on the right side of (6.4), we conclude that $2S=n(n+1)$ or $S=n(n+1)/2$. Hence

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

You might think that the proof of Result 6.3 (and Result 6.4) that we gave by mathematical induction is longer (and more complicated) than the one we just gave, and this may very well be true. But, in general, mathematical induction is a technique that can be used to prove a wide range of statements. In this chapter, we will see a variety of statements where mathematical induction is a natural technique used in verifying their truth. We begin with an example that leads to a problem involving mathematical induction.

Suppose that an $n\times n$ square S is composed of n^2 1×1 squares. For all integers k with $1\le k\le n$, how many different $k\times k$ squares does S contain? (See Figure 6.1 for the case where $n=3$.) For $n=3$, the square S contains the 3×3 square S itself, four 2×2 squares, and nine 1×1 squares (see Figure 6.1). Therefore, the number of different squares that S contains is $1+4+9=1^2+2^2+3^2=14$.

In order to determine the number of different $k\times k$ squares in an $n\times n$ square S, we place S in the first quadrant of the coordinate plane so that the lower-left corner of S is at the origin $(0,0)$. (See Figure 6.2.) Then the upper-right corner of S is at the point

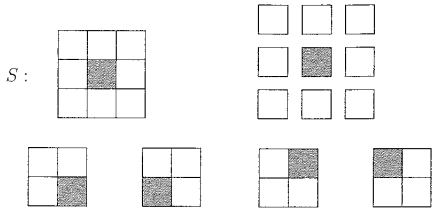


Figure 6.1 The squares in a 3×3 square

(n, n). Consequently, the lower-left corner of a $k \times k$ square S^*, where $1 \le k \le n$, is at some point (x, y), while the upper-right corner of S^* is at $(x + k, y + k)$. Necessarily, x and y are nonnegative integers with $x + k \le n$ and $y + k \le n$ (again see Figure 6.2). Since $0 \le x \le n - k$ and $0 \le y \le n - k$, the number of choices for each of x and y is $n - k + 1$ and so the number of possibilities for (x, y) is $(n - k + 1)^2$. Because k is any of the integers $1, 2, \ldots, n$, the total number of different squares in S is

$$\sum_{k=1}^{n}(n - k + 1)^2 = n^2 + (n - 1)^2 + \cdots + 2^2 + 1^2$$

$$= 1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2.$$

Is there a compact formula for the expression

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2?$$

For the problem we are describing, it would be very helpful to know the answer to this question. Since we brought up this question, you might have already guessed that the answer is yes. A formula is given next, along with a proof by induction.

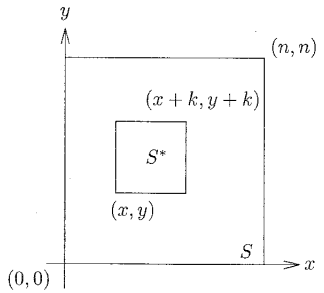


Figure 6.2 A $k \times k$ square in an $n \times n$ square

Result 6.5 *For every positive integer n,*

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}.$$

Proof We proceed by induction. Since $1^2 = (1 \cdot 2 \cdot 3)/6 = 1$, the statement is true when $n = 1$. Assume that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k + 1)(2k + 1)}{6}$$

for an arbitrary positive integer k. We show that

$$1^2 + 2^2 + \cdots + (k + 1)^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}.$$

Observe that

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k + 1)^2 &= [1^2 + 2^2 + \cdots + k^2] + (k + 1)^2 \\ &= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\ &= \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} \\ &= \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6} \\ &= \frac{(k + 1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k + 1)(k + 2)(2k + 3)}{6}, \end{aligned}$$

as desired.

By the Principle of Mathematical Induction,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$$

for every positive integer n. ∎

Strictly speaking, the last sentence in the proof of Result 6.5 is typical of the last sentence of every proof using mathematical induction, for the idea is to show that the hypothesis of the Principle of Mathematical Induction is satisfied and so the conclusion follows. Some therefore omit this final sentence since it is understood that once properties (1) and (2) of Theorem 6.2 are satisfied, we have a proof. For emphasis, we will continue to include this concluding sentence, however.

There is another question that might occur to you. We explained why $1 + 2 + \cdots + n$ equals $n(n + 1)/2$, but how did we know that $1^2 + 2^2 + \cdots + n^2$ equals $n(n + 1)(2n + 1)/6$? We can actually show that $1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6$ by using the formula $1 + 2 + \cdots + n = n(n + 1)/2$. We begin by solving

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1$$

for k^2. Since $3k^2 = (k+1)^3 - k^3 - 3k - 1$, it follows that

$$k^2 = \frac{1}{3}\left[(k+1)^3 - k^3\right] - k - \frac{1}{3}$$

and so

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}\left[\sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n} k^3\right] - \sum_{k=1}^{n} k - \frac{1}{3}\sum_{k=1}^{n} 1.$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{1}{3}\left[(n+1)^3 - 1^3\right] - \frac{1}{2}n(n+1) - \frac{1}{3}n \\ &= \frac{n^3+3n^2+3n}{3} - \frac{n^2+n}{2} - \frac{n}{3} \\ &= \frac{2n^3+6n^2+6n-3n^2-3n-2n}{6} = \frac{2n^3+3n^2+n}{6} \\ &= \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}.\end{aligned}$$

This is actually an alternative proof that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for every positive integer n, but of course this proof depends on knowing that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every positive integer n.

We have now used mathematical induction to establish the formulas

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{6.5}$$

and

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{6.6}$$

for every positive integer n. We saw that (6.6) gives the number of different squares in an $n \times n$ square composed of n^2 1×1 squares. Actually, (6.5) gives the number of intervals in an interval of length n composed of n intervals of length 1. You can probably guess what $1^3 + 2^3 + \cdots + n^3$ counts. Exercise 6.8 deals with this expression.

We now present a formula for

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)(n+2)}$$

for every positive integer n.

Result 6.6 *For every positive integer n,*

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}.$$

Proof We use induction. Since

$$\frac{1}{2\cdot 3} = \frac{1}{2\cdot 1+4} = \frac{1}{6},$$

the formula holds for $n = 1$. Assume that

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2k+4}$$

for a positive integer k. We show that

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+1)+4} = \frac{k+1}{2k+6}.$$

Observe that

$$\begin{aligned}&\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+2)(k+3)} \\ &\quad= \left[\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)}\right] + \frac{1}{(k+2)(k+3)} \\ &\quad= \frac{k}{2k+4} + \frac{1}{(k+2)(k+3)} = \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} \\ &\quad= \frac{k(k+3)+2}{2(k+2)(k+3)} = \frac{k^2+3k+2}{2(k+2)(k+3)} \\ &\quad= \frac{(k+1)(k+2)}{2(k+2)(k+3)} = \frac{k+1}{2(k+3)} = \frac{k+1}{2k+6},\end{aligned}$$

giving us the desired result. By the Principle of Mathematical Induction,

$$\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$$

for every positive integer n. ∎

PROOF ANALYSIS Each of the examples of mathematical induction proofs that we have seen involves a certain amount of algebra. We'll need to recall even more algebra soon. Many mistakes in these proofs are due to algebra errors. Therefore, care must be taken. For example, in the proof of Result 6.6, we encountered the sum

$$\frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)}.$$

To add these fractions, we needed to find a common denominator (actually a *least* common denominator), which is $2(k+2)(k+3)$. This was used to obtain the next fraction, that is,

$$\frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k(k+3)}{2(k+2)(k+3)} + \frac{2}{2(k+2)(k+3)} = \frac{k(k+3)+2}{2(k+2)(k+3)}.$$

When we expanded and factored the numerator and then cancelled the term $k + 2$, this was actually expected since the final result we were looking for was

$$\frac{k+1}{2k+6} = \frac{k+1}{2(k+3)},$$

which does not contain $k + 2$ as a factor in the denominator. ♦

6.2 A More General Principle of Mathematical Induction

The Principle of Mathematical Induction, described in the preceding section, gives us a technique for proving that a statement of the type

For every positive integer n, $P(n)$.

is true. There are situations, however, when the domain of $P(n)$ consists of those integers greater than or equal to some fixed integer m different from 1. We now describe an analogous technique to verify the truth of a statement of the following type where m denotes some fixed integer:

For every integer $n \geq m$, $P(n)$.

According to the Well-Ordering Principle, the set $\mathbf{N}$ of natural numbers is well-ordered; that is, every nonempty subset of $\mathbf{N}$ has a least element. As a consequence of the Well-Ordering Principle, other sets are also well-ordered.

Theorem 6.7 *For each integer m, the set*

$$S = \{i \in \mathbf{Z} : i \geq m\}$$

is well-ordered.

The proof of Theorem 6.7 is left as an exercise (see Exercise 6.12). The following is a consequence of Theorem 6.7. This is a slightly more general form of the Principle of Mathematical Induction. Consequently, it is commonly referred to by the same name.

Theorem 6.8 **(The Principle of Mathematical Induction)** *For a fixed integer m, let $S = \{i \in \mathbf{Z} : i \geq m\}$. For each integer $n \in S$, let $P(n)$ be a statement. If*

(1) *$P(m)$ is true and*
(2) *the implication*

If $P(k)$, then $P(k + 1)$.

is true for every integer $k \in S$,

then $P(n)$ is true for every integer $n \in S$.

The proof of Theorem 6.8 is similar to the proof of Theorem 6.2. We also state Theorem 6.8 symbolically.

The Principle of Mathematical Induction *For a fixed integer m, let $S = \{i \in \mathbf{Z} : i \geq m\}$. For each $n \in S$, let $P(n)$ be a statement. If*

(1) *$P(m)$ is true and*
(2) *$\forall k \in S$, $P(k) \Rightarrow P(k + 1)$ is true,*

then $\forall n \in S$, $P(n)$ is true.

This (more general) Principle of Mathematical Induction can be used to prove that certain quantified statements of the type $\forall n \in S$, $P(n)$ are true when $S = \{i \in \mathbf{Z} : i \geq m\}$ for a prescribed integer m. Of course, if $m = 1$, then $S = \mathbf{N}$. We now consider several examples.

Result 6.9 *For every nonnegative integer n,*

$$2^n > n.$$

Proof We proceed by induction. The inequality holds for $n = 0$ since $2^0 > 0$. Assume that $2^k > k$, where k is a nonnegative integer. We show that $2^{k+1} > k + 1$. When $k = 0$, we have $2^{k+1} = 2 > 1 = k + 1$. We therefore assume that $k \geq 1$. Then

$$2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1.$$

By the Principle of Mathematical Induction, $2^n > n$ for every nonnegative integer n. ■

PROOF ANALYSIS Let's review the proof of Result 6.9. First, since Result 6.9 concerns nonnegative integers, we are applying Theorem 6.8 with $m = 0$. We began by observing that $2^n > n$ when $n = 0$. Next we assumed that $2^k > k$, where k is a nonnegative integer. Our goal was to show that $2^{k+1} > k + 1$. It seems logical to observe that $2^{k+1} = 2 \cdot 2^k$. Since we knew that $2^k > k$, we have $2^{k+1} = 2 \cdot 2^k > 2k$. If we could show that $2k \geq k + 1$, then we have a proof. However, when $k = 0$, the inequality $2k \geq k + 1$ doesn't hold. That's why we handled $k = 0$ separately in the proof. This allowed us to assume that $k \geq 1$ and then conclude that $2k \geq k + 1$.

We could have proved Result 6.9 a bit differently. We could have observed first that $2^n > n$ when $n = 0$ and then proved that $2^n > n$ for $n \geq 1$ by induction. ♦

Our next example is to show that $2^n > n^2$ if n is a sufficiently large integer. We begin by trying a few values of n, as shown in Figure 6.3. It appears that $2^n > n^2$ whenever $n \geq 5$.

Result to Prove For every integer $n \geq 5$,

$$2^n > n^2.$$

n	2^n	n^2
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16
5	32	25
6	64	36

Figure 6.3 Comparing 2^n and n^2

PROOF STRATEGY Let's see what an induction proof of this result might look like. Of course, $2^n > n^2$ when $n = 5$. We assume that $2^k > k^2$, where $k \geq 5$ (and k is an integer) and we want to prove that $2^{k+1} > (k+1)^2$. We start with

$$2^{k+1} = 2 \cdot 2^k > 2k^2.$$

We would have a proof if we could show that $2k^2 \geq (k+1)^2$ or that $2k^2 \geq k^2 + 2k + 1$. There are several convincing ways to show that $2k^2 \geq k^2 + 2k + 1$ for integers $k \geq 5$. Here's one way:

Observe that $2k^2 = k^2 + k^2 = k^2 + k \cdot k \geq k^2 + 5k$ since $k \geq 5$. Also $k^2 + 5k = k^2 + 2k + 3k \geq k^2 + 2k + 3 \cdot 5 = k^2 + 2k + 15$, again since $k \geq 5$. Finally, $k^2 + 2k + 15 > k^2 + 2k + 1$. We now present a formal proof. (Here we are using the Principle of Mathematical Induction with $m = 5$.) ♦

Result 6.10 *For every integer $n \geq 5$,*

$$2^n > n^2.$$

Proof We proceed by induction. Since $2^5 > 5^2$, the inequality holds for $n = 5$. Assume that $2^k > k^2$, where $k \geq 5$. We show that $2^{k+1} > (k+1)^2$. Observe that

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k > 2k^2 = k^2 + k^2 \geq k^2 + 5k \\ &= k^2 + 2k + 3k \geq k^2 + 2k + 15 \\ &> k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

Therefore, $2^{k+1} > (k+1)^2$. By the Principle of Mathematical Induction, $2^n > n^2$ for every integer $n \geq 5$. ■

Result 6.11 *For every nonnegative integer n,*

$$3 \mid \left(2^{2n} - 1\right).$$

Proof We proceed by induction. The result is true when $n = 0$ since in this case $2^{2n} - 1 = 0$ and $3 \mid 0$. Assume that $3 \mid \left(2^{2k} - 1\right)$, where k is a nonnegative integer. We show that $3 \mid \left(2^{2k+2} - 1\right)$. Since $3 \mid \left(2^{2k} - 1\right)$, there exists an integer x such that $2^{2k} - 1 = 3x$ and so $2^{2k} = 3x + 1$. Now

$$2^{2k+2} - 1 = 4 \cdot 2^{2k} - 1 = 4(3x+1) - 1 = 12x + 3 = 3(4x+1).$$

Since $4x + 1$ is an integer, $3 \mid \left(2^{2k+2} - 1\right)$.

By the Principle of Mathematical Induction, $3 \mid \left(2^{2n} - 1\right)$ for every nonnegative integer n. ■

PROOF ANALYSIS Let's review the preceding proof. As expected, to establish the inductive step, we assumed that $3 \mid \left(2^{2k} - 1\right)$ for an arbitrary nonnegative integer k and attempted to show that $3 \mid \left(2^{2k+2} - 1\right)$. To verify that $3 \mid \left(2^{2k+2} - 1\right)$, it was necessary to show that $2^{2k+2} - 1$ is a multiple of 3, that is, we needed to show that $2^{2k+2} - 1$ can be expressed as $3z$ for some integer z. Since our goal was to show that $2^{2k+2} - 1$ can be expressed in a certain form, it is natural to consider $2^{2k+2} - 1$ and see how we might write it. Since we knew that $2^{2k} - 1 = 3x$, where $x \in \mathbf{Z}$, it was logical to rewrite $2^{2k+2} - 1$ so that 2^{2k} appears. Actually, this is quite easy since

$$2^{2k+2} = 2^2 \cdot 2^{2k} = 4 \cdot 2^{2k}.$$

Therefore, $2^{2k+2} - 1 = 4 \cdot 2^{2k} - 1$. At this point, we need to be a bit careful because the expression we are currently considering is $4 \cdot 2^{2k} - 1$, not $4(2^{2k} - 1)$. That is, it would be incorrect to say that $4 \cdot 2^{2k} - 1 = 4(3x)$. Hence we need to substitute for 2^{2k} in this case, not for $2^{2k} - 1$. This is the reason that in the proof we rewrote $2^{2k} - 1 = 3x$ as $2^{2k} = 3x + 1$. ♦

We reinforce this kind of proof with another example.

Result 6.12 *For every nonnegative integer n,*

$$9 \mid \left(4^{3n} - 1\right).$$

Proof We proceed by induction. When $n = 0$, $4^{3n} - 1 = 0$. Since $9 \mid 0$, the statement is true when $n = 0$. Assume that $9 \mid \left(4^{3k} - 1\right)$, where k is a nonnegative integer. We now show that $9 \mid \left(4^{3k+3} - 1\right)$. Since $9 \mid \left(4^{3k} - 1\right)$, it follows that $4^{3k} - 1 = 9x$ for some integer x. Hence $4^{3k} = 9x + 1$. Now observe that

$$\begin{aligned} 4^{3k+3} - 1 &= 4^3 \cdot 4^{3k} - 1 = 64(9x+1) - 1 \\ &= 64 \cdot 9x + 64 - 1 = 64 \cdot 9x + 63 \\ &= 9(64x + 7). \end{aligned}$$

Since $64x + 7$ is an integer, $9 \mid \left(4^{3k+3} - 1\right)$.

By the Principle of Mathematical Induction, $9 \mid \left(4^{3n} - 1\right)$ for every nonnegative integer n. ■

As a final comment regarding the preceding proof, notice that we did not multiply 64 and 9 since we were about to factor 9 from the expression in the next step in any case.

We saw in Theorem 3.10 that for an integer x, its square x^2 is even if and only if x is even. This is actually a consequence of Theorem 3.15, which states that for integers

a and b, their product ab is even if and only if a or b is even. We now present a generalization of Theorem 3.10.

Result 6.13 *Let $x \in \mathbf{Z}$. For every integer $n \geq 2$, x^n is even if and only if x is even.*

Proof Assume, first, that x is even. Then $x = 2y$ for some integer y. Hence

$$x^n = x \cdot x^{n-1} = (2y)x^{n-1} = 2\left(yx^{n-1}\right).$$

Since yx^{n-1} is an integer, x^n is even.

We now verify the converse, namely, if x^n is even, where $n \geq 2$, then x is even. We proceed by induction. If x^2 is even, then we have already seen that x is even. Hence the statement is true for $n = 2$. Assume that if x^k is even for some integer $k \geq 2$, then x is even. We show that if x^{k+1} is even, then x is even. Let x^{k+1} be an even integer. Then $x \cdot x^k$ is even. By Theorem 3.15, x is even or x^k is even. If x is even, then the result is proved. On the other hand, if x^k is even, then, by the induction hypothesis, x is even as well. By the Principle of Mathematical Induction, it follows, for every integer $n \geq 2$, that if x^n is even, then x is even. ∎

Although it is impossible to illustrate every type of result where induction can be used, we give two examples that are considerably different than those we have seen.

One of De Morgan's laws (see Theorem 4.19) states that

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

for every two sets A and B. It is possible to use this law to show that

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$$

for every three sets A, B, and C. We show how induction can be used to prove De Morgan's law for any finite number of sets.

Theorem 6.14 *If $A_1, A_2, \ldots, A_n$ are $n \geq 2$ sets, then*

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Proof We proceed by induction. For $n = 2$, the result *is* De Morgan's Law and is therefore true. Assume that the result is true for any k sets, where $k \geq 2$; that is, assume that if $B_1, B_2, \ldots, B_k$ are any k sets, then

$$\overline{B_1 \cup B_2 \cup \cdots \cup B_k} = \overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_k}.$$

We prove that the result is true for any $k + 1$ sets. Let $S_1, S_2, \ldots, S_{k+1}$ be $k + 1$ sets. We show that

$$\overline{S_1 \cup S_2 \cup \cdots \cup S_{k+1}} = \overline{S_1} \cap \overline{S_2} \cap \cdots \cap \overline{S_{k+1}}.$$

Let $T = S_1 \cup S_2 \cup \cdots \cup S_k$. Then

$$\overline{S_1 \cup S_2 \cup \cdots \cup S_{k+1}} = \overline{(S_1 \cup S_2 \cup \cdots \cup S_k) \cup S_{k+1}} = \overline{T \cup S_{k+1}}.$$

Now, by De Morgan's Law,

$$\overline{T \cup S_{k+1}} = \overline{T} \cap \overline{S_{k+1}}.$$

By the definition of T and by the inductive hypothesis, we have

$$\overline{T} = \overline{S_1 \cup S_2 \cup \cdots \cup S_k} = \overline{S_1} \cap \overline{S_2} \cap \cdots \cap \overline{S_k}.$$

Therefore,

$$\begin{aligned}\overline{S_1 \cup S_2 \cup \cdots \cup S_{k+1}} &= \overline{T \cup S_{k+1}} = \overline{T} \cap \overline{S_{k+1}} \\ &= \overline{S_1} \cap \overline{S_2} \cap \cdots \cap \overline{S_k} \cap \overline{S_{k+1}}.\end{aligned}$$

By the Principle of Mathematical Induction, for every $n \geq 2$ sets $A_1, A_2, \ldots, A_n$,

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n},$$

as desired. ∎

PROOF ANALYSIS A few comments may be useful concerning the notation used in the statement and the proof of Theorem 6.14. First, the sets $A_1, A_2, \ldots, A_n$ were used in the statement of Theorem 6.14 only as an aid to describe the result. Theorem 6.14 could have also been stated as:

> *For every integer $n \geq 2$, the complement of the union of any n sets equals the intersection of the complements of these sets.*

To verify the inductive step in the proof of Theorem 6.14, we assumed that the statement is true for any $k \geq 2$ sets, which we denoted by $B_1, B_2, \ldots, B_k$. The fact that we used $A_1, A_2, \ldots, A_n$ to describe the statement of Theorem 6.14 did not mean that we should use $A_1, A_2, \ldots, A_k$ for the k sets in the inductive hypothesis. In fact, it is probably better that we do not use this notation. In the inductive step, we now need to show that the result is true for any $k + 1$ sets. We used $S_1, S_2, \ldots, S_{k+1}$ for these sets. It would have been a bad idea to denote the $k + 1$ sets by $B_1, B_2, \ldots, B_{k+1}$ because that would have (improperly) suggested that k of the $k + 1$ sets must specifically be the sets mentioned in the inductive hypothesis. ♦

We are now able to prove another well-known theorem concerning sets, to which we earlier referred.

Theorem 6.15 *If A is a finite set of cardinality $n \geq 0$, then the cardinality of its power set $\mathcal{P}(A)$ is 2^n.*

Proof We proceed by induction. If A is a set with $|A| = 0$, then $A = \emptyset$. Thus $\mathcal{P}(A) = \{\emptyset\}$ and so $|\mathcal{P}(A)| = 1 = 2^0$. Therefore, the theorem is true for $n = 0$. Assume that if B is any set with $|B| = k$ for some nonnegative integer k, then $|\mathcal{P}(B)| = 2^k$. We show that if C is a set with $|C| = k + 1$, then $|\mathcal{P}(C)| = 2^{k+1}$. Let

$$C = \{c_1, c_2, \ldots, c_{k+1}\}.$$

By the inductive hypothesis, there are 2^k subsets of the set $\{c_1, c_2, \ldots, c_k\}$, that is, there are 2^k subsets of C not containing c_{k+1}. Any subset of C containing c_{k+1} can be expressed as $D \cup \{c_{k+1}\}$, where $D \subseteq \{c_1, c_2, \ldots, c_k\}$. Again, by the inductive hypothesis, there are 2^k such subsets D. Therefore, there are $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets of C.

By the Principle of Mathematical Induction, it follows for every nonnegative integer n that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. ∎

6.3 Proof by Minimum Counterexample

For each positive integer n, let $P(n)$ be a statement. We have seen that induction is a natural proof technique to verify the truth of the quantified statement

$$\forall n \in \mathbf{N},\ P(n). \tag{6.7}$$

There are certainly such statements where induction does not work, or does not work well. If we would attempt to prove (6.7) using a proof by contradiction, then we would begin such a proof by assuming that the statement $\forall n \in \mathbf{N}$, $P(n)$ is false. Consequently, there are positive integers n such that $P(n)$ is a false statement. By the Well-Ordering Principle, there exists a smallest positive integer n such that $P(n)$ is a false statement. Denote this integer by m. Therefore, $P(m)$ is a false statement and for any integer k with $1 \leq k < m$, the statement $P(k)$ is true. The integer m is referred to as a **minimum counterexample** of the statement (6.7). If a proof (by contradiction) of $\forall n \in \mathbf{N}$, $P(n)$ can be given using the fact that m is a minimum counterexample, then such a proof is called a **proof by minimum counterexample**.

We now illustrate this proof technique. For the example we are about to describe, it is useful to recall from algebra that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Suppose that we wish to prove that $6 \mid (n^3 - n)$ for every positive integer n. An induction proof would probably start like this:

If $n = 1$, then $n^3 - n = 0$. Since $6 \mid 0$, the result is true for $n = 1$. Assume that $6 \mid (k^3 - k)$, where k is a positive integer. We wish to prove that $6 \mid [(k+1)^3 - (k+1)]$. Since $6 \mid (k^3 - k)$, it follows that $k^3 - k = 6x$ for some integer x. Then

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= (k^3 - k) + 3k^2 + 3k\\ &= 6x + 3k(k+1).\end{aligned}$$

If we can show that $6 \mid 3k(k+1)$, we have a proof. Thus we need to show that $k(k+1)$ is even for every positive integer k. A lemma could be introduced to verify this. This lemma could be proved in two cases (k is even and k is odd) or induction could be used. Although such a lemma would not be difficult to prove, we give an alternative proof that avoids the need for a lemma.

Result 6.16 *For every positive integer n,*

$$6 \mid (n^3 - n).$$

Proof Assume, to the contrary, that there are positive integers n such that $6 \nmid (n^3 - n)$. Then there is a smallest positive integer n such that $6 \nmid (n^3 - n)$. Let m be this integer. If $n = 1$, then $n^3 - n = 0$; while if $n = 2$, then $n^3 - n = 6$. Since $6 \mid 0$ and $6 \mid 6$, it follows that $6 \mid (n^3 - n)$ for $n = 1$ and $n = 2$. Therefore $m \geq 3$. So we can write $m = k + 2$, where $1 \leq k < m$. Observe that

$$\begin{aligned}m^3 - m = (k+2)^3 - (k+2) &= (k^3 + 6k^2 + 12k + 8) - (k+2)\\ &= (k^3 - k) + (6k^2 + 12k + 6).\end{aligned}$$

Since $k < m$, it follows that $6 \mid (k^3 - k)$. Hence $k^3 - k = 6x$ for some integer x. So we have

$$m^3 - m = 6x + 6(k^2 + 2k + 1) = 6(x + k^2 + 2k + 1).$$

Since $x + k^2 + 2k + 1$ is an integer, $6 \mid (m^3 - m)$, which produces a contradiction. ■

PROOF ANALYSIS Let's see how this proof was constructed. In this proof, m is a positive integer such that $6 \nmid (m^3 - m)$; while for every positive integer n with $n < m$, we have $6 \mid (n^3 - n)$. We are trying to determine just how large m needs to be to obtain a contradiction. We saw that $6 \mid (1^3 - 1)$ and $6 \mid (2^3 - 2)$; so $m \geq 3$. Knowing that $m \geq 3$ allowed us to write m as $k + 2$, where $1 \leq k < m$. Because $1 \leq k < m$, we know that $6 \mid (k^3 - k)$ and so $k^3 - k = 6x$, where $x \in \mathbf{Z}$. So, in the proof, we wrote

$$\begin{aligned}m^3 - m = (k+2)^3 - (k+2) &= (k^3 + 6k^2 + 12k + 8) - (k+2)\\ &= (k^3 - k) + (6k^2 + 12k + 6) = 6x + 6k^2 + 12k + 6.\end{aligned}$$

The fact that we can factor 6 from $6x + 6k^2 + 12k + 6$ is what allowed us to conclude that $6 \mid (m^3 - m)$ and obtain a contradiction. But how did we know that we wanted $m \geq 3$? If we had observed only that $6 \mid (1^3 - 1)$ and not that $6 \mid (2^3 - 2)$, then we would have known only that $m \geq 2$, which would have allowed us to write $m = k + 1$, where $1 \leq k < m$. Of course, we would still know that $6 \mid (k^3 - k)$ and so $k^3 - k = 6x$, where $x \in \mathbf{Z}$. However, when we consider $m^3 - m$, we would have

$$\begin{aligned}m^3 - m = (k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1)\\ &= (k^3 - k) + 3k^2 + 3k = (k^3 - k) + 3k(k+1)\\ &= 6x + 3k(k+1).\end{aligned}$$

As it stands, we can factor 3 from $6x + 3k(k+1)$ but cannot factor 6 unless we can prove that $k(k+1)$ is even. This is the same difficulty we encountered when we were considering an induction proof. In any case, no contradiction is obtained. ♦

If a result can be proved by induction, then it can also be proved by minimum counterexample. It is not difficult to use induction to prove that $3 \mid (2^{2n} - 1)$ for every nonnegative integer n. We also give a proof by minimum counterexample of this statement.

Result 6.17 *For every nonnegative integer n,*

$$3 \mid (2^{2n} - 1).$$

Proof Assume, to the contrary, that there are nonnegative integers n for which $3 \nmid (2^{2n} - 1)$. By Theorem 6.7, there is a smallest nonnegative integer n such that $3 \nmid (2^{2n} - 1)$. Denote this integer by m. Thus $3 \nmid (2^{2m} - 1)$ and $3 \mid (2^{2n} - 1)$ for all nonnegative integers n for which $0 \leq n < m$. Since $3 \mid (2^{2n} - 1)$ when $n = 0$, it follows that $m \geq 1$. Hence $m = k + 1$, where $0 \leq k < m$. Thus $3 \mid (2^{2k} - 1)$, which implies that $2^{2k} - 1 = 3x$ for

some integer x. Consequently, $2^{2k} = 3x + 1$. Observe that

$$2^{2m} - 1 = 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^2 \cdot 2^{2k} - 1$$
$$= 4(3x + 1) - 1 = 12x + 3 = 3(4x + 1).$$

Since $4x + 1$ is an integer, $3 \mid (2^{2m} - 1)$, which produces a contradiction. ∎

We give one additional example of proof by minimum counterexample.

Result 6.18 *For every positive integer n,*

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof Assume, to the contrary, that

$$1 + 2 + 3 + \cdots + n \neq \frac{n(n+1)}{2}$$

for some positive integers n. By the Well-Ordering Principle, there is a smallest positive integer n such that

$$1 + 2 + 3 + \cdots + n \neq \frac{n(n+1)}{2}.$$

Denote this integer by m. Therefore,

$$1 + 2 + 3 + \cdots + m \neq \frac{m(m+1)}{2},$$

while

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every integer n with $1 \leq n < m$. Since $1 = 1(1 + 1)/2$, it follows that $m \geq 2$. Hence we can write $m = k + 1$, where $1 \leq k < m$. Consequently,

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Observe that

$$1 + 2 + 3 + \cdots + m = 1 + 2 + 3 + \cdots + (k + 1) = (1 + 2 + 3 + \cdots + k) + (k + 1)$$
$$= \frac{k(k+1)}{2} + (k + 1) = \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2} = \frac{m(m+1)}{2},$$

which produces a contradiction. ∎

6.4 The Strong Principle of Mathematical Induction

We close with one last form of mathematical induction. This principle goes by many names: the Strong Principle of Mathematical Induction, the Strong Form of Induction, the Alternate Form of Mathematical Induction, and the Second Principle of Mathematical Induction are common names.

Theorem 6.19 **(The Strong Principle of Mathematical Induction)** *For each positive integer n, let $P(n)$ be a statement. If*

(1) *$P(1)$ is true and*
(2) *the implication*

If $P(i)$ for every integer i with $1 \leq i \leq k$, then $P(k + 1)$.

is true for every positive integer k,

then $P(n)$ is true for every positive integer n.

As with the Principle of Mathematical Induction (Theorem 6.2), the Strong Principle of Mathematical Induction is also a consequence of the Well-Ordering Principle. The Strong Principle of Mathematical Induction is now stated more symbolically below.

The Strong Principle of Mathematical Induction

For each positive integer n, let $P(n)$ be a statement. If

(1) *$P(1)$ is true and*
(2) *$\forall k \in \mathbf{N},\ P(1) \wedge P(2) \wedge \cdots \wedge P(k) \Rightarrow P(k + 1)$ is true,*

then $\forall n \in \mathbf{N},\ P(n)$ is true.

The difference in the statements of the Principle of Mathematical Induction and the Strong Principle of Mathematical Induction lies in the inductive step (condition 2). To prove that $\forall n \in \mathbf{N},\ P(n)$ is true by the Principle of Mathematical Induction, we are required to show that $P(1)$ is true and to verify the implication:

$$\text{If } P(k), \text{ then } P(k + 1). \tag{6.8}$$

is true for every positive integer k. On the other hand, to prove $\forall n \in \mathbf{N},\ P(n)$ is true by the Strong Principle of Mathematical Induction, we are required to show that $P(1)$ is true and to verify the implication:

$$\text{If } P(i) \text{ for every } i \text{ with } 1 \leq i \leq k, \text{ then } P(k + 1). \tag{6.9}$$

is true for every positive integer k. If we were to give direct proofs of the implications (6.8) and (6.9), then we are permitted to assume more in the inductive step (6.9) of the Strong Principle of Mathematical Induction than in the induction step (6.8) of the Principle of Mathematical Induction and yet obtain the same conclusion. If the assumption that $P(k)$ is true is insufficient to verify the truth of $P(k + 1)$ for an arbitrary positive integer k, but the assumption that all of the statements $P(1), P(2), \ldots, P(k)$ are true is sufficient to verify the truth of $P(k + 1)$, then this suggests that we should use the Strong Principle of Mathematical Induction. Indeed, any result that can be proved by the Principle of Mathematical Induction can also be proved by the Strong Principle of Mathematical Induction.

Just as there is a more general version of the Principle of Mathematical Induction (namely, Theorem 6.8), there is a more general version of the Strong Principle of Mathematical Induction. We shall also refer to this as the Strong Principle of Mathematical Induction.

Theorem 6.20 **(The Strong Principle of Mathematical Induction)** *For a fixed integer m, let $S = \{i \in \mathbf{Z} : i \geq m\}$. For each $n \in S$, let $P(n)$ be a statement. If*

(1) *$P(m)$ is true and*
(2) *the implication*

If $P(i)$ for every integer i with $m \leq i \leq k$, then $P(k+1)$.

is true for every integer $k \in S$,

then $P(n)$ is true for every integer $n \in S$.

We now consider a class of problems where the Strong Principle of Mathematical Induction is commonly the appropriate proof technique.

Suppose that we are considering a sequence $a_1, a_2, a_3, \ldots$ of numbers. One way of defining a sequence $\{a_n\}$ is to specify explicitly the nth term a_n (as a function of n). For example, we might have $a_n = \frac{1}{n}$, $a_n = \frac{(-1)^n}{n^2}$, or $a_n = n^3 + n$ for each $n \in \mathbf{N}$. A sequence can also be **defined recursively**. In a **recursively defined sequence** $\{a_n\}$, only the first term or perhaps the first few terms are defined specifically, say $a_1, a_2, \ldots, a_k$ for some fixed $k \in \mathbf{N}$. These are called the **initial values**. Then a_{k+1} is expressed in terms of $a_1, a_2, \ldots, a_k$ and, more generally, for $n > k$, a_n is expressed in terms of $a_1, a_2, \ldots, a_{n-1}$. This is called the **recurrence relation**.

A specific example of this is the sequence $\{a_n\}$ defined by $a_1 = 1, a_2 = 3$, and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 3$. In this case, there are two initial values, namely $a_1 = 1$ and $a_2 = 3$. The recurrence relation here is

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Letting $n = 3$, we find that $a_3 = 2a_2 - a_1 = 5$; while letting $n = 4$, we have $a_4 = 2a_3 - a_2 = 7$. Similarly, $a_5 = 9$ and $a_6 = 11$. It appears that $a_n = 2n - 1$ for each $n \in \mathbf{N}$. Using the Strong Principle of Mathematical Induction, we prove that this is, in fact, the case.

Result 6.21 *A sequence $\{a_n\}$ is defined recursively by*

$$a_1 = 1, a_2 = 3, \text{ and } a_n = 2a_{n-1} - a_{n-2} \text{ for } n \geq 3.$$

Then $a_n = 2n - 1$ for all $n \in \mathbf{N}$.

Proof We proceed by induction. Since $a_1 = 2 \cdot 1 - 1 = 1$, the formula holds for $n = 1$. Assume for an arbitrary positive integer k that $a_i = 2i - 1$ for all integers i with $1 \leq i \leq k$. We show that $a_{k+1} = 2(k+1) - 1 = 2k + 1$. If $k = 1$, then $a_{k+1} = a_2 = 2 \cdot 1 + 1 = 3$.

Since $a_2 = 3$, it follows that $a_{k+1} = 2k + 1$ when $k = 1$. Hence we may assume that $k \geq 2$. Since $k + 1 \geq 3$, it follows that

$$a_{k+1} = 2a_k - a_{k-1} = 2(2k-1) - (2k-3) = 2k + 1,$$

which is the desired result. By the Strong Principle of Mathematical Induction, $a_n = 2n - 1$ for all $n \in \mathbf{N}$. ∎

PROOF ANALYSIS A few comments about the proof of Result 6.21 are in order. At one point, we assumed for an arbitrary positive integer k that $a_i = 2i - 1$ for all integers i with $1 \leq i \leq k$. Our goal was to show that $a_{k+1} = 2k + 1$. Since k is a positive integer, it may occur that $k = 1$ or $k \geq 2$. If $k = 1$, then we need to show that $a_{k+1} = a_2 = 2 \cdot 1 + 1 = 3$. We know that $a_2 = 3$ because this is one of the initial values. If $k \geq 2$, then $k + 1 \geq 3$ and a_{k+1} can be expressed as $2a_k - a_{k-1}$ by the recurrence relation. In order to show that $a_{k+1} = 2k + 1$ when $k \geq 2$, it was necessary to know that $a_k = 2k - 1$ and that $a_{k-1} = 2(k-1) - 1 = 2k - 3$. Because we were using the Strong Principle of Mathematical Induction, we knew both pieces of information. If we had used the Principle of Mathematical Induction, then we would have assumed (and therefore knew) that $a_k = 2k - 1$ but we would not have known that $a_{k-1} = 2k - 3$, and so we would have been unable to establish the desired expression for a_{k+1}. ♦

Problem 6.22 *A sequence $\{a_n\}$ is defined recursively by*

$$a_1 = 1, a_2 = 4, \text{ and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for } n \geq 3.$$

Conjecture a formula for a_n and verify that your conjecture is correct.

Solution We begin by finding a few more terms of the sequence. Observe that $a_3 = 2a_2 - a_1 + 2 = 9$, while $a_4 = 2a_3 - a_2 + 2 = 16$ and $a_5 = 2a_4 - a_3 + 2 = 25$. The obvious conjecture is that $a_n = n^2$ for every positive integer n. We verify that this conjecture is correct in the next result. ♦

Result 6.23 *A sequence $\{a_n\}$ is defined recursively by*

$$a_1 = 1, a_2 = 4, \text{ and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for } n \geq 3.$$

Then $a_n = n^2$ for all $n \in \mathbf{N}$.

Proof We proceed by induction. Since $a_1 = 1 = 1^2$, the formula holds for $n = 1$. Assume for an arbitrary positive integer k that $a_i = i^2$ for every integer i with $1 \leq i \leq k$. We show that $a_{k+1} = (k+1)^2$. Since $a_2 = 4$, it follows that $a_{k+1} = (k+1)^2$ when $k = 1$. Thus we may assume that $k \geq 2$. Hence $k + 1 \geq 3$ and so

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} + 2 = 2k^2 - (k-1)^2 + 2 \\ &= 2k^2 - (k^2 - 2k + 1) + 2 = k^2 + 2k + 1 = (k+1)^2. \end{aligned}$$

By the Strong Principle of Mathematical Induction, $a_n = n^2$ for all $n \in \mathbf{N}$. ∎

Although we mentioned that problems involving recurrence relations are commonly solved with the aid of the Strong Principle of Mathematical Induction, it is by no means the only kind of problem where the Strong Principle of Mathematical Induction can be applied. Although the best examples require a knowledge of mathematics beyond what we have covered thus far, we do present another type of example.

Result 6.24 *For each integer $n \geq 8$, there are nonnegative integers a and b such that $n = 3a + 5b$.*

Proof We proceed by induction. Since $8 = 3 \cdot 1 + 5 \cdot 1$, the statement is true for $n = 8$. Assume for each integer i with $8 \leq i \leq k$, where $k \geq 8$ is an arbitrary integer, that there are nonnegative integers s and t such that $i = 3s + 5t$. Consider the integer $k + 1$. We show that there are nonnegative integers x and y such that $k + 1 = 3x + 5y$. Since $9 = 3 \cdot 3 + 5 \cdot 0$ and $10 = 3 \cdot 0 + 5 \cdot 2$, this is true if $k + 1 = 9$ and $k + 1 = 10$. Hence we may assume that $k + 1 \geq 11$. Thus $8 \leq (k + 1) - 3 < k$. By the induction hypothesis, there are nonnegative integers a and b such that

$$(k + 1) - 3 = 3a + 5b \text{ and so } k + 1 = 3(a + 1) + 5b.$$

Letting $x = a + 1$ and $y = b$, we have the desired conclusion.

By the Strong Principle of Mathematical Induction, for every integer $n \geq 8$, there are nonnegative integers a and b such that $n = 3a + 5b$. ∎

EXERCISES FOR CHAPTER 6

Section 6.1: The Principle of Mathematical Induction

6.1. Which of the following sets are well-ordered?

(a) $S = \{x \in \mathbf{Q} : x \geq -10\}$
(b) $S = \{-2, -1, 0, 1, 2\}$
(c) $S = \{x \in \mathbf{Q} : -1 \leq x \leq 1\}$
(d) $S = \{p : p \text{ is a prime}\} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$

6.2. Prove that if A is any well-ordered set of real numbers and B is a nonempty subset of A, then B is also well-ordered.

6.3. Prove that every nonempty set of negative integers has a largest element.

6.4. Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n

(1) by mathematical induction.
(2) by adding $1 + 3 + 5 + \cdots + (2n - 1)$ and $(2n - 1) + (2n - 3) + \cdots + 1$.

6.5. Use mathematical induction to prove that

$$1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n$$

for every positive integer n.

6.6. Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integers n, and then verify your formula by mathematical induction.

6.7. Find another formula suggested by Exercises 6.4 and 6.5, and verify your formula by mathematical induction.

6.8. (a) We have seen that $1^2 + 2^2 + \cdots + n^2$ is the number of squares in an $n \times n$ square composed of n^2 1×1 squares. What does $1^3 + 2^3 + 3^3 + \cdots + n^3$ represent geometrically?
(b) Use mathematical induction to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for every positive integer n.

6.9. Prove that $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{n(n+1)(2n+7)}{6}$ for every positive integer n.

6.10. Let $r \neq 1$ be a real number. Use induction to prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for every positive integer n.

6.11. Prove that $\frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{(n+2)(n+3)} = \frac{n}{3n+9}$ for every positive integer n.

Section 6.2: A More General Principle of Mathematical Induction

6.12. Prove Theorem 6.7: For each integer m, the set $S = \{i \in \mathbf{Z} : i \geq m\}$ is well-ordered. [Hint: For every subset T of S, either $T \subseteq \mathbf{N}$ or $T - \mathbf{N}$ is a finite nonempty set.]

6.13. Prove that $2^n > n^3$ for every integer $n \geq 10$.

6.14. Prove that $n! > 2^n$ for every integer $n \geq 4$.

6.15. Prove that $3^n > n^2$ for every positive integer n.

6.16. Prove that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for every positive integer n.

6.17. Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer n,

$$(1 + x)^n \geq 1 + nx.$$

6.18. Prove that $4 \mid (5^n - 1)$ for every nonnegative integer n.

6.19. Prove that $81 \mid (10^{n+1} - 9n - 10)$ for every nonnegative integer n.

6.20. Prove that $7 \mid \left(3^{2n} - 2^n\right)$ for every nonnegative integer n.

6.21. In Exercise 4.6, you were asked to prove that if $3 \mid 2a$, where $a \in \mathbf{Z}$, then $3 \mid a$. Assume that this result is true. Prove the following generalization: Let $a \in \mathbf{Z}$. For every positive integer n, if $3 \mid 2^n a$, then $3 \mid a$.

6.22. Prove that if $A_1, A_2, \ldots, A_n$ are any $n \geq 2$ sets, then

$$\overline{A_1 \cap A_2 \cap \cdots \cap A_n} = \overline{A_1} \cup \overline{A_2} \cup \cdots \cup \overline{A_n}.$$

6.23. Recall for integers $n \geq 2$, a, b, c, d, that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then both $a + c \equiv b + d \pmod{n}$ and $ac \equiv bd \pmod{n}$. Use these results and mathematical induction to prove the following: For any $2m$ integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ for which $a_i \equiv b_i \pmod{n}$ for $1 \leq i \leq m$,

(a) $a_1 + a_2 + \cdots + a_m \equiv b_1 + b_2 + \cdots + b_m \pmod{n}$ and
(b) $a_1 a_2 \ldots a_m \equiv b_1 b_2 \cdots b_m \pmod{n}$.

6.24. Prove the following implication for every integer $n \geq 2$: If $x_1, x_2, \ldots, x_n$ are any n real numbers such that $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 0$, then at least one of the numbers $x_1, x_2, \ldots, x_n$ is 0. (Use the fact that if the product of two real numbers is 0, then at least one of the numbers is 0.)

6.25. (a) Use mathematical induction to prove that every finite nonempty set of real numbers has a largest element.
(b) Use (a) to prove that every finite nonempty set of real numbers has a smallest element.

Section 6.3: Proof by Minimum Counterexample

6.26. Use proof by minimum counterexample to prove that $6 \mid 7n\left(n^2-1\right)$ for every positive integer n.

6.27. Use the method of minimum counterexample to prove that $3 \mid \left(2^{2n}-1\right)$ for every positive integer n.

6.28. Prove that $12 \mid \left(n^4-n^2\right)$ for every positive integer n.

6.29. Prove that $5 \mid \left(n^5-n\right)$ for every integer n.

6.30. Use proof by minimum counterexample to prove that $3 \mid \left(2^n+2^{n+1}\right)$ for every nonnegative integer n.

6.31. Let $S=\{2^n : n \in \mathbf{Z}, n \geq 0\}$. Use proof by minimum counterexample to prove that for every $n \in \mathbf{N}$, there exists a subset S_n of S such that $\sum_{i \in S_n} i = n$.

Section 6.4: The Strong Principle of Mathematical Induction

6.32. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 2$. Conjecture a formula for a_n and verify that your conjecture is correct.

6.33. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for a_n and verify that your conjecture is correct.

6.34. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3)$$

for $n \geq 4$. Conjecture a formula for a_n and prove that your conjecture is correct.

6.35. Consider the sequence $F_1, F_2, F_3, \ldots$, where

$$F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5, \text{ and } F_6 = 8.$$

The terms of this sequence are called **Fibonacci numbers**.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.
(b) Prove that $2 \mid F_n$ if and only if $3 \mid n$.

6.36. Consider the following sequence of equalities:

$1 = 0 + 1$
$2 + 3 + 4 = 1 + 8$
$5 + 6 + 7 + 8 + 9 = 8 + 27$
$10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$
$\ldots$

(a) What is the next equality in this sequence?
(b) Now develop a general conjecture and prove that your conjecture is correct by induction.

6.37. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 12$, there are nonnegative integers a and b such that $n = 3a + 7b$.

ADDITIONAL EXERCISES FOR CHAPTER 6

6.38. By Result 6.5,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \textbf{(6.10)}$$

for every positive integer n.

(a) Use (6.10) to determine a formula for $2^2 + 4^2 + 6^2 + \cdots + (2n)^2$ for every positive integer n.
(b) Use (6.10) and (a) to determine a formula for $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2$ for every positive integer n.
(c) Use (a) and (b) to determine a formula for

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1} n^2$$

for every positive integer n.
(d) Use mathematical induction to verify the formulas in (b) and (c).

6.39. Prove that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for every positive integer n.

6.40. Prove that $4^n > n^3$ for every positive integer n.

6.41. Prove that $24 \mid (5^{2n} - 1)$ for every positive integer n.

6.42. Use the Strong Principle of Mathematical Induction to prove the following. Let $S = \{i \in \mathbf{Z} : i \geq 2\}$ and let P be a subset of S with the properties that $2, 3 \in P$ and if $n \in S$, then either $n \in P$ or $n = ab$, where $a, b \in S$. Then every element of S either belongs to P or can be expressed as a product of elements of P.

6.43. Use the Strong Principle of Mathematical Induction to prove that for each integer $n \geq 28$, there are nonnegative integers x and y such that $n = 5x + 8y$.

6.44. Find a positive integer m such that for each integer $n \geq m$, there are positive integers x and y such that $n = 3x + 5y$. Use the Principle of Mathematical Induction to prove this.

6.45. Find a positive integer m such that for each integer $n \geq m$, there are integers $x, y \geq 2$ such that $n = 2x + 3y$. Use the Principle of Mathematical Induction to prove this.

6.46. Consider the sequence $a_1 = 2$, $a_2 = 5$, $a_3 = 9$, $a_4 = 14$, etc.

(a) Find a recurrence relation that expresses a_n in terms of a_{n-1} for every integer $n \geq 2$.
(b) Conjecture an explicit formula for a_n and then prove that your conjecture is correct.

6.47. The following theorem allows one to prove certain quantified statements over some finite sets.
The Principle of Finite Induction *For a fixed positive integer m, let $S = \{1, 2, \ldots, m\}$. For each $n \in S$, let $P(n)$ be a statement. If*

(1) *$P(1)$ is true and*
(2) *the implication*

If $P(k)$, then $P(k+1)$

is true for every integer k with $1 \leq k < m$,

then $P(n)$ is true for every integer $n \in S$.
Use the Principle of Finite Induction to prove the following result.
Let $S = \{1, 2, \ldots, 24\}$. For every integer t with $1 \leq t \leq 300$, there exists a subset $S_t \subseteq S$ such that $\sum_{i \in S_t} i = t$.

6.48. Below is given a proof of a result. What result is being proved and which proof technique is being used?

Proof First observe that $a_1 = 8 = 3 \cdot 1 + 5$, and $a_2 = 11 = 3 \cdot 2 + 5$. Thus $a_n = 3n + 5$ for $n = 1$ and $n = 2$. Assume that $a_i = 3i + 5$ for all integers i with $1 \leq i \leq k$, where $k \geq 2$. Since $k + 1 \geq 3$, it follows that

$$\begin{aligned} a_{k+1} &= 5a_k - 4a_{k-1} - 9 = 5(3k+5) - 4(3k+2) - 9 \\ &= 15k + 25 - 12k - 8 - 9 = 3k + 8 = 3(k+1) + 5. \end{aligned}$$

■

6.49. A proof of a result is given below. What result is being proved and which proof technique is being used?

Proof Assume, to the contrary, that there is some positive integer n such that $8 \nmid (3^{2n} - 1)$. Let m be the smallest positive integer such that $8 \nmid (3^{2m} - 1)$. For $n = 1$, $3^{2n} - 1 = 8$. Since $8 \mid 8$, it follows that $m \geq 2$. Let $m = k + 1$. Since $1 \leq k < m$, it follows that $8 \mid (3^{2k} - 1)$. Therefore, $3^{2k} - 1 = 8x$ for some integer x and so $3^{2k} = 8x + 1$. Hence

$$\begin{aligned} 3^{2m} - 1 = 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 = 9 \cdot 3^{2k} - 1 \\ &= 9(8x + 1) - 1 = 72x + 8 = 8(9x + 1). \end{aligned}$$

Since $9x + 1$ is an integer, $8 \mid (3^{2m} - 1)$, which produces a contradiction. ∎

6.50. Evaluate the proposed proof of the following result.

Result For every positive integer n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Proof We proceed by induction. Since $2 \cdot 1 - 1 = 1^2$, the formula holds for $n = 1$. Assume that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ for a positive integer k. We prove that $1 + 3 + 5 + \cdots + (2k + 1) = (k + 1)^2$. Observe that

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k + 1) &= (k + 1)^2 \\ 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= (k + 1)^2 \\ k^2 + (2k + 1) &= (k + 1)^2 \\ (k + 1)^2 &= (k + 1)^2. \end{aligned}$$

∎

6.51. By an n-gon, we mean an n-sided polygon. So a 3-gon is a triangle and a 4-gon is a quadrilateral. It is well known that the sum of the interior angles of a triangle is 180°. Use induction to prove that for every integer $n \geq 3$, the sum of the interior angles of an n-gon is $(n - 2) \cdot 180°$.

7

Prove or Disprove

In every mathematical statement that you have seen so far, you have been informed of its truth value. If the statement was true, then we have either provided a proof for you or have asked you to provide one of your own. What you didn't know (perhaps) was how we or you were to verify its truth. If the statement was false, then here too we either verified this or asked you to verify that it was false. As you proceed further into the world of mathematics, you will more and more often encounter statements whose truth is in question. Consequently, each such statement presents two problems for you: (1) Determine the truth or falseness of the statement and (2) Verify the correctness of your belief.

7.1 Conjectures in Mathematics

In mathematics, when we don't know whether a certain statement is true but there is good reason to believe that it is, then we refer to the statement as a **conjecture**. So the word "conjecture" is used in mathematics as a sophisticated synonym for an intelligent guess (or perhaps just a guess). Once a conjecture is proved, then the conjecture becomes a theorem. If, on the other hand, the conjecture is shown to be false, then we made an incorrect guess. This is the way mathematics develops – by guessing and showing that our guess is correct or wrong, and then possibly making a new conjecture and then repeating the process (possibly often). As we learn what's true and what's false about the mathematics we're studying, this influences the questions we ask and the conjectures we make.

Let's consider an example of a conjecture (although there is always the possibility that someone has settled the conjecture between the time it was written here and the moment you read it). A word is called a **palindrome** if it reads the same forward and backward (such as *deed, noon,* and *radar*). Indeed, a sentence is a palindrome if it reads the same forward and backward, ignoring spaces (*Name no one man*). A positive integer is called a **palindrome** if it is the same number when its digits are reversed. (It is *considerably easier* to give an example of a number that is a palindrome than a word that is a palindrome.) For example, 1221 and 47374 are palindromes. Consider the integer 27. It is not a palindrome. Reverse its digits and we obtain 72. Needless to say, 72 is not

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a palindrome either. Adding 27 and 72, we have:

$$\begin{array}{r} 27 \\ +72 \\ \hline 99 \end{array}$$

A palindrome results. Consider another positive integer, say 59. It is not a palindrome. Reverse its digits and add:

$$\begin{array}{r} 59 \\ +95 \\ \hline 154 \end{array}$$

The result is not a palindrome either. Reverse its digits and add:

$$\begin{array}{r} 154 \\ +451 \\ \hline 605 \end{array}$$

Once again we arrive at a number that is not a palindrome. But reverse its digits and add:

$$\begin{array}{r} 605 \\ +506 \\ \hline 1111 \end{array}$$

This time the result *is* a palindrome. It has been conjectured that if we begin with any positive integer and apply the technique described above to it, then we will eventually arrive at a palindrome. However, no one knows if this is true. (It is known to be true for all two-digit numbers.)

Some conjectures have become famous because it has taken years, decades, or *even centuries* to establish their truth or falseness. Other conjectures remain undecided still today. We now consider four conjectures in mathematics, each of which has a long history.

In 1852, a question occurred to the British student Francis Guthrie when he was coloring a map of the counties of England. Suppose that some country (real or imaginary) has been divided into counties in some manner. Is it possible to color the counties in this map with four or fewer colors such that one color is used for each county and two counties that share a common boundary (not simply a single point) are colored differently? For example, the map of the "country" shown in Figure 7.1 has eight "counties", which are

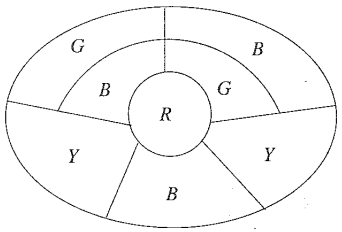

Figure 7.1 Coloring the counties in a country with four colors

colored with the four colors red (R), blue (B), green (G), and yellow (Y), according to the rules described above. This map can also be colored with more than four colors but not less than four.

Within a few years, some of the best known mathematicians of the time had become aware of Francis Guthrie's question, and eventually a famous conjecture developed from this.

The Four Color Conjecture

Every map can be colored with four or fewer colors.

Many attempted to settle this conjecture. In fact, in 1879 an article was published containing a reported proof of the conjecture. However, in 1890, an error was discovered in the proof, and the "theorem" returned to its conjecture status. It was not until 1976 when an actual proof by Kenneth Appel and Wolfgang Haken, combining both mathematics and computers, was presented. The period between the origin of the problem and its solution covered some 124 years. This is now a theorem.

The Four Color Theorem

Every map can be colored with four or fewer colors.

We now describe a conjecture with an even longer history. One of the famous mathematicians of the 17th century was Pierre Fermat. He is undoubtedly best known for one particular assertion he made. He wrote that for each integer $n \geq 3$, there are no nonzero integers x, y, and z such that $x^n + y^n = z^n$. Of course, there are many nonzero integer solutions to the equation $x^2 + y^2 = z^2$. For example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, and $8^2 + 15^2 = 17^2$. A triple (x, y, z) of positive integers such that $x^2 + y^2 = z^2$ is often called a **Pythagorean triple**. Therefore, (3, 4, 5), (5, 12, 13), and (8, 15, 17) are Pythagorean triples. Indeed, if (a, b, c) is a Pythagorean triple and $k \in \mathbf{N}$, then (ka, kb, kc) is also a Pythagorean triple. Fermat's assertion was discovered, unproved, in a margin of a book of Fermat's after his death. In the margin it was written that there was insufficient space to contain his "truly remarkable demonstration". Consequently, this statement became known as Fermat's Last Theorem. It would have been more appropriate, however, to have referred to this statement as Fermat's Last Conjecture as the truth or falseness of this statement remained in question for approximately 350 years. However, in 1993, the British mathematician Andrew Wiles settled the conjecture by giving a truly remarkable proof of it. Hence Fermat's Last Theorem is at last a theorem.

Fermat's Last Theorem

For each integer $n \geq 3$, there are no nonzero integers x, y, and z such that $x^n + y^n = z^n$.

The final two conjectures we mention concern primes. Although we have mentioned primes from time to time, we have not yet presented a formal definition. We do this now. An integer $p \geq 2$ is a **prime** if its only positive integer divisors are 1 and p. A **Fermat number** (Yes, the same Fermat!) is an integer of the form $F_t = 2^{2^t} + 1$, where t is a nonnegative integer. The first five Fermat numbers are

$$F_0 = 3,\ F_1 = 5,\ F_2 = 17,\ F_3 = 257,\ F_4 = 65,537,$$

all of which happen to be primes.

In 1640 Fermat wrote to many (including to the famous mathematician Blaise Pascal) that he believed *every* such number (He didn't call them Fermat numbers.) was a prime, but he was unable to prove this. Hence we have the following.

Fermat's Conjecture Every Fermat number is a prime.

Nearly one century later (in 1739), the famous mathematician Leonhard Euler proved that $F_5 = 4{,}294{,}967{,}297$ is divisible by 641, thereby disproving Fermat's Conjecture. More specifically, Euler proved the following.

Euler's Theorem If p is a prime factor of F_t, then $p = 2^{t+1}k + 1$ for some positive integer k.

Letting $t = 5$ in Euler's Theorem, we see that each prime factor of F_5 is of the form $64k + 1$. The first five primes of this form are 193, 257, 449, 577, and 641, the last of which divides F_5.

In recent decades, other Fermat numbers have been studied and have been shown not to be prime. Indeed, many students of this topic now lean toward the opposing viewpoint (and conjecture): Except for the Fermat numbers $F_0, F_1, \cdots, F_4$ (all of which were observed to be prime by Fermat), *no* Fermat number is prime.

The last conjecture we describe here has its origins around 1742. The German mathematician Christian Goldbach conjectured that every even integer exceeding 2 is the sum of two primes. Of course, this is easy to see for small even integers. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, and $10 = 7 + 3 = 5 + 5$. The major difference between this conjecture and the three preceding conjectures is that this conjecture has never been resolved. Hence we conclude with the following.

Goldbach's Conjecture Every even integer exceeding 2 is the sum of two primes.

7.2 Revisiting Quantified Statements

Many (in fact, most) of the statements we have encountered are quantified statements. Indeed, for an open sentence $P(x)$ over a domain S, we have often considered a quantified statement with a universal quantifier, namely

$$\forall x \in S, P(x): \text{ For every } x \in S, P(x). \text{ or If } x \in S, \text{ then } P(x).$$

or a quantified statement with an existential quantifier, namely

$$\exists x \in S, P(x): \text{ There exists } x \in S \text{ such that } P(x).$$

Recall that $\forall x \in S, P(x)$ is a true statement if $P(x)$ is true for every $x \in S$; while $\exists x \in S, P(x)$ is a true statement if $P(x)$ is true for at least one $x \in S$.

Example 7.1 *Let $S = \{1, 3, 5, 7\}$ and consider*

$$P(n): n^2 + n + 1 \textit{ is prime.}$$

for each $n \in S$. Then both

$$\forall n \in S, P(n): \textit{For every } n \in S, n^2 + n + 1 \textit{ is prime.}$$

and

$$\exists n \in S, P(n): \textit{There exists } n \in S \textit{ such that } n^2 + n + 1 \textit{ is prime.}$$

are quantified statements. Since

$$\begin{aligned} P(1): & \ 1^2 + 1 + 1 = 3 \textit{ is prime.} && \textit{is true,} \\ P(3): & \ 3^2 + 3 + 1 = 13 \textit{ is prime.} && \textit{is true,} \\ P(5): & \ 5^2 + 5 + 1 = 31 \textit{ is prime.} && \textit{is true,} \\ P(7): & \ 7^2 + 7 + 1 = 57 \textit{ is prime.} && \textit{is false,} \end{aligned}$$

it follows that $\forall n \in S, P(n)$ is false and $\exists n \in S, P(n)$ is true. On the other hand, the statement

$$Q: 323 \textit{ is prime.}$$

is not a quantified statement, but Q is false (as $323 = 17 \cdot 19$ is not prime). ♦

Let $P(x)$ be a statement for each x in some domain S. Recall that the negation of $\forall x \in S, P(x)$ is

$$\sim(\forall x \in S, P(x)) \equiv \exists x \in S, \sim P(x).$$

and the negation of $\exists x \in S, P(x)$ is

$$\sim(\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x).$$

Again, consider

$$P(n): n^2 + n + 1 \text{ is prime.}$$

from Example 7.1, which is a statement for each n in $S = \{1, 3, 5, 7\}$. The negation of $\forall n \in S, P(n)$ is

$$\exists n \in S, \sim P(n): \text{ There exists } n \in S \text{ such that } n^2 + n + 1 \text{ is not prime.}$$

is true as $7 \in S$ but $7^2 + 7 + 1 = 57$ is not prime. On the other hand, the negation of $\exists n \in S, P(n)$ is

$$\forall n \in S, \sim P(n): \text{ If } n \in S, \text{ then } n^2 + n + 1 \text{ is not prime.}$$

is false since, for example, $1 \in S$ and $1^2 + 1 + 1 = 3$ *is* prime.

In Chapter 2 we began a discussion of quantified statements containing two quantifiers. The following example concerns two quantifiers.

Example 7.2 *Consider*

$$P(s, t): 2^s + 3^t \textit{ is prime.}$$

where s is a positive even integer and t is a positive odd integer. If we let S denote the set of positive even integers and T the set of positive odd integers, then the quantified statement

$$\exists s \in S, \exists t \in T, P(s,t)$$

can be expressed in words as

There exist a positive even integer s and a positive odd integer t such that $2^s + 3^t$ is prime. (7.1)

The statement (7.1) is true since

$$P(2,1): 2^2 + 3^1 = 7 \text{ is prime.}$$

is true. On the other hand, the quantified statement

$$\forall s \in S, \forall t \in T, P(s,t)$$

can be expressed in words as

For every positive even integer s and every positive odd integer t, $2^s + 3^t$ is prime. (7.2)

The statement (7.2) is false since

$$P(6,3): 2^6 + 3^3 = 91 \text{ is a prime.}$$

is false, as $91 = 7 \cdot 13$ is not a prime. ♦

Let $P(s,t)$ be an open sentence, where the domain of the variable s is S and the domain of the variable t is T. Recall that the negations of the quantified statements $\exists s \in S, \exists t \in T, P(s,t)$ and $\forall s \in S, \forall t \in T, P(s,t)$ are

$$\sim(\exists s \in S, \exists t \in T, P(s,t)) \equiv \forall s \in S, \forall t \in T, \sim P(s,t)$$

and

$$\sim(\forall s \in S, \forall t \in T, P(s,t)) \equiv \exists s \in S, \exists t \in T, \sim P(s,t).$$

Therefore, the negation of the statement (7.1) is

For every positive even integer s and every positive odd integer t, $2^s + 3^t$ is not prime.

which is a false statement. On the other hand, the negation of the statement (7.2) is

There exist a positive even integer s and a positive odd integer t such that $2^s + 3^t$ is not prime.

which is a true statement.

Quantified statements may also contain different kinds of quantifiers. For example, it follows by the definition of an even integer that for every even integer n, there exists an integer k such that $n = 2k$. There is another mathematical symbol with which you should be familiar. The symbol $\ni$ denotes the phrase *such that* (although some mathematicians simply write s.t. for "such that"). For example, let S denote the set of even integers again. Then

$$\forall n \in S, \ \exists k \in \mathbf{Z} \ni n = 2k \tag{7.3}$$

states:

For every even integer n, there exists an integer k such that $n = 2k$.

This statement can be reworded as:

If n is an even integer, then $n = 2k$ for some integer k.

If we interchange the two quantifiers in (7.3), we obtain, in words:

There exists an even integer n such that for every integer k, $n = 2k$.

This statement can also be reworded as

There exists an even integer n such that $n = 2k$ for every integer k.

This statement can be expressed in symbols as

$$\exists n \in S, \forall k \in \mathbf{Z}, n = 2k. \tag{7.4}$$

Certainly, the statements (7.3) and (7.4) say something totally different. Indeed, (7.3) is true and (7.4) is false.

Another such example of this is

For every real number x, there exists an integer n such that $|x - n| < 1$. (7.5)

This statement can also be expressed as

If x is a real number, then there exists an integer n such that $|x - n| < 1$.

In order to state (7.5) in symbols, let

$$P(x,n): \ |x - n| < 1$$

where the domain of the variable x is $\mathbf{R}$ and the domain of the variable n is $\mathbf{Z}$. Thus (7.5) can be expressed in symbols as

$$\forall x \in \mathbf{R}, \exists n \in \mathbf{Z}, P(x;n).$$

The statement (7.5) is true, as we now verify.

Result 7.3 *For every real number x, there exists an integer n such that $|x - n| < 1$.*

Proof Let x be a real number. If we let $n = \lceil x \rceil$, where, recall, $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x, then $|x - n| = |x - \lceil x \rceil| = \lceil x \rceil - x < 1$. ■

Another example of a quantified statement containing two different quantifiers is

There exists a positive even integer m such that for every positive integer n, $\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{2}$. (7.6)

Let S denote the set of positive even integers and let

$$P(m,n): \ \left|\tfrac{1}{m} - \tfrac{1}{n}\right| \leq \tfrac{1}{2}.$$

where the domain of the variable m is S and the domain of the variable n is $\mathbf{N}$. Thus, (7.6) can be expressed in symbols as

$$\exists m \in S, \forall n \in \mathbf{N}, P(m,n).$$

The truth of the statement (7.6) is now verifed.

Result 7.4 *There exists a positive even integer m such that for every positive integer n,*

$$\left|\frac{1}{m}-\frac{1}{n}\right| \le \frac{1}{2}.$$

Proof Consider $m = 2$. Let n be a positive integer. We consider three cases.

Case 1. $n = 1$. Then $\left|\frac{1}{m}-\frac{1}{n}\right| = \left|\frac{1}{2}-\frac{1}{1}\right| = \frac{1}{2}$.

Case 2. $n = 2$. Then $\left|\frac{1}{m}-\frac{1}{n}\right| = \left|\frac{1}{2}-\frac{1}{2}\right| = 0 < \frac{1}{2}$.

Case 3. $n \ge 3$. Then $\left|\frac{1}{m}-\frac{1}{n}\right| = \left|\frac{1}{2}-\frac{1}{n}\right| = \frac{1}{2}-\frac{1}{n} < \frac{1}{2}$.

Thus $\left|\frac{1}{2}-\frac{1}{n}\right| \le \frac{1}{2}$ for every $n \in \mathbf{N}$. ∎

Let $P(s, t)$ be an open sentence, where the domain of the variable s is S and the domain of the variable t is T. The negation of the quantified statement $\forall s \in S, \exists t \in T, P(s, t)$ is

$$\begin{aligned}\sim(\forall s \in S, \exists t \in T, P(s,t)) &\equiv \exists s \in S, \sim(\exists t \in T, P(s,t))\\ &\equiv \exists s \in S, \forall t \in T, \sim P(s,t);\end{aligned}$$

while the negation of the quantified statement $\exists s \in S, \forall t \in T, P(s, t)$ is

$$\begin{aligned}\sim(\exists s \in S, \forall t \in T, P(s,t)) &\equiv \forall s \in S, \sim(\forall t \in T, P(s,t))\\ &\equiv \forall s \in S, \exists t \in T, \sim P(s,t).\end{aligned}$$

Consequently, the negation of the statement (7.5) is

There exists a real number x such that for every integer n, $|x - n| \ge 1$.

This statement is therefore false. The negation of the statement (7.6) is

For every positive even integer m, there exists a positive integer n such that $\left|\frac{1}{m}-\frac{1}{n}\right| > \frac{1}{2}$.

This too is false.

Let's consider the following statement, which has more than two quantifiers.

For every positive real number e, there exists a positive real number d such that for every real number x, $|x| < d$ implies that $|2x| < e$. (7.7)

If we let

$$P(x, d):\ |x| < d \ \text{ and } \ Q(x, e):\ |2x| < e$$

where the domain of the variables e and d is $\mathbf{R}^+$ and the domain of the variable x is $\mathbf{R}$, then (7.7) can be expressed in symbols as

$$\forall e \in \mathbf{R}^+, \exists d \in \mathbf{R}^+, \forall x \in \mathbf{R}, P(x, d) \Rightarrow Q(x, e).$$

The statement (7.7) is in fact true, which we now verify.

Result 7.5 *For every positive real number e, there exists a positive real number d such that if x is a real number with $|x| < d$, then $|2x| < e$.*

Proof Let e be a positive real number. Now choose $d = e/2$. Let x be a real number with $|x| < d = e/2$. Then

$$|2x| = 2|x| < 2\left(\frac{e}{2}\right) = e,$$

as desired. ∎

7.3 Testing Statements

We now turn our attention to the main topic of this chapter. For a given statement whose truth value is not provided to us, our task is to determine the truth or falseness of the statement and, in addition, show that our conclusion is correct by proving or disproving the statement, as appropriate.

Example 7.6 *Prove or disprove: There is a real number solution of the equation*

$$x^6 + 2x^2 + 1 = 0.$$

Strategy Observe that x^6 and x^2 are even powers of x. Thus if x is any real number, then $x^6 \ge 0$ and $x^2 \ge 0$, so $2x^2 \ge 0$. Adding 1 to $x^6 + 2x^2$ shows that $x^6 + 2x^2 + 1 \ge 1$. Hence it is impossible for $x^6 + 2x^2 + 1$ to be 0. These thoughts lead us to our solution. We begin by informing the reader that the statement is false, so the reader knows what we will be trying to do. ♦

Solution of Example 7.6 The statement is false. Let $x \in \mathbf{R}$. Since $x^6 \ge 0$ and $x^2 \ge 0$, it follows that $x^6 + 2x^2 + 1 \ge 1$ and so $x^6 + 2x^2 + 1 \ne 0$. ♦

For the preceding example, we wrote "Strategy" rather than "Proof Strategy" for two reasons: (1) Since the statement may be false, there may be no proof in this case. (2) We are essentially "thinking out loud", trying to convince ourselves whether the statement is true or false. Of course, if the statement turns out to be true, then our strategy may very well turn into a proof strategy.

Example 7.7 *Prove or disprove: Let $x, y, z \in \mathbf{Z}$. Then two of the integers x, y, and z are of the same parity.*

Strategy For any three given integers, either two are even or two are odd. So it certainly seems as if the statement is true. The only question appears to be whether what we said in the preceding sentence is convincing enough to all readers. We try another approach. ♦

Solution The statement is true.

Proof Consider x and y. If x and y are of the same parity, then the proof is complete. Thus we may assume that x and y are of opposite parity, say x is even and y is odd. If z is even, then x and z are of the same parity; while if z is odd, then y and z are of the same parity. ∎

Of course, the preceding proof could have been done by cases as well.

Example 7.8 *Prove or disprove: Let A, B, and C be sets. If $A \times C = B \times C$, then $A = B$.*

Strategy The elements of the set $A \times C$ are ordered pairs of elements, namely, they are of the form (x, y), where $x \in A$ and $y \in C$. Let $(x, y) \in A \times C$. If $A \times C = B \times C$, then it follows that (x, y) must be an element of $B \times C$ as well. This says that $x \in B$ and $y \in C$. Conversely, if $(x, y) \in B \times C$, then $(x, y) \in A \times C$, which implies that $x \in A$ as well. These observations certainly seem to suggest that it should be possible to show that $A = B$ under these conditions. However, this argument depends on $A \times C$ containing an element (x, y). Could it happen that $A \times C$ contains no elements? If A or C is empty, this would happen. However, if $C \neq \emptyset$ and $A \times C = \emptyset$, then A must be empty. But $B \times C = A \times C = \emptyset$ would mean that B must also be empty and so $A = B$. This suggests a different response. ♦

Solution of Example 7.8 The statement is false. Let $A = \{1\}$, $B = \{2\}$, and $C = \emptyset$. Then $A \times C = B \times C = \emptyset$, but $A \neq B$. Hence these sets A, B, and C form a counterexample. ♦

In some instances, we might consider modifying a false statement so that the revised statement is true. Our preceding discussion seems to suggest that if the set C were required to be nonempty, then the statement would have been true.

Result 7.9 *Let A, B, and C be sets, where $C \neq \emptyset$. If $A \times C = B \times C$, then $A = B$.*

Proof Assume that $A \times C = B \times C$. Since $C \neq \emptyset$, the set C contains some element c. Let $x \in A$. Then $(x, c) \in A \times C$. Since $A \times C = B \times C$, it follows that $(x, c) \in B \times C$. Hence $x \in B$ and so $A \subseteq B$. By a similar argument, it follows that $B \subseteq A$. Thus $A = B$. ■

Example 7.10 *Prove or disprove: There exists a real number x such that $x^3 < x < x^2$.*

Strategy If there is a real number x such that $x^3 < x < x^2$, then this number is certainly not 0. Consequently, any real number x with this property is either positive or negative. If $x > 0$, then we can divide $x^3 < x < x^2$ by x, obtaining $x^2 < 1 < x$. However, if $x > 1$, then $x^2 > 1$. Therefore, there is no positive real number x for which $x^3 < x < x^2$. Hence any real number x satisfying $x^3 < x < x^2$ must be negative. Dividing $x^3 < x < x^2$ by x gives us $x^2 > 1 > x$ or $x < 1 < x^2$. Experimenting with some negative numbers tells us that any number less than -1 has the desired property. ♦

Solution of Example 7.10 The statement is true.

Proof Consider $x = -2$. Then $x^3 = -8$ and $x^2 = 4$. Thus $x^3 < x < x^2$. ■

Example 7.11 *Prove or disprove: For every positive irrational number b, there exists an irrational number a such that $0 < a < b$.*

Strategy We begin with a positive irrational number b. If this statement is true, then we must show that there is an irrational number a such that $0 < a < b$. If we let $a = b/2$, then certainly $0 < a < b$. The only question is whether $b/2$ is necessarily irrational. We have seen, however, that $b/2$ is irrational (in Exercise 5.13). ♦

Solution of Example 7.11 The statement is true.

Proof Let b be a positive irrational number. Now let $a = b/2$. Then $0 < a < b$ and a is irrational by Exercise 5.13. ■

Example 7.12 *Prove or disprove: Every even integer is the sum of three distinct even integers.*

Strategy This statement can be reworded in a variety of ways. One rewording of this statement is: If n is an even integer, then there exist three distinct even integers a, b, and c such that $n = a + b + c$. What this statement does *not* say is that the sum of three distinct even integers is even; that is, we do not begin with three distinct even integers and show that their sum is even. We begin with an even integer n and ask whether we can find three distinct even integers a, b, and c such that $n = a + b + c$. This is certainly true for $n = 0$ since $0 = (-2) + 0 + 2$. It is also true for $n = 2$ since $2 = (-2) + 0 + 4$. If $n = 4$, then $4 = (-2) + 2 + 4$. This last example may suggest a proof in general. For every even integer n, we can write $n = 2 + (-2) + n$. Certainly, n, 2, and -2 are even. But are they distinct? They are not distinct if $n = 2$ or $n = -2$. This provides a plan for a proof. ♦

Solution of Example 7.12 The statement is true.

Proof Let n be an even integer. We show that n is the sum of three distinct even integers by considering the following three cases.

Case 1. $n = 2$. Observe that $2 = (-2) + 0 + 4$.

Case 2. $n = -2$. Observe that $-2 = (-4) + 0 + 2$.

Case 3. $n \neq 2, -2$. Then $n = 2 + (-2) + n$. ■

Example 7.13 *Prove or disprove: Let $k \in \mathbf{N}$. If $k^2 + 5k$ is odd, then $(k + 1)^2 + 5(k + 1)$ is odd.*

Strategy One idea that might occur to us is to assume that $k^2 + 5k$ is an odd integer, where $k \in \mathbf{N}$, and see if we can show that $(k + 1)^2 + 5(k + 1)$ is also odd. If $k^2 + 5k$ is odd, then we can write $k^2 + 5k = 2\ell + 1$ for some integer ℓ. Then

$$\begin{aligned}(k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 = (k^2 + 5k) + (2k + 6)\\ &= (2\ell + 1) + (2k + 6) = (2\ell + 2k + 6) + 1\\ &= 2(\ell + k + 3) + 1,\end{aligned}$$

which is an odd integer and we have a proof. ♦

Solution of Example 7.13 The statement is true.

Proof Assume that $k^2 + 5k$ is an odd integer, where $k \in \mathbf{N}$. Then $k^2 + 5k = 2\ell + 1$ for some integer ℓ. Hence

$$\begin{aligned}(k+1)^2 + 5(k+1) &= k^2 + 2k + 1 + 5k + 5 = (k^2 + 5k) + (2k + 6)\\ &= (2\ell + 1) + (2k + 6) = (2\ell + 2k + 6) + 1\\ &= 2(\ell + k + 3) + 1.\end{aligned}$$

Since $\ell + k + 3$ is an integer, $(k + 1)^2 + 5(k + 1)$ is an odd integer. ■

Example 7.14 *Prove or disprove: For every positive integer n, $n^2 + 5n$ is an odd integer.*

Strategy It seems like the reasonable thing to do is to investigate $n^2 + 5n$ for a few values of n. For $n = 1$, we have $n^2 + 5n = 1 + 5 \cdot 1 = 6$. We have already solved the problem! For $n = 1$, $n^2 + 5n$ is not an odd integer. We have discovered a counterexample. ♦

Solution of Example 7.14 The statement is false. For $n = 1$, $n^2 + 5n = 1 + 5 \cdot 1 = 6$, which is even. Thus $n = 1$ is a counterexample. ♦

Looking at Examples 7.13 and 7.14 again, we might be wondering what exactly is going on. Certainly, these two examples seem to be related. Perhaps this thought may occur to us. For each positive integer n, let

$$P(n): \text{ The integer } n^2 + 5n \text{ is odd.}$$

and consider the (quantified) statement

$$\text{For every positive integer } n,\ n^2 + 5n \text{ is odd.}$$

or in symbols,

$$\forall n \in \mathbf{N},\ P(n). \tag{7.8}$$

We might ask whether (7.8) is true. Because of the domain, a proof by induction seems appropriate. In fact, the statement in Example 7.13 is the inductive step in an induction proof of (7.8). By Example 7.13, the inductive step is true. On the other hand, the statement (7.8) is false, as $n = 1$ is a counterexample. This emphasizes the importance of verifying both the basis step and the inductive step in an induction proof. Returning to Example 7.13 once again, we can show (using a proof by cases) that $k^2 + 5k$ is even for every $k \in \mathbf{N}$, which would provide a vacuous proof of the statement in Example 7.13.

In this chapter we have discussed analyzing statements, particularly understanding statements, determining whether they are true or false, and proving or disproving them. All of the statements that we have analyzed were provided to us. But how do we obtain statements to analyze for ourselves? This is an important question and concerns the creative aspect of mathematics – how new mathematics is discovered. Obviously, there is no rule or formula for creativity, but creating new statements often comes from studying old statements.

Let's illustrate how we might create statements to analyze. In Exercise 4.6 in Chapter 4, you were asked to prove the following:

$$\text{Let } a \in \mathbf{Z}. \text{ If } 3 \mid 2a, \text{ then } 3 \mid a. \tag{7.9}$$

What other statements does this suggest? For example, is its converse true? (The answer is yes, but the converse is not very interesting.) Is (7.9) true if 3 and 2 are interchanged? What integers can we replace 2 by in (7.9) and obtain a true statement? That is, for which positive integers k is it true that if $3 \mid ka$, then $3 \mid a$? Of course, this is true for $k = 1$. And we know that it's true for $k = 2$. It is not true for $k = 3$; that is, it is not true that if $3 \mid 3a$, then $3 \mid a$. The integer $a = 1$ is a counterexample. On the other hand, it is possible to prove that if $3 \mid 4a$, where $a \in \mathbf{Z}$, then $3 \mid a$. What we are attempting to do is to **extend** the result in (7.9) so that we have a result of the type:

$$\text{Let } a \in \mathbf{Z}. \text{ If } 3 \mid ka, \text{ then } 3 \mid a. \tag{7.10}$$

for a fixed integer k greater than 2. We would like to find a set S of positive integers such that the following is true:

$$\text{Let } a \in \mathbf{Z}. \text{ If } 3 \mid ka, \text{ where } k \in S, \text{ then } 3 \mid a. \tag{7.11}$$

Surely $2 \in S$. Result (7.9) then becomes a special case and a corollary of (7.11). For this reason (7.11) is called a **generalization** of (7.9). Ideally, we would like S to have the added property that (7.11) is true if $k \in S$, while (7.11) is false if $k \notin S$. We are thus seeking a set S of integers such that the following is true:

$$\text{Let } a \in \mathbf{Z}. \text{ Then } 3 \mid ka \text{ implies that } 3 \mid a \text{ if and only if } k \in S.$$

If we were successful in finding this set S, then we might start all over again by replacing 3 in (7.9) by some other positive integer.

In mathematics it is often the case that a new result is obtained by looking at an old result in a new way and extending it to obtain a generalization of the old result. Hence it frequently happens that: *Today's theorem becomes tomorrow's corollary.*

7.4 A Quiz of "Prove or Disprove" Problems

We conclude this chapter with a quiz. Solutions are given following the quiz.

Quiz

Prove or disprove each of the following statements.

1. If n is a positive integer and s is an irrational number, then n/s is an irrational number.
2. For every integer b, there exists a positive integer a such that $|a - |b|| \leq 1$.
3. If x and y are integers of the same parity, then xy and $(x + y)^2$ are of the same parity.
4. Let $a, b \in \mathbf{Z}$. If $6 \nmid ab$, then either (1) $2 \nmid a$ and $3 \nmid b$ or (2) $3 \nmid a$ and $2 \nmid b$.
5. For every positive integer n, $2^{2^n} \geq 4^{n!}$.
6. If A, B, and C are sets, then $(A - B) \cup (A - C) = A - (B \cup C)$.
7. Let $n \in \mathbf{N}$. If $(n + 1)(n + 4)$ is odd, then $(n + 1)(n + 4) + 3^n$ is odd.
8. (a) There exist distinct rational numbers a and b such that $(a - 1)(b - 1) = 1$.
   (b) There exist distinct rational numbers a and b such that $\frac{1}{a} + \frac{1}{b} = 1$.

9. Let $a, b, c \in \mathbf{Z}$. If every two of a, b, and c are of the same parity, then $a + b + c$ is even.
10. If n is a nonnegative integer, then 5 divides $2 \cdot 4^n + 3 \cdot 9^n$.

Solutions for Quiz

1. The statement is true.

   Proof Assume, to the contrary, that there exist a positive integer n and an irrational number s such that n/s is a rational number. Then $n/s = a/b$, where $a, b \in \mathbf{Z}$ and $a, b \neq 0$. Therefore, $s = nb/a$, where $nb, a \in \mathbf{Z}$ and $a \neq 0$. Thus s is rational, producing a contradiction. ■

2. The statement is true.

   Proof Let $b \in \mathbf{Z}$. Now let $a = |b| + 1$. Thus $a \in \mathbf{N}$ and $|a - |b|| = |(|b| + 1) - |b|| = 1$. ■

3. The statement is false. Observe that $x = 1$ and $y = 3$ are of the same parity. Then $xy = 3$ and $(x + y)^2 = 16$ are of opposite parity. Hence $x = 1$ and $y = 3$ produce a counterexample. ♦

4. The statement is false. Let $a = b = 2$. So $ab = 4$. Hence $6 \nmid ab$. Since $2 \mid a$ and $2 \mid b$, both (1) and (2) are false. Thus $a = b = 2$ produces a counterexample. ♦

5. The statement is false. For $n = 3$, $2^{2^n} = 2^8 = 256$ while $4^{n!} = 4^{3!} = 4^6 = 4096$. Thus $2^{2^3} < 4^{3!}$ and so $n = 3$ is a counterexample. ♦

6. The statement is false. Let $A = \{1, 2, 3\}$, $B = \{2\}$, and $C = \{3\}$. Thus $B \cup C = \{2, 3\}$. Hence $A - B = \{1, 3\}$, $A - C = \{1, 2\}$, and $A - (B \cup C) = \{1\}$. Therefore, $(A - B) \cup (A - C) = \{1, 2, 3\} \neq A - (B \cup C)$. So $A = \{1, 2, 3\}$, $B = \{2\}$, and $C = \{3\}$ constitute a counterexample. ♦

7. The statement is true.

   Proof Let $n \in \mathbf{N}$ and consider $(n + 1)(n + 4)$. We show that $(n + 1)(n + 4)$ is even, thereby giving a vacuous proof.

   There are two cases.

   Case 1. *n is even.* Then $n = 2k$ for some integer k. Thus

   $$(n + 1)(n + 4) = (2k + 1)(2k + 4) = 2(2k + 1)(k + 2).$$

   Since $(2k + 1)(k + 2) \in \mathbf{Z}$, it follows that $(n + 1)(n + 4)$ is even.

   Case 2. *n is odd.* Then $n = 2\ell + 1$ for some integer ℓ. Thus

   $$(n + 1)(n + 4) = (2\ell + 2)(2\ell + 5) = 2(\ell + 1)(2\ell + 5).$$

   Since $(\ell + 1)(2\ell + 5) \in \mathbf{Z}$, it follows that $(n + 1)(n + 4)$ is even. ■

8. (a) The statement is true.

   Proof Let $a = 3$ and $b = \frac{3}{2}$. Then $(a - 1)(b - 1) = 2\left(\frac{1}{2}\right) = 1$. ■

   (b) The statement is true.

   Proof Let $a = \frac{1}{2}$ and $b = -1$. Then

   $$\frac{1}{a} + \frac{1}{b} = \frac{1}{\frac{1}{2}} + \frac{1}{-1} = 2 - 1 = 1.$$ ■

   Proof Analysis Observe that if a and b are two (distinct) rational numbers that satisfy $\frac{1}{a} + \frac{1}{b} = 1$, then $\frac{a+b}{ab} = 1$ and so $a + b = ab$. Thus $ab - a - b = 0$, which is equivalent to $ab - a - b + 1 = 1$ and so $(a - 1)(b - 1) = 1$. Therefore, two distinct rational numbers a and b satisfy

   $$(a - 1)(b - 1) = 1$$

   if and only if a and b satisfy

   $$\frac{1}{a} + \frac{1}{b} = 1$$

   if and only if a and b satisfy

   $$a + b = ab.$$ ♦

9. The statement is false. Let $a = 1$, $b = 3$, and $c = 5$. Then every two of a, b, and c are of the same parity; yet $a + b + c$ is odd. Hence $a = 1$, $b = 3$, and $c = 5$ produce a counterexample. ♦

10. The statement is true.

    Proof We proceed by induction. For $n = 0$, $2 \cdot 4^n + 3 \cdot 9^n = 2 \cdot 1 + 3 \cdot 1 = 5$. Thus $5 \mid (2 \cdot 4^0 + 3 \cdot 9^0)$ and the statement is true for $n = 0$.

    Assume that $5 \mid (2 \cdot 4^k + 3 \cdot 9^k)$ for a nonnegative integer k. We show that $5 \mid (2 \cdot 4^{k+1} + 3 \cdot 9^{k+1})$. Since $5 \mid (2 \cdot 4^k + 3 \cdot 9^k)$, it follows that $2 \cdot 4^k + 3 \cdot 9^k = 5x$ for some integer x. Thus $2 \cdot 4^k = 5x - 3 \cdot 9^k$. Hence

    $$\begin{aligned} 2 \cdot 4^{k+1} + 3 \cdot 9^{k+1} &= 4(2 \cdot 4^k) + 3 \cdot 9^{k+1} \\ &= 4(5x - 3 \cdot 9^k) + 3 \cdot 9^{k+1} \\ &= 20x - 12 \cdot 9^k + 27 \cdot 9^k \\ &= 20x + 15 \cdot 9^k = 5(4x + 3 \cdot 9^k). \end{aligned}$$

    Since $4x + 3 \cdot 9^k \in \mathbf{Z}$, it follows that $5 \mid (2 \cdot 4^{k+1} + 3 \cdot 9^{k+1})$. By the Principle of Mathematical Induction, 5 divides $2 \cdot 4^n + 3 \cdot 9^n$ for every nonnegative integer n. ■

EXERCISES FOR CHAPTER 7

Section 7.2: Revisiting Quantified Statements

7.1. (a) Express the following quantified statement in symbols:
For every odd integer n, the integer $3n + 1$ is even.
(b) Prove that the statement in (a) is true.

7.2. (a) Express the following quantified statement in symbols:
There exists a positive even integer n such that $3n + 2^{n-2}$ is odd.
(b) Prove that the statement in (a) is true.

7.3. (a) Express the following quantified statement in symbols:
For every positive integer n, the integer n^{n-1} is even.
(b) Show that the statement in (a) is false.

7.4. (a) Express the following quantified statement in symbols:
There exists an integer n such that $3n^2 - 5n + 1$ is an even integer.
(b) Show that the statement in (a) is false.

7.5. (a) Express the following quantified statement in symbols:
For every integer $n \geq 2$, there exists an integer m such that $n < m < 2n$.
(b) Prove that the statement in (a) is true.

7.6. (a) Express the following quantified statement in symbols:
There exists an integer n such that $m(n-3) < 1$ for every integer m.
(b) Prove that the statement in (a) is true.

7.7. (a) Express the following quantified statement in symbols:
For every integer n, there exists an integer m such that $(n-2)(m-2) > 0$.
(b) Express in symbols the negation of the statement in (a).
(c) Show that the statement in (a) is false.

7.8. (a) Express the following quantified statement in symbols:
There exists a positive integer n such that $-nm < 0$ for every integer m.
(b) Express in symbols the negation of the statement in (a).
(c) Show that the statement in (a) is false.

7.9. (a) Express the following quantified statement in symbols:
For every positive integer a, there exists an integer b with $|b| < a$ such that $|bx| < a$ for every real number x.
(b) Prove that the statement in (a) is true.

7.10. (a) Express the following quantified statement in symbols:
For every real number x, there exist integers a and b such that $a \leq x \leq b$ and $b - a = 1$.
(b) Prove that the statement in (a) is true.

7.11. (a) Express the following quantified statement in symbols:
There exists an integer n such that for two real numbers x and y, $x^2 + y^2 \geq n$.
(b) Prove that the statement in (a) is true.

7.12. (a) Express the following quantified statement in symbols:
For every even integer a and odd integer b, there exists a rational number c such that either $a < c < b$ or $b < c < a$.
(b) Prove that the statement in (a) is true.

7.13. (a) Express the following quantified statement in symbols:
There exist two integers a and b such that for every positive integer n, $a < \frac{1}{n} < b$.
(b) Prove that the statement in (a) is true.

7.14. (a) Express the following quantified statement in symbols:
There exist odd integers a, b, and c such that $a + b + c = 1$.
(b) Prove that the statement in (a) is true.

7.15. (a) Express the following quantified statement in symbols:
For every three odd integers a, b, and c, their product abc is odd.
(b) Prove that the statement in (a) is true.

7.16. Consider the following statement.

R : There exists a real number L such that for every positive real number e, there exists a positive real number d such that if x is a real number with $|x| < d$, then $|3x - L| < e$.

(a) Use $P(x, d)$: $|x| < d$ and $Q(x, L, e)$: $|3x - L| < e$ to express the statement R in symbols.
(b) Prove that the statement R is true.

Section 7.3: Testing Statements

7.17. For the set $S = \{1, 2, 3, 4\}$, let

$$P(n) : 2^{n+1} + (-1)^{n+1}\left(2^n + 2^{n-1}\right) \text{ is prime. and } Q(n) : 2n + 3 \text{ is prime.}$$

Prove or disprove: $\forall n \in S$, $P(n) \Rightarrow Q(n)$.

7.18. Let $P(n) : n^2 + 3n + 1$ is even. Prove or disprove:
(a) $\forall k \in \mathbf{N}$, $P(k) \Rightarrow P(k+1)$.
(b) $\forall n \in \mathbf{N}$, $P(n)$.

For Exercises 7.19–7.67, the directions are: Prove or disprove.

7.19. Let $x \in \mathbf{Z}$. If $4x + 7$ is odd, then x is even.

7.20. For every nonnegative integer n, there exists a nonnegative integer k such that $k < n$.

7.21. Every even integer is the sum of two odd integers.

7.22. If $x, y, z \in \mathbf{Z}$ such that $x + y + z = 101$, then two of the integers x, y, and z are of opposite parity.

7.23. For every two sets A and B, $(A \cup B) - B = A$.

7.24. Let A be a set. If $A \cap B = \emptyset$ for every set B, then $A = \emptyset$.

7.25. There exists an odd integer, the sum of whose digits is even, and the product of whose digits is odd.

7.26. For every nonempty set A, there exists a set B such that $A \cup B = \emptyset$.

7.27. If x and y are real numbers, then $|x + y| = |x| + |y|$.

7.28. Let S be a nonempty set. For every proper subset A of S, there exists a nonempty subset B of S such that $A \cup B = S$ and $A \cap B = \emptyset$.

7.29. There exists a real number x such that $x^2 < x < x^3$.

7.30. There exists an integer a such that $a \cdot c \geq 0$ for every integer c.

7.31. There exist real numbers a, b, and c such that $\frac{a+b}{a+c} = \frac{b}{c}$.

7.32. The equation $x^3 + x^2 - 1 = 0$ has a real number solution between $x = 0$ and $x = 1$.

7.33. There is a real number solution of the equation $x^4 + x^2 + 1 = 0$.

7.34. If $x, y \in \mathbf{R}$ and $x^2 < y^2$, then $x < y$.

7.35. If $x \in \mathbf{Z}$, then $\frac{x^3+x}{x^4-1} = \frac{x}{x^2-1}$.

7.36. Let A and B be sets. If $A - B = B - A$, then $A - B = \emptyset$.

7.37. Let $x, y, z \in \mathbf{Z}$. If $z = x - y$ and z is even, then x and y are odd.

7.38. For every positive rational number b, there exists an irrational number a with $0 < a < b$.

7.39. Let A be a set. If $A - B = \emptyset$ for every set B, then $A = \emptyset$.

7.40. Every odd integer is the sum of three odd integers.

7.41. For every nonempty set A, there exists a set B such that $|A - B| = |B - A|$.

7.42. Let A, B, and C be sets. If $A \cap B = A \cap C$, then $B = C$.

7.43. Let $x, y, z \in \mathbf{Z}$. If $z = x + y$ and x is odd, then y is even and z is odd.

7.44. For every two rational numbers a and b with $a < b$, there exists a rational number r such that $a < r < b$.

7.45. For every rational number a/b, where $a, b \in \mathbf{N}$, there exists a rational number c/d, where c and d are positive odd integers, such that

$$0 < \frac{c}{d} < \frac{a}{b}.$$

7.46. Let A be a set. If $A \cup B \neq \emptyset$ for every set B, then $A \neq \emptyset$.

7.47. Every even integer is the sum of two even integers.

7.48. There exists a real number solution of the equation $x^2 + x + 1 = 0$.

7.49. Let A, B, C, and D be sets with $A \subseteq C$ and $B \subseteq D$. If A and B are disjoint, then C and D are disjoint.

7.50. Every nonzero rational number is the product of two irrational numbers.

7.51. There exist an irrational number a and a rational number b such that a^b is irrational.

7.52. For every odd integer a, there exist integers b and c of opposite parity such that $a + b = c$.

7.53. Let S be a set containing at least two elements. For every proper nonempty subset A of S, there exists a proper nonempty subset B of S such that A and B are disjoint.

7.54. Let A and B be sets. If $A \cup B \neq \emptyset$, then both A and B are nonempty.

7.55. For every two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

7.56. Let S be a nonempty set and let T be a collection of subsets of S. If $A \cap B \neq \emptyset$ for all pairs A, B of elements of T, then there exists an element $x \in S$ such that $x \in C$ for all $C \in T$.

7.57. Let A, B, and C be sets. Then $A \cup (B - C) = (A \cup B) - (A \cup C)$.

7.58. Let $a, b, c \in \mathbf{Z}$. If ab, ac, and bc are even, then a, b, and c are even.

7.59. Let $a, b, c \in \mathbf{Z}$. Then at least one of the numbers $a + b$, $a + c$, and $b + c$ is even.

7.60. For every two integers a and c, there exists an integer b such that $a + b = c$.

7.61. For every two *positive* integers a and c, there exists a *positive* integer b such that $a + b = c$.

7.62. There exist three distinct integers a, b, and c such that $a^b = b^c$.

7.63. Let $n \in \mathbf{Z}$. If $n^3 + n$ is even, then n is even.

7.64. Every integer can be expressed as the sum of two unequal integers.

7.65. There exist positive integers x and y such that $x^2 - y^2 = 101$.

7.66. For every positive integer n, $n^2 - n + 11$ is a prime.

7.67. For every odd prime p, there exist positive integers a and b such that $a^2 - b^2 = p$.

ADDITIONAL EXERCISES FOR CHAPTER 7

7.68. (a) Show that the following statement is false: For every natural number x, there exists a natural number y such that $x < y < x^2$.

(b) Make a small addition to the statement in (a) so that the new statement is true. Prove the new statement.

7.69. (a) Show that the following statement is false: Every positive integer is the sum of two distinct positive odd integers.

(b) Make a small addition to the statement in (a) so that the new statement is true. Prove the new statement.

7.70. (a) Prove or disprove: There exist two distinct positive integers whose sum exceeds their product.

(b) Your solution to (a) should suggest another problem to you. State and solve this new problem.

7.71. (a) Prove or disprove: If a and b are positive integers, then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

(b) Prove or disprove: There exist positive real numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

(c) Complete the following statement so that it's true and provide a proof:
Let $a, b \in \mathbf{R}^+ \cup \{0\}$. Then $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ if and only if _____.

7.72. In Exercise 4.6, you were asked to prove the statement

$$P: \text{ Let } a \in \mathbf{Z}. \text{ If } 3 \mid 2a, \text{ then } 3 \mid a.$$

(a) Prove that the converse of P is true. Now state P and its converse in a more familiar manner.

(b) Is the statement obtained by interchanging 2 and 3 in P true?

(c) Find a set S of positive integers with $2 \in S$ and $|S| \geq 3$ such that the following is true:

Let $a \in \mathbf{Z}$. If $3 \mid ka$, where $k \in S$, then $3 \mid a$.

Prove this generalization of the statement P.

(d) In Exercise 4.50, it was shown for integers a and b that $3 \mid ab$ if and only if $3 \mid a$ or $3 \mid b$. How can this be used to answer (c)?

7.73. In Exercise 5.17 in Chapter 5, you were asked to prove that $\sqrt{2} + \sqrt{3}$ is irrational.

(a) Prove that $\sqrt{2} + \sqrt{5}$ is irrational.

(b) Determine, with proof, another positive integer a such that $\sqrt{2} + \sqrt{a}$ is irrational.

(c) State, and prove, a generalization of the result in (a).

7.74. In Exercise 3.20, you were asked to prove the statement:

$$P: \text{ If } n \in \mathbf{Z}, \text{ then } n^3 - n \text{ is even.}$$

This can be restated as:

$$P: \text{ If } n \in \mathbf{Z}, \text{ then } 2 \mid (n^3 - n).$$

(a) Find a positive integer $a \neq 2$ such that

$$\text{If } n \in \mathbf{Z}, \text{ then } a \mid (n^3 - n).$$

is true and prove this statement.

(b) Find a positive integer $k \neq 3$ such that

$$\text{If } n \in \mathbf{Z}, \text{ then } 2 \mid (n^k - n).$$

is true and prove this statement.

(c) Ask a question of your own dealing with P and provide an answer.

7.75. Let A denote the set of odd integers. Investigate the truth (or falseness) of the following statements.

(a) For all $x, y \in A$, $2 \mid (x^2 + 3y^2)$.

(b) There exist $x, y \in A$ such that $4 \mid (x^2 + 3y^2)$.

(c) For all $x, y \in A$, $4 \mid (x^2 + 3y^2)$.

(d) There exist $x, y \in A$ such that $8 \mid (x^2 + 3y^2)$.

(e) There exist $x, y \in A$ such that $6 \mid (x^2 + 3y^2)$.

(f) Provide a related statement of your own, and determine whether it is true or false.

7.76. Evaluate the proof of the following statement.

Result Every even integer can be expressed as the sum of three distinct even integers.

Proof Let n be an even integer. Since $n + 2$, $n - 2$, and $-n$ are distinct even integers and

$$n = (n + 2) + (n - 2) + (-n),$$

the desired result follows. ∎

7.77. (a) Prove or disprove the following: There exist four positive integers a, b, c, and d such that $a^2 + b^2 + c^2 = d^2$.

(b) Prove or disprove the following: There exist four *distinct* positive integers a, b, c, and d such that $a^2 + b^2 + c^2 = d^2$.

(c) The problems in (a) or (b) above should suggest another problem that you can solve. State and solve such a problem.

(d) The problems in (a) or (b) above should suggest a conjecture to you (that you probably cannot solve). State such a conjecture.

7.78. It is known (although challenging to prove) that for every nonnegative integer m, the integer $8m + 3$ can be expressed as $a^2 + b^2 + c^2$ for positive integers a, b, and c.

(a) For every integer m with $0 \leq m \leq 10$, find positive integers a, b, and c such that $8m + 3 = a^2 + b^2 + c^2$.

(b) Prove or disprove: If a, b, and c are positive integers such that $a^2 + b^2 + c^2 = 8m + 3$ for some integer m, then all of a, b, and c are odd.

8

Equivalence Relations

There are many common examples of relations in mathematics. For instance, three different ways that a real number x can be related to a real number y are:

$$(1)\ x < y,\ (2)\ y = x^2 + 1,\ \text{or}\ (3)\ x = y.$$

Three different ways that an integer a can be related to an integer b are:

$$(1)\ a \mid b,\ (2)\ a \text{ and } b \text{ are of opposite parity, or } (3)\ a \equiv b \pmod 3.$$

In the area of geometry, three different ways that a line ℓ in 3-space can be related to a plane Π in 3-space are:

(1) ℓ lies on Π, (2) ℓ is parallel to Π, or (3) ℓ intersects Π in exactly one point.

Three different ways that a triangle T can be related to a triangle T' are:

(1) T is congruent to T', (2) T is similar to T', or (3) T has the same area as T'.

All of the preceding examples concern two sets A and B (possibly $A \neq B$), where elements of A are related to elements of B in some manner. We now study this idea in a more general setting.

8.1 Relations

Let A and B be two sets. By a **relation R from A to B** we mean a subset of $A \times B$. That is, R is a set of ordered pairs, where the first coordinate of the pair belongs to A and the second coordinate belongs to B. If $(a, b) \in R$, then we say that a is **related** to b by R and write $a\ R\ b$. If $(a, b) \notin R$, then a is *not* related to b by R and we write $a \not R\ b$. For the sets $A = \{x, y, z\}$ and $B = \{1, 2\}$, the set

$$R = \{(x, 2), (y, 1), (y, 2)\} \tag{8.1}$$

is a subset of $A \times B$ and is therefore a relation from A to B. Thus, $x\ R\ 2$ (x is related to 2) and $x \not R\ 1$ (x is not related to 1). For two given sets A and B, it is always the case that $\emptyset$ and $A \times B$ are subsets of $A \times B$. Therefore, $\emptyset$ and $A \times B$ are both examples of relations from A to B. (Indeed, these are the extreme examples.) For the relation $\emptyset$, no

element of A is related to any element of B, while for the relation $A \times B$, each element of A is related to every element of B. Simply said then, a relation from a set A to a set B tells us which elements of A are related to which elements of B. Although this may seem like a fairly simple idea, it is very important that we have a thorough understanding of it.

Let R be a relation from A to B. The **domain** of R, denoted by dom R, is the subset of A defined by

$$\text{dom } R = \{a \in A : (a, b) \in R \text{ for some } b \in B\};$$

while the **range** of R, denoted by ran R, is the subset of B defined by

$$\text{ran } R = \{b \in B : (a, b) \in R \text{ for some } a \in A\}.$$

Hence dom R is that set of elements of A that occur as first coordinates among the ordered pairs in R, and ran R is the set of elements of B that occur as second coordinates among the ordered pairs in R. The domain and range of the relation R given in (8.1) are dom $R = \{x, y\}$ and ran $R = \{1, 2\}$. The reason that $z \notin \text{dom } R$ is because there is no ordered pair in R whose first coordinate is z.

By a **relation on a set** A, we mean a relation from A to A. That is, a relation on a single set A is a collection of ordered pairs whose first *and* second coordinates belong to A. Therefore, $\{(1, 2), (1, 3), (2, 2), (2, 3)\}$ is an example of a relation on the set $A = \{1, 2, 3, 4\}$.

If $A = \{1, 2\}$, then

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

Since $|A \times A| = 4$, the number of subsets of $A \times A$ is $2^4 = 16$. However, a relation on A is, by definition, a subset of $A \times A$. Consequently, there are 16 relations on A. Six of these 16 relations are

$$\emptyset, \{(1, 2)\}, \{(1, 1), (1, 2)\}, \{(1, 2), (2, 1)\}, \{(1, 1), (1, 2), (2, 2)\}, A \times A.$$

8.2 Properties of Relations

For a relation defined on a single set, there are three properties that a relation may possess and which will be of special interest to us. A relation R defined on a set A is called **reflexive** if $x \; R \; x$ for every $x \in A$. That is, R is reflexive if $(x, x) \in R$ for every $x \in A$. Let $S = \{a, b, c\}$ and consider the following six relations defined on S:

$$R_1 = \{(a, b), (b, a), (c, a)\}$$
$$R_2 = \{(a, b), (b, b), (b, c), (c, b), (c, c)\}$$
$$R_3 = \{(a, a), (a, c), (b, b), (c, a), (c, c)\}$$
$$R_4 = \{(a, a), (a, b), (b, b), (b, c), (a, c)\}$$
$$R_5 = \{(a, a), (a, b)\}$$
$$R_6 = \{(a, b), (a, c)\}.$$

The relation R_1 is not reflexive since $(a, a) \notin R_1$, for example. Since $(a, a) \notin R_2$, it follows that R_2 is not reflexive either. Because $(a, a), (b, b), (c, c) \in R_3$, the relation R_3 is reflexive. None of the relations R_4, R_5, R_6 is reflexive.

A relation R defined on a set A is called **symmetric** if whenever $x \; R \; y$, then $y \; R \; x$ for all $x, y \in A$. Hence for a relation R on A to be "not symmetric", there must be some ordered pair (w, z) in R for which $(z, w) \notin R$. Certainly, if such an ordered pair (w, z) exists, then $w \neq z$. The relation R_1 is not symmetric since $(c, a) \in R_1$ but $(a, c) \notin R_1$. Notice that $(a, b) \in R_1$ and $(b, a) \in R_1$, but this does not mean that R_1 is symmetric. Recall that the definition of a symmetric relation R on a set A says that whenever $x \; R \; y$, then $y \; R \; x$ *for all* $x, y \in A$.

The relation R_3 *is* symmetric, however, since both (a, c) and (c, a) belong to R_3. None of the ordered pairs $(a, a), (b, b), (c, c)$ in R_3 are relevant as to whether R_3 is symmetric. None of the relations R_2, R_4, R_5, R_6 is symmetric.

A relation R defined on a set A is called **transitive** if whenever $x \; R \; y$ and $y \; R \; z$, then $x \; R \; z$, for all $x, y, z \in A$. Notice that in this definition, it is *not* required that x, y, and z be distinct. Hence for a relation R on A to be "not transitive", there must exist two ordered pairs (u, v) and (v, w) in R such that $(u, w) \notin R$. If this should occur, then necessarily $u \neq v$ and $v \neq w$ (although perhaps $u = w$). For example, the relation R_2 is not transitive since $(a, b), (b, c) \in R_2$, but $(a, c) \notin R_2$. Actually, R_1 is not transitive either because $(a, b), (b, a) \in R_1$ but $(a, a) \notin R_1$. The example (counterexample) that shows that R_1 is not transitive illustrates the fact that showing a relation is not transitive may not be easy. All of the relations R_3, R_4, R_5, R_6 are transitive. It is not always easy to convince oneself that a relation *is* transitive either. Let's give a careful argument as to why the relations R_5 and R_6 are transitive.

For R_5 to be transitive, it is required that (x, z) belongs to R_5 whenever (x, y) and (y, z) belong to R_5 for all $x, y, z \in A$. To verify that the transitive property holds in R_5, we must consider *all* possible pairs of ordered pairs of the type (x, y) and (y, z). We have two choices for (x, y) in R_5, namely, (a, a) and (a, b), that is, $x = a$ and $y = a$, or $x = a$ and $y = b$. If $(x, y) = (a, a)$, then $y = a$ and so either $(y, z) = (a, a)$ or $(y, z) = (a, b)$. In the first case, we have

$$(a, a) \in R_5 \text{ and } (a, a) \in R_5,$$

and $(x, z) = (a, a)$ belongs to R_5. In the second case,

$$(a, a) \in R_5 \text{ and } (a, b) \in R_5,$$

and $(x, z) = (a, b)$ belongs to R_5. This example suggests (correctly!) that if (x, y) and (y, z) belong to some relation R and $x = y$, then certainly $(x, z) \in R$. The same could be said if $y = z$. Thus, when checking transitivity, we need only consider ordered pairs (x, y) and (y, z) for which $x \neq y$ and $y \neq z$. Suppose next that $(x, y) = (a, b)$, so that $y = b$. Here there is no ordered pair of the type (y, z) in R_5; that is, there is no ordered pair of R_5 whose first coordinate is b. Thus, there is nothing to check when $(x, y) = (a, b)$. For R_5, there are only two possibilities for two ordered pairs of the type (x, y), (y, z) and in each case, $(x, z) \in R_5$. Thus R_5 is transitive.

Let's turn to R_6 now. The relation R_6 does not contain two ordered pairs of the type (x, y), (y, z) since if $(x, y) = (a, b)$, no ordered pair has b as its first coordinate; while if $(x, y) = (a, c)$, no ordered pair has c as its first coordinate. Consequently, the hypothesis of the transitive property is false and the implication "If $(x, y) \in R_6$ and $(y, z) \in R_6$, then $(x, z) \in R_6$." is satisfied vacuously. Hence, R_6 is transitive. Another way to convince yourself that R_6 is transitive is to think of what must happen if R_6

is not transitive; namely, there must be two ordered pairs (x, y), (y, z) in R_6 such that $(x, z) \notin R_6$. But there are no such ordered pairs (x, y) and (y, z)!

In the preceding discussions, we have made use of an important point when testing a relation for transitivity. It bears repeating here.

> *When we are attempting to determine whether a relation R is transitive and, consequently, checking all pairs of the type (x, y) and (y, z), we need not consider the situation where $x = y$ or $y = z$.*

In this case, the ordered pair (x, z) will always be present in R. If a relation R is not transitive, then there must exist ordered pairs (x, y) and (y, z) in R, where $x \neq y$ and $y \neq z$, such that (x, z) is not in R. That is, (x, y) and (y, z) constitute a counterexample to the implication "If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$." which is the definition of R being transitive.

We already mentioned that relations occur frequently in mathematics. Let R be the relation defined on the set $\mathbf{Z}$ of integers by $a\ R\ b$ if $a \leq b$; that is, R is the relation $\leq$. Since $x \leq x$ for every integer x, it follows that $x\ R\ x$ for every $x \in \mathbf{Z}$; that is, R is reflexive. Certainly, $2\ R\ 3$ since $2 \leq 3$. However, $3 > 2$; so $3\ \not R\ 2$. Therefore, R is not symmetric. On the other hand, it is a well-known property of integers that if $a \leq b$ and $b \leq c$, then $a \leq c$. Therefore, if $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$. So R is transitive.

Another relation R we could consider on the set $\mathbf{Z}$ is defined by $a\ R\ b$ if $a \neq b$. In this case $1\ \not R\ 1$ since $1 = 1$. Consequently, this relation is not reflexive. If a and b are integers such that $a \neq b$, then we also have $b \neq a$. So if $a\ R\ b$, then $b\ R\ a$. This says that this relation is symmetric. Notice that $2 \neq 3$ and $3 \neq 2$, but $2 = 2$. That is, $2\ R\ 3$ and $3\ R\ 2$, but $2\ \not R\ 2$. Therefore, R is not transitive.

The **distance** between two real numbers a and b is $|a - b|$. So the distance between 2 and 4.5 is $|2 - 4.5| = |-2.5| = 2.5$. Define a relation R on the set $\mathbf{R}$ of real numbers by $a\ R\ b$ if $|a - b| \leq 1$, that is, a is related to b if the distance between a and b is at most 1. Certainly, the distance from a real number to itself is 0, that is, $|a - a| = 0 \leq 1$ for every $a \in \mathbf{R}$. So $a\ R\ a$ and R is reflexive. If the distance between two real numbers a and b is at most 1, then the distance between b and a is at most 1. In symbols, if $|a - b| \leq 1$, then $|b - a| = |a - b| \leq 1$; that is, if $a\ R\ b$, then $b\ R\ a$. Therefore, R is symmetric. Now to the transitive property. If $a\ R\ b$ and $b\ R\ c$, is $a\ R\ c$? That is, if the distance between a and b is at most 1 and the distance between b and c is at most 1, does it follow that the distance between a and c is at most 1? The answer is no. For example, $3\ R\ 2$ and $2\ R\ 1$ since $|3 - 2| \leq 1$ and $|2 - 1| \leq 1$. However, $|3 - 1| = 2$. So $3\ \not R\ 1$ and R is not transitive.

8.3 Equivalence Relations

Perhaps the most familiar relation that we have encountered in mathematics is the equals relation. For example, let R be the relation defined on $\mathbf{Z}$ by $a\ R\ b$ if $a = b$. For every integer a, we have $a = a$ and so $a\ R\ a$. If $a = b$, then $b = a$. Hence if $a\ R\ b$, then $b\ R\ a$. Also, if $a = b$ and $b = c$, then $a = c$. So if $a\ R\ b$ and $b\ R\ c$, then $a\ R\ c$. These observations tell us that the equals relation on the set of integers possesses all three of the properties reflexive, symmetric, and transitive. This suggests the question of asking what other relations (on the set $\mathbf{Z}$ or indeed on any set) have these same three properties

possessed by the equals relation. These are the relations that will be our primary focus in this chapter.

A relation R on a set A is called an **equivalence relation** if R is reflexive, symmetric, and transitive. Of course, then, the equals relation R defined on $\mathbf{Z}$ by $a\ R\ b$ if $a = b$ is an equivalence relation on $\mathbf{Z}$. For another example, consider the set $A = \{1, 2, 3, 4, 5, 6\}$ and the relation

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 3), (1, 6), (6, 1), (6, 3), (3, 1), (3, 6), (2, 4), (4, 2)\} \tag{8.2}$$

defined on A. This relation has all three of the properties reflexive, symmetric, and transitive and is consequently an equivalence relation.

Suppose that R is an equivalence relation on some set A. If $a \in A$, then a is related to a since R is reflexive. Quite possibly, other elements of A are related to a as well. The set of all elements that are related to a given element of A will turn out to be important and, for this reason, these sets are given special names. For an equivalence relation R defined on a set A and for $a \in A$, the set

$$[a] = \{x \in A : x\ R\ a\}$$

consisting of all elements in A that are related to a, is called an **equivalence class**, in fact, the equivalence class containing a since $a \in [a]$ (because R is reflexive). Loosely speaking, then, $[a]$ consists of the "relatives" of a. For the equivalence relation R defined in (8.2), the resulting equivalence classes are

$$[1] = \{1, 3, 6\}, \quad [2] = \{2, 4\}, \quad [3] = \{1, 3, 6\},$$
$$[4] = \{2, 4\}, \quad [5] = \{5\}, \quad [6] = \{1, 3, 6\}. \tag{8.3}$$

Since $[1] = [3] = [6]$ and $[2] = [4]$, there are only three distinct equivalence classes in this case, namely, [1], [2], and [5].

Let's return to the equals relation R defined on $\mathbf{Z}$ by $a\ R\ b$ if $a = b$ and determine the equivalence classes for this equivalence relation. For $a \in \mathbf{Z}$,

$$[a] = \{x \in \mathbf{Z} : x\ R\ a\} = \{x \in \mathbf{Z} : x = a\} = \{a\};$$

that is, every integer is in an equivalence class by itself.

As another illustration, define a relation R on the set L of straight lines in the plane by $\ell_1\ R\ \ell_2$ if either $\ell_1 = \ell_2$ (the lines coincide) or ℓ_1 is parallel to ℓ_2. Since every line coincides with itself, R is reflexive. If a line ℓ_1 is parallel to a line ℓ_2 (or they coincide), then ℓ_2 is parallel to ℓ_1 (or they coincide). Thus R is symmetric. Finally, if ℓ_1 is parallel to ℓ_2 and ℓ_2 is parallel to ℓ_3 (including the possibility that such pairs of lines may coincide), then either ℓ_1 is parallel to ℓ_3 or they coincide. Indeed, it may very well occur that ℓ_1 and ℓ_2 are distinct parallel lines, as are ℓ_2 and ℓ_3, but ℓ_1 and ℓ_3 coincide. In any case, though, this relation is transitive. Therefore, R is an equivalence relation. Hence for $\ell \in L$, the equivalence class

$$[\ell] = \{x \in L : x\ R\ \ell\} = \{x \in L : x = \ell \text{ or } x \text{ is parallel to } \ell\};$$

that is, the equivalence class $[\ell]$ consists of ℓ and all lines in the plane parallel to ℓ.

To describe additional examples of relations from geometry, let $\mathcal{T}$ be the set of all triangles in a plane. For two triangles T and T' in $\mathcal{T}$, define relations R_1 and R_2 on $\mathcal{T}$ by

$T\ R_1\ T'$ if T is congruent to T' and $T\ R_2\ T'$ if T is similar to T'. Then both R_1 and R_2 are equivalence relations. For a triangle T and the relation R_1, $[T]$ is the set of triangles in $\mathcal{T}$ that are congruent to T, while for R_2, $[T]$ is the set of triangles in $\mathcal{T}$ that are similar to T.

The relation R defined on $\mathbf{Z}$ by $x\ R\ y$ if $|x| = |y|$ is also an equivalence relation. In this case, for $a \in \mathbf{Z}$, the equivalence class $[a]$ consists of the two integers a and $-a$, unless $a = 0$, in which case $[0] = \{0\}$. We now consider an example that requires more thought and explanation.

Result 8.1 *A relation R is defined on $\mathbf{Z}$ by $x\ R\ y$ if $x + 3y$ is even. Then R is an equivalence relation.*

Before proving this result, let's be certain that we understand this relation. First, notice that $5\ R\ 7$ since $5 + 3 \cdot 7 = 26$ is even. However, $8 \not R\ 9$ since $8 + 3 \cdot 9 = 35$ is not even. On the other hand, $4\ R\ 4$ because $4 + 3 \cdot 4 = 16$ is even.

Proof of Result 8.1 First we show that R is reflexive. Let $a \in \mathbf{Z}$. Then $a + 3a = 4a = 2(2a)$ is even since $2a \in \mathbf{Z}$. Therefore $a\ R\ a$ and R is reflexive.

Next we show that R is symmetric. Assume that $a\ R\ b$. Thus $a + 3b$ is even. Hence $a + 3b = 2k$ for some integer k. So $a = 2k - 3b$. Therefore,

$$b + 3a = b + 3(2k - 3b) = b + 6k - 9b = 6k - 8b = 2(3k - 4b).$$

Since $3k - 4b$ is an integer, $b + 3a$ is even. Therefore, $b\ R\ a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a\ R\ b$ and $b\ R\ c$. Hence $a + 3b$ and $b + 3c$ are even; so $a + 3b = 2k$ and $b + 3c = 2\ell$ for some integers k and ℓ. Adding these two equations, we obtain $(a + 3b) + (b + 3c) = 2k + 2\ell$. So $a + 4b + 3c = 2k + 2\ell$, and $a + 3c = 2k + 2\ell - 4b = 2(k + \ell - 2b)$. Since $k + \ell - 2b$ is an integer, $a + 3c$ is even. Hence $a\ R\ c$ and so R is transitive. Therefore, R is an equivalence relation. ∎

PROOF ANALYSIS A few remarks concerning the preceding proof are in order. Recall that a relation R defined on a set A is reflexive if $x\ R\ x$ for every $x \in A$. The reflexive property may also be reworded to read: For every $x \in A$, $x\ R\ x$, or: If $x \in A$, then $x\ R\ x$. Hence when we proved that R is reflexive in Result 8.1, we began by assuming that a was an arbitrary element of $\mathbf{Z}$. (We're giving a direct proof.) We were then required to show that $a + 3a$ is even, which we did. It would be incorrect, however, to assume that $a + 3a$ is even or that $a\ R\ a$. This, in fact, is what we want to prove. ♦

Since the relation defined in Result 8.1 is an equivalence relation, there are equivalence classes, namely an equivalence class $[a]$ for each $a \in \mathbf{Z}$. Let's start with 0, say. The equivalence class $[0]$ is the set of all integers related to 0. In symbols, this equivalence class is

$$\begin{aligned}[0] &= \{x \in \mathbf{Z} :\ x\ R\ 0\} = \{x \in \mathbf{Z} :\ x + 3 \cdot 0 \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x \text{ is even}\} = \{0, \pm 2, \pm 4, \ldots\};\end{aligned}$$

that is, $[0]$ is the set of even integers. It shouldn't be difficult to see that if a is an even integer, say $a = 2k$, where $k \in \mathbf{Z}$, then

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} :\ x\ R\ a\} = \{x \in \mathbf{Z} :\ x + 3a \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x + 3(2k) \text{ is even}\} = \{x \in \mathbf{Z} :\ x + 6k \text{ is even}\}\end{aligned}$$

is also the set of even integers. On the other hand, the equivalence class consisting of those integers related to 1 is

$$\begin{aligned}[1] &= \{x \in \mathbf{Z} :\ x\ R\ 1\} = \{x \in \mathbf{Z} :\ x + 3 \cdot 1 \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x + 3 \text{ is even}\} = \{\pm 1, \pm 3, \pm 5, \ldots\},\end{aligned}$$

which is the set of odd integers. In fact, if a is an odd integer, then $a = 2\ell + 1$ for some integer ℓ and

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} :\ x + 3a \text{ is even}\} = \{x \in \mathbf{Z} :\ x + 3(2\ell + 1) \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x + 6\ell + 3 \text{ is even}\}\end{aligned}$$

is the set of odd integers. Therefore, if a and b are even, then $[a] = [b]$ is the set of even integers; while if a and b are odd, then $[a] = [b]$ is the set of odd integers. Hence there are only two distinct equivalence classes, namely, $[0]$ and $[1]$, the sets of even and odd integers, respectively. We will soon see that there is a good reason for this observation.

8.4 Properties of Equivalence Classes

You may have noticed that in the preceding examples of equivalence relations, we have seen several situations where two equivalence classes are equal. It is possible to determine exactly when this happens.

Theorem 8.2 *Let R be an equivalence relation on a nonempty set A, and let a and b be elements of A. Then $[a] = [b]$ if and only if $a\ R\ b$.*

Proof Assume that $a\ R\ b$. We show that the sets $[a]$ and $[b]$ are equal by verifying that $[a] \subseteq [b]$ and $[b] \subseteq [a]$. First, we show that $[a] \subseteq [b]$. Let $x \in [a]$. Then $x\ R\ a$. Since $a\ R\ b$ and R is transitive, $x\ R\ b$. Therefore, $x \in [b]$ and so $[a] \subseteq [b]$. Next, let $y \in [b]$. Thus, $y\ R\ b$. Since $a\ R\ b$ and R is symmetric, $b\ R\ a$. Again, by the transitivity of R, we have $y\ R\ a$. Therefore, $y \in [a]$ and so $[b] \subseteq [a]$. Hence $[a] = [b]$.

For the converse, assume that $[a] = [b]$. Because R is reflexive, $a \in [a]$. But, since $[a] = [b]$, it follows that $a \in [b]$. Consequently, $a\ R\ b$. ∎

According to Theorem 8.2 then, if R is an equivalence relation on a set A and a is related to b, then the set $[a]$ of elements of A related to a and the set $[b]$ of elements of A related to b are equal, that is, $[a] = [b]$. Because the theorem characterizes when $[a] = [b]$, we know that if $a \not R\ b$, then $[a] \neq [b]$.

Let's return once more to the equivalence relation defined in (8.2) on the set $A = \{1, 2, 3, 4, 5, 6\}$ and the equivalence classes given in (8.3). We observed earlier that $[1] = [3] = [6]$. Since every two of the integers 1, 3, 6 are related to each other (according to the definition of R), Theorem 8.2 tells us that the equality of $[1]$, $[3]$, and $[6]$ is expected. The same can be said of $[2]$ and $[4]$. However, since $(5, 6) \notin R$, for example, Theorem 8.2 tells us that $[5] \neq [6]$, which is the case. Therefore, as we also observed earlier, there are only three distinct equivalence classes, namely,

$$[1] = [3] = [6] = \{1, 3, 6\}, \quad [2] = [4] = \{2, 4\}, \quad [5] = \{5\}. \tag{8.4}$$

Now, you might have noticed one other thing. Every element of A belongs to exactly one equivalence class. This observation might remind you of a concept we discussed earlier.

Recall that a **partition** P of a nonempty set S is a collection of nonempty subsets of S with the property that every element of S belongs to exactly one of these subsets; that is, P is a collection of pairwise disjoint, nonempty subsets of S whose union is S. Hence the set of the distinct equivalence classes in (8.4) is a partition of the set $A = \{1, 2, 3, 4, 5, 6\}$. We now show that this too is expected.

Theorem 8.3 *Let R be an equivalence relation defined on a nonempty set A. Then the set*

$$P = \{[a] : a \in A\}$$

of equivalence classes resulting from R is a partition of A.

Proof Certainly, each equivalence class $[a]$ is nonempty since $a \in [a]$, and so each element of A belongs to at least one equivalence class. We show that every element of A belongs to exactly one equivalence class. Assume that some element x of A belongs to two equivalence classes, say $[a]$ and $[b]$. Since $x \in [a]$ and $x \in [b]$, it follows that $x \; R \; a$ and $x \; R \; b$. Because R is symmetric, $a \; R \; x$. Thus $a \; R \; x$ and $x \; R \; b$. Since R is transitive, $a \; R \; b$. By Theorem 8.2, it follows that $[a] = [b]$. So any two equivalence classes to which x belongs are equal. Hence x belongs to a unique equivalence class. ∎

In the proof of Theorem 8.3, we were required to show that each element $x \in A$ belongs to a unique equivalence class. During this proof, we assumed that x belongs to two equivalence classes $[a]$ and $[b]$. Observe that we made no assumption whether $[a]$ and $[b]$ are distinct. Later we learned that $[a] = [b]$; so x can only belong to one equivalence class. With a very small change, we could have reached the same conclusion by a different proof technique. We could have said: Assume, to the contrary, that x belongs to two *distinct* equivalence classes $[a]$ and $[b]$. By the same argument as above, we can show that $[a] = [b]$. However, *now*, this produces a contradiction, and we have just given a proof by contradiction.

According to Theorem 8.3 then, whenever we have an equivalence relation R defined on a nonempty set A, a partition of A into the associated equivalence classes of R results. Perhaps unexpectedly, the converse is also true. That is, if we are given a partition of A, then there is a corresponding equivalence relation that can be defined on A, whose resulting equivalence classes are the elements of the given partition. For example, let

$$P = \{\{1, 3, 4\}, \{2, 7\}, \{5, 6\}\}$$

be a given partition of the set $A = \{1, 2, 3, 4, 5, 6, 7\}$. (Notice that every element of A belongs to exactly one subset in P.) Then

$$R = \{(1, 1), (1, 3), (1, 4), (2, 2), (2, 7), (3, 1), (3, 3), (3, 4), (4, 1), (4, 3), (4, 4), (5, 5), (5, 6), (6, 5), (6, 6), (7, 2), (7, 7)\}$$

is an equivalence relation on A, whose distinct equivalence classes are

$$[1] = \{1, 3, 4\}, \quad [2] = \{2, 7\}, \quad \text{and} \quad [5] = \{5, 6\},$$

and so $P = \{[1], [2], [5]\}$. We now establish this result in general; that is, if we have a nonempty set A and a partition P of A, then it is possible to create an equivalence relation R on A such that the distinct equivalence classes of R are precisely the subsets in P. Since we are trying to verify this in general (and not for a specific example), we need to describe the subsets in P with the aid of an index set. Since we will want every subset in P to be an equivalence class, we will need every two elements in the same subset to be related. On the other hand, since we will want two different subsets in P to be different equivalence classes, we will need elements in distinct subsets not to be related.

Theorem 8.4 *Let $P = \{A_\alpha : \alpha \in I\}$ be a partition of a nonempty set A. Then there exists an equivalence relation R on A such that P is the set of equivalence classes determined by R, that is, $P = \{[a] : a \in A\}$.*

Proof Define a relation R on A by $x \; R \; y$ if x and y belong to the same subset in P; that is, $x \; R \; y$ if $x, y \in A_\alpha$ for some $\alpha \in I$. We now show that R is an equivalence relation. Let $a \in A$. Since P is a partition of A, it follows that $a \in A_\beta$ for some $\beta \in I$. Trivially, a and a belong to A_β; so $a \; R \; a$ and R is reflexive.

Next, let $a, b \in A$, and assume that $a \; R \; b$. Then a and b belong to A_γ for some $\gamma \in I$. Hence b and a belong to A_γ; so $b \; R \; a$ and R is symmetric.

Finally, let a, b, and c be elements of A such that $a \; R \; b$ and $b \; R \; c$. Therefore, $a, b \in A_\beta$ and $b, c \in A_\gamma$ for some $\beta, \gamma \in I$. Since P is a partition of A, the element b belongs to only one set in P. Hence $A_\beta = A_\gamma$ and so $a, c \in A_\beta$. Thus $a \; R \; c$ and R is transitive. Therefore, R is an equivalence relation on A.

We now consider the equivalence classes resulting from R. Let $a \in A$. Then $a \in A_\alpha$ for some $\alpha \in I$. The equivalence class $[a]$ consists of all elements of A related to a. From the way that R is defined, however, the only elements related to a are those elements belonging to the same subset in P to which a belongs; that is, $[a] = A_\alpha$. Hence

$$\{[a] : a \in A\} = \{A_\alpha : \alpha \in I\} = P. \quad ∎$$

We now give an additional example of an equivalence relation. Although this example is similar to the one described in Result 8.1, it is different enough to require some thought.

Result to Prove A relation R is defined on $\mathbf{Z}$ by $x \; R \; y$ if $11x - 5y$ is even. Then R is an equivalence relation.

PROOF STRATEGY Since we want to verify that R is an equivalence relation, we need to show that R is reflexive, symmetric, and transitive. Let's start with the first of these. We begin with an integer a. To show that $a \; R \; a$, we need to show that $11a - 5a$ is even. However, $11a - 5a = 6a = 2(3a)$, so this shouldn't cause any difficulties.

To verify that R is symmetric, we begin with $a \; R \; b$ (where $a, b \in \mathbf{Z}$, of course) and attempt to show that $b \; R \; a$. Since $a \; R \; b$, it follows that $11a - 5b$ is even. To show that $b \; R \; a$, we need to show that $11b - 5a$ is even. Since $11a - 5b$ is even, we can write $11a - 5b = 2k$ for some integer k. At first, though, it might seem like a good idea to solve for a in terms of b or solve for b in terms of a. However, because neither the

coefficient of a nor the coefficient of b is 1 or -1 in $11a - 5b = 2k$, fractions would be introduced. We need another approach. Notice that if we write

$$11b - 5a = (11a - 5b) + (?\ a + ?\ b),$$

then we have

$$\begin{aligned} 11b - 5a &= (11a - 5b) + (-16a + 16b) \\ &= 2k - 16a + 16b = 2(k - 8a + 8b). \end{aligned}$$

This will work.

To verify that R is transitive, we begin by assuming that $a\ R\ b$ and $b\ R\ c$ (and attempt to show that $a\ R\ c$). Thus $11a - 5b$ and $11b - 5c$ are even and so

$$11a - 5b = 2k \quad \text{and} \quad 11b - 5c = 2\ell, \tag{8.5}$$

for integers k and ℓ. To show that $a\ R\ c$, we must verify that $11a - 5c$ is even. We need to work the expression $11a - 5c$ into the discussion. However, this can be done by adding the expressions in (8.5). We're ready to give a proof now. ♦

Result 8.5 *A relation R is defined on $\mathbf{Z}$ by $x\ R\ y$ if $11x - 5y$ is even. Then R is an equivalence relation.*

Proof First, we show that R is reflexive. Let $a \in \mathbf{Z}$. Then $11a - 5a = 6a = 2(3a)$. Since $3a$ is an integer, $11a - 5a$ is even. Thus $a\ R\ a$ and R is reflexive.

Next we show that R is symmetric. Assume that $a\ R\ b$ (where, of course, $a, b \in \mathbf{Z}$). Thus $11a - 5b$ is even. Therefore $11a - 5b = 2k$, where $k \in \mathbf{Z}$. Observe that

$$\begin{aligned} 11b - 5a &= (11a - 5b) + (-16a + 16b) \\ &= 2k - 16a + 16b = 2(k - 8a + 8b). \end{aligned}$$

Since $k - 8a + 8b$ is an integer, $11b - 5a$ is even. Hence $b\ R\ a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a\ R\ b$ and $b\ R\ c$. Hence $11a - 5b$ and $11b - 5c$ are even. Therefore, $11a - 5b = 2k$ and $11b - 5c = 2\ell$, where $k, \ell \in \mathbf{Z}$. Adding these equations, we obtain $(11a - 5b) + (11b - 5c) = 2k + 2\ell$. Solving for $11a - 5c$, we have

$$11a - 5c = 2k + 2\ell - 6b = 2(k + \ell - 3b).$$

Since $k + \ell - 3b$ is an integer, $11a - 5c$ is even. Hence $a\ R\ c$ and R is transitive. Therefore, R is an equivalence relation. ■

We now determine the equivalence classes for the equivalence relation just discussed. Let's begin with the equivalence class containing 0, say. Then

$$\begin{aligned} [0] &= \{x \in \mathbf{Z} :\ x\ R\ 0\} = \{x \in \mathbf{Z} :\ 11x \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x \text{ is even}\} = \{0, \pm 2, \pm 4, \ldots\}. \end{aligned}$$

Recall that the distinct equivalence classes always produce a partition of the set involved (in this case $\mathbf{Z}$). Since the class $[0]$ does not consist of all integers, there is at least one other equivalence class. To determine another equivalence class, we look for an element that does not belong to $[0]$. Since $1 \notin [0]$, the equivalence class $[1]$ is distinct (and disjoint) from $[0]$. Thus

$$\begin{aligned} [1] &= \{x \in \mathbf{Z} :\ x\ R\ 1\} = \{x \in \mathbf{Z} :\ 11x - 5 \text{ is even}\} \\ &= \{x \in \mathbf{Z} :\ x \text{ is odd}\} = \{\pm 1, \pm 3, \pm 5, \ldots\}. \end{aligned}$$

Since $[0]$ and $[1]$ produce a partition of $\mathbf{Z}$ (that is, every integer belongs to exactly one of $[0]$ and $[1]$), these are the only equivalence classes in this case.

8.5 Congruence Modulo n

Next we describe one of the most important equivalence relations. If you have more mathematics in your future, it is likely that you will see the equivalence relation we are about to describe again – indeed often. Recall again that for integers a and b, where $a \neq 0$, the integer a is said to **divide** b, written as $a \mid b$, if there exists an integer c such that $b = ac$. Also, for integers a, b, and $n \geq 2$, a is said to be **congruent to b modulo n**, written $a \equiv b \pmod{n}$, if $n \mid (a - b)$. For example, $24 \equiv 6 \pmod{9}$ since $9 \mid (24 - 6)$, while $1 \equiv 5 \pmod{2}$ since $2 \mid (1 - 5)$. Also, $4 \equiv 4 \pmod{5}$ since $5 \mid (4 - 4)$. However, $8 \not\equiv 2 \pmod{4}$ since $4 \nmid (8 - 2)$. These concepts were introduced in Chapter 4.

Let's consider a few examples of pairs a, b of integers such that $a \equiv b \pmod{5}$. Notice that $7 \equiv 7 \pmod{5}$, $-1 \equiv -1 \pmod{5}$, and $0 \equiv 0 \pmod{5}$. Also, $2 \equiv -8 \pmod{5}$ and $-8 \equiv 2 \pmod{5}$. Notice also that $2 \equiv 17 \pmod{5}$. Therefore, both $-8 \equiv 2 \pmod{5}$ and $2 \equiv 17 \pmod{5}$. Furthermore, $-8 \equiv 17 \pmod{5}$. These examples might suggest that the reflexive, symmetric, and transitive properties are satisfied here, a fact which we are about to verify. This is the important equivalence relation we referred to at the beginning of this section, not just for $n = 5$ but for any integer $n \geq 2$.

Theorem 8.6 *Let $n \in \mathbf{Z}$, where $n \geq 2$. Then congruence modulo n (that is, the relation R defined on $\mathbf{Z}$ by $a\ R\ b$ if $a \equiv b \pmod{n}$) is an equivalence relation on $\mathbf{Z}$.*

Proof Let $a \in \mathbf{Z}$. Since $n \mid 0$, it follows that $n \mid (a - a)$ and so $a \equiv a \pmod{n}$. Thus, $a\ R\ a$, implying that R is reflexive.

Next, we show that R is symmetric. Assume that $a\ R\ b$, where $a, b \in \mathbf{Z}$. Since $a\ R\ b$, it follows that $a \equiv b \pmod{n}$ and so $n \mid (a - b)$. Hence, there exists $k \in \mathbf{Z}$ such that $a - b = nk$. Thus,

$$b - a = -(a - b) = -(nk) = n(-k).$$

Since $-k \in \mathbf{Z}$, it follows that $n \mid (b - a)$, and so $b \equiv a \pmod{n}$. Therefore, $b\ R\ a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a\ R\ b$ and $b\ R\ c$, where $a, b, c \in \mathbf{Z}$. We show that $a\ R\ c$. Since $a\ R\ b$ and $b\ R\ c$, we know that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. Thus, $n \mid (a - b)$ and $n \mid (b - c)$. Consequently,

$$a - b = nk \quad \text{and} \quad b - c = n\ell \tag{8.6}$$

for some integers k and ℓ. Adding the equations in (8.6), we obtain

$$(a - b) + (b - c) = nk + n\ell = n(k + \ell);$$

so $a - c = n(k + \ell)$. Since $k + \ell \in \mathbf{Z}$, we have $n \mid (a - c)$, and so $a \equiv c \pmod{n}$. Therefore, $a \; R \; c$ and R is transitive. ∎

PROOF ANALYSIS Theorem 8.6 describes a well-known equivalence relation. Let's review how we verified this. The proof we gave to show that congruence modulo n is an equivalence relation is a common proof technique for this kind of result, and we need to be familiar with it. To prove that R is reflexive, we began with an arbitrary element of $\mathbf{Z}$. We called this element a. Our goal was to show that $a \; R \; a$. By definition, $a \; R \; a$ if and only if $a \equiv a \pmod{n}$. However, $a \equiv a \pmod{n}$ if and only if $n \mid (a - a)$, which is the same as the statement $n \mid 0$. Clearly, $n \mid 0$ and this is where we decided to start.

To prove that R is symmetric, we started (as always) by assuming that $a \; R \; b$. Our goal was to show that $b \; R \; a$. Since $a \; R \; b$, the definition of the relation R tells us that $a \equiv b \pmod{n}$. From this, we knew that $n \mid (a - b)$ and $a - b = nk$ for some integer k. However, to show that $b \; R \; a$, we needed to verify that $b \equiv a \pmod{n}$. But this can be done only if we can show that $n \mid (b - a)$ or, equivalently, that $b - a = n\ell$ for some integer ℓ. Hence we needed to verify that $b - a$ can be expressed as the product of n and some other integer. Since $b - a$ is the negative of $a - b$ and we have a convenient expression for $a - b$, this provided us with a key step.

Finally, to prove that R is transitive, we began by assuming that $a \; R \; b$ and $b \; R \; c$, which led us to the expressions $a - b = nk$ and $b - c = n\ell$, where $k, \ell \in \mathbf{Z}$. Since our goal was to show that $a \; R \; c$, we were required to show that $a - c$ is a multiple of n. Somehow then, we needed to work the term $a - c$ into the problem, knowing that $a - b = nk$ and $b - c = n\ell$. The key step here was to observe that $a - c = (a - b) + (b - c)$. ♦

According to Theorem 8.6 then, congruence modulo 3 is an equivalence relation. In other words, if we define a relation R on $\mathbf{Z}$ by $a \; R \; b$ if $a \equiv b \pmod{3}$, then it follows that R is an equivalence relation. Let's determine the distinct equivalence classes in this case. First, select an integer, say 0. Then $[0]$ is an equivalence class. Indeed,

$$\begin{aligned}[0] &= \{x \in \mathbf{Z} : \; x \; R \; 0\} = \{x \in \mathbf{Z} : \; x \equiv 0 \pmod{3}\} \\ &= \{x \in \mathbf{Z} : \; 3 \mid x\} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}.\end{aligned}$$

Hence the class $[0]$ consists of the multiples of 3. This class could be denoted by $[3]$ or $[6]$ or even $[-300]$. Since there is an integer that is not in $[0]$, there must be at least one equivalence class distinct from $[0]$. In particular, since $1 \notin [0]$, it follows that $[1] \neq [0]$; in fact, necessarily, $[1] \cap [0] = \emptyset$. The equivalence class

$$\begin{aligned}[1] &= \{x \in \mathbf{Z} : \; x \; R \; 1\} = \{x \in \mathbf{Z} : \; x \equiv 1 \pmod{3}\} \\ &= \{x \in \mathbf{Z} : \; 3 \mid (x - 1)\} = \{1, -2, 4, -5, 7, -8, \ldots\}.\end{aligned}$$

Since $2 \notin [0]$ and $2 \notin [1]$, the equivalence class $[2]$ is different from both $[0]$ and $[1]$. By definition,

$$\begin{aligned}[2] &= \{x \in \mathbf{Z} : \; x \; R \; 2\} = \{x \in \mathbf{Z} : \; x \equiv 2 \pmod{3}\} \\ &= \{x \in \mathbf{Z} : \; 3 \mid (x - 2)\} = \{2, -1, 5, -4, 8, -7, \ldots\}.\end{aligned}$$

Since every integer belongs to (exactly) one of these classes, we have exactly three

distinct equivalence classes in this case, namely:

$$\begin{aligned}[0] &= \{0, \pm 3, \pm 6, \pm 9, \ldots\}, \\ [1] &= \{1, -2, 4, -5, 7, -8, \ldots\}, \\ [2] &= \{2, -1, 5, -4, 8, -7, \ldots\}.\end{aligned}$$

These equivalence classes have a connection with some very familiar mathematical concepts: division and remainders. If m and $n \geq 2$ are integers and m is divided by n, then we can express this division as $m = nq + r$, where q is the quotient and r is the remainder. The remainder r has the requirement that $0 \leq r < n$. With this requirement, q and r are unique and the result that we have just referred to is called the **Division Algorithm**. (The Division Algorithm will be studied in considerable detail in Chapter 11.) As we saw in Chapter 4, every integer m can be expressed as $3q + r$, where $0 \leq r < 3$, that is, r has one of the values 0, 1, 2. Hence, every integer can be expressed as $3q$, $3q + 1$, or $3q + 2$ for some integer q. In this case, the equivalence class $[0]$ consists of the multiples of 3, and so every integer having a remainder of 0 when divided by 3 belongs to $[0]$. Furthermore, every integer having a remainder of 1 when divided by 3 belongs to $[1]$, while every integer having a remainder of 2 when divided by 3 belongs to $[2]$. Since

$$73 = 24 \cdot 3 + 1 \quad \text{and} \quad -22 = (-8) \cdot 3 + 2,$$

for example, it follows that $73 \in [1]$ and $-22 \in [2]$. In fact, $[73] = [1]$ and $[-22] = [2]$.

In general, for $n \geq 2$, the equivalence relation congruence modulo n results in n distinct equivalence classes. In other words, if we define $a \; R \; b$ by $a \equiv b \pmod{n}$, then there are n distinct equivalence classes: $[0], [1], \ldots, [n - 1]$. In fact, for an integer r with $0 \leq r < n$, an integer m belongs to the set $[r]$ if and only if there is an integer q (the quotient) such that $m = nq + r$. Thus the equivalence class $[r]$ consists of all integers having a remainder of r when divided by n.

Let's consider another equivalence relation defined on $\mathbf{Z}$ involving congruence, but which is seemingly different from the class of examples we have just described.

Result to Prove Let R be the relation defined on $\mathbf{Z}$ by $a \; R \; b$ if $2a + b \equiv 0 \pmod{3}$. Then R is an equivalence relation.

PROOF STRATEGY To prove that R is reflexive, we must show that $x \; R \; x$ for every $x \in \mathbf{Z}$. This means that we must show that $2x + x \equiv 0 \pmod{3}$ or that $3x \equiv 0 \pmod{3}$. This is equivalent to showing that $3 \mid 3x$, which is obvious. This tells us where to begin the proof of the reflexive property.

Proving that R is symmetric is somewhat more subtle. Of course, we know where to begin. We assume that $x \; R \; y$. From this, we have $2x + y \equiv 0 \pmod{3}$. So $3 \mid (2x + y)$, or $2x + y = 3r$ for some integer r. Our goal is to show that $y \; R \; x$ or, equivalently, that $2y + x \equiv 0 \pmod{3}$. Eventually, then, we must show that $2y + x = 3s$ for some integer s. We cannot assume this of course. Since $2x + y = 3r$, it follows that $y = 3r - 2x$. So

$$2y + x = 2(3r - 2x) + x = 6r - 3x = 3(2r - x).$$

Since $2r - x \in \mathbf{Z}$, we have $3 \mid (2y + x)$, and the verification of symmetry is nearly complete.

Proving that R is transitive should be as expected. ♦

Result 8.7 *Let R be the relation defined on $\mathbf{Z}$ by $a\ R\ b$ if $2a + b \equiv 0 \pmod 3$. Then R is an equivalence relation.*

Proof Let $x \in \mathbf{Z}$. Since $3 \mid 3x$, it follows that $3x \equiv 0 \pmod 3$. So $2x + x \equiv 0 \pmod 3$. Thus, $x\ R\ x$ and R is reflexive.

Next we verify that R is symmetric. Assume that $x\ R\ y$, where $x, y \in \mathbf{Z}$. Thus $2x + y \equiv 0 \pmod 3$, and so $3 \mid (2x + y)$. Therefore, $2x + y = 3r$ for some integer r. Hence $y = 3r - 2x$. So

$$2y + x = 2(3r - 2x) + x = 6r - 3x = 3(2r - x).$$

Since $2r - x$ is an integer, $3 \mid (2y + x)$. So $2y + x \equiv 0 \pmod 3$. Therefore, $y\ R\ x$ and R is symmetric.

Finally, we show that R is transitive. Assume that $x\ R\ y$ and $y\ R\ z$, where $x, y, z \in \mathbf{Z}$. Then $2x + y \equiv 0 \pmod 3$ and $2y + z \equiv 0 \pmod 3$. Thus, $3 \mid (2x + y)$ and $3 \mid (2y + z)$. From this, it follows that $2x + y = 3r$ and $2y + z = 3s$ for some integers r and s. Adding these two equations, we obtain

$$2x + 3y + z = 3r + 3s;$$

so

$$2x + z = 3r + 3s - 3y = 3(r + s - y).$$

Since $r + s - y$ is an integer, $3 \mid (2x + z)$; so $2x + z \equiv 0 \pmod 3$. Hence $x\ R\ z$ and R is transitive. ∎

PROOF ANALYSIS A few additional comments about the proof of the symmetric property in Result 8.7 might be helpful. At one point in the proof we knew that $2x + y = 3r$ for some integer r, and we wanted to show that $2y + x = 3s$ for some integer s. If we added these two equations, then we would obtain $3x + 3y = 3r + 3s$. Of course, we can't add these because we don't know that $2y + x = 3s$. But this does suggest another idea.

Assume that $x\ R\ y$. Thus $2x + y \equiv 0 \pmod 3$. Hence $3 \mid (2x + y)$; so $2x + y = 3r$ for some integer r. Observe that

$$3x + 3y = (2x + y) + (2y + x) = 3r + (2y + x).$$

Therefore,

$$2y + x = 3x + 3y - 3r = 3(x + y - r).$$

Because $x + y - r \in \mathbf{Z}$, it follows that $3 \mid (2y + x)$. Consequently, $2y + x \equiv 0 \pmod 3$, $y\ R\ x$, and R is symmetric. ♦

The distinct equivalence classes for the equivalence relation described in Result 8.7 are

$$\begin{aligned}[0] &= \{x \in \mathbf{Z} :\ x\ R\ 0\} = \{x \in \mathbf{Z} :\ 2x \equiv 0 \pmod 3\} \\ &= \{x \in \mathbf{Z} :\ 3 \mid 2x\} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}, \\ [1] &= \{x \in \mathbf{Z} :\ x\ R\ 1\} = \{x \in \mathbf{Z} :\ 2x + 1 \equiv 0 \pmod 3\} \\ &= \{x \in \mathbf{Z} :\ 3 \mid (2x + 1)\} = \{1, -2, 4, -5, 7, -8, \ldots\},\end{aligned}$$

$$\begin{aligned}[2] &= \{x \in \mathbf{Z} :\ x\ R\ 2\} = \{x \in \mathbf{Z} :\ 2x + 2 \equiv 0 \pmod 3\} \\ &= \{x \in \mathbf{Z} :\ 3 \mid (2x + 2)\} = \{2, -1, 5, -4, 8, -7, \ldots\}.\end{aligned}$$

Let's discuss how we obtained these equivalence classes. We started with the integer 0 and saw that $[0] = \{x \in \mathbf{Z} :\ 3 \mid 2x\}$. By trying various values of x (namely, 0, 1, 2, 3, 4, 5, etc. and -1, -2, -3, -4, etc.), we see that we are obtaining the multiples of 3. (Exercise 4.6 of Chapter 4 asks you to show that if $3 \mid 2x$, then x is a multiple of 3.) The contents of [1] and [2] can be justified, if necessary, in a similar manner.

We have seen that if we define a relation R_1 on $\mathbf{Z}$ by $a\ R_1\ b$ if $a \equiv b \pmod 3$, then we have three distinct equivalence classes; while if we define a relation R_2 on $\mathbf{Z}$ by $a\ R_2\ b$ if $2a + b \equiv 0 \pmod 3$, then we also have three distinct classes – in fact, the *same* equivalence classes. Let's see why this is true.

Result 8.8 *Let $a, b \in \mathbf{Z}$. Then $a \equiv b \pmod 3$ if and only if $2a + b \equiv 0 \pmod 3$.*

Proof First, assume that $a \equiv b \pmod 3$. Then $3 \mid (a - b)$ and so $a - b = 3x$ for some integer x. Thus $a = 3x + b$. Now

$$2a + b = 2(3x + b) + b = 6x + 3b = 3(2x + b).$$

Since $2x + b$ is an integer, $3 \mid (2a + b)$ and so $2a + b \equiv 0 \pmod 3$.

For the converse, assume that $2a + b \equiv 0 \pmod 3$. Hence $3 \mid (2a + b)$, which implies that $2a + b = 3y$ for some integer y. Thus $b = 3y - 2a$. Observe that

$$a - b = a - (3y - 2a) = 3a - 3y = 3(a - y).$$

Since $a - y$ is an integer, $3 \mid (a - b)$ and so $a \equiv b \pmod 3$. ∎

We shouldn't jump to the conclusion that just because we are dealing with an equivalence relation defined in terms of the integers modulo 3, we will necessarily have three distinct equivalence classes. For example, suppose that we define a relation R on $\mathbf{Z}$ by $a\ R\ b$ if $a^2 \equiv b^2 \pmod 3$. Then, here too, R is an equivalence relation. In this case, however, there are only *two* distinct equivalence classes, namely,

$$[0] = \{0, \pm 3, \pm 6, \pm 9, \ldots\} \text{ and } [1] = \{\pm 1, \pm 2, \pm 4, \pm 5, \ldots\},$$

since whenever an integer n has a remainder 1 or 2 when it is divided by 3, then n^2 has a remainder of 1 when it is divided by 3.

8.6 The Integers Modulo n

We have already seen that for each integer $n \geq 2$, the relation R defined on $\mathbf{Z}$ by $a\ R\ b$ if $a \equiv b \pmod n$ is an equivalence relation. Furthermore, this equivalence relation results in the n distinct equivalence classes $[0], [1], \ldots, [n - 1]$. We denote the set of these equivalence classes by $\mathbf{Z}_n$ and refer to this set as the **integers modulo n**. Thus, $\mathbf{Z}_3 = \{[0], [1], [2]\}$ and, in general,

$$\mathbf{Z}_n = \{[0], [1], \ldots, [n - 1]\}.$$

Hence each element $[r]$ of $\mathbf{Z}_n$, where $0 \leq r < n$, is a set that contains infinitely many integers; indeed, as we have noted, $[r]$ consists of all those integers having the remainder r when divided by n. For this reason, the elements of $\mathbf{Z}_n$ are sometimes called **residue classes.**

Although it makes perfectly good sense to take the union and intersection of two elements of $\mathbf{Z}_n$ since these elements are sets (in fact, subsets of $\mathbf{Z}$), it doesn't make sense at this point to add or multiply two elements of $\mathbf{Z}_n$. However, since the elements of $\mathbf{Z}_n$ have the appearance of integers, say $[a]$ and $[b]$, where $a, b \in \mathbf{Z}$, it does suggest the possibility of defining addition and multiplication in $\mathbf{Z}_n$. We now discuss how these operations can be defined on the set $\mathbf{Z}_n$.

Of course, we have seen addition and multiplication defined many times before. When we speak of addition and multiplication being *operations* on a set S, we mean that for $x, y \in S$, the sum $x + y$ and the product xy should both belong to S. For example, in the set $\mathbf{Q}$ of rational numbers, the sum and product of two rational numbers a/b and c/d (so $a, b, c, d \in \mathbf{Z}$ and $b, d \neq 0$) are defined by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

both of which are rational numbers and so belong to $\mathbf{Q}$.

As we mentioned, if addition and multiplication are operations on a set S, then $x + y \in S$ and $xy \in S$ for all $x, y \in S$. Therefore, if T is a nonempty subset of S and $x, y \in T$, then $x + y \in S$ and $xy \in S$. The set T is **closed under addition** if $x + y \in T$ whenever $x, y \in T$. Similarly, T is **closed under multiplication** if $xy \in T$ whenever $x, y \in T$. Necessarily, if addition and multiplication are operations on a set S, then S is closed under addition and multiplication.

For example, addition and multiplication are operations on $\mathbf{Z}$. If A and B denote the sets of even integers and odd integers, respectively, then A is closed under both addition and multiplication but B is closed under multiplication only.

However addition and multiplication might be defined in $\mathbf{Z}_n$, we would certainly expect that the sum and product of two elements of $\mathbf{Z}_n$ to be an element of $\mathbf{Z}_n$. There appears to be a natural definition of addition and multiplication in $\mathbf{Z}_n$; namely, for two equivalence classes $[a]$ and $[b]$ in $\mathbf{Z}_n$, we define

$$[a] + [b] = [a + b] \quad \text{and} \quad [a] \cdot [b] = [ab]. \tag{8.7}$$

Let's suppose that we are considering $\mathbf{Z}_6$, for example, where then $\mathbf{Z}_6 = \{[0], [1], \ldots, [5]\}$. From the definitions of addition and multiplication that we just gave, $[1] + [3] = [1 + 3] = [4]$ and $[1] \cdot [3] = [1 \cdot 3] = [3]$. This certainly seems harmless enough, but let's consider adding and multiplying two other equivalence classes, say $[2]$ and $[3]$. Again, according to the definitions in (8.7), $[2] + [3] = [2 + 3] = [5]$ and $[2] \cdot [3] = [2 \cdot 3] = [6]$. However, we have been expressing the elements of $\mathbf{Z}_6$ by $[0]$, $[1]$, $[2]$, $[3]$, $[4]$, and $[5]$ and we don't explicitly see $[2 \cdot 3] = [6]$ among these elements. Since $6 \equiv 0 \pmod 6$, it follows that $6 \in [0]$; that is, $[6] = [0]$. (Also, the remainder is 0 when 6 is divided by 6, and so $[6] = [0]$.) Therefore, $[2] \cdot [3] = [0]$. By similar reasoning, $[3] + [5] = [2]$ and $[3] \cdot [5] = [3]$. In fact, the complete addition and multiplication tables for $\mathbf{Z}_6$ are given in Figure 8.1.

If we add $[1]$ to $[0]$, add $[1]$ to $[1]$, and continue in this manner, then we obtain $[0] + [1] = [1], [1] + [1] = [2], [2] + [1] = [3], \ldots, [5] + [1] = [6] = [0], [6] + [1] = [0] + [1] = [1]$, and so forth; that is, we return to $[0]$ and cycle through all the classes of $\mathbf{Z}_6$ again (and again). If, instead of $\mathbf{Z}_6$, we were dealing with $\mathbf{Z}_{12}$, we would have $[0] + [1] = [1]$, $[1] + [1] = [2]$, $[2] + [1] = [3]$, $\ldots$, $[11] + [1] = [12] = [0]$, $[12] + [1] = [0] + [1] = [1]$, and so forth, and this should remind you of what occurs when a certain number of hours is added to a time (in hours), where, of course, 12 o'clock is represented here as 0 o'clock. (For example, if it is 11 o'clock now, what time will it be 45 hours from now?)

$+$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[0]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[1]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$
$[2]$	$[2]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$
$[3]$	$[3]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$
$[4]$	$[4]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$
$[5]$	$[5]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$

$\cdot$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$	$[0]$
$[1]$	$[0]$	$[1]$	$[2]$	$[3]$	$[4]$	$[5]$
$[2]$	$[0]$	$[2]$	$[4]$	$[0]$	$[2]$	$[4]$
$[3]$	$[0]$	$[3]$	$[0]$	$[3]$	$[0]$	$[3]$
$[4]$	$[0]$	$[4]$	$[2]$	$[0]$	$[4]$	$[2]$
$[5]$	$[0]$	$[5]$	$[4]$	$[3]$	$[2]$	$[1]$

Figure 8.1 The Addition and Multiplication Tables for $\mathbf{Z}_6$

Although the definitions of addition and multiplication in $\mathbf{Z}_n$ that we gave in (8.7) should seem quite reasonable and expected, there is a possible point of concern here that needs to be addressed. According to the definition of addition in $\mathbf{Z}_6$, $[4] + [5] = [3]$. However, the class $[4]$, which consists of all integers x such that $x \equiv 4 \pmod 6$, need not be represented this way. Since $10 \in [4]$, it follows that $[10] = [4]$. Also, $[16] = [4]$ and $[-2] = [4]$, for example. Moreover, $[11] = [5]$, $[17] = [5]$, and $[-25] = [5]$. Hence adding the equivalence classes $[4]$ and $[5]$ is the same as adding $[10]$ and $[-25]$, say, since $[10] = [4]$ and $[-25] = [5]$. But, according to the definition we have given, $[10] + [-25] = [-15]$. Luckily, $[-15] = [3]$ and so we obtain the same sum as before. But will this happen every time? That is, does the definition of the sum of the equivalence classes $[a]$ and $[b]$ that we gave in (8.7) depend on the representatives a and b of these classes? If the sum (or product) of two equivalence classes does not depend on the representatives, then we say that this sum (or product) is **well-defined**. We certainly would want this to be the case, which, fortunately, it is. More precisely, addition and multiplication in $\mathbf{Z}_n$ are **well-defined** if whenever $[a] = [b]$ and $[c] = [d]$ in $\mathbf{Z}_n$, then $[a + c] = [b + d]$ and $[ac] = [bd]$.

Theorem 8.9 *Addition in $\mathbf{Z}_n$, $n \geq 2$, is well-defined.*

Proof The set $\mathbf{Z}_n$ is the set of equivalence classes resulting from the equivalence relation R defined on $\mathbf{Z}$ by $a \, R \, b$ if $a \equiv b \pmod n$. Let $[a], [b], [c], [d] \in \mathbf{Z}_n$, where $[a] = [b]$ and $[c] = [d]$. We prove that $[a + c] = [b + d]$. Since $[a] = [b]$, it follows by Theorem 8.2 that $a \, R \, b$. Similarly, $c \, R \, d$. Therefore, $a \equiv b \pmod n$ and $c \equiv d \pmod n$. Thus, $n \mid (a - b)$ and $n \mid (c - d)$. Hence, there exist integers x and y so that

$$a - b = nx \text{ and } c - d = ny. \tag{8.8}$$

Adding the equations in (8.8), we obtain

$$(a - b) + (c - d) = nx + ny = n(x + y);$$

so $(a + c) - (b + d) = n(x + y)$. This implies that $n \mid [(a + c) - (b + d)]$. Thus, $(a + c) \equiv (b + d) \pmod{n}$. From this, we conclude that $(a + c)\ R\ (b + d)$, which implies that $[a + c] = [b + d]$. ∎

If the proof of Theorem 8.9 looks a bit familiar, review Result 4.10 and its proof. As an example, in $\mathbf{Z}_7$, $[118] + [26] = [144]$. Since the remainder is 4 when 144 is divided by 7, it follows that $[118] + [26] = [4]$. Furthermore, $[118] = [6]$ and $[26] = [5]$; so $[118] + [26] = [6] + [5] = [11] = [4]$.

As we have mentioned, the multiplication in $\mathbf{Z}_n$ that we described in (8.7) is also well-defined. The verification of this fact has been left as an exercise (Exercise 8.41).

Addition and multiplication in $\mathbf{Z}_n$ satisfy many familiar properties. Among these are:

Commutative Properties

$$[a] + [b] = [b] + [a] \text{ and } [a] \cdot [b] = [b] \cdot [a] \quad \text{for all } a, b \in \mathbf{Z};$$

Associative Properties

$$([a] + [b]) + [c] = [a] + ([b] + [c]) \text{ and}$$
$$([a] \cdot [b]) \cdot [c] = [a] \cdot ([b] \cdot [c]) \quad \text{for all } a, b, c \in \mathbf{Z};$$

Distributive Property

$$[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c] \quad \text{for all } a, b, c \in \mathbf{Z}.$$

Although we defined multiplication in $\mathbf{Z}_n$ in a manner that was probably expected, this is not the only way it could have been defined. For example, suppose that we are considering the set $\mathbf{Z}_3$ of integers modulo 3. For equivalence classes $[a]$ and $[b]$ in $\mathbf{Z}_3$, define the "product" $[a] \cdot [b]$ to equal $[q]$, where $[q]$ is the quotient when ab is divided by 3. Since the "product" of every two elements of $\mathbf{Z}_3$ is an element of $\mathbf{Z}_3$, this operation is closed. In particular, $[2] \cdot [2] = [1]$ since the quotient is 1 when $2 \cdot 2 = 4$ is divided by 3. However, $[2] = [5]$ but $[5] \cdot [5] = [8] = [2]$. Notice also that $[5] \cdot [2] = [3] = [0]$. Hence *this* multiplication is not well-defined.

EXERCISES FOR CHAPTER 8

Section 8.1: Relations

8.1. Let $A = \{a, b, c\}$ and $B = \{r, s, t, u\}$. Furthermore, let $R = \{(a, s), (a, t), (b, t)\}$ be a relation from A to B. Determine dom R and ran R.

8.2. Let A be a nonempty set and $B \subseteq \mathcal{P}(A)$. Define a relation R from A to B by $x\ R\ Y$ if $x \in Y$. Give an example of two sets A and B that illustrate this. What is R for these two sets?

8.3. Let $A = \{0, 1\}$. Determine all the relations on A.

8.4. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then $R_1 = \{(a, 2), (a, 3), (b, 1), (b, 3), (c, 4)\}$ is a relation from A to B, while $R_2 = \{(1, b), (1, c), (2, a), (2, b), (3, c), (4, a), (4, c)\}$ is a relation from B to A. A relation R is defined on A by $x\ R\ y$ if there exists $z \in B$ such that $x\ R_1\ z$ and $z\ R_2\ y$. Express R by listing its elements.

Section 8.2: Properties of Relations

8.5. Let $A = \{a, b, c, d\}$, and let $R = \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, c), (b, d), (c, c), (c, d), (d, d)\}$ be a relation on A. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

8.6. Let $S = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on S. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

8.7. Let $S = \{a, b, c\}$. Then $R = \{(a, b)\}$ is a relation on S. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

8.8. Let $A = \{a, b, c, d\}$. Give an example (with justification) of a relation R on A that has none of the following properties: reflexive, symmetric, transitive.

8.9. A relation R is defined on $\mathbf{Z}$ by $a\ R\ b$ if $|a - b| \leq 2$. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

8.10. Let $A = \{a, b, c, d\}$. How many relations defined on A are reflexive, symmetric, and transitive and contain the ordered pairs (a, b), (b, c), (c, d)?

8.11. Let $R = \emptyset$ be the empty relation on a nonempty set A. Which of the properties reflexive, symmetric, and transitive does R possess?

8.12. Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric, but not transitive.
(b) reflexive and transitive, but not symmetric.
(c) symmetric and transitive, but not reflexive.
(d) reflexive, but neither symmetric nor transitive.
(e) symmetric, but neither reflexive nor transitive.
(f) transitive, but neither reflexive nor symmetric.

8.13. A relation R is defined on $\mathbf{Z}$ by $x\ R\ y$ if $xy \geq 0$. Prove or disprove the following: (a) R is reflexive, (b) R is symmetric, (c) R is transitive.

Section 8.3: Equivalence Relations

8.14. Let R be an equivalence relation on $A = \{a, b, c, d, e, f, g\}$ such that $a\ R\ c$, $c\ R\ d$, $d\ R\ g$, and $b\ R\ f$. If there are three distinct equivalence classes resulting from R, then determine these equivalence classes and determine all elements of R.

8.15. Let R be a relation defined on $\mathbf{Z}$ by $a\ R\ b$ if $a^3 = b^3$. Show that R is an equivalence relation on $\mathbf{Z}$ and determine the distinct equivalence classes.

8.16. (a) Let R be the relation defined on $\mathbf{Z}$ by $a\ R\ b$ if $a + b$ is even. Show that R is an equivalence relation and determine the distinct equivalence classes.
(b) Suppose that "even" is replaced by "odd" in (a). Which of the properties reflexive, symmetric, and transitive does R possess?

8.17. Let $A = \{1, 2, 3, 4, 5, 6\}$. The relation

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$$

is an equivalence relation on A. Determine the distinct equivalence classes.

8.18. Let $A = \{1, 2, 3, 4, 5, 6\}$. The distinct equivalence classes resulting from an equivalence relation R on A are $\{1, 4, 5\}$, $\{2, 6\}$, and $\{3\}$. What is R?

8.19. Let R be an equivalence relation defined on a set A containing the elements a, b, c, and d. Prove that if $a \mathrel{R} b$, $c \mathrel{R} d$, and $a \mathrel{R} d$, then $b \mathrel{R} c$.

8.20. A relation R on a nonempty set A is defined to be **circular** if whenever $x \mathrel{R} y$ and $y \mathrel{R} z$, then $z \mathrel{R} x$ for all $x, y, z \in A$. Prove that a relation R on A is an equivalence relation if and only if R is circular and reflexive.

Section 8.4: Properties of Equivalence Classes

8.21. Give an example of an equivalence relation R on the set $A = \{v, w, x, y, z\}$ such that there are exactly three distinct equivalence classes. What are the equivalence classes for your example?

8.22. A relation R is defined on $\mathbf{N}$ by $a \mathrel{R} b$ if $a^2 + b^2$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

8.23. Let R be a relation defined on the set $\mathbf{N}$ by $a \mathrel{R} b$ if either $a \mid b$ or $b \mid a$. Prove or disprove: R is an equivalence relation.

8.24. Let S be a nonempty subset of $\mathbf{Z}$, and let R be a relation defined on S by $x \mathrel{R} y$ if $3 \mid (x + 2y)$.

(a) Prove that R is an equivalence relation.

(b) If $S = \{-7, -6, -2, 0, 1, 4, 5, 7\}$, then what are the distinct equivalence classes in this case?

8.25. A relation R is defined on $\mathbf{Z}$ by $x \mathrel{R} y$ if $3x - 7y$ is even. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

8.26. (a) Prove that the intersection of two equivalence relations on a nonempty set is an equivalence relation.

(b) Consider the equivalence relations R_2 and R_3 defined on $\mathbf{Z}$ by $a \mathrel{R_2} b$ if $a \equiv b \pmod 2$ and $a \mathrel{R_3} b$ if $a \equiv b \pmod 3$. By (a), $R_1 = R_2 \cap R_3$ is an equivalence relation on $\mathbf{Z}$. Determine the distinct equivalence classes in R_1.

8.27. Prove or disprove: The union of two equivalence relations on a nonempty set is an equivalence relation.

Section 8.5: Congruence Modulo n

8.28. Classify each of the following statements as true or false.

(a) $25 \equiv 9 \pmod 8$, (b) $-17 \equiv 9 \pmod 8$, (c) $-14 \equiv -14 \pmod 4$, (d) $25 \equiv -3 \pmod{11}$.

8.29. A relation R is defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $3a + 5b \equiv 0 \pmod 8$. Prove that R is an equivalence relation.

8.30. Let R be the relation defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $a + b \equiv 0 \pmod 3$. Show that R is not an equivalence relation.

8.31. The relation R on $\mathbf{Z}$ defined by $a \mathrel{R} b$ if $a^2 \equiv b^2 \pmod 4$ is known to be an equivalence relation. Determine the distinct equivalence classes.

8.32. The relation R defined on $\mathbf{Z}$ by $x \mathrel{R} y$ if $x^3 \equiv y^3 \pmod 4$ is known to be an equivalence relation. Determine the distinct equivalence classes.

8.33. A relation R is defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $5a \equiv 2b \pmod 3$. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

8.34. A relation R is defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $2a + 2b \equiv 0 \pmod 4$. Prove that R is an equivalence relation. Determine the distinct equivalence classes.

8.35. Let R be the relation defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $2a + 3b \equiv 0 \pmod 5$. Prove that R is an equivalence relation, and determine the distinct equivalence classes.

8.36. Let R be the relation defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $a^2 \equiv b^2 \pmod 5$. Prove that R is an equivalence relation, and determine the distinct equivalence classes.

Section 8.6: The Integers Modulo n

8.37. Construct the addition and multiplication tables in $\mathbf{Z}_4$ and $\mathbf{Z}_5$.

8.38. In $\mathbf{Z}_8$, express the following sums and products as $[r]$, where $0 \le r < 8$.

(a) $[2] + [6]$ (b) $[2] \cdot [6]$ (c) $[-13] + [138]$ (d) $[-13] \cdot [138]$

8.39. In $\mathbf{Z}_{11}$, express the following sums and products as $[r]$, where $0 \le r < 11$.

(a) $[7] + [5]$ (b) $[7] \cdot [5]$ (c) $[-82] + [207]$ (d) $[-82] \cdot [207]$

8.40. (a) Let $[a], [b] \in \mathbf{Z}_8$. If $[a] \cdot [b] = [0]$, does it follow that $[a] = [0]$ or $[b] = [0]$?

(b) How is the question in (a) answered if $\mathbf{Z}_8$ is replaced by $\mathbf{Z}_9$? by $\mathbf{Z}_{10}$? by $\mathbf{Z}_{11}$?

(c) For which integers $n \ge 2$ is the following statement true? (You are asked only to make a conjecture, not to provide a proof.) Let $[a], [b] \in \mathbf{Z}_n$, $n \ge 2$. If $[a] \cdot [b] = [0]$, then $[a] = [0]$ or $[b] = [0]$.

8.41. Prove that the multiplication in $\mathbf{Z}_n$, $n \ge 2$, defined by $[a][b] = [ab]$ is well-defined. (See Result 4.11.)

ADDITIONAL EXERCISES FOR CHAPTER 8

8.42. Prove or disprove:

(a) There exists an integer a such that $ab \equiv 0 \pmod 3$ for every integer b.

(b) If $a \in \mathbf{Z}$, then $ab \equiv 0 \pmod 3$ for every $b \in \mathbf{Z}$.

(c) For every integer a, there exists an integer b such that $ab \equiv 0 \pmod 3$.

8.43. Let k and ℓ be integers such that $k + \ell \equiv 0 \pmod 3$, and let $a, b \in \mathbf{Z}$. Prove that if $a \equiv b \pmod 3$, then $ka + \ell b \equiv 0 \pmod 3$.

8.44. State and prove a generalization of Exercise 8.43.

8.45. In Exercise 8.13, a relation R was defined on $\mathbf{Z}$ by $x \mathrel{R} y$ if $xy \ge 0$, and we were asked to determine which of the properties reflexive, symmetric, and transitive are satisfied.

(a) How would our answers have changed if $xy \ge 0$ was replaced by: (i) $xy \le 0$, (ii) $xy > 0$, (iii) $xy \ne 0$, (iv) $xy \ge 1$, (v) xy is odd, (vi) xy is even, (vii) $xy \not\equiv 2 \pmod 3$?

(b) What are some additional questions you could ask?

8.46. For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Every symmetric and transitive relation on a nonempty set is an equivalence relation.

Proof Let R be a symmetric and transitive relation defined on a nonempty set A. We need only show that R is reflexive. Let $x \in A$. We show that $x \mathrel{R} x$. Let $y \in A$ such that $x \mathrel{R} y$. Since R is symmetric, $y \mathrel{R} x$. Now $x \mathrel{R} y$ and $y \mathrel{R} x$. Since R is transitive, $x \mathrel{R} x$. Thus R is reflexive. ∎

8.47. Evaluate the proposed proof of the following result.

Result A relation R is defined on $\mathbf{Z}$ by $a \mathrel{R} b$ if $3 \mid (a + 2b)$. Then R is an equivalence relation.

Proof Assume that $a \mathrel{R} a$. Then $3 \mid (a + 2a)$. Since $a + 2a = 3a$ and $a \in \mathbf{Z}$, it follows that $3 \mid 3a$ or $3 \mid (a + 2a)$. Therefore, $a \mathrel{R} a$ and R is reflexive.

Next, we show that R is symmetric. Assume that $a \mathrel{R} b$. Then $3 \mid (a + 2b)$. So $a + 2b = 3x$, where $x \in \mathbf{Z}$. Hence $a = 3x - 2b$. Therefore,

$$b + 2a = b + 2(3x - 2b) = b + 6x - 4b = 6x - 3b = 3(2x - b).$$

Since $2x - b$ is an integer, $3 \mid (b + 2a)$. So $b \; R \; a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a \; R \; b$ and $b \; R \; c$. Then $3 \mid (a + 2b)$ and $3 \mid (b + 2c)$. So $a + 2b = 3x$ and $b + 2c = 3y$, where $x, y \in \mathbf{Z}$. Adding, we have $(a + 2b) + (b + 2c) = 3x + 3y$. So

$$a + 2c = 3x + 3y - 3b = 3(x + y - b).$$

Since $x + y - b$ is an integer, $3 \mid (a + 2c)$. Hence $a \; R \; c$ and R is transitive. ∎

8.48. A relation R is defined on $\mathbf{Z}$ by $a \; R \; b$ if $|a - 2| = |b - 2|$. Prove that R is an equivalence relation and determine the distinct equivalence classes.

8.49. A relation R is defined on $\mathbf{R}$ by $a \; R \; b$ if $a - b \in \mathbf{Z}$. Prove that R is an equivalence relation and determine the equivalence classes $[1/2]$ and $[\sqrt{2}]$.

8.50. Determine each of the following.

(a) $[4]^3 = [4][4][4]$ in $\mathbf{Z}_5$ (b) $[7]^5$ in $\mathbf{Z}_{10}$

8.51. Let A be a nonempty set and B a fixed subset of A. A relation R is defined on $\mathcal{P}(A)$ by $X \; R \; Y$ if $X \cap B = Y \cap B$.

(a) Prove that R is an equivalence relation.

(b) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 4\}$. For $X = \{2, 3, 4\}$, determine $[X]$.

8.52. Let R_1 and R_2 be relations on a nonempty set A. Prove or disprove each of the following.

(a) If $R_1 \cap R_2$ is reflexive, then so are R_1 and R_2.

(b) If $R_1 \cap R_2$ is symmetric, then so are R_1 and R_2.

(c) If $R_1 \cap R_2$ is transitive, then so are R_1 and R_2.

8.53. Let R be an equivalence relation on a set A. The inverse relation R^{-1} is defined on A as follows: For $a, b \in A$, $a \; R^{-1} \; b$ if $b \; R \; a$. Prove that R^{-1} is an equivalence relation on A.

8.54. Let R_1 and R_2 be equivalence relations on a nonempty set A. A relation $R = R_1 R_2$ is defined on A as follows: For $a, b \in A$, $a \; R \; b$ if there exists $c \in A$ such that $a \; R_1 \; c$ and $c \; R_2 \; b$. Prove or disprove: R is an equivalence relation on A.

8.55. A relation R is defined on $\mathbf{Z}$ by $a \; R \; b$ if $3 \mid (a^3 - b)$. Prove or disprove the following:

(a) R is reflexive.

(b) R is transitive.

8.56. A relation R is defined on $\mathbf{Z}$ by $a \; R \; b$ if $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$. Prove or disprove: R is an equivalence relation on $\mathbf{Z}$.

8.57. A relation R is defined on $\mathbf{Z}$ by $a \; R \; b$ if $a \equiv b \pmod 2$ or $a \equiv b \pmod 3$. Prove or disprove: R is an equivalence relation on $\mathbf{Z}$.

8.58. A relation R on a nonempty set S is called **sequential** if for every sequence x, y, z of elements of S (distinct or not), at least one of the ordered pairs (x, y) and (y, z) belongs to R. Prove or disprove: Every symmetric, sequential relation on a nonempty set is an equivalence relation.

8.59. Let $S = \{(a, b) : a, b \in \mathbf{R}, a \neq 0\}$.

(a) Show that the relation R defined on S by $(a, b) \; R \; (c, d)$ if $ad = bc$ is an equivalence relation.

(b) Describe geometrically the elements of the equivalence classes $[(1, 2)]$ and $[(3, 0)]$.

8.60. (a) Show that the relation R defined on $\mathbf{R} \times \mathbf{R}$ by $(a, b) \; R \; (c, d)$ if $|a| + |b| = |c| + |d|$ is an equivalence relation.

(b) Describe geometrically the elements of the equivalence classes $[(1, 2)]$ and $[(3, 0)]$.

9 Functions

If R is a relation from a set A to a set B and x is an element of A, then either x is related to no elements of B or x is related to at least one element of B. In the latter case, it may occur that x is related to all elements of B or perhaps to exactly one element of B. If every element of A is related to no elements of B, then R is the empty set $\emptyset$. If every element of A is related to all elements of B, then R is the Cartesian product $A \times B$. However, if every element of A is related to exactly one element of B, then we have the most studied relation of all: a function. Surely, you have encountered functions before, at least in calculus and precalculus. But it is likely that you have not studied functions in the manner we are about to describe here.

9.1 The Definition of Function

Let A and B be nonempty sets. By a **function** f from A to B, written $f : A \to B$, we mean a relation from A to B with the property that every element a in A is the first coordinate of exactly one ordered pair in f. Since f is a relation, the set A in this case is the **domain** of f, denoted by $\text{dom } f$. The set B is called the **codomain** of f.

For a function $f : A \to B$, let $(a, b) \in f$. Since f contains only one ordered pair whose first coordinate is a, it follows that b is the unique second coordinate of an ordered pair whose first coordinate is a; that is, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. If $(a, b) \in f$, then we write $b = f(a)$ and refer to b as the **image** of a. Sometimes f is said to **map** a into b. Indeed, f itself is sometimes called a **mapping**. The set

$$\text{ran } f = \{b \in B : b \text{ is an image under } f \text{ of some element of } A\} = \{f(x) : x \in A\}$$

is the **range** of f and consists of the second coordinates of the elements of f. If A is a finite set, then the function f is a finite set, and the number of elements in f is $|A|$ since there is exactly one ordered pair in f corresponding to each element of A. Throughout this chapter, as with earlier chapters, whenever we refer to cardinalities of sets, we are concerned with finite sets only.

Suppose that $f : A \to B$ and $g : A \to B$ are two functions from A to B and $a \in A$. Then f and g contain exactly one ordered pair having a as its first coordinate, say $(a, x) \in f$ and $(a, y) \in g$. If the sets f and g are equal, then (a, x) belongs to g as well.

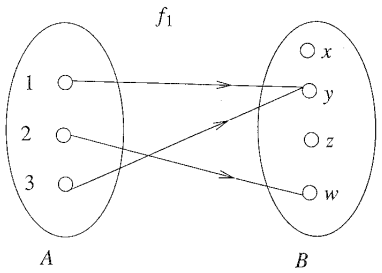

Figure 9.1 A function $f_1 : A \to B$

Since g contains only one ordered pair whose first coordinate is a, it follows that $(a, x) = (a, y)$. But this implies that $x = y$, that is, $f(a) = g(a)$. Hence it is natural to define two functions $f : A \to B$ and $g : A \to B$ to be **equal**, written $f = g$, if $f(a) = g(a)$ for all $a \in A$.

Let $A = \{1, 2, 3\}$ and $B = \{x, y, z, w\}$. Then $f_1 = \{(1, y), (2, w), (3, y)\}$ is a function from A to B, and so we may write $f_1 : A \to B$. On the other hand, $f_2 = \{(1, x), (2, z), (3, y), (2, x)\}$ is not a function since there are two ordered pairs whose first coordinate is 2. In addition, $f_3 = \{(1, z), (3, x)\}$ is not a function from A to B either because $\text{dom } f_3 \neq A$. On the other hand, f_3 *is* a function from $A - \{2\}$ to B.

It is often convenient to "visualize" a function $f : A \to B$ by representing the two sets A and B by diagrams and drawing an arrow (a directed line segment) from an element $x \in A$ to its image $f(x) \in B$. This is illustrated for the function f_1 described above in Figure 9.1. Therefore, in order to represent a function in this way, exactly one directed line segment must leave each element of A and proceed to an element of B.

In calculus, "functions" such as $f(x) = x^2$ are considered. This function f is from $\mathbf{R}$ to $\mathbf{R}$, that is, $A = \mathbf{R}$ and $B = \mathbf{R}$. Although $f(x) = x^2$ is commonly referred to as a "function" in calculus and elsewhere, strictly speaking, $f(x)$ is the image of a real number x under f. The function f itself is actually the set

$$f = \{(x, x^2) : x \in \mathbf{R}\}.$$

So $(2, 4)$ and $(-3, 9)$, for example, belong to f. The set $\{(x, x^2) : x \in \mathbf{R}\}$ of points in the plane is the graph of f. In this case, the graph is a parabola. Here the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ can also be thought of as defined by a rule, namely the rule that associates the number x^2 with each real number x.

Another function encountered in calculus is $g(x) = e^x$. As we mentioned above, this function is actually the set

$$g = \{(x, e^x) : x \in \mathbf{R}\}.$$

More precisely, this is the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = e^x$ for all $x \in \mathbf{R}$. In general, we will follow this latter convention for defining functions that are often described by some rule or formula. Consequently, the function $h(x) = \frac{1}{x-1}$ from calculus

is the function $h : \mathbf{R} - \{1\} \to \mathbf{R}$ defined by $h(x) = \frac{1}{x-1}$ for all $x \in \mathbf{R}$, $x \neq 1$, and the function $\phi(x) = \ln x$ is the function $\phi : \mathbf{R}^+ \to \mathbf{R}$ defined by $\phi(x) = \ln x$ for all $x \in \mathbf{R}^+$, where, recall, $\mathbf{R}^+$ is the set of all positive real numbers.

Among the many classes of functions encountered in calculus are the polynomial functions, rational functions, and exponential functions. The function f defined above is a polynomial function, h is a rational function, and g is an exponential function. Other important classes of functions encountered often in calculus are continuous functions and differentiable functions.

The definition of function that we have given is most likely not the definition you recall from calculus; in fact, you may not recall the definition of function given in calculus at all. If this is the case, then it is not surprising. The evolution of what is meant by a function has spanned hundreds of years. It was in the development of calculus that the necessity of a formal definition of function became apparent.

Early in the 18th century, the Swiss mathematician Johann Bernoulli wrote:

> *I call a function of a variable magnitude a quantity composed in any manner whatsoever from this variable magnitude and from constants.*

Later in the 18th century, the famous Swiss mathematician Leonhard Euler studied calculus as a theory of functions and did not appeal to diagrams and geometric interpretations, as many of his predecessors had done. The definition of function that Euler gave in his work on calculus is:

> *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Early in the 19th century, the German mathematician Peter Dirichlet developed a more modern definition of function:

> *y is a function of x when to each value of x in a given interval there corresponds a unique value of y.*

Dirichlet said that it didn't matter whether y depends on x according to some formula, law, or mathematical operation. He emphasized this by considering the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Later in the 19th century, the German mathematician Richard Dedekind wrote:

> *A function ϕ on a set S is a law according to which to every determinate element s of S there belongs a determinate thing which is called the transform of s and is denoted by $\phi(s)$.*

So, by this time, the modern definition of function had nearly arrived.

9.2 The Set of All Functions from A to B

For nonempty sets A and B, we denote the set of all functions from A to B by B^A. That is, $B^A = \{f : f \text{ is a function from } A \text{ to } B\}$ or, more simply,

$$B^A = \{f : f : A \to B\}.$$

Although this may seem like peculiar notation, it is actually quite logical. In particular, let's determine B^A for $A = \{a, b\}$ and $B = \{x, y, z\}$. Each function f from A to B is necessarily of the form

$$f = \{(a, \alpha), (b, \beta)\},$$

where $\alpha, \beta \in B$. Since there are 3 choices for α and 3 choices for β, the total number of such functions f is $3 \cdot 3 = 3^2 = 9$. These nine functions are listed below:

$$\begin{array}{lll} f_1 = \{(a, x), (b, x)\}, & f_2 = \{(a, x), (b, y)\}, & f_3 = \{(a, x), (b, z)\}, \\ f_4 = \{(a, y), (b, x)\}, & f_5 = \{(a, y), (b, y)\}, & f_6 = \{(a, y), (b, z)\}, \\ f_7 = \{(a, z), (b, x)\}, & f_8 = \{(a, z), (b, y)\}, & f_9 = \{(a, z), (b, z)\}. \end{array}$$

Hence the number of elements in B^A is 3^2. In general, for finite sets A and B, the number of functions from A to B is

$$|B^A| = |B|^{|A|}.$$

If $B = \{0, 1\}$, then it is common to represent the set of all functions from A to B by 2^A.

9.3 One-to-One and Onto Functions

We now consider two important properties that a function may possess. A function f from a set A to a set B is called **one-to-one** or **injective** if every two distinct elements of A have distinct images in B. In symbols, a function $f : A \to B$ is one-to-one if whenever $x, y \in A$ and $x \neq y$, then $f(x) \neq f(y)$. Thus, if a function $f : A \to B$ is not one-to-one, then there exist distinct elements w and z in A such that $f(w) = f(z)$.

Let $A = \{a, b, c, d\}$, $B = \{r, s, t, u, v\}$, and $C = \{x, y, z\}$. Then

$$f_1 = \{(a, s), (b, u), (c, v), (d, r)\}$$

is a one-to-one function from A to B since distinct elements of A have distinct images in B; while the function

$$f_2 = \{(a, s), (b, t), (c, s), (d, u)\}$$

from A to B is not one-to-one since a and c have the same image, namely s. There is no one-to-one function from A to C, however.

In general, for a function $f : A \to B$ to be one-to-one, where A and B are finite sets, every two elements of A must have distinct images in B, and so there must be at least as many elements in B as in A, that is, $|A| \leq |B|$.

At times, the definition of a one-to-one function is difficult to work with since it deals with *unequal* elements. However, there is a useful equivalent formulation of the definition using the contrapositive:

A function $f : A \to B$ is **one-to-one** if
whenever $f(x) = f(y)$, where $x, y \in A$, then $x = y$.

We show how this formulation can be applied to functions defined by formulas.

Result 9.1 *Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x - 5$. Then f is one-to-one.*

Proof Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R}$. Then $3a - 5 = 3b - 5$. Adding 5 to both sides, we obtain $3a = 3b$. Dividing by 3, we have $a = b$, and so f is one-to-one. ∎

Example 9.2 *Let the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 - 3x - 2$. Determine whether f is one-to-one.*

Solution Since $f(0) = -2$ and $f(3) = -2$, it follows that f is not one-to-one. ♦

Analysis Thus to show that the function f defined in Example 9.2 is not one-to-one, we must show that there exist two distinct real numbers having the same image under f. This was accomplished by showing that $f(0) = f(3)$. But what if we can't find two real numbers with this property? Naturally, if we can't find two such numbers, then we might think that f is one-to-one. In that case, we should be trying to prove that f is one-to-one. We would probably begin such a proof by assuming that $f(a) = f(b)$, that is, $a^2 - 3a - 2 = b^2 - 3b - 2$. We would then try to show that $a = b$. We can simplify $a^2 - 3a - 2 = b^2 - 3b - 2$ by adding 2 to both sides, producing $a^2 - 3a = b^2 - 3b$. When attempting to solve an equation, it is often convenient to collect all terms on one side of the equation with 0 on the other side. Rewriting this equation, we obtain $a^2 - 3a - b^2 + 3b = 0$. Rearranging some terms and factoring, we have

$$\begin{aligned} a^2 - 3a - b^2 + 3b &= (a^2 - b^2) - 3(a - b) \\ &= (a - b)(a + b) - 3(a - b) = (a - b)(a + b - 3) = 0. \end{aligned}$$

Hence if $f(a) = f(b)$, then $(a - b)(a + b - 3) = 0$. Since $(a - b)(a + b - 3) = 0$, it follows that either $a - b = 0$ (and so $a = b$) or $a + b - 3 = 0$. Therefore, $f(a) = f(b)$ does *not* imply that $a = b$. It only implies that $a = b$ *or* $a + b = 3$. Since $0 + 3 = 3$, we now see why $f(0) = f(3)$. In fact, if a and b are *any* two real numbers where $a + b = 3$, then $f(a) = f(b)$. This tells us how to find all possible counterexamples to the statement: f is one-to-one. Looking at $f(x) = x^2 - 3x - 2$ once again, we see that $f(x) = x(x - 3) - 2$. Since $x(x - 3) = 0$ if $x = 0$ or $x = 3$, it is now more apparent why 0 and 3 are numbers for which $f(0) = f(3)$. ♦

A function $f : A \to B$ is called **onto** or **surjective** if every element of the codomain B is the image of some element of A.

A function we considered earlier was $f_1 : A \to B$, where $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$, and $f_1 = \{(1, y), (2, w), (3, y)\}$. This function f_1 is *not* onto since neither x nor z is an image of some element of A. You might notice that for these two sets A and B, there is *no* function from A to B that is onto since any such function has exactly three ordered pairs but B has four elements. Thus for finite sets A and B, if $f : A \to B$ is a surjective function, then $|B| \leq |A|$. The function $g : B \to A$, where

$g = \{(x, 3), (y, 1), (z, 3), (w, 2)\}$ *is* a surjective function, however, since each of the elements 1, 2, and 3 is an image of some element of B. Next, we determine which of the functions defined in Result 9.1 and Example 9.2 are onto.

Result to Prove The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is onto.

PROOF STRATEGY Let's make a few observations before we begin the proof. To show that f is onto, we must show that every element in the codomain $B = \mathbf{R}$ is the image of some element in the domain $A = \mathbf{R}$. Since $f(0) = -5$ and $f(1) = -2$, certainly -5 and -2 are images of elements of $\mathbf{R}$. The real number 10 is an image as well since $f(5) = 10$. Is π an image of some real number? To answer this question, we need to determine whether there is a real number x such that $f(x) = \pi$. Since $f(x) = 3x - 5$, we need only find a solution for x to the equation $3x - 5 = \pi$. Solving this equation for x, we find that $x = (\pi + 5)/3$, which, of course, is a real number. Finally, observe that

$$f(x) = f\left(\frac{\pi + 5}{3}\right) = 3\left(\frac{\pi + 5}{3}\right) - 5 = \pi.$$

This discussion, however, gives us the information we need to prove that f is onto since for an arbitrary real number r, say, we need to find a real number x such that $f(x) = r$. However, then, $3x - 5 = r$ and $x = (r + 5)/3$. ♦

Result 9.3 *The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is onto.*

Proof Let $r \in \mathbf{R}$. We show that there exists $x \in \mathbf{R}$ such that $f(x) = r$. Choose $x = (r + 5)/3$. Then $x \in \mathbf{R}$ and

$$f(x) = f\left(\frac{r + 5}{3}\right) = 3\left(\frac{r + 5}{3}\right) - 5 = r.$$

■

PROOF ANALYSIS Notice that the proof itself of Result 9.3 does not include consideration of the equation $3x - 5 = r$. Our goal was to show that some real number x exists such that $f(x) = r$. How we obtain this number, though possibly interesting, is *not* part of the proof. On the other hand, it may be a good idea to accompany the proof with this information. ♦

Let $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$, and $C = \{a, b, c\}$. Four functions $g_1 : A \to B$, $g_2 : B \to C$, $g_3 : A \to C$, and $g_4 : A \to C$ are defined as follows:

$$g_1 = \{(1, y), (2, w), (3, x)\},$$
$$g_2 = \{(x, b), (y, a), (z, c), (w, b)\},$$
$$g_3 = \{(1, a), (2, c), (3, b)\},$$
$$g_4 = \{(1, b), (2, b), (3, b)\}.$$

The functions g_1 and g_3 are one-to-one, while g_2 and g_4 are not one-to-one since $g_2(x) = g_2(w) = b$ and $g_4(1) = g_4(2) = b$. Both g_2 and g_3 are onto. The function g_1 is not onto because z is not an image of any element of A, while g_4 is not onto since neither a nor c is an image of an element of A.

9.4 Bijective Functions

We have already mentioned, for finite sets A and B, that if $f : A \to B$ is a surjective function, then $|A| \geq |B|$. Also, we mentioned that if $f : A \to B$ is one-to-one, then $|A| \leq |B|$. Hence if A and B are finite sets and there is a function $f : A \to B$ that is both one-to-one and onto, then $|A| = |B|$. What happens when A and B are infinite sets will be dealt with in detail in Chapter 10.

A function $f : A \to B$ is called **bijective** or a **one-to-one correspondence** if it is both one-to-one and onto. From what we mentioned earlier, if a function $f : A \to B$ is bijective and A and B are finite sets, then $|A| = |B|$. Perhaps it is also clear that if A and B are finite sets with $|A| = |B|$, then there exists a bijective function $f : A \to B$. A bijective function from a set A to a set B creates a pairing of the elements of A with the elements of B.

There is another interesting fact concerning the existence of bijective functions $f : A \to B$ for finite sets A and B with $|A| = |B|$.

Theorem 9.4 *Let A and B be finite nonempty sets such that $|A| = |B|$, and let f be a function from A to B. Then f is one-to-one if and only if f is onto.*

Proof Let $|A| = |B| = n$. Assume first that f is one-to-one. Since the n elements of A have distinct images, there are n distinct images. Thus $\operatorname{ran} f = B$ and so f is onto.

For the converse, assume that f is onto. Thus each of the n elements of B is an image of some element of A. Consequently, the n elements of A have n distinct images in B, which implies that no two distinct elements of A can have the same image and so f is one-to-one. ■

Theorem 9.4 concerns finite sets A and B with $|A| = |B|$. Even though we have not defined cardinality for infinite sets, we would certainly expect that $|A| = |A|$ for every infinite set A. With this understanding, Theorem 9.4 is false for infinite sets A and B, even when $A = B$. For example, the function $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = 2n$ is one-to-one; yet its range is the set of all even integers. That is, f is not onto, even though f is a one-to-one function from $\mathbf{Z}$ to $\mathbf{Z}$. The function $f : \mathbf{N} \to \mathbf{N}$ defined by $g(n) = n - 1$ when $n \geq 2$ and $g(1) = 1$ is onto but not one-to-one since $g(1) = g(2) = 1$.

For the sets $A = \{1, 2, 3\}$, $B = \{x, y, z, w\}$, and $C = \{a, b, c\}$ described above then, no function from A to B or from B to C can be bijective. It is possible to have a bijective function from A to C, however, since $|A| = |C|$. In fact, g_3 is such a function, although other bijective functions from A to C exist. Certainly, not every function from A to C is bijective, as g_4 illustrates.

For a nonempty set A, the function $i_A : A \to A$ defined by $i_A(a) = a$ for each $a \in A$ is called the **identity function** on A. If the set A under discussion is clear, we write the identity function i_A by i. For $S = \{1, 2, 3\}$, the identity function is

$$i_S = i = \{(1, 1), (2, 2), (3, 3)\}.$$

Not only is *this* identity function bijective, the identity function i_A is bijective for *every* nonempty set A. Identity functions are important and we will see them again soon.

We give one additional example of a bijective function.

Result 9.5 *The function* $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{3\}$ *defined by*

$$f(x) = \frac{3x}{x-2}$$

is bijective.

Proof Here it is necessary to show that f is both one-to-one and onto. We begin with the first of these. Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R} - \{2\}$. Then $\frac{3a}{a-2} = \frac{3b}{b-2}$. Multiplying both sides by $(a-2)(b-2)$, we obtain $3a(b-2) = 3b(a-2)$. Simplifying, we have $3ab - 6a = 3ab - 6b$. Adding $-3ab$ to both sides and dividing by -6, we obtain $a = b$. Thus f is one-to-one.

To show that f is onto, let $r \in \mathbf{R} - \{3\}$. We show that there exists $x \in \mathbf{R} - \{2\}$ such that $f(x) = r$. Choose $x = \frac{2r}{r-3}$. Then

$$f(x) = f\left(\frac{2r}{r-3}\right) = \frac{3\left(\frac{2r}{r-3}\right)}{\frac{2r}{r-3} - 2}$$
$$= \frac{6r}{2r - 2(r-3)} = \frac{6r}{6} = r,$$

implying that f is onto. Therefore f is bijective. ∎

PROOF ANALYSIS Some remarks concerning the proof that the function f in Result 9.5 is onto may be useful. For a given real number r in $\mathbf{R} - \{3\}$, we need to find a real number x in $\mathbf{R} - \{2\}$ such that $f(x) = r$. Since we wanted $f(x) = \frac{3x}{x-2} = r$, it was required to solve this equation for x. This can be done by rewriting this equation as $3x = r(x-2)$ and then simplifying it to obtain $rx - 3x = 2r$. Now, factoring x from $rx - 3x$ and dividing by $r - 3$, we have the desired choice of x, namely $x = 2r/(r-3)$. Incidentally, it was perfectly permissible to divide by $r - 3$ since $r \in \mathbf{R} - \{3\}$ and so $r \neq 3$. Notice also that $x \in \mathbf{R} - \{2\}$, for if $x = 2r/(r-3) = 2$, then $2r = 2r - 6$, which is impossible. Although solving $\frac{3x}{x-2} = r$ for x is *not* part of the proof, again it may be useful to include this work in addition to the proof. ♦

Of course, if $f(x) = f(y)$ implies that $x = y$ for all $x, y \in A$, then f is one-to-one. It may seem obvious that if $x = y$, then $f(x) = f(y)$ for all $x, y \in A$ since this is simply a requirement of a function.

In order for a relation f from a set A to a set B to be a function from A to B, the following two conditions must be satisfied:

(1) For each element $a \in A$, there is an element $b \in B$ such that $(a, b) \in f$.
(2) If $(a, b), (a, c) \in f$, then $b = c$.

Condition (1) states that the domain of f is A, that is, every element of A has an image in B; while condition (2) says that if an element of A has an image in B, then this image is unique.

Occasionally, a function f that satisfies condition (2) is called **well-defined**. Since (2) is a requirement of every function however, it follows that every function must be well-defined. There are situations though when the definition of a function f may make it unclear whether f is well-defined. This can often occur when a function is defined on the set of equivalence classes of an equivalence relation. The next result illustrates this with the equivalence classes for the relation congruence modulo 4 on the set of integers.

Result to Prove The function $f : \mathbf{Z}_4 \to \mathbf{Z}_4$ defined by $f([x]) = [3x+1]$ is a well-defined bijective function.

PROOF STRATEGY To prove that this function is well-defined, we are required to prove that if $[a] = [b]$, then $f([a]) = f([b])$, that is, $[3a+1] = [3b+1]$. It seems reasonable to use a direct proof, so we assume that $[a] = [b]$. Since $[a]$ and $[b]$ are elements of $\mathbf{Z}_4$, to say that $[a] = [b]$ means that $a \equiv b \pmod 4$. Since $a \equiv b \pmod 4$, it follows that $4 \mid (a-b)$ and so $a - b = 4k$ for some integer k. To verify that $[3a+1] = [3b+1]$, we are required to show that $3a + 1 \equiv 3b + 1 \pmod 4$ or, equivalently, that $(3a+1) - (3b+1) = 3a - 3b = 3(a-b)$ is a multiple of 4.

Since $\mathbf{Z}_4$ consists only of four elements, namely, $[0]$, $[1]$, $[2]$, $[3]$, to prove that f is bijective, we need only observe that the elements $f([0])$, $f([1])$, $f([2])$, $f([3])$ are distinct. ♦

Result 9.6 *The function* $f : \mathbf{Z}_4 \to \mathbf{Z}_4$ *defined by* $f([x]) = [3x+1]$ *is a well-defined bijective function.*

Proof First, we verify that this function is well-defined; that is, if $[a] = [b]$, then $f([a]) = f([b])$. Assume then that $[a] = [b]$. Thus $a \equiv b \pmod 4$ and so $4 \mid (a-b)$. Hence $a - b = 4k$ for some integer k. Therefore,

$$(3a+1) - (3b+1) = 3(a-b) = 3(4k) = 4(3k).$$

Since $3k$ is an integer, $4 \mid [(3a+1) - (3b+1)]$. Thus $3a + 1 \equiv 3b + 1 \pmod 4$ and $[3a+1] = [3b+1]$, so $f([a]) = f([b])$. Hence f is well-defined. Since $f([0]) = [1]$, $f([1]) = [0]$, $f([2]) = [3]$, and $f([3]) = [2]$, it follows that f is both one-to-one and onto; that is, f is bijective. ∎

9.5 Composition of Functions

As it is common to define operations on certain sets of numbers (and on the set $\mathbf{Z}_n$ of equivalence classes, as we described in Chapter 8), it is possible to define operations on certain sets of functions, under suitable circumstances. For example, for functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$, you might recall from calculus that the sum $f + g$ and product fg of f and g are defined by

$$(f+g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x) \cdot g(x) \qquad \textbf{(9.1)}$$

for all $x \in \mathbf{R}$. So if f is defined by $f(x) = x^2$ and g is defined by $g(x) = \sin x$, then $(f+g)(x) = x^2 + \sin x$ and $(fg)(x) = x^2 \sin x$ for all $x \in \mathbf{R}$. In calculus we are especially interested in these operations because once we have learned how to determine the

derivatives of f and g, we want to know how to use this information to find the derivatives of $f + g$ and fg. The derivative of fg, for example, gives rise to the well-known product rule for derivatives:

$$(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

This later led us to study the quotient rule for derivatives.

The definitions in (9.1) of the sum $f + g$ and product fg of the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ depend on the fact that the codomain of these two functions is $\mathbf{R}$, whose elements can be added and multiplied, and so $f(x) + g(x)$ and $f(x) \cdot g(x)$ make sense. On the other hand, if $f : A \to B$ and $g : A \to B$, where $B = \{a, b, c\}$, say, then $f(x) + g(x)$ and $f(x) \cdot g(x)$ have no meaning.

There is an operation that can be defined on pairs of functions satisfying appropriate conditions that has no connection with numbers. For nonempty sets A, B, and C and functions $f : A \to B$ and $g : B \to C$, it is possible to create a new function from f and g, called their composition. The **composition** $g \circ f$ of f and g is the function from A to C defined by

$$(g \circ f)(a) = g(f(a)) \quad \text{for all } a \in A.$$

To illustrate this definition, let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$, and $C = \{r, s, t, u, v\}$, and define the functions $f : A \to B$ and $g : B \to C$ by

$$f = \{(1, b), (2, d), (3, a), (4, a)\},$$
$$g = \{(a, u), (b, r), (c, r), (d, s)\}.$$

We now have the correct arrangement of sets and functions to consider the composition $g \circ f$. Since $g \circ f$ is a function from A to C, it follows that $g \circ f$ has the following appearance:

$$g \circ f = \{(1, \alpha), (2, \beta), (3, \gamma), (4, \delta)\},$$

where $\alpha, \beta, \gamma, \delta \in C$. It remains only to determine the image of each element of A. First, we find the image of 1. According to the definition of $g \circ f$,

$$(g \circ f)(1) = g(f(1)) = g(b) = r,$$

so $(1, r) \in g \circ f$. Similarly, $(g \circ f)(2) = g(f(2)) = g(d) = s$, and so $(2, s) \in g \circ f$. Continuing in this manner, we obtain

$$g \circ f = \{(1, r), (2, s), (3, u), (4, u)\}.$$

A diagram that illustrates how $g \circ f$ is determined is shown in Figure 9.2. To find the image of 1 under $g \circ f$, we follow the arrow from 1 to b and then from b to r. The function $g \circ f$ is basically found by removing the set B. The fact that $g \circ f$ is defined does not necessarily imply that $f \circ g$ is also defined. Since g is a function from B to C and f is a function from A to B, the only way that $f \circ g$ would be defined is if $\operatorname{ran} g \subseteq A$. In the example we have just seen, $f \circ g$ is not defined since $\operatorname{ran} g = \{r, s, u\} \not\subseteq A$.

Composition of functions is also encountered in calculus. Let's consider an example of composition that you might have seen in calculus. Again, suppose that the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x^2$ and $g(x) = \sin x$. In

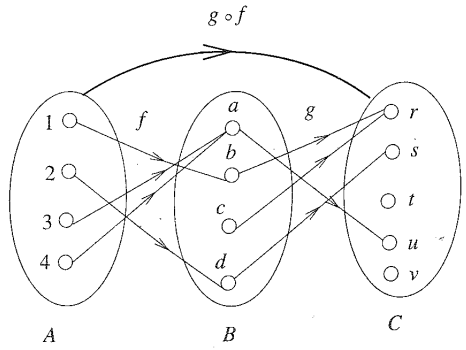

Figure 9.2 The composition function $g \circ f$

this case, we can determine both $g \circ f$ and $f \circ g$; namely,

$$(g \circ f)(x) = g(f(x)) = g(x^2) = \sin(x^2)$$
$$(f \circ g)(x) = f(g(x)) = f(\sin x) = (\sin x)^2 = \sin^2 x.$$

This example also serves to illustrate that even when $g \circ f$ and $f \circ g$ are both defined, they need not be equal.

The study of composition of functions in calculus led us to the well-known chain rule for differentiation:

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x).$$

There are two facts concerning properties of composition of functions that will be especially useful to us. First, if f and g are injective functions such that $g \circ f$ is defined, then $g \circ f$ is injective. The corresponding statement is also true for surjective functions.

Result to Prove Let $f : A \to B$ and $g : B \to C$ be two functions.

(a) If f and g are injective, then so is $g \circ f$.
(b) If f and g are surjective, then so is $g \circ f$.

PROOF STRATEGY To verify (a), we use a direct proof and begin by assuming that f and g are one-to-one. To show that $g \circ f$ is one-to-one, we prove that whenever $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$. However, $(g \circ f)(a_1) = (g \circ f)(a_2)$ means that $g(f(a_1)) = g(f(a_2))$. But g is one-to-one, so $g(x) = g(y)$ implies that $x = y$. The form $g(x) = g(y)$ is exactly what we have, where $x = f(a_1)$ and $y = f(a_2)$. This leads us to $f(a_1) = f(a_2)$. But we also know that f is one-to-one.

To verify (b), we need to prove that if f and g are onto, then $g \circ f$ is onto. To show that $g \circ f$ is onto, it is necessary to show that every element of C is an image of some element of A under the function $g \circ f$. So we begin with an element $c \in C$. Since g is

onto, there is an element $b \in B$ such that $g(b) = c$. But f is onto, so there is an element $a \in A$ such that $f(a) = b$. This suggests considering $(g \circ f)(a)$. ♦

Theorem 9.7 *Let $f : A \to B$ and $g : B \to C$ be two functions.*

(a) *If f and g are injective, then so is $g \circ f$.*
(b) *If f and g are surjective, then so is $g \circ f$.*

Proof Let $f : A \to B$ and $g : B \to C$ be injective functions. Assume that $(g \circ f)(a_1) = (g \circ f)(a_2)$, where $a_1, a_2 \in A$. By definition, $g(f(a_1)) = g(f(a_2))$. Since g is injective, it follows that $f(a_1) = f(a_2)$. However, since f is injective, it follows that $a_1 = a_2$. This implies that $g \circ f$ is injective.

Next let $f : A \to B$ and $g : B \to C$ be surjective functions, and let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. On the other hand, since f is surjective, it follows that there exists $a \in A$ such that $f(a) = b$. Hence $(g \circ f)(a) = g(f(a)) = g(b) = c$, implying that $g \circ f$ is also surjective. ■

Combining the two parts of Theorem 9.7 produces an immediate corollary.

Corollary 9.8 *If $f : A \to B$ and $g : B \to C$ are bijective functions, then $g \circ f$ is bijective.*

For nonempty sets A, B, C, and D, let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then the compositions $g \circ f : A \to C$ and $h \circ g : B \to D$ are defined, as are the compositions $h \circ (g \circ f) : A \to D$ and $(h \circ g) \circ f : A \to D$. Composition of the functions f, g, and h is **associative** if the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal. This is, in fact, the case.

Theorem 9.9 *For nonempty sets A, B, C, and D, let $f : A \to B$, $g : B \to C$, and $h : C \to D$ be functions. Then $(h \circ g) \circ f = h \circ (g \circ f)$.*

Proof Let $a \in A$ and suppose that $f(a) = b$, $g(b) = c$, and $h(c) = d$. Then

$$((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = (h \circ g)(b) = h(g(b)) = h(c) = d;$$

while

$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = h(g(b)) = h(c) = d.$$

Thus $(h \circ g) \circ f = h \circ (g \circ f)$. ■

As we have mentioned, it is common, when considering the composition of functions, to begin with two functions f and g, where $f : A \to B$ and $g : B \to C$ and arrive at the function $g \circ f : A \to C$. Strictly speaking, however, all that is needed is for the domain of g to be a set B' where ran f is a subset of B'. In other words, if f and g are functions with $f : A \to B$ and $g : B' \to C$, where ran $f \subseteq B'$, then the composition $g \circ f : A \to C$ is defined.

Example 9.10 *For the sets $A = \{-3, -2, \ldots, 3\}$ and $B = \{0, 1, \ldots, 10\}$, $B' = \{0, 1, 4, 5, 8, 9\}$, and $C = \{1, 2, \ldots, 10\}$, let $f : A \to B$ and $g : B' \to C$ be functions defined by $f(n) = n^2$ for all $n \in A$ and $g(n) = n + 1$ for all $n \in B'$.*

(a) *Show that the composition $g \circ f : A \to C$ is defined.*
(b) *For $n \in A$, determine $(g \circ f)(n)$.*

Solution (a) Since ran $f = \{0, 1, 4, 9\}$ and ran $f \subseteq B'$, it follows that the composition $g \circ f : A \to C$ is defined.
(b) For $n \in A$, $(g \circ f)(n) = g(f(n)) = g(n^2) = n^2 + 1$. ♦

9.6 Inverse Functions

Next we describe a property possessed by all bijective functions. In preparation for doing this, we return to relations to recall a concept introduced in Chapter 8. For a relation R from a set A to a set B, the **inverse relation** R^{-1} from B to A is defined as

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

For example, if $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, and

$$R = \{(a, 1), (a, 3), (c, 2), (c, 3), (d, 1)\}$$

is a relation from A to B, then

$$R^{-1} = \{(1, a), (3, a), (2, c), (3, c), (1, d)\}$$

is the inverse relation of R. Of course, every function $f : A \to B$ is also a relation from A to B, and so there is an inverse relation f^{-1} from B to A. This brings up a natural question: Under what conditions is the inverse relation f^{-1} from B to A also a function from B to A? If the inverse relation f^{-1} is a function from B to A, then certainly dom $f^{-1} = B$. This implies that f must be onto. If f is not one-to-one, then $f(a_1) = f(a_2) = b$ for some $a_1, a_2 \in A$ and $b \in B$, where $a_1 \neq a_2$. But then $(b, a_1), (b, a_2) \in f^{-1}$, which cannot occur if f^{-1} is a function. This leads us to the following theorem. In the proof, two basic facts are used repeatedly, namely

(1) *$f(a) = b$ if and only if $(a, b) \in f$, and*
(2) *if f^{-1} is a function and $f(a) = b$, then $(b, a) \in f^{-1}$.*

Theorem 9.11 *Let $f : A \to B$ be a function. Then the inverse relation f^{-1} is a function from B to A if and only if f is bijective. Furthermore, if f is bijective, then f^{-1} is also bijective.*

Proof First, assume that f^{-1} is a function from B to A. Then we show that f is both one-to-one and onto. Assume that $f(a_1) = f(a_2) = y$, where $y \in B$. Then $(a_1, y), (a_2, y) \in f$, implying that $(y, a_1), (y, a_2) \in f^{-1}$. Since f^{-1} is a function from B to A, every element of B has a unique image under f^{-1}. Thus, in particular, y has a unique image under f^{-1}. Since $f^{-1}(y) = a_1$ and $f^{-1}(y) = a_2$, it now follows that $a_1 = a_2$, and so f is one-to-one.

To show that f is onto, let $b \in B$. Since f^{-1} is a function from B to A, there exists a unique element $a \in A$ such that $f^{-1}(b) = a$. Hence $(b, a) \in f^{-1}$, implying that $(a, b) \in f$, that is, $f(a) = b$. Therefore, f is onto.

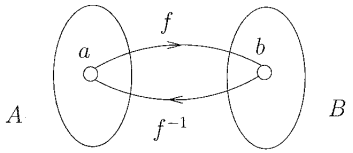

Figure 9.3 A bijective function and its inverse

For the converse, assume that the function $f : A \to B$ is bijective. We show that f^{-1} is a function from B to A. Let $b \in B$. Since f is onto, there exists $a \in A$ such that $(a, b) \in f$. Hence $(b, a) \in f^{-1}$. It remains to show that (b, a) is the unique element of f^{-1} whose first coordinate is b. Assume that (b, a) and (b, a') are both in f^{-1}. Then $(a, b), (a', b) \in f$, which implies that $f(a) = f(a') = b$. Since f is one-to-one, $a = a'$. Therefore, we have shown that for every $b \in B$ there exists a unique element $a \in A$ such that $(b, a) \in f^{-1}$; that is, f^{-1} is a function from B to A.

Finally, we show that if f is bijective, then f^{-1} is bijective. Assume that f is bijective. We have just seen that f^{-1} is a function from B to A. First, we show that f^{-1} is one-to-one.

Assume that $f^{-1}(b_1) = f^{-1}(b_2) = a$. Then $(b_1, a), (b_2, a) \in f^{-1}$, and so $(a, b_1), (a, b_2) \in f$. Since f is a function, $b_1 = b_2$ and f^{-1} is one-to-one. To show that f^{-1} is onto, let $a \in A$. Since f is a function, there is an element $b \in B$ such that $(a, b) \in f$. Consequently, $(b, a) \in f^{-1}$ so that $f^{-1}(b) = a$, and f^{-1} is onto. Therefore, f^{-1} is bijective. ∎

Let $f : A \to B$ be a bijective function. By Theorem 9.11 then, $f^{-1} : B \to A$ is a bijective function, which is referred to as the **inverse function** or simply the **inverse** of f. Hence both composition functions $f^{-1} \circ f$ and $f \circ f^{-1}$ are defined. In fact, $f^{-1} \circ f$ is a function from A to A and $f \circ f^{-1}$ is a function from B to B. As we are about to learn, $f^{-1} \circ f$ and $f \circ f^{-1}$ are functions we've visited earlier. Let $a \in A$ and suppose that $f(a) = b$. So $(a, b) \in f$ and therefore $(b, a) \in f^{-1}$, that is, $f^{-1}(b) = a$. Thus $\left(f^{-1} \circ f\right)(a) = f^{-1}(f(a)) = f^{-1}(b) = a$, and $\left(f \circ f^{-1}\right)(b) = f\left(f^{-1}(b)\right) = f(a) = b$. So it follows that

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B$$

are the identity functions on the sets A and B. (See Figure 9.3.)

If a bijective function f has a relatively small number of ordered pairs, then it is easy to find f^{-1}. But what if f is a bijective function that one might encounter in calculus, say? We illustrate this next with the function described in Result 9.5.

Example 9.12 *The function* $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{3\}$ *defined by*

$$f(x) = \frac{3x}{x - 2}$$

is known to be bijective. Determine $f^{-1}(x)$*, where* $x \in \mathbf{R} - \{3\}$.

Solution Since $\left(f \circ f^{-1}\right)(x) = x$ for all $x \in \mathbf{R} - \{3\}$, it follows that

$$\left(f \circ f^{-1}\right)(x) = f\left(f^{-1}(x)\right) = \frac{3f^{-1}(x)}{f^{-1}(x) - 2} = x.$$

Thus $3f^{-1}(x) = x(f^{-1}(x) - 2)$ and $3f^{-1}(x) = xf^{-1}(x) - 2x$. Collecting the terms involving $f^{-1}(x)$ on the same side of the equation and then factoring out the term $f^{-1}(x)$, we have

$$xf^{-1}(x) - 3f^{-1}(x) = 2x;$$

so

$$f^{-1}(x)(x - 3) = 2x.$$

Solving for $f^{-1}(x)$, we obtain

$$f^{-1}(x) = \frac{2x}{x - 3}.$$ ♦

Analysis You might very well have dealt with the problem of finding the inverse of a function before and might recall a somewhat different approach than the one we just gave. Let's look at this example again, but from a different perspective.

When we consider functions from calculus, rather than writing $f(x) = x^2$, $g(x) = 5x + 1$, or $h(x) = x + \frac{1}{x}$, we sometimes write these as $y = x^2$, $y = 5x + 1$, or $y = x + \frac{1}{x}$. In Example 9.12, we were given $f(x) = \frac{3x}{x-2}$ and found that $f^{-1}(x) = \frac{2x}{x-3}$. Let's write the inverse as $y = \frac{2x}{x-3}$ instead. That is, $(x, y) \in f^{-1}$, where $y = \frac{2x}{x-3}$. Of course, initially, we don't know what y is. But if $(x, y) \in f^{-1}$, then $(y, x) \in f$ and we know that $x = f(y) = \frac{3y}{y-2}$. Solving this equation for y, we have $x(y - 2) = 3y$, so $xy - 2x = 3y$. Collecting the terms with y on the same side of the equation and factoring out the term y, we obtain

$$xy - 3y = 2x \quad \text{and} \quad y(x - 3) = 2x.$$

Solving for y, we obtain $y = \dfrac{2x}{x - 3}$; that is,

$$f^{-1}(x) = \frac{2x}{x - 3}.$$

In short, to find f^{-1} if $f(x) = \frac{3x}{x-2}$, we replace $f(x)$ by x and x by y, and then solve for y. The result is $f^{-1}(x)$. Of course, the procedure we have described for finding $f^{-1}(x)$ is exactly the same as before. The only difference is the notation. You might have also noticed that the algebra performed to determine $f^{-1}(x)$ in Example 9.12 is exactly the same as the algebra performed in proving f is onto in Result 9.5. ♦

Finding the inverse of a bijective function is not always possible by algebraic manipulation. For example, the function $f : \mathbf{R} \to (0, \infty)$ defined by $f(x) = e^x$ is bijective, but $f^{-1}(x) = \ln x$. Indeed, the function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = 3x^7 + 5x^3 + 4x - 1$ is bijective but there is no way to find an expression for $g^{-1}(x)$.

If $f : A \to B$ is a one-to-one function from A to B that is not onto, then, of course, f is not bijective, and, according to Theorem 9.11, f does not have an inverse (from B

to A). On the other hand, if we define a new function $g : A \to \text{ran } f$ by $g(x) = f(x)$ for all $x \in A$, then g is a bijective function and so its inverse function $g^{-1} : \text{ran } f \to A$ exists. For example, let E denote the set of all even integers and consider the function $f : \mathbf{Z} \to \mathbf{Z}$ by $f(n) = 2n$. Then this function f is injective but not surjective, and so there is no inverse function of f from $\mathbf{Z}$ to $\mathbf{Z}$. Observe that $\text{ran } f = E$. If we define $g : \mathbf{Z} \to E$ by $g(n) = f(n)$ for all $n \in \mathbf{Z}$, then g is bijective and $g^{-1} : E \to \mathbf{Z}$ is a (bijective) function. In fact, $g^{-1}(n) = n/2$ for all $n \in E$.

9.7 Permutations

We have already mentioned that the identity function i_A defined on a nonempty set A is bijective. Normally, there are many bijective functions that can be defined on nonempty sets. These types of functions occur often in mathematics, especially in the area of mathematics called abstract (or modern) algebra.

A **permutation** of (or on) a nonempty set A is a bijective function on A, that is, a function from A to A that is both one-to-one and onto. By Results 9.1 and 9.3, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is a permutation of $\mathbf{R}$. Let's consider an even simpler example. For $A = \{1, 2, 3\}$, let f be a permutation of A. Then f is completely determined once we know the images of 1, 2, and 3 under f. There are three possible choices for $f(1)$, two choices for $f(2)$ once $f(1)$ has been specified, and one choice for $f(3)$ once $f(1)$ and $f(2)$ have been specified. From this, it follows that there are $3 \cdot 2 \cdot 1 = 3! = 6$ different permutations f of the set $A = \{1, 2, 3\}$.

One of these functions is the identity function defined on $\{1, 2, 3\}$, which we denote by α_1; that is,

$$\alpha_1 = \{(1, 1), (2, 2), (3, 3)\}.$$

Another permutation of $\{1, 2, 3\}$ is

$$\alpha_2 = \{(1, 1), (2, 3), (3, 2)\}.$$

There are other common ways to represent these permutations. A permutation of $\{1, 2, 3\}$ is also written as

$$\begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix},$$

where the numbers immediately below 1, 2, and 3 are their images. Hence α_1, α_2, and the other four permutations of $\{1, 2, 3\}$ can be expressed as:

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Since each permutation α_i $(1 \le i \le 6)$ is a bijective function from $\{1, 2, 3\}$ to $\{1, 2, 3\}$, it follows from Corollary 9.8 that the composition of any two permutations of $\{1, 2, 3\}$ is again a permutation of $\{1, 2, 3\}$. For example, let's consider

$$\alpha_2 \circ \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ - & - & - \end{pmatrix}.$$

Since $(\alpha_2 \circ \alpha_5)(1) = \alpha_2(\alpha_5(1)) = \alpha_2(2) = 3$, $(\alpha_2 \circ \alpha_5)(2) = 2$, and $(\alpha_2 \circ \alpha_5)(3) = 1$, it follows that

$$\alpha_2 \circ \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \alpha_3.$$

By Theorem 9.9, it follows that composition of permutations on the same nonempty set A is associative. Hence for every three integers $i, j, k \in \{1, 2, \cdots, 6\}$,

$$(\alpha_i \circ \alpha_j) \circ \alpha_k = \alpha_i \circ (\alpha_j \circ \alpha_k).$$

Also, by Theorem 9.11, since a permutation is a bijective function, each permutation has an inverse, which is also a permutation. Thus for each i $(1 \le i \le 6)$, $\alpha_i^{-1} = \alpha_j$ for some j $(1 \le j \le 6)$. The inverse of a permutation can be found by interchanging the two rows and then re-ordering the columns so that the top row is in the natural order $1, 2, 3, \ldots$. Thus

$$\alpha_5^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \alpha_6.$$

In general, there are $n!$ permutations of the set $\{1, 2, \cdots, n\}$. The set of all such permutations is denoted by $\mathcal{S}_n$. Thus

$$\mathcal{S}_3 = \{\alpha_1, \alpha_2, \cdots, \alpha_6\}.$$

As we have seen with $\mathcal{S}_3$, the elements of $\mathcal{S}_n$ satisfy the properties of closure, associativity, and the existence of inverses for every positive integer n. This will be revisited in Chapter 13.

EXERCISES FOR CHAPTER 9

Section 9.1: The Definition of Function

9.1. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. Then $f = \{(a, y), (b, z), (c, y), (d, z)\}$ is a function from A to B. Determine dom f and ran f.

9.2. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Give an example of a relation R from A to B containing exactly three elements such that R is *not* a function from A to B. Explain why R is not a function.

9.3. Let A be a nonempty set. If R is a relation from A to A that is both an equivalence relation and a function, then what familiar function is R? Justify your answer.

9.4. For the given subset A_i of $\mathbf{R}$ and the relation R_i $(1 \le i \le 3)$ from A_i to $\mathbf{R}$, determine whether R_i is a function from A_i to $\mathbf{R}$.

(a) $A_1 = \mathbf{R}$, $R_1 = \{(x, y) : x \in A_1, y = 4x - 3\}$

(b) $A_2 = [0, \infty)$, $R_2 = \{(x, y) : x \in A_2, (y + 2)^2 = x\}$

(c) $A_3 = \mathbf{R}$, $R_3 = \{(x, y) : x \in A_3, (x + y)^2 = 4\}$

9.5. Let A and B be nonempty sets and let R be a nonempty relation from A to B. Show that there exists a subset A' of A and a subset f of R such that f is a function from A' to B.

9.6. In each of the following, a function $f_i : A_i \to \mathbf{R}$ $(1 \le i \le 5)$ is defined, where the domain A_i consists of all real numbers x for which $f_i(x)$ is defined. In each case, determine the domain A_i and the range of f_i.

(a) $f_1(x) = 1 + x^2$
(b) $f_2(x) = 1 - \frac{1}{x}$
(c) $f_3(x) = \sqrt{3x - 1}$
(d) $f_4(x) = x^3 - 8$
(e) $f_5(x) = \frac{x}{x-3}$.

Section 9.2: The Set of All Functions from A to B

9.7. Let $A = \{1, 2, 3\}$ and $B = \{x, y\}$. Determine B^A.

9.8. For sets $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, give an example of a function $g \in B^A$ and a function $h \in B^B$.

9.9. For $A = \{a, b, c\}$, determine 2^A.

9.10. (a) Give an example of two sets A and B such that $|B^A| = 8$.
(b) For the sets A and B given in (a), provide an example of an element in B^A.

Section 9.3: One-to-One and Onto Functions

9.11. Let $A = \{w, x, y, z\}$ and $B = \{r, s, t\}$. Give an example of a function $f : A \to B$ that is neither one-to-one nor onto. Explain why f fails to have these properties.

9.12. Give an example of two finite sets A and B and two functions $f : A \to B$ and $g : B \to A$ such that f is one-to-one but not onto and g is onto but not one-to-one.

9.13. A function $f : \mathbf{Z} \to \mathbf{Z}$ is defined by $f(n) = 2n + 1$. Determine whether f is (a) injective, (b) surjective.

9.14. A function $f : \mathbf{Z} \to \mathbf{Z}$ is defined by $f(n) = n - 3$. Determine whether f is (a) injective, (b) surjective.

9.15. A function $f : \mathbf{Z} \to \mathbf{Z}$ is defined by $f(n) = 5n + 2$. Determine whether f is (a) injective, (b) surjective.

9.16. Prove or disprove: For every nonempty set A, there exists an injective function $f : A \to \mathcal{P}(A)$.

9.17. Determine whether the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 + 4x + 9$ is (a) one-to-one, (b) onto.

9.18. Is there a function $f : \mathbf{R} \to \mathbf{R}$ that is onto but not one-to-one? Explain your answer.

9.19. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is
(a) one-to-one and onto
(b) one-to-one but not onto
(c) onto but not one-to-one
(d) neither one-to-one nor onto.

Section 9.4: Bijective Functions

9.20. Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 7x - 2$ is bijective.

9.21. Prove that the function $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

9.22. Let $f : \mathbf{Z}_5 \to \mathbf{Z}_5$ be a function defined by $f([a]) = [2a + 3]$.

(a) Show that f is well-defined.
(b) Determine whether f is bijective.

9.23. For two finite nonempty sets A and B, let R be a relation such that $\operatorname{ran} R = B$. Define the domination number $\gamma(R)$ of R as the smallest cardinality of a subset $S \subseteq A$ such that for every element y of B, there is an element $x \in S$ such that x is related to y.

(a) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{a, b, c, d, e, f, g\}$ and let $R = \{(1, c), (1, e), (2, c), (2, f), (2, g), (3, b), (3, f), (4, a), (4, c), (4, g), (5, a), (5, b), (5, c), (6, d), (6, e), (7, a), (7, g)\}$. Determine $\gamma(R)$.

(b) If R is an equivalence relation defined on a finite nonempty set A (and so $B = A$), then what is $\gamma(R)$?
(c) If f is a bijective function from A to B, where both A and B are finite and nonempty, then what is $\gamma(f)$?

9.24. Let $A = [0, 1]$ denote the closed interval of real numbers between 0 and 1. Give an example of two different bijective functions f_1 and f_2 from A to A, neither of which is the identity function.

9.25. Let A be a nonempty set and let $f : A \to A$ be a function. Prove that if $f \circ f = i_A$, then f is bijective.

Section 9.5: Composition of Functions

9.26. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $C = \{w, x, y, z\}$. Consider the functions $f : A \to B$ and $g : B \to C$, where $f = \{(1, b), (2, c), (3, c), (4, a)\}$ and $g = \{(a, x), (b, y), (c, x)\}$. Determine $g \circ f$.

9.27. Two functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = 3x^2 + 1$ and $g(x) = 5x - 3$ for all $x \in \mathbf{R}$. Determine $(g \circ f)(1)$ and $(f \circ g)(1)$.

9.28. For nonempty sets A, B, and C, let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove:

If $g \circ f$ is one-to-one, then f is one-to-one.

using as many of the following proof techniques as possible: direct proof, proof by contrapositive, proof by contradiction.
(b) Disprove: If $g \circ f$ is one-to-one, then g is one-to-one.

9.29. Prove or disprove the following:

(a) If two functions $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.
(b) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is onto, then $g \circ f : A \to C$ is onto.
(c) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is one-to-one, then $g \circ f : A \to C$ is one-to-one.
(d) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.
(e) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.

9.30. Let A and B be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

9.31. Let A denote the set of integers that are multiples of 4, let B denote the set of integers that are multiples of 8, and let B' denote the set of even integers. Thus

$$A = \{4k : k \in \mathbf{Z}\},\ B = \{8k : k \in \mathbf{Z}\},\ \text{and}\ B' = \{2k : k \in \mathbf{Z}\}.$$

Let $f : A \times A \to B$ and $g : B' \to \mathbf{Z}$ be functions defined by $f(x, y) = xy$ for $x, y \in A$ and $g(n) = n/2$ for $n \in B'$.

(a) Show that the composition function $g \circ f : A \times A \to \mathbf{Z}$ is defined.
(b) For $k, \ell \in \mathbf{Z}$, determine $(g \circ f)(4k, 4\ell)$.

Section 9.6: Inverse Functions

9.32. Let $A = \{a, b, c\}$. Give an example of a function $f : A \to A$ such that the inverse (relation) f^{-1} is not a function.

9.33. Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 4x - 3$ is bijective, and determine $f^{-1}(x)$ for $x \in \mathbf{R}$.

9.34. Show that the function $f : \mathbf{R} - \{3\} \to \mathbf{R} - \{5\}$ defined by $f(x) = \frac{5x}{x-3}$ is bijective, and determine $f^{-1}(x)$ for $x \in \mathbf{R} - \{5\}$.

9.35. Let the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3$ and $g(x) = -3x + 5$.

(a) Show that f is one-to-one and onto.
(b) Show that g is one-to-one and onto.
(c) Determine the composition function $g \circ f$.
(d) Determine the inverse functions f^{-1} and g^{-1}.
(e) Determine the inverse function $(g \circ f)^{-1}$ of $g \circ f$ and the composition $f^{-1} \circ g^{-1}$.

9.36. Let $A = \mathbf{R} - \{1\}$ and define $f : A \to A$ by $f(x) = \dfrac{x}{x-1}$ for all $x \in A$.

(a) Prove that f is bijective.
(b) Determine f^{-1}.
(c) Determine $f \circ f \circ f$.

9.37. Let A, B, and C be nonempty sets and let f, g, and h be functions such that $f : A \to B$, $g : B \to C$, and $h : B \to C$. For each of the following, prove or disprove:

(a) If $g \circ f = h \circ f$, then $g = h$.
(b) If f is one-to-one and $g \circ f = h \circ f$, then $g = h$.

Section 9.7: Permutations

9.38. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ be permutations in $\mathcal{S}_5$. Determine $\alpha \circ \beta$ and β^{-1}.

9.39. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 6 & 2 & 1 & 4 \end{pmatrix}$ be elements of $\mathcal{S}_6$.

(a) Determine α^{-1} and β^{-1}.
(b) Determine $\alpha \circ \beta$ and $\beta \circ \alpha$.

ADDITIONAL EXERCISES FOR CHAPTER 9

9.40. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^2 + 3x + 4$.

(a) Show that f is not injective.
(b) Find all pairs r_1, r_2 of real numbers such that $f(r_1) = f(r_2)$.
(c) Show that f is not surjective.
(d) Find the set S of all real numbers such that if $s \in S$, then there is no real number x such that $f(x) = s$.
(e) What well-known set is the set S in (d) related to?

9.41. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^2 + ax + b$, where $a, b \in \mathbf{R}$. Show that f is not one-to-one. [Hint: It might be useful to consider the cases $a \neq 0$ and $a = 0$ separately.]

9.42. In Result 9.1, we saw that the (linear) function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3x - 5$ is one-to-one. In fact, we have seen that other linear functions are one-to-one. Prove the following generalization of this result: The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = ax + b$, where $a, b \in \mathbf{R}$ and $a \neq 0$, is one-to-one.

9.43. Evaluate the proposed proof of the following result.

Result The function $f : \mathbf{R} - \{1\} \to \mathbf{R} - \{3\}$ defined by $f(x) = \frac{3x}{x-1}$ is bijective.

Proof First, we show that f is one-to-one. Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R} - \{1\}$. Then $\frac{3a}{a-1} = \frac{3b}{b-1}$. Crossmultiplying, we obtain $3a(b-1) = 3b(a-1)$. Simplifying, we have

$3ab - 3a = 3ab - 3b$. Subtracting $3ab$ from both sides and dividing by -3, we have $a = b$. Thus f is one-to-one.

Next, we show that f is onto. Let $f(x) = r$. Then $\frac{3x}{x-1} = r$; so $3x = r(x-1)$. Simplifying, we have $3x = rx - r$ and so $3x - rx = -r$. Therefore, $x(3-r) = -r$. Since $r \in \mathbf{R} - \{3\}$, we can divide by $3 - r$ and obtain $x = \frac{-r}{3-r} = \frac{r}{r-3}$. Therefore,

$$f(x) = f\left(\frac{r}{r-3}\right) = \frac{3\left(\frac{r}{r-3}\right)}{\frac{r}{r-3} - 1} = \frac{3r}{r-(r-3)} = \frac{3r}{3} = r.$$

Thus f is onto. ∎

9.44. Let $A = \{a, b, c, d, e\}$. Then $f = \{(a, c), (b, e), (c, d), (d, b), (e, a)\}$ is a function from A to A.

(a) Show that it is possible to list the five elements of A in such a way that the image of each of the first four elements on the list is to the immediate right of the element and that the image of the last element on the list is the first element on the list.
(b) Show that it is not possible to list elements of A as in (a) for every function from A to A.

9.45. Let S be a nonempty set. Show that there exists an injective function from $\mathcal{P}(S)$ to $\mathcal{P}(\mathcal{P}(S))$.

9.46. For each of the following functions, determine, with explanation, whether the function is one-to-one and whether it is onto.

(a) $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, where $f(x, y) = (3x - 2, 5y + 7)$
(b) $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$, where $g(m, n) = (n + 6, 2 - m)$
(c) $h : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$, where $h(r, s) = (2r + 1, 4s + 3)$
(d) $\phi : \mathbf{Z} \times \mathbf{Z} \to S = \{a + b\sqrt{2} : a, b \in \mathbf{Z}\}$, where $\phi(a, b) = a + b\sqrt{2}$
(e) $\alpha : \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, where $\alpha(x) = (x^2, 2x + 1)$.

9.47. Let $\mathcal{U}$ be some universal set and A a subset of $\mathcal{U}$. A function $g_A : \mathcal{U} \to \{0, 1\}$ is defined by

$$g_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Verify each of the following.

(a) $g_{\mathcal{U}}(x) = 1$ for all $x \in \mathcal{U}$.
(b) $g_{\emptyset}(x) = 0$ for all $x \in \mathcal{U}$.
(c) For $\mathcal{U} = \mathbf{R}$ and $A = [0, \infty)$, $(g_A \circ g_A)(x) = 1$ for $x \in \mathbf{R}$.
(d) For subsets A and B of $\mathcal{U}$ and $C = A \cap B$,

$$g_C = (g_A) \cdot (g_B)$$

where $((g_A) \cdot (g_B))(x) = g_A(x) \cdot g_B(x)$.
(e) For $A \subseteq \mathcal{U}$,

$$g_{\overline{A}}(x) = 1 - g_A(x) \text{ for each } x \in \mathcal{U}.$$

9.48. For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$, suppose that $g \circ f = i_A$, the identity function on A.

(a) Prove that f is one-to-one and g is onto.
(b) Show that f need not be onto.
(c) Show that g need not be one-to-one.
(d) Prove that if f is onto, then g is one-to-one.

(e) Prove that if g is one-to-one, then f is onto.

(f) Combine the results in (d) and (e) into a single statement.

9.49. Let $A = \mathbf{R} - \{0\}$ and let $f : A \to A$ be defined by $f(x) = 1 - \frac{1}{x}$ for all $x \in \mathbf{R}$.

(a) Show that $f \circ f \circ f = i_A$.

(b) Determine f^{-1}.

9.50. Give an example of a nonempty set A and a bijective function $f : A \to A$ such that (1) $f \neq i_A$, (2) $f \circ f \neq i_A$, and (3) $f \circ f \circ f = i_A$.

9.51. Let $A = \{1, 2\}$, $B = \{1, -1, 2, -2\}$, and $C = \{1, 2, 3, 4\}$. Then $f = \{(1, 1), (1, -1), (2, 2), (2, -2)\}$ is a relation from A to B, while $g = \{(1, 1), (-1, 1), (2, 4), (-2, 4)\}$ is a relation from B to C. Furthermore,

$$gf = \{(x, z) : (x, y) \in f \text{ and } (y, z) \in g \text{ for some } y \in B\}$$

is a relation from A to C. Observe that even though the relation f is not a function from A to B, the relation gf *is* a function from A to C. Explain why.

9.52. A relation f on $\mathbf{R}$ is defined by $f = \{(x, y) : x \in \mathbf{R} \text{ and } y = x \text{ or } y = -x\}$ and a function $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(x) = x^2$. Then

$$gf = \{(x, z) : (x, y) \in f \text{ and } (y, z) \in g \text{ for some } y \in B\}.$$

(a) Explain why f is not a function from $\mathbf{R}$ to $\mathbf{R}$.

(b) Show that gf is a function from $\mathbf{R}$ to $\mathbf{R}$ and explicitly determine it.

(c) Even though the relation f is not a function from $\mathbf{R}$ to $\mathbf{R}$, the relation gf is a function from $\mathbf{R}$ to $\mathbf{R}$. Explain why.

9.53. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, and $C = \{1, 2, 3, 4, 5, 6\}$. Give an example of a function f from A to B and a relation g from B to C that is not a function from B to C such that

$$gf = \{(x, z) : (x, y) \in f \text{ and } (y, z) \in g \text{ for some } y \in B\}$$

is a function from A to C.

9.54. Let $\mathcal{F}$ be the set of all functions with domain and codomain $\mathbf{R}$. Define a relation R on $\mathcal{F}$ by $f \; R \; g$ if there exists a constant C such that $f(x) = g(x) + C$ for all $x \in \mathbf{R}$.

(a) Show that R is an equivalence relation.

(b) If $f \in \mathcal{F}$ and its derivative $h(x)$ is defined for all $x \in \mathbf{R}$, use this information to describe the elements in the equivalence class $[f]$.

9.55. (a) Let $S = \{a, b, c, d\}$ and let T be the set of all six 2-element subsets of S. Show that there exists an injective function $f : S \to \{0, 1, 2, \ldots, |T|\}$ such that the function $g : T \to \{1, 2, \ldots, |T|\}$ defined by $g(\{i, j\}) = |f(i) - f(j)|$ is bijective.

(b) Let $S = \{a, b, c, d, e\}$ and let T be the set of all ten 2-element subsets of S. Show that there exists no injective function $f : S \to \{0, 1, 2, \ldots, |T|\}$ such that the function $g : T \to \{1, 2, \ldots, |T|\}$ defined by $g(\{i, j\}) = |f(i) - f(j)|$ is bijective.

(c) For the sets S and T in (b), show that there exists an injective function $f : S \to \{0, 1, 2, \ldots, |T| + 2\}$ such that the function $g : T \to \{1, 2, \ldots, |T| + 2\}$ defined by $g(\{i, j\}) = |f(i) - f(j)|$ is injective.

(d) The results in (b) and (c) should suggest a question to you. Ask and answer such a question.

9.56. Let S be the set of odd positive integers. A function $F : \mathbf{N} \to S$ is defined by $F(n) = k$ for each $n \in \mathbf{N}$, where k is that odd positive integer for which $3n + 1 = 2^m k$ for some nonnegative integer m. Prove or disprove the following:

(a) F is one-to-one.

(b) F is onto.

9.57. A function $F : \mathbf{N} \to \mathbf{N} \cup \{0\}$ is defined by $F(n) = m$ for each $n \in \mathbf{N}$, where m is that nonnegative integer for which $3n + 1 = 2^m k$ and k is an odd integer. Prove or disprove the following:

(a) F is one-to-one.

(b) F is onto.

9.58. Recall that the derivative of $\ln x$ is $1/x$ and that the derivative of x^n is nx^{n-1} for every integer n. In symbols,

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \text{ and } \frac{d}{dx}\left(x^n\right) = nx^{n-1}.$$

Let $f : \mathbf{R}^+ \to \mathbf{R}$ be defined by $f(x) = \ln x$ for every $x \in \mathbf{R}^+$. Prove that the nth derivative of $f(x)$ is given by

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

for every positive integer n.

9.59. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = xe^{-x}$ for every $x \in \mathbf{R}$. Prove that the nth derivative of $f(x)$ is given by

$$f^{(n)}(x) = (-1)^n e^{-x}(x - n)$$

for every positive integer n.

For Exercises 9.60–9.62, use the following definition. Let $f : A \to B$ be a function. For a subset C of A, the **image of C under f** is the set

$$f(C) = \{f(c) : c \in C\}.$$

(Thus $f(A)$ is the range of f.)

9.60. Let $A_1, A_2 \subseteq A$. Prove the following.

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(c) If f is one-to-one, then $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

9.61. If $g : \mathbf{Q} \to \mathbf{Q}$ is defined by $g(r) = 4r + 1$ for each $r \in \mathbf{Q}$, then determine $g(\mathbf{Z})$ and $g(E)$, where E is the set of even integers.

9.62. Define the function $h : \mathbf{Z}_{16} \to \mathbf{Z}_{24}$ by $h([a]) = [3a]$ for each $a \in \mathbf{Z}$.

(a) Prove that the function h is well-defined, that is, prove that if $[a] = [b]$ in $\mathbf{Z}_{16}$, then $h([a]) = h([b])$ in $\mathbf{Z}_{24}$.

(b) For the subsets $A = \{[0], [3], [6], [9], [12], [15]\}$ and $B = \{[0], [8]\}$ of $\mathbf{Z}_{16}$, determine the subsets $h(A)$ and $h(B)$ of $\mathbf{Z}_{24}$.

10

Cardinalities of Sets

Many consider the Italian mathematician and scientist Galileo Galilei to be the founder of modern physics. Among his major contributions was his mathematical view of the laws of motion. Early in the 17th century, Galileo applied mathematics to study the motion of the earth. He was convinced that the earth revolved about the sun, an opinion not shared by the Catholic Church at that time. This led him to be imprisoned for the last nine years of his life.

Galileo's two main scientific writings were *Dialogue Concerning the Two Chief World Systems* and *Discourses and Mathematical Demonstrations Concerning Two New Sciences*, the first published before he went to prison and the second published (in the Netherlands) while he was in prison. In these two works, he would often discuss scientific theories by means of a dialogue among fictional characters. It is in this manner that he could state his positions on various theories.

One topic that intrigued Galileo was infinite sets. Galileo observed that there is a one-to-one correspondence (that is, a bijective function) between the set **N** of positive integers and the subset S of **N** consisting of the squares of positive integers. This led Galileo to observe that even though there are many positive integers that are not squares, there are as many squares as there are positive integers. This led Galileo to be faced with a property of an infinite set that he found bothersome: There can be a one-to-one correspondence between a set and a proper subset of the set. While Galileo concluded correctly that the number of squares of positive integers is not less than the number of positive integers, he could not bring himself to say that these sets have the same number of elements.

Bernhard Bolzano was a Bohemian priest, philosopher, and mathematician. Although best known for his work in calculus during the first half of the 19th century, he too was interested in infinite sets. His *Paradoxes of the Infinite*, published two years after his death and unnoticed for twenty years, contained many ideas of the modern theory of sets. He noted that one-to-one correspondences between an infinite set and a proper subset of itself are common and was comfortable with this fact, contrary to Galileo's feelings. The German mathematician Richard Dedekind studied under the brilliant Carl Friedrich Gauss. Dedekind had a long and productive career in mathematics and made many contributions to the study of irrational numbers. What had confused Galileo and

interested Bolzano gave rise to a definition of an infinite set by Dedekind during the last part of the 19th century: A set S is infinite if it contains a proper subset that can be put in one-to-one correspondence with S. Certainly, then, understanding infinite sets was not an easy task, even among well-known mathematicians of the past.

We mentioned in Chapter 1 that the cardinality $|S|$ of a set S is the number of elements in S and, for the present, we would use the notation $|S|$ only when S is a finite set. A set S is **finite** if either $S = \emptyset$ or $|S| = n$ for some $n \in \mathbf{N}$; while a set is **infinite** if it is not finite. It may seem that we should write $|S| = \infty$ if S is infinite, but we will see later that this is not particularly informative. Indeed, it is considerably more difficult to give a meaning to $|S|$ if S is an infinite set; however, it is precisely this topic that we are about to explore.

10.1 Numerically Equivalent Sets

It is rather obvious that the sets $A = \{a, b, c\}$ and $B = \{x, y, z\}$ have the same cardinality since each has exactly three elements. That is, if we count the number of elements in two sets and arrive at the same value, then these two sets have the same cardinality. There is, however, another way to see that the sets A and B described above have the same cardinality without counting the elements in each set. Observe that we can pair off the elements of A and B, say as (a, x), (b, y), and (c, z). This implies that A and B have the same number of elements, that is, $|A| = |B|$. What we have actually done is describe a bijective function $f : A \to B$, namely $f = \{(a, x), (b, y), (c, z)\}$. Although it is much easier to see that $|A| = |B|$ by observing that each set has three elements than by constructing a bijective function from A to B, it is this latter method of showing that $|A| = |B|$ that can be generalized to the situation where A and B are infinite sets.

Two sets A and B (finite or infinite) are said to have the **same cardinality**, written $|A| = |B|$, if either A and B are both empty or there is a bijective function f from A to B. Two sets having the same cardinality are also referred to as **numerically equivalent sets**. Two finite sets are therefore numerically equivalent if they are both empty or if both have n elements for some positive integer n. Consequently, two nonempty sets A and B are not numerically equivalent, written $|A| \neq |B|$, if there is no bijective function f from one set to the other. The study of numerically equivalent infinite sets is more challenging but considerably more interesting than the study of numerically equivalent finite sets.

The justification for the term "numerically equivalent sets" lies in the following theorem, which combines the major concepts of Chapters 8 and 9.

Theorem 10.1 *Let $\mathcal{S}$ be a nonempty collection of nonempty sets. A relation R is defined on $\mathcal{S}$ by $A \; R \; B$ if there exists a bijective function from A to B. Then R is an equivalence relation.*

Proof Let $A \in \mathcal{S}$. Since the identity function $i_A : A \to A$ is bijective, it follows that $A \; R \; A$. Thus R is reflexive. Next, assume that $A \; R \; B$, where $A, B \in \mathcal{S}$. Then there is a bijective function $f : A \to B$. By Theorem 9.11, f has an inverse function $f^{-1} : B \to A$ and, furthermore, f^{-1} is bijective. Therefore, $B \; R \; A$ and R is symmetric.

Finally, assume that $A \; R \; B$ and $B \; R \; C$, where $A, B, C \in \mathcal{S}$. Then there are bijective functions $f : A \to B$ and $g : B \to C$. It follows by Corollary 9.8 that the composition $g \circ f : A \to C$ is bijective as well, and so $A \; R \; C$. Therefore, R is transitive. Consequently, R is an equivalence relation. ∎

According to the equivalence relation defined in Theorem 10.1, if A is a nonempty set, then the equivalence class $[A]$ consists of all those elements of $\mathcal{S}$ having the same cardinality as A; hence the term "numerically equivalent sets" is natural for two sets having the same cardinality.

10.2 Denumerable Sets

In order to start to gain an understanding of the cardinality of an infinite set, we begin with a particular class of infinite sets. A set A is called **denumerable** if $|A| = |\mathbf{N}|$, that is, if A has the same cardinality as the set of natural numbers. Certainly, if A is denumerable, then A is infinite. By definition, if A is a denumerable set, then there is a bijective function $f : \mathbf{N} \to A$ and so $f = \{(1, f(1)), (2, f(2)), (3, f(3)), \ldots\}$. Consequently, $A = \{f(1), f(2), f(3), \ldots\}$, that is, we can list the elements of A as $f(1), f(2), f(3), \ldots$. Equivalently, we can list the elements of A as $a_1, a_2, a_3, \ldots$, where then $a_i = f(i)$ for $i \in \mathbf{N}$. Conversely, if the elements of A can be listed as $a_1, a_2, a_3, \ldots$, where $a_i \neq a_j$ for $i \neq j$, then A is denumerable since the function $g : \mathbf{N} \to A$ defined by $g(n) = a_n$ for each $n \in \mathbf{N}$ is certainly bijective. Therefore, A is a denumerable set if and only if it is possible to list the elements of A as $a_1, a_2, a_3, \ldots$ and so $A = \{a_1, a_2, a_3, \ldots\}$.

A set is **countable** if it is either finite or denumerable. **Countably infinite** sets are then precisely the denumerable sets. Hence, if A is a nonempty countable set, then we can either write $A = \{a_1, a_2, a_3, \ldots, a_n\}$ for some $n \in \mathbf{N}$ or $A = \{a_1, a_2, a_3, \ldots\}$. A set that is not countable is called **uncountable**. An uncountable set is necessarily infinite. It may not be clear whether any set is uncountable, but we will soon see that such sets do exist.

Let's look at a few examples of denumerable sets. Certainly, $\mathbf{N}$ itself is denumerable since the identity function $i_{\mathbf{N}} : \mathbf{N} \to \mathbf{N}$ is bijective. However, not only is the set of positive integers denumerable, the set of *all* integers is denumerable. The proof of this fact that we give illustrates a common technique for showing that a set is denumerable, namely, if we can list the elements of a set A as $a_1, a_2, a_3, \ldots$ such that every element of A appears exactly once in the list, then A is denumerable.

Result 10.2 *The set $\mathbf{Z}$ of integers is denumerable.*

Proof Observe that the elements of $\mathbf{Z}$ can be listed as $0, 1, -1, 2, -2, \ldots$. Thus the function $f : \mathbf{N} \to \mathbf{Z}$ described in Figure 10.1 is bijective, and so $\mathbf{Z}$ is denumerable. ∎

The function $f : \mathbf{N} \to \mathbf{Z}$ given in Figure 10.1 can be also defined by

$$f(n) = \frac{1 + (-1)^n(2n - 1)}{4}. \tag{10.1}$$

$$f: \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

Figure 10.1 A bijective function $f : \mathbf{N} \to \mathbf{Z}$

Although we have already observed that this function f is bijective, Exercise 10.4. asks for a formal proof of this fact.

The fact that $\mathbf{Z}$ is denumerable illustrates what Galileo had observed centuries ago: It is possible for two sets to have the same cardinality where one is a proper subset of the other. (Such a situation could never occur with finite sets, however.) For example, $\mathbf{N} \subset \mathbf{Z}$ and $|\mathbf{N}| = |\mathbf{Z}|$. This fact serves as an illustration of a result, the proof of which is a bit intricate.

Theorem to Prove Every infinite subset of a denumerable set is denumerable.

PROOF STRATEGY In the proof, we begin with two sets, which we'll call A and B, where A is denumerable, $B \subseteq A$, and B is infinite. Because A is denumerable, we can write $A = \{a_1, a_2, a_3, \ldots\}$. Since our goal is to show that B is denumerable, we need to show that we can write $B = \{b_1, b_2, b_3, \ldots\}$. The question, of course, is how to do it.

Because B is an infinite subset of A, some of the elements of A belong to B (in fact, infinitely many elements of A belong to B); while, most likely, some elements of A do not belong to B. We can keep track of the elements of A that belong to B by means of a set, which we'll denote by S. If $a_1 \in B$, then $1 \in S$; if $a_1 \notin B$, then $1 \notin S$. In general, $n \in S$ if and only if $a_n \in B$. Certainly, $S \subseteq \mathbf{N}$. Since $\mathbf{N}$ is a well-ordered set (by the Well-Ordering Principle), S contains a smallest element, say s. That is, $a_s \in B$. Furthermore, if r is an integer such that $1 \leq r < s$, then $a_r \notin B$. It is the element a_s that we will call b_1. It is now logical to look at the (infinite) set $S - \{s\}$ and consider its smallest element, say t. Thus $t > s$. The element a_t will become b_2. And so on.

Since we want to give a precise and careful proof, we are already faced with two problems. First, denoting the smallest element of S by s and denoting the smallest element of $S - \{s\}$ by t will present difficulties to us. We need to use better notation. So let us denote the smallest element of S by i_1 (so $b_1 = a_{i_1}$) and the smallest element of $S - \{i_1\}$ by i_2 (so $b_2 = a_{i_2}$). This is much better notation. The other problem we have is when we wrote "And so on." Once we have the positive integers i_1 and i_2, it will follow that the positive integer i_3 is the least element of $S - \{i_1, i_2\}$. In general, once we have determined the positive integers $i_1, i_2, \ldots, i_k$, where $k \in \mathbf{N}$, the positive integer i_{k+1} is the smallest element of $S - \{i_1, i_2, \ldots, i_k\}$. In fact, this suggests that the elements $b_1, b_2, b_3, \ldots$ can be located in A using induction.

After using induction to construct the set $\{b_1, b_2, b_3, \ldots\}$, which we will denote by B', say, then we still have one more concern. Are we certain that $B' = B$? Because each element of B' belongs to B, we know that $B' \subseteq B$. To show that $B' = B$, we must also be sure that $B \subseteq B'$. As we know, the standard way to show that $B \subseteq B'$ is to take a typical element $b \in B$ and show that $b \in B'$.

Let's now write a complete proof. ♦

Theorem 10.3 *Every infinite subset of a denumerable set is denumerable.*

Proof Let A be a denumerable set and let B be an infinite subset of A. Since A is denumerable, we can write $A = \{a_1, a_2, a_3, \ldots\}$. Let $S = \{i \in \mathbf{N} : a_i \in B\}$; that is, S consists of all those positive integers that are subscripts of the elements in A that also belong to B. Since B is infinite, S is infinite. First we use induction to show that B contains a denumerable subset. Since S is a nonempty subset of $\mathbf{N}$, it follows from the Well-Ordering Principle that S has a least element, say i_1. Let $b_1 = a_{i_1}$. Let $S_1 = S - \{i_1\}$. Since $S_1 \neq \emptyset$ (indeed, S_1 is infinite), S_1 has a least element, say i_2. Let $b_2 = a_{i_2}$, which, of course, is distinct from b_1. Assume that for an arbitrary integer $k \geq 2$, the (distinct) elements $b_1, b_2, \ldots, b_k$ have been defined by $b_j = a_{i_j}$ for each integer j with $1 \leq j \leq k$, where i_1 is the smallest element in S and i_j is the minimum element in $S_{j-1} = S - \{i_1, i_2, \ldots, i_{j-1}\}$ for $2 \leq j \leq k$. Now let i_{k+1} be the minimum element of $S_k = S - \{i_1, i_2, \ldots, i_k\}$ and let $b_{k+1} = a_{i_{k+1}}$. Hence it follows that for each integer $n \geq 2$, an element b_n belongs to B that is distinct from $b_1, b_2, \ldots, b_{n-1}$. Thus we have exhibited the elements $b_1, b_2, b_3, \ldots$ in B.

Let $B' = \{b_1, b_2, b_3, \ldots\}$. Certainly $B' \subseteq B$. We claim, in fact, that $B = B'$. It remains only to show that $B \subseteq B'$. Let $b \in B$. Since $B \subseteq A$, it follows that $b = a_n$ for some $n \in \mathbf{N}$ and so $n \in S$. If $n = i_1$, then $b = b_1 = a_n$ and so $b \in B'$. Thus we may assume that $n > i_1$. Let S' consist of those positive integers less than n that belong to S. Since $n > i_1$ and $i_1 \in S$, it follows that $S' \neq \emptyset$. Certainly, $1 \leq |S'| \leq n - 1$; so S' is finite. Thus $|S'| = m$ for some $m \in \mathbf{N}$. The set S' therefore consists of the m smallest integers of S, that is, $S' = \{i_1, i_2, \ldots, i_m\}$. The smallest integer that belongs to S and is greater than i_m must be i_{m+1}, of course, and $i_{m+1} \geq n$. But $n \in S$, so $n = i_{m+1}$ and $b = a_n = a_{i_{m+1}} = b_{m+1} \in B'$. Hence $B = B' = \{b_1, b_2, b_3, \ldots\}$, which is denumerable. ■

In order to use Theorem 10.3 to describe other denumerable sets, it is convenient to introduce some additional notation. Let $k \in \mathbf{N}$. Then the set $k\mathbf{Z}$ is defined by

$$k\mathbf{Z} = \{kn : n \in \mathbf{Z}\}.$$

Similarly,

$$k\mathbf{N} = \{kn : n \in \mathbf{N}\}.$$

Thus $1\mathbf{Z} = \mathbf{Z}$ and $1\mathbf{N} = \mathbf{N}$, while $2\mathbf{Z}$ is the set of even integers. An immediate consequence of Theorem 10.3 is stated next.

Result 10.4 *The set $2\mathbf{Z}$ of even integers is denumerable.*

Proof Since $2\mathbf{Z}$ is infinite and $2\mathbf{Z} \subseteq \mathbf{Z}$, it follows by Theorem 10.3 that $2\mathbf{Z}$ is denumerable. ■

Of course, $k\mathbf{Z}$ is denumerable for *every* nonzero integer k. We now describe a denumerable set that can be obtained from two given sets. Recall, for sets A and B, that the Cartesian product $A \times B = \{(a, b) : a \in A, b \in B\}$.

Result 10.5 *If A and B are denumerable sets, then $A \times B$ is denumerable.*

Proof Since A and B are denumerable sets, we can write $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$. Consider the table shown in Figure 10.2(a), which has an infinite (denumerable) number of rows and columns, where the elements $a_1, a_2, a_3, \ldots$ are written along the side and $b_1, b_2, b_3, \ldots$ are written across the top. In row i, column j of the table, we place the ordered pair (a_i, b_j). Certainly, every element of $A \times B$ appears exactly once in this table. This table is reproduced in Figure 10.2(b), where the directed lines indicate the order in which we will encounter the entries in the table. That is, we

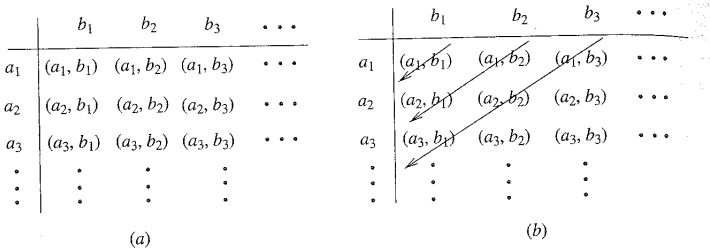

Figure 10.2 Constructing a bijective function $f : \mathbf{N} \to A \times B$

encounter the elements of $A \times B$ in the order

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), \ldots.$$

Since every element of $A \times B$ occurs in this list exactly once, this describes a bijective function $f : \mathbf{N} \to A \times B$, where

$$f(1) = (a_1, b_1), f(2) = (a_1, b_2), f(3) = (a_2, b_1), f(4) = (a_1, b_3), f(5) = (a_2, b_2), \ldots.$$

Therefore, $A \times B$ is denumerable. ∎

We can use a technique similar to that used in proving Result 10.5 to show that another familiar set is denumerable.

Result 10.6 *The set $\mathbf{Q}^+$ of positive rational numbers is denumerable.*

Proof Consider the table shown in Figure 10.3(a). In row i, column j, we place the rational number j/i. Certainly, then, every positive rational number appears in the table of Figure 10.2(a); indeed, it appears infinitely often. For example, the number $1/2$ appears in row 2, column 1, as well as in row 4, column 2.

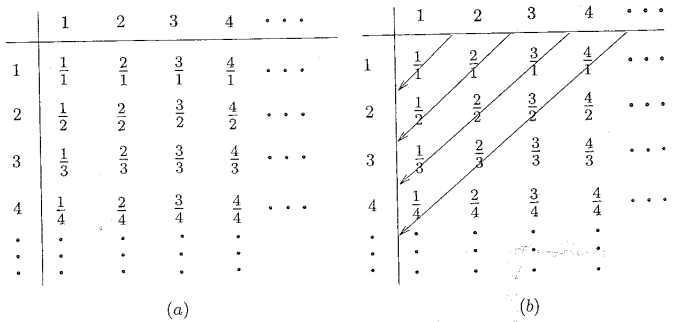

Figure 10.3 A table used to show that $\mathbf{Q}^+$ is denumerable

The table of Figure 10.3(a) is reproduced in Figure 10.3(b), where the arrows indicate the order in which we will consider the entries in the table. That is, we now consider the positive rational numbers in the order

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \ldots.$$

With the aid of this list, we can describe a bijective function $f : \mathbf{N} \to \mathbf{Q}^+$. In particular, we define $f(1) = 1/1 = 1$, $f(2) = 2/1 = 2$, $f(3) = 1/2$, and $f(4) = 3/1 = 3$ as expected. However, since $2/2 = 1$ and we have already defined $f(1) = 1$, we do not define $f(5) = 1$ (since f must be one-to-one). We bypass $2/2 = 1$ and, following the arrows, go directly to the next number on the list, namely $1/3$. In fact, whenever we encounter a number on the list that we have previously seen, we move to the next number on the list. In this manner, the function f being described will be one-to-one. The function f is shown in Figure 10.4.

Because every element of $\mathbf{Q}^+$ is encountered eventually, f is onto as well and so f is bijective. Consequently, $\mathbf{Q}^+$ is denumerable. ∎

The function f described in Figure 10.4 is by no means unique. There are many ways to traverse the positive rational numbers in the table described in Figure 10.3(a). The tables shown in Figure 10.5 indicate two additional methods.

Some care must be taken when proceeding about the entries in the table of Figure 10.3(a). For example, traversing the positive rational numbers by rows (see Figure 10.6) just won't do. Since the first row never ends, we will only encounter the positive integers.

The set $\mathbf{Q}^+$ can also be shown to be denumerable with the aid of the table in Figure 10.7. In the first row, all positive rational numbers j/i with $i = 1$ are shown. In the second row, all positive rational numbers j/i with $i = 2$ and such that j/i has been reduced to lowest terms are shown. This results in the rational number $(2j - 1)/2$ in row 2,

$$f: \quad \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & \frac{1}{2} & 3 & \frac{1}{3} & \cdots \end{array}$$

Figure 10.4 A bijective function $f : \mathbf{N} \to \mathbf{Q}^+$

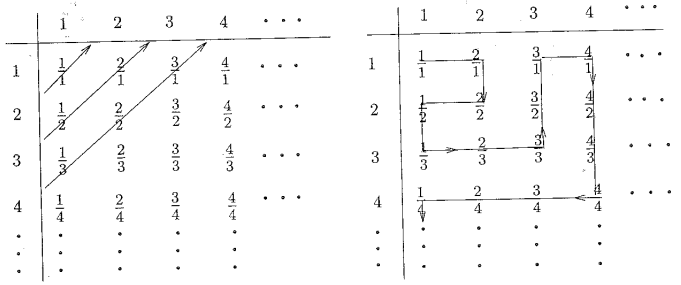

Figure 10.5 Traversing the positive rational numbers

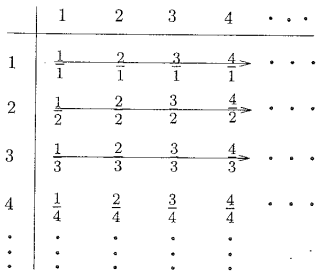

Figure 10.6 How not to traverse the positive rational numbers

column j. We continue in this manner with all other rows. In this way, every positive rational number appears *exactly once* in the table. Thus when we proceed through the entries as the arrows indicate, we obtain the positive rational numbers in the order

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{3}{2}, \frac{1}{3}, \frac{4}{1}, \cdots$$

and the corresponding bijective function $g : \mathbf{N} \to \mathbf{Q}^+$. Therefore, $g(1) = 1$, $g(2) = 2$, $g(3) = 1/2$, $g(4) = 3$, $g(5) = 3/2$, and so on.

Now that we have shown that $\mathbf{Q}^+$ is denumerable, it is not difficult to show that the set $\mathbf{Q}$ of *all* rational numbers is denumerable.

Result 10.7 *The set* $\mathbf{Q}$ *of all rational numbers is denumerable.*

Proof Since $\mathbf{Q}^+$ is denumerable, we can write $\mathbf{Q}^+ = \{q_1, q_2, q_3, \ldots\}$. Thus, $\mathbf{Q} = \{0\} \cup \{q_1, q_2, q_3, \ldots\} \cup \{-q_1, -q_2, -q_3, \ldots\}$. Therefore, $\mathbf{Q} = \{0, q_1, -q_1, q_2, -q_2, \ldots\}$ and the function $f : \mathbf{N} \to \mathbf{Q}$ shown in Figure 10.8 is bijective and so $\mathbf{Q}$ is denumerable. ∎

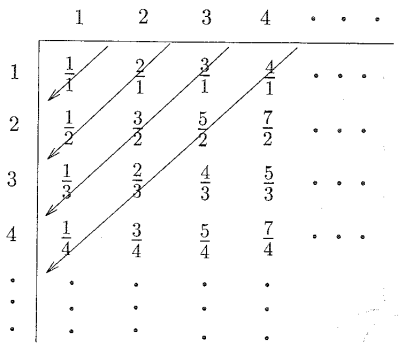

Figure 10.7 Another bijective function $g : \mathbf{N} \to \mathbf{Q}^+$

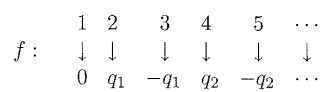

Figure 10.8 A bijective function $f : \mathbf{N} \to \mathbf{Q}$

10.3 Uncountable Sets

Although we have now given several examples of denumerable sets (and consequently countably infinite sets), we have yet to give an example of an uncountable set. We will do this next. First though, let's review a few facts about decimal expansions of real numbers. Every irrational number has a unique decimal expansion and this expansion is nonrepeating, while every rational number has a repeating decimal expansion. For example, $\frac{3}{11} = 0.272727\cdots$. Some rational numbers, however, have two (repeating) decimal expansions. For example, $\frac{1}{2} = 0.5000\cdots$ and $\frac{1}{2} = 0.4999\cdots$. (The number $\frac{3}{11}$ has only one decimal expansion.) In particular, a rational number a/b, where $a, b \in \mathbf{N}$, that is reduced to lowest terms has two decimal expansions if and only if the only primes that divide b are 2 or 5. If a rational number has two decimal expansions, then one of the expansions repeats the digit 0 from some point on (that is, the decimal expansion terminates), while the alternate expansion repeats the digit 9 from some point on.

We are now prepared to give an example of an uncountable set. Recall that for real numbers a and b with $a < b$, the open interval (a, b) is defined by

$$(a, b) = \{x \in \mathbf{R} : a < x < b\}.$$

Although, as it will turn out, all open intervals (a, b) of real numbers are uncountable, we will prove now only that $(0, 1)$ is uncountable.

Theorem to Prove The open interval $(0, 1)$ of real numbers is uncountable.

PROOF STRATEGY Since uncountable means not countable, it is not surprising that we should try a proof by contradiction here. So the proof would begin by assuming that $(0, 1)$ is countable. Since $(0, 1)$ is an infinite set, this means that we are assuming that $(0, 1)$ is denumerable, which implies that there must exist a bijective function $f : \mathbf{N} \to (0, 1)$. Therefore, for each $n \in \mathbf{N}$, $f(n)$ is a number in the set $(0, 1)$. It might be convenient to introduce some notation for the number $f(n)$, say $f(n) = a_n$, where then $0 < a_n < 1$. Since f is assumed to be one-to-one, it follows that $a_i \neq a_j$ for distinct positive integers i and j. Each number a_n has a decimal expansion, say $a_n = 0.a_{n1}a_{n2}a_{n3}\cdots$, where a_{n1} is the first digit in the expansion, a_{n2} is the second digit in the expansion, and so on. We have to be a bit careful here, however, for as we have seen, some real numbers have two decimal expansions. To avoid possible confusion, we can choose the decimal expansion that repeats the digit 0 from some point on. That is, no real number a_n has a decimal expansion that repeats 9 from some point on.

But where does this lead to a contradiction? From what we have said, $(0, 1) = \{a_1, a_2, a_3, \ldots\}$. If we can think of some real number $b \in (0, 1)$ such that $b \notin \{a_1, a_2, a_3, \ldots\}$, then this would give us a contradiction because this would say that f is not onto. So we

need to find a number $b \in (0, 1)$ such that $b \neq a_n$ for each $n \in \mathbf{N}$. Since $b \in (0, 1)$, the number b has a decimal expansion, say $b = 0.b_1b_2b_3\cdots$. How can we choose the digits $b_1, b_2, b_3, \ldots$ so that $b \neq a_n$ for every $n \in \mathbf{N}$? We could choose $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, etc. But would this mean that $b \neq a_1$, $b \neq a_2$, etc.? We must be careful here. For example, $0.500\cdots$ and $0.499\cdots$ are two equal numbers whose first digits in their expansions are not equal. Of course, the reason for this is that one is the alternate decimal expansion of the other. Thus, provided we can avoid selecting a decimal expansion for b that is the alternate decimal expansion for some number a_n, where $n \in \mathbf{N}$, we will have found a number $b \in (1, 0)$ such that $b \notin \{a_1, a_2, a_3, \ldots\}$. This will give us a contradiction. ♦

Theorem 10.8 *The open interval* $(0, 1)$ *of real numbers is uncountable.*

Proof Assume, to the contrary, that $(0, 1)$ is countable. Since $(0, 1)$ is infinite, it is denumerable. Therefore, there exists a bijective function $f : \mathbf{N} \to (0, 1)$. For $n \in \mathbf{N}$, let $f(n) = a_n$. Since $a_n \in (0, 1)$, the number a_n has a decimal expansion, say $0.a_{n1}a_{n2}a_{n3}\cdots$, where $a_{ni} \in \{0, 1, 2, \ldots, 9\}$ for all $i \in \mathbf{N}$. If a_n is irrational, then its decimal expansion is unique. If $a_n \in \mathbf{Q}$, then the expansion *may* be unique. If it is not unique, then, without loss of generality, we assume that the digits of the decimal expansion $0.a_{n1}a_{n2}a_{n3}\cdots$ are 0 from some position on. For example, since f is bijective, $2/5$ is the image of exactly one positive integer and this image is written as $0.4000\cdots$ (rather than as $0.3999\cdots$). To summarize, we have

$$\begin{aligned} f(1) &= a_1 = 0.a_{11}a_{12}a_{13}\cdots \\ f(2) &= a_2 = 0.a_{21}a_{22}a_{23}\cdots \\ f(3) &= a_3 = 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \quad\quad \vdots \quad\quad \vdots \end{aligned}$$

We show that the function f is not onto, however. Define the number $b = 0.b_1b_2b_3\cdots$, where $b_i \in \{0, 1, 2, \ldots, 9\}$ for all $i \in \mathbf{N}$, by

$$b_i = \begin{cases} 4 & \text{if } a_{ii} = 5 \\ 5 & \text{if } a_{ii} \neq 5. \end{cases}$$

(For example, let's suppose that $a_1 = 0.31717\cdots$, $a_2 = 0.151515\cdots$, and $a_3 = 0.04000\cdots$. Then the first three digits in the decimal expansion of b are 5, 4, and 5, that is, $b = 0.545\cdots$.)

For each $i \in \mathbf{N}$, the digit $b_i \neq a_{ii}$, implying that $b \neq a_n$ for all $n \in \mathbf{N}$ since b is not the alternate expansion of any rational number, as no digit in the expansion of b is 9. Thus, b is not an image of any element of $\mathbf{N}$. Therefore, f is not onto and, consequently, not bijective, producing a contradiction. ■

In the proof of Theorem 10.8, each digit in the decimal expansion of the number b constructed is 4 or 5. We could have selected any two distinct digits that did not use 9. It is now easy to give examples of other uncountable sets with the aid of the following result.

Theorem 10.9 *Let A and B be sets such that $A \subseteq B$. If A is uncountable, then B is uncountable.*

Proof Let A and B be two sets such that $A \subseteq B$ and A is uncountable. Necessarily then A and B are infinite. Assume, to the contrary, that B is denumerable. Since A is an infinite subset of a denumerable set, it follows by Theorem 10.3 that A is denumerable, producing a contradiction. ■

Corollary 10.10 *The set* $\mathbf{R}$ *of real numbers is uncountable.*

Proof Since $(0, 1)$ is uncountable by Theorem 10.8 and $(0, 1) \subseteq \mathbf{R}$, it follows by Theorem 10.9 that $\mathbf{R}$ is uncountable. ■

Let's pause for a moment to review a few facts that we've discovered about infinite sets (at least about certain infinite sets). First, recall that two nonempty sets A and B are defined to have the same cardinality (same number of elements) if there exists a bijective function from A to B. We're especially interested in the situation when A and B are infinite. One family of infinite sets we've introduced is the class of denumerable sets. Recall too that a set S is denumerable if there exists a bijective function from $\mathbf{N}$ to S.

Suppose that A and B are two denumerable sets. Then there exist bijective functions $f : \mathbf{N} \to A$ and $g : \mathbf{N} \to B$. Since f is bijective, f has an inverse function $f^{-1} : A \to \mathbf{N}$, where f^{-1} is also bijective. Since $f^{-1} : A \to \mathbf{N}$ and $g : \mathbf{N} \to B$ are bijective functions, it follows that the composition function $g \circ f^{-1} : A \to B$ is also bijective. This tells us that $|A| = |B|$, that is, A and B have the same number of elements. We state this as a theorem for emphasis.

Theorem 10.11 *Every two denumerable sets are numerically equivalent.*

Next, let B be an uncountable set. So B is an infinite set that is not denumerable. Also, let A be a denumerable set. Therefore, there exists a bijective function $f : \mathbf{N} \to A$. We claim that $|A| \neq |B|$, that is, A and B do *not* have the same number of elements. Let's prove this. Assume, to the contrary, that $|A| = |B|$. Hence there exists a bijective function $g : A \to B$. Since the functions $f : \mathbf{N} \to A$ and $g : A \to B$ are bijective, the composition function $g \circ f : \mathbf{N} \to B$ is bijective. But this means that B is a denumerable set, which is a contradiction. We also state this fact as a theorem.

Theorem 10.12 *If A is a denumerable set and B is an uncountable set, then A and B are not numerically equivalent.*

Theorems 10.11 and 10.12 can also be considered as consequences of Theorem 10.1. In particular, Theorem 10.12 says that $\mathbf{Z}$ and $\mathbf{R}$ are not numerically equivalent and so $|\mathbf{Z}| \neq |\mathbf{R}|$. So, here are two infinite sets that do *not* have the same number of elements. In other words, there are different sizes of infinity. This now brings up a number of questions, one of which is: Do there exist *three* infinite sets so that no two of them have the same number of elements? Also, if A is a denumerable set and B is an uncountable set, is one of these sets "bigger" than the other in some sense? In other words, we would like to be able to compare $|A|$ and $|B|$ in some precise manner. Since $|\mathbf{Z}| \neq |\mathbf{R}|$ and $\mathbf{Z} \subset \mathbf{R}$, it is tempting to conclude that $|\mathbf{Z}| < |\mathbf{R}|$ but we have yet to give a meaning to $|A| < |B|$ for sets A and B. This idea will be addressed in Section 10.4. We should remind ourselves, however, that for infinite sets C and D, it is possible that both $C \subset D$

and $|C| = |D|$. For example, $\mathbf{Z} \subset \mathbf{Q}$ and $|\mathbf{Z}| = |\mathbf{Q}|$ since $\mathbf{Z}$ and $\mathbf{Q}$ are both denumerable. Before leaving our discussion of $\mathbf{Z}$ and $\mathbf{R}$, one other observation is useful. Recall that, according to Theorem 10.3, if B is an infinite subset of a denumerable set A, then B is also denumerable. But what if A is uncountable? That is, if B is an infinite subset of an uncountable set A, can we conclude that B is uncountable? The sets $\mathbf{Z}$ and $\mathbf{R}$ answer this question since $\mathbf{Z}$ is infinite, $\mathbf{R}$ is uncountable, and $\mathbf{Z} \subset \mathbf{R}$. However, $\mathbf{Z}$ is not uncountable.

We have now seen two examples of uncountable sets, namely, the open interval $(0, 1)$ of real numbers and the set $\mathbf{R}$ of all real numbers. Neither of these sets have the same number of elements as any denumerable set. But how do these sets compare with each other? We show that these two sets actually have the same number of elements. To verify this, we show that there is a bijective function $f : (0, 1) \to \mathbf{R}$ or equivalently, a bijective function $g : \mathbf{R} \to (0, 1)$. We'll show the first of these.

Theorem to Prove The sets $(0, 1)$ and $\mathbf{R}$ are numerically equivalent.

PROOF STRATEGY We show that there is a bijective function $f : (0, 1) \to \mathbf{R}$. We are faced with two problems here. First, we must discover a function f that we believe has this property and, second, we must show that our function f is, in fact, bijective. Of course, if we can't think of such a function f, then we don't have to worry about the second problem.

Let's review what properties we want our function f to have. The domain of f is $(0, 1)$ and the range is $\mathbf{R}$. So every real number needs to be the image of a number in the set $(0, 1)$; in fact, every real number needs to be the image of *exactly one* number in the set $(0, 1)$. Let's think back to calculus of what the graph of such function might look like. See Figure 10.9.

The graph of $y = f(x)$ for our trial function f in Figure 10.9 actually looks a bit like one branch of the graph of $y = \tan x$, but perhaps this seems too complicated. Another property of the graph of $y = f(x)$ for our still-unknown function f is that it appears that $x = 0$ and $x = 1$ are vertical asymptotes. You might recall that when a rational function (the ratio of two polynomials) has $x = a$ as a vertical asymptote, where $a \in \mathbf{R}$, then $x - a$ occurs as a factor in the denominator of the rational function. So perhaps $x(x - 1)$ should be a factor in the denominator. We also have the graph of our function intersecting the x-axis around $x = 1/2$; so this suggests a factor of $x - \frac{1}{2}$ or

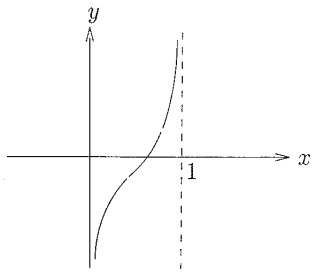

Figure 10.9 Constructing a bijective function $f : (0, 1) \to \mathbf{R}$

$2x - 1$ in the numerator. Consequently, we seem to be arriving at a function f defined by $f(x) = (2x - 1)/(x^2 - x)$ for $x \in (0, 1)$. If we draw the graph of this function, then we see that f is decreasing in the interval $(0, 1)$, not increasing as we have indicated in Figure 10.9. Although we could just as well use this function, we can define a function whose graph is essentially that in Figure 10.9 by multiplying by -1, that is, by defining

$$f(x) = -\frac{2x - 1}{x^2 - x} = \frac{1 - 2x}{x^2 - x}. \tag{10.2}$$

Let's choose this as our function f. Of course, we will only know that we made a good choice once we verify that the function $f : (0, 1) \to \mathbf{R}$ defined in (10.2) is actually bijective. Let's attempt to prove this.

It will turn out that showing that f is one-to-one is relatively straightforward. But showing that f is onto is another story. Let's see what we're faced with for this function. To prove that the function $f : (0, 1) \to \mathbf{R}$ defined in (10.2) is onto, we must show that for each $r \in \mathbf{R}$, there exists $x \in (0, 1)$ such that $f(x) = r$. This is certainly true if $r = 0$ since $f(\frac{1}{2}) = 0$, so we can assume that $r \neq 0$. Since we want

$$f(x) = \frac{1 - 2x}{x^2 - x} = r,$$

this gives us the equation $rx^2 + (-r + 2)x - 1 = 0$. What must x equal? Think of this equation as a quadratic equation (which it is), that is, $ax^2 + bx + c = 0$, where $a = r$, $b = -r + 2$, and $c = -1$. By the quadratic formula,

$$x = \frac{r - 2 \pm \sqrt{r^2 + 4}}{2r}. \tag{10.3}$$

We need to analyze (10.3) a bit now and decide which sign should be chosen in $\pm$. Notice that if $x = \frac{r-2-\sqrt{r^2+4}}{2r}$ and we let $r = 2$, then $x < 0$; while if we let $r = -2$, then $x > 1$. Since for each $r \in \mathbf{R}$, we want $x \in (0, 1)$ such that $f(x) = r$, we choose $x = \frac{r-2+\sqrt{r^2+4}}{2r}$.

Let $r > 0$. Then

$$2 < \sqrt{r^2 + 4} < \sqrt{r^2 + 4r + 4} = r + 2.$$

So

$$\frac{1}{2} = \frac{r}{2r} = \frac{(r-2)+2}{2r} < \frac{(r-2)+\sqrt{r^2+4}}{2r} < \frac{(r-2)+(r+2)}{2r} = \frac{2r}{2r} = 1.$$

Therefore, when $r > 0$, we have $\frac{1}{2} < x < 1$.

Next, let $r < 0$. First, notice that since $\sqrt{(r-2)^2} \geq 0$ and $r < 0$, we must have $\sqrt{(r-2)^2} = -(r-2)$. Now

$$(r-2) + \sqrt{r^2+4} < (r-2) + \sqrt{r^2 - 4r + 4} = (r-2) - (r-2) = 0.$$

Since $r < 0$,

$$\frac{(r-2)+\sqrt{r^2+4}}{2r} > 0.$$

Also, $(r-2) + \sqrt{r^2+4} > (r-2) + 2 = r$. So

$$0 < \frac{(r-2)+\sqrt{r^2+4}}{2r} < \frac{r}{2r} = \frac{1}{2}.$$

So, as desired, when $r < 0$, we have $0 < x < \frac{1}{2}$. With this choice of x, it remains only to show that $f(x) = r$. ♦

Theorem 10.13 *The sets* $(0, 1)$ *and* $\mathbf{R}$ *are numerically equivalent.*

Proof Consider the function $f : (0, 1) \to \mathbf{R}$ defined by

$$f(x) = \frac{1 - 2x}{x^2 - x}.$$

We show that f is bijective. First, we verify that f is one-to-one. Let $f(a) = f(b)$, where $a, b \in (0, 1)$. Then $\frac{1-2a}{a^2-a} = \frac{1-2b}{b^2-b}$. Crossmultiplying and simplifying, we obtain

$$(a - b)(a + b - 1 - 2ab) = 0. \tag{10.4}$$

To show that $a = b$, we need to show that $a + b - 1 - 2ab \neq 0$. Assume, to the contrary, that $a + b - 1 - 2ab = 0$. This is equivalent, however, to $ab - a - b + 1 = -ab$ or $(a - 1)(b - 1) = -ab$. Since $0 < a < 1$ and $0 < b < 1$, it follows that $(a - 1)(b - 1) > 0$ and $-ab < 0$, which is impossible. Thus, as claimed, $a + b - 1 - 2ab \neq 0$. Therefore, $a - b = 0$ and so $a = b$. Hence f is one-to-one.

Next we verify that f is onto. Let $r \in \mathbf{R}$. We show that there exists $x \in (0, 1)$ such that $f(x) = r$. Since $f(\frac{1}{2}) = 0$, we can assume that $r \neq 0$. Let $x = \frac{r-2+\sqrt{r^2+4}}{2r}$. Then $x \in (0, 1)$ and

$$f(x) = \frac{1 - 2\left(\frac{r-2+\sqrt{r^2+4}}{2r}\right)}{\left(\frac{r-2+\sqrt{r^2+4}}{2r}\right)^2 - \left(\frac{r-2+\sqrt{r^2+4}}{2r}\right)} = \frac{r(8 - 4\sqrt{r^2+4})}{8 - 4\sqrt{r^2+4}} = r.$$

Therefore, the function f is onto and consequently f is bijective. ■

Actually, the function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = \tan\left(\frac{\pi}{2}(2x - 1)\right)$ is bijective as well. However, the proof of this relies on knowing certain facts about this function.

10.4 Comparing Cardinalities of Sets

As we know, two nonempty sets A and B have the same cardinality if there exists a bijective function $f : A \to B$. Let's illustrate this concept one more time by showing that two familiar sets associated with a given set are numerically equivalent. Recall that the power set $\mathcal{P}(A)$ of a set A is the set of all subsets of A and that 2^A is the set of all functions from A to $\{0, 1\}$. If $A = \{a, b, c\}$, then $|\mathcal{P}(A)| = 2^3 = 8$. Also, the set 2^A contains $2^{|A|} = 2^3 = 8$ functions. So in this case, $\mathcal{P}(A)$ and 2^A have the same number of elements. This is not a coincidence.

Theorem to Prove For every nonempty set A, the sets $\mathcal{P}(A)$ and 2^A are numerically equivalent.

PROOF STRATEGY To prove that $\mathcal{P}(A)$ and 2^A are numerically equivalent, it is necessary to construct a bijective function $\phi : \mathcal{P}(A) \to 2^A$. We use ϕ for this function since 2^A is a set of functions

and it is probably better to use more standard notation, such as f, to denote the elements of 2^A. But how can such a function ϕ be defined? Let's take a look at $\mathcal{P}(A)$ and 2^A for $A = \{a, b\}$. In this case,

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\};$$

while $2^A = \{f_1, f_2, f_2, f_4\}$, where

$$f_1 = \{(a, 0), (b, 0)\}, \quad f_2 = \{(a, 1), (b, 0)\},$$
$$f_3 = \{(a, 0), (b, 1)\}, \quad f_4 = \{(a, 1), (b, 1)\}.$$

Since each of $\mathcal{P}(A)$ and 2^A has four elements, we can easily find a bijective function from $\mathcal{P}(A)$ to 2^A. But this is not the question. What we are looking for is a bijective function $\phi : \mathcal{P}(A) \to 2^A$ for $A = \{a, b\}$ that suggests a way for us to define a bijective function from $\mathcal{P}(A)$ to 2^A for *any* set A (finite or infinite). Notice, for $A = \{a, b\}$, the connection between the following pairs of elements, the first element belonging to $\mathcal{P}(A)$ and the second belonging to 2^A:

$$\begin{array}{ll} \emptyset & f_1 = \{(a, 0), (b, 0)\} \\ \{a\} & f_2 = \{(a, 1), (b, 0)\} \\ \{b\} & f_3 = \{(a, 0), (b, 1)\} \\ \{a, b\} & f_4 = \{(a, 1), (b, 1)\}. \end{array}$$

For example, the subset $\{a\}$ of $\{a, b\}$ contains a but not b, while f_2 maps a to 1 and b to 0. For an arbitrary set A, this suggests defining ϕ so that a subset S of A is mapped into the function in which 1 is the image of elements of A that belong to S and 0 is the image of elements of A that do not belong to S. ♦

Theorem 10.14 *For every nonempty set* A*, the sets* $\mathcal{P}(A)$ *and* 2^A *are numerically equivalent.*

Proof We show that there exists a bijective function ϕ from $\mathcal{P}(A)$ to 2^A. Define $\phi : \mathcal{P}(A) \to 2^A$ such that for $S \in \mathcal{P}(A)$, we have $\phi(S) = f_S$, where, for $x \in A$,

$$f_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Certainly, $f_S \in 2^A$. First, we show that ϕ is one-to-one. Let $\phi(S) = \phi(T)$. Thus, $f_S = f_T$, which implies that $f_S(x) = f_T(x)$ for every $x \in A$. Therefore, $f_S(x) = 1$ if and only if $f_T(x) = 1$ for every $x \in A$, that is, $x \in S$ if and only if $x \in T$, and so $S = T$.

It remains to show that ϕ is onto. Let $f \in 2^A$. Define

$$S = \{x \in A : f(x) = 1\}.$$

Hence $f_S = f$, and so $\phi(S) = f$. Thus ϕ is onto and, consequently, ϕ is bijective. ■

It is clear that $A = \{x, y, z\}$ has fewer elements than $B = \{a, b, c, d, e\}$, that is, $|A| < |B|$. And it certainly seems that $|B| < |\mathbf{N}|$ and that, in general, any finite set has fewer elements than any denumerable set (or than any infinite set). Also, our discussion about countable and uncountable sets appears to suggest that uncountable sets have more elements than countable sets. But these assertions are based on intuition. We now make this more precise.

A set A is said to have **smaller cardinality** than a set B, written as $|A| < |B|$, if there exists a one-to-one function from A to B but no bijective function from A to B. That is, $|A| < |B|$ if it is possible to pair off the elements of A with some of the elements of B but not with all of the elements of B. If $|A| < |B|$, then we also write $|B| > |A|$. For example, since $\mathbf{N}$ is denumerable and $\mathbf{R}$ is uncountable, there is no bijective function from $\mathbf{N}$ to $\mathbf{R}$. Since the function $f : \mathbf{N} \to \mathbf{R}$ defined by $f(n) = n$ for all $n \in \mathbf{N}$ is injective, it follows that $|\mathbf{N}| < |\mathbf{R}|$. Moreover, $|A| \leq |B|$ means that $|A| = |B|$ or $|A| < |B|$. Hence to verify that $|A| \leq |B|$, we need only show the existence of a one-to-one function from A to B.

The cardinality of the set $\mathbf{N}$ of natural numbers is denoted by $\aleph_o$ (often read "aleph null"); so $|\mathbf{N}| = \aleph_o$. Actually, $\aleph$ is the first letter of Hebrew alphabet. Indeed, if A is any denumerable set, then $|A| = \aleph_o$. The set $\mathbf{R}$ of real numbers is also referred to as the **continuum** and its cardinality is denoted by c. Hence $|\mathbf{R}| = c$ and from what we have seen, $\aleph_o < c$. It was the German mathematician Georg Cantor who helped to put the theory of sets on a firm foundation. An interesting conjecture of his became known as:

The Continuum Hypothesis There exists no set S such that

$$\aleph_o < |S| < c.$$

Of course, if the Continuum Hypothesis were true, then this would imply that every subset of $\mathbf{R}$ is either countable or is numerically equivalent to $\mathbf{R}$. However, in 1931 the Austrian mathematician Kurt Gödel proved that it was impossible to disprove the Continuum Hypothesis from the axioms on which the theory of sets is based. In 1963 the American mathematician Paul Cohen took it one step further by showing that it was also impossible to *prove* the Continuum Hypothesis from these axioms. Thus the Continuum Hypothesis is independent of the axioms of set theory.

Another question that might occur to you is the following: Is there a set S such that $|S| > c$? This is a question we can answer, however, and the answer might be surprising.

Theorem to Prove If A is a set, then $|A| < |\mathcal{P}(A)|$.

PROOF STRATEGY First, it is not surprising that $|A| < |\mathcal{P}(A)|$ if A is finite, for if A has n elements, where $n \in \mathbf{N}$, then $\mathcal{P}(A)$ has 2^n elements and $2^n > n$ (which was proved by induction in Result 6.15). Of course, we must still show that $|A| < |\mathcal{P}(A)|$ when A is infinite. First we show that there exists a one-to-one function $f : A \to \mathcal{P}(A)$ for every set A. Let's give ourselves an example, say $A = \{a, b\}$. Then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Although there are many injective functions from A to $\mathcal{P}(A)$, there is one natural injective function:

$$f = \{(a, \{a\}), (b, \{b\})\};$$

in other words, define $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$.

Once we have verified that this function is one-to-one, then we know that $|A| \leq |\mathcal{P}(A)|$. To show that the inequality is strict, however, we must prove that there is no bijective function from A to $\mathcal{P}(A)$. The natural technique to use for this proof is proof by contradiction. ♦

Theorem 10.15 *If A is a set, then $|A| < |\mathcal{P}(A)|$.*

Proof If $A = \emptyset$, then $|A| = 0$ and $|\mathcal{P}(A)| = 1$; so $|A| < |\mathcal{P}(A)|$. Hence we may assume that $A \neq \emptyset$. First, we show that there is a one-to-one function from A to $\mathcal{P}(A)$. Define the function $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$ for each $x \in A$. Let $f(x_1) = f(x_2)$. Then $\{x_1\} = \{x_2\}$. So $x_1 = x_2$ and f is one-to-one.

To prove that $|A| < |\mathcal{P}(A)|$, it remains to show that there is no bijective function from A to $\mathcal{P}(A)$. Assume, to the contrary, that there exists a bijective function $g : A \to \mathcal{P}(A)$. For each $x \in A$, let $g(x) = A_x$, where $A_x \subseteq A$. We show that there is a subset of A that is distinct from A_x for each $x \in A$. Define the subset B of A by

$$B = \{x \in A : x \notin A_x\}.$$

By assumption, there exists an element $y \in A$ such that $B = A_y$. If $y \in A_y$, then $y \notin B$ by the definition of B. On the other hand, if $y \notin A_y$, then, according to the definition of the set B, it follows that $y \in B$. In either case, y belongs to exactly one of A_y and B. Hence $B \neq A_y$, producing a contradiction. ■

According to Theorem 10.15, there is no largest set. In particular, there is a set S with $|S| > c$.

10.5 The Schröder–Bernstein Theorem

For two nonempty sets A and B, let f be a function from A to B, and let D be a nonempty subset of A. By the **restriction f_1 of f to D**, we mean the function

$$f_1 = \{(x, y) \in f : x \in D\}.$$

Hence a restriction of f refers to restricting the domain of f. For example, for the sets $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$, let $f = \{(a, 2), (b, 1), (c, 3), (d, 2)\}$ be a function from A to B. For $D = \{a, c\}$, the restriction of f to D is the function $f_1 : D \to B$ given by $\{(a, 2), (c, 3)\}$. Sometimes, we might also consider a new codomain B' for such a restriction f_1 of f. Of course, we must have $\text{ran } f_1 \subseteq B'$. Next, consider the function $g : \mathbf{R} \to [0, \infty)$ defined by $g(x) = x^2$ for $x \in \mathbf{R}$. Although g is onto, g is not one-to-one since $g(1) = g(-1) = 1$, for example. On the other hand, the restriction g_1 of g to $[0, \infty)$ is one-to-one and so the restricted function $g_1 : [0, \infty) \to [0, \infty)$ defined by $g_1(x) = g(x) = x^2$ for all $x \in [0, \infty)$ is bijective. Furthermore, if $f : A \to B$ is a one-to-one function, then any restriction of f to a subset of A is also one-to-one.

Let $f : A \to B$ and $g : C \to D$ be functions, where A and C are disjoint sets. We define a function h from $A \cup C$ to $B \cup D$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C. \end{cases}$$

Recalling that a function is a *set* of ordered pairs, we see that h is the union of the two sets f and g. Of course, it is essential for A and C to be disjoint in order to be guaranteed that h is a function. If f and g are onto, then h must be onto as well; however, if f and g are one-to-one, then h need not be one-to-one. The following result does provide a sufficient condition for h to be one-to-one, however.

Lemma 10.16 *Let $f : A \to B$ and $g : C \to D$ be one-to-one functions, where $A \cap C = \emptyset$, and define*

$h : A \cup C \to B \cup D$ *by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C. \end{cases}$$

If $B \cap D = \emptyset$, then h is also a one-to-one function. Consequently, if f and g are bijective functions, then h is a bijective function.

Proof Assume that $h(x_1) = h(x_2) = y$, where $x_1, x_2 \in A \cup B$. Then $y \in B \cup D$. So $y \in B$ or $y \in D$, say the former. Since $B \cap D = \emptyset$, it follows that $y \notin D$. Hence $x_1, x_2 \in A$ and so $h(x_1) = f(x_1)$ and $h(x_2) = f(x_2)$. Since $f(x_1) = f(x_2)$ and f is one-to-one, it follows that $x_1 = x_2$. ∎

Let A and B be nonempty sets such that $B \subseteq A$ and let $f : A \to B$. Thus for $x \in A$, the element $f(x) \in B$. Since $B \subseteq A$, it follows, of course, that $f(x) \in A$ and so $f(f(x)) \in B$. It is convenient to introduce some notation in this case. Let $f^1(x) = f(x)$ and let $f^2(x) = f(f(x))$. In general, for an integer $k \geq 2$, let $f^k(x) = f(f^{k-1}(x))$. Hence $f^1(x), f^2(x), f^3(x), \ldots$ is a recursively defined sequence of elements of B (and of A as well). Thus $f^n(x)$ is defined for every positive integer n.

For example, consider the function $f : \mathbf{Z} \to 2\mathbf{Z}$ defined by $f(n) = 4n$ for all $n \in \mathbf{Z}$. Then $f^1(3) = f(3) = 4 \cdot 3 = 12$ and $f^2(3) = f(f(3)) = f(12) = 4 \cdot 12 = 48$.

If A and B are nonempty sets such that $B \subseteq A$, then the function $\phi : B \to A$ defined by $\phi(x) = x$ for all $x \in B$ is injective. This gives us the expected result that $|B| \leq |A|$. On the other hand, if there is an injective function from A to B, a more interesting consequence results.

Theorem 10.17 *Let A and B be nonempty sets such that $B \subseteq A$. If there exists an injective function from A to B, then there exists a bijective function from A to B.*

Proof If $B = A$, then the identity function $i_A : A \to B = A$ is bijective. Thus we can assume that $B \subset A$, and so $A - B \neq \emptyset$. Let $f : A \to B$ be an injective function. If f is bijective, then the proof is complete. Therefore, we can assume that f is not onto. Hence ran $f \subset B$, and so $B - \text{ran } f \neq \emptyset$.

Consider the subset B' of B defined by

$$B' = \{f^n(x) : x \in A - B, n \in \mathbf{N}\}.$$

Thus $B' \subseteq \text{ran } f$. Hence, for each $x \in A - B$, its image $f(x)$ belongs to B'. Moreover, for $x \in A - B$, the element $f^2(x) = f(f(x)) \in B'$, $f^3(x) = f(f^2(x)) \in B'$, and so on.

Let $C = (A - B) \cup B'$, and consider the restriction $f_1 : C \to B'$ of f to C. We show that f_1 is onto. Let $y \in B'$. Then $y = f^n(x)$ for some $x \in A - B$ and some $n \in \mathbf{N}$. This implies that $y = f(x)$ for some $x \in A - B$, or $y = f(x)$ for some $x \in B'$. Therefore, $f_1(x) = y$ for some $x \in C$, and so f_1 is onto. Furthermore, since f is one-to-one, the function f_1 is also one-to-one. Hence $f_1 : C \to B'$ is bijective.

Let $D = B - B'$. Since $B - \text{ran } f \neq \emptyset$ and $B - \text{ran } f \subseteq B - B'$, it follows that $D \neq \emptyset$. Also, D and B' are disjoint, as are D and C. Certainly, the identity function

$i_D : D \to D$ is bijective. Let $h : C \cup D \to B' \cup D$ be defined by

$$h(x) = \begin{cases} f_1(x) & \text{if } x \in C \\ i_D(x) & \text{if } x \in D. \end{cases}$$

By Lemma 10.16, h is bijective. However, $C \cup D = A$ and $B' \cup D = B$; so h is a bijective function from A to B. ∎

From what we know of inequalities (of real numbers), it might seem that if A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. This is indeed the case. This theorem is often referred to as the Schröder–Bernstein Theorem.

Theorem 10.18 **(The Schröder–Bernstein Theorem)** *If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof Since $|A| \leq |B|$ and $|B| \leq |A|$, there are injective functions $f : A \to B$ and $g : B \to A$. Thus $g_1 : B \to \text{ran } g$ defined by $g_1(x) = g(x)$ for all $x \in B$ is a bijective function. By Theorem 9.10, g_1^{-1} exists and $g_1^{-1} : \text{ran } g \to B$ is a bijective function.

Since $f : A \to B$ and $g_1 : B \to \text{ran } g$ are injective functions, it follows by Theorem 9.7 that $g_1 \circ f : A \to \text{ran } g$ is an injective function. Because ran $g \subseteq A$, we have by Theorem 10.17 that there exists a bijective function $h : A \to \text{ran } g$. Thus $h : A \to \text{ran } g$ and $g_1^{-1} : \text{ran } g \to B$ are bijective functions. By Corollary 9.8,

$$g_1^{-1} \circ h : A \to B$$

is a bijective function and $|A| = |B|$. ∎

The Schröder–Bernstein Theorem is referred to by some as the Cantor-Schröder-Bernstein Theorem. Although the history of this theorem has never been fully documented, there are several substantiated facts.

A mathematician who will forever be associated with the theory of sets is Georg Cantor (1845–1918). Born in Russia, Cantor studied for and obtained his Ph.D in mathematics from the University of Berlin in 1867. In 1869 he became a faculty member at the University of Halle in Germany. It was while he was there that he became interested in set theory.

In 1873 Cantor proved that the set of rational numbers is denumerable. Shortly afterwards, he proved that the set of real numbers is uncountable. In this paper, he essentially introduced the idea of a one-to-one correspondence (bijective function). During the next several years, he made numerous contributions to set theory–studying sets of equal cardinality. There were, however, a number of problems that proved difficult for Cantor.

Consider the following two theorems:

Theorem A *For every two sets A and B, exactly one of the following occurs:* (1) $|A| = |B|$, (2) $|A| < |B|$, (3) $|A| > |B|$.

Theorem B *If A and B are two sets for which there exist a one-to-one function from A to B and a one-to-one function from B to A, then $|A| = |B|$.*

Cantor observed that once Theorem A had been proved, Theorem B could be proved. On the other hand, there has never been any evidence that Cantor was able to prove Theorem A. Ernst Zermelo (1871–1953) was able to prove Theorem A in 1904. However, Zermelo's proof made use of an axiom formulated by Zermelo. This axiom, which was controversal in the mathematical world for many years, is known as the **Axiom of Choice**.

> **The Axiom of Choice.** *For every collection of pairwise disjoint nonempty sets, there exists at least one set that contains exactly one element of each of these nonempty sets.*

As it turned out, not only can the Axiom of Choice be used to prove Theorem A, but Theorem A is true if and only if the Axiom of Choice is true.

Ernst Schröder (1841–1902), a German mathematician, was one of the important figures in mathematical logic. During 1897–1898 Schröder presented a "proof" of Theorem B, which contained a defect however. About the same time, Felix Bernstein (1878–1956) gave his own proof of Theorem B in his doctoral dissertation, which became the first complete proof of Theorem B. His proof did not require knowledge of Theorem A.

You may be surprised to learn that $\mathbf{R}$ and the power set of $\mathbf{N}$ are numerically equivalent. But how could one ever find a bijective function between these two sets? Theorem 10.18 tells us that discovering such a function is unnecessary.

Theorem 10.19 *The sets $\mathcal{P}(\mathbf{N})$ and $\mathbf{R}$ are numerically equivalent.*

Proof First we show that there is a one-to-one function $f : (0, 1) \to \mathcal{P}(\mathbf{N})$. Recall that a real number $a \in (0, 1)$ can be expressed uniquely as $a = 0.a_1a_2a_3\cdots$, where each $a_i \in \{0, 1, \ldots, 9\}$ and there is no positive integer N such that $b_n = 9$ for all $n \geq N$. Thus we define

$$f(a) = \{10^{n-1}a_n : n \in \mathbf{N}\} = A.$$

For example, $f(0.1234) = \{1, 20, 300, 4000\}$ and $f(1/3) = \{3, 30, 300, \ldots\}$. We now show that f is one-to-one. Assume that $f(a) = f(b)$, where $a, b \in (0, 1)$ and $a = 0.a_1a_2a_3\cdots$ and $b = 0.b_1b_2b_3\cdots$ with $a_i, b_i \in \{0, 1, \ldots, 9\}$ for each $i \in \mathbf{N}$ such that the decimal expansion of neither a nor b is 9 from some point on. Therefore,

$$A = \{10^{n-1}a_n : n \in \mathbf{N}\} = \{10^{n-1}b_n : n \in \mathbf{N}\} = B.$$

Consider the ith digit, namely a_i, in the decimal expansion of a. Then $10^{i-1}a_i \in A$. If $a_i \neq 0$, then $10^{i-1}a_i$ is the unique number in the interval $[10^{i-1}, 9 \cdot 10^{i-1}]$ belonging to A. Since $A = B$, it follows that $10^{i-1}a_i \in B$. However, $10^{i-1}b_i$ is the unique number in the interval $[10^{i-1}, 9 \cdot 10^{i-1}]$ belonging to B; so $10^{i-1}a_i = 10^{i-1}b_i$. Thus $a_i = b_i$. If $a_i = 0$, then $0 \in A$ and there is no number in the interval $[10^{i-1}, 9 \cdot 10^{i-1}]$ belonging to A. Since $A = B$, it follows that $0 \in B$ and there is no number in the interval $[10^{i-1}, 9 \cdot 10^{i-1}]$ belonging to B. Thus $b_i = 0$ and so $a_i = b_i$. Hence $a_i = b_i$ for all $i \in \mathbf{N}$, and so $a = b$. Therefore, f is one-to-one and $|(0, 1)| \leq |\mathcal{P}(\mathbf{N})|$.

Next we define a function $g : \mathcal{P}(\mathbf{N}) \to (0, 1)$. For $S \subseteq \mathbf{N}$, define $g(S) = 0.s_1s_2s_3\cdots$, where

$$s_n = \begin{cases} 1 & \text{if } n \in S \\ 2 & \text{if } n \notin S. \end{cases}$$

Thus $g(S)$ is a real number in $(0, 1)$, whose decimal expansion consists only of 1s and 2s. We show that g is one-to-one. Assume that $g(S) = g(T)$, where $S, T \subseteq \mathbf{N}$. Thus

$$g(S) = s = 0.s_1s_2s_3\cdots = 0.t_1t_2t_3\cdots = t = g(T),$$

where

$$s_n = \begin{cases} 1 & \text{if } n \in S \\ 2 & \text{if } n \notin S \end{cases} \text{ and } t_n = \begin{cases} 1 & \text{if } n \in T \\ 2 & \text{if } n \notin T. \end{cases}$$

Since the decimal expansions of s and t contain no 0s or 9s, both s and t have unique decimal expansions. We show that $S = T$. First, we verify that $S \subseteq T$. Let $k \in S$. Then $s_k = 1$. Since $s = t$, it follows that $t_k = 1$, which implies that $k \in T$. Hence $S \subseteq T$. The proof that $T \subseteq S$ is similar and is therefore omitted. Thus $S = T$ and g is one-to-one. Therefore, $|\mathcal{P}(\mathbf{N})| \leq |(0, 1)|$. By the Schröder–Bernstein Theorem, $|\mathcal{P}(\mathbf{N})| = |(0, 1)|$. By Theorem 10.13, $|(0, 1)| = |\mathbf{R}|$. Thus, $|\mathcal{P}(\mathbf{N})| = |\mathbf{R}|$. ∎

As a corollary to Theorems 10.14 and 10.19, we have the following result.

Corollary 10.20 *The sets $2^{\mathbf{N}}$ and $\mathbf{R}$ are numerically equivalent.*

We have already mentioned that $|A| = \aleph_o$ for every denumerable set A and that $|\mathbf{R}| = c$. If A is denumerable, then we represent the cardinality of the set 2^A by $2^{\aleph_o}$. By Corollary 10.20, $2^{\aleph_o} = c$.

EXERCISES FOR CHAPTER 10

Section 10.2: Denumerable Sets

10.1. Prove that if A and B are disjoint denumerable sets, then $A \cup B$ is denumerable.

10.2. Let $\mathbf{R}^+$ denote the set of positive real numbers and let A and B be denumerable subsets of $\mathbf{R}^+$. Define $C = \{x \in \mathbf{R} : -x \in B\}$. Show that $A \cup C$ is denumerable.

10.3. Let

$$S = \left\{x \in \mathbf{R} : x = \frac{n^2 + \sqrt{2}}{n}, n \in \mathbf{N}\right\}.$$

Define $f : \mathbf{N} \to S$ by $f(n) = \frac{n^2+\sqrt{2}}{n}$.

(a) List three elements that belong to S.

(b) Show that f is one-to-one.

(c) Show that f is onto.

(d) Is S denumerable? Explain.

10.4. Prove that the function $f : \mathbf{N} \to \mathbf{Z}$ defined in (10.1) by

$$f(n) = \frac{1 + (-1)^n(2n - 1)}{4}$$

is bijective.

10.5. Let A be a denumerable set and let $B = \{x, y\}$. Prove that $A \times B$ is denumerable.

10.6. Let B be a denumerable set and let A be a nonempty set of unspecified cardinality. If $f : A \to B$ is a one-to-one function, then what can be said about the cardinality of A? Explain.

10.7. Prove that $S = \{(a, b) : a, b \in \mathbf{N} \text{ and } b \geq 2a\}$ is denumerable.

10.8. Let $S \subseteq \mathbf{N} \times \mathbf{N}$ be defined by $S = \{(i, j) : i \leq j\}$. Show that S is denumerable.

10.9. Prove that the set of all 2-element subsets of $\mathbf{N}$ is denumerable.

10.10. A **Gaussian integer** is a complex number of the form $a + bi$, where $a, b \in \mathbf{Z}$ and $i = \sqrt{-1}$. Show that the set $\mathcal{G}$ of Gaussian integers is denumerable.

10.11. Let $A_1, A_2, A_3, \ldots$ be pairwise disjoint denumerable sets. Prove that $\cup_{i=1}^{\infty} A_i$ is denumerable.

10.12. Let $A = \{a_1, a_2, a_3, \ldots\}$. Define $B = A - \{a_{n^2} : n \in \mathbf{N}\}$. Prove that $|A| = |B|$.

10.13. Prove that $|\mathbf{Z}| = |\mathbf{Z} - \{2\}|$.

10.14. (a) Prove that the function $f : \mathbf{R} - \{1\} \to \mathbf{R} - \{2\}$ defined by $f(x) = \frac{2x}{x-1}$ is bijective.
(b) Explain why $|\mathbf{R} - \{1\}| = |\mathbf{R} - \{2\}|$.

Section 10.3: Uncountable Sets

10.15. Prove that the set of irrational numbers is uncountable.

10.16. Prove that the set of complex numbers is uncountable.

10.17. Prove that the open interval $(0, 2)$ and $\mathbf{R}$ are numerically equivalent by finding a bijective function $f : (0, 2) \to \mathbf{R}$. (Show that your function is, in fact, bijective.)

10.18. (a) Prove that the function $f : (0, 1) \to (0, 2)$, mapping the open interval $(0, 1)$ into the open interval $(0, 2)$ and defined by $f(x) = 2x$, is bijective.
(b) Explain why $(0, 1)$ and $(0, 2)$ have the same cardinality.
(c) Let $a, b \in \mathbf{R}$, where $a < b$. Prove that $(0, 1)$ and (a, b) have the same cardinality.

Section 10.4: Comparing Cardinalities of Sets

10.19. Prove or disprove the following:

(a) If A is an uncountable set, then $|A| = |\mathbf{R}|$.
(b) There exists a bijective function $f : \mathbf{Q} \to \mathbf{R}$.
(c) If A, B, and C are sets such that $A \subseteq B \subseteq C$, and A and C are denumerable, then B is denumerable.
(d) The set $S = \left\{\frac{\sqrt{2}}{n} : n \in \mathbf{N}\right\}$ is denumerable.
(e) There exists a denumerable subset of the set of irrational numbers.
(f) Every infinite set is a subset of some denumerable set.
(g) If A and B are sets with the property that there exists an injective function $f : A \to B$, then $|A| = |B|$.

10.20. Prove or disprove: If A and B are two sets such that A is countable and $|A| < |B|$, then B is uncountable.

10.21. How do the cardinalities of the sets $[0, 1]$ and $[1, 3]$ compare? Justify your answer.

10.22. Let $A = \{a, b, c\}$. Then $\mathcal{P}(A)$ consists of the following subsets of A:

$$A_a = \emptyset, A_b = A, A_c = \{a, b\}, A_d = \{a, c\},$$
$$A_e = \{b, c\}, A_f = \{a\}, A_g = \{b\}, A_h = \{c\}.$$

In one part of the proof of Theorem 10.15, it was established (using a contradiction argument) that $|A| < |\mathcal{P}(A)|$ for every nonempty set A. In this argument, the existence of a bijective function $g : A \to \mathcal{P}(A)$ is assumed, where $g(x) = A_x$ for each $x \in A$. Then a subset B of A is defined by

$$B = \{x \in A : x \notin A_x\}.$$

(a) For the sets A and $\mathcal{P}(A)$ described above, what is the set B?
(b) What does the set B in (a) illustrate?

10.23. Let A and B be nonempty sets. Prove that $|A| \leq |A \times B|$.

Section 10.5: The Schröder–Bernstein Theorem

10.24. Prove that if A, B, and C are nonempty sets such that $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.

10.25. Use the Schröder–Bernstein Theorem to prove that $|(0, 1)| = |[0, 1]|$.

10.26. Prove that $|\mathbf{Q} - \{q\}| = \aleph_0$ for every rational number q and $|\mathbf{R} - \{r\}| = c$ for every real number r.

10.27. Let $f : \mathbf{Z} \to 2\mathbf{Z}$ be defined by $f(k) = 4k$ for all $k \in \mathbf{Z}$.

(a) Prove that $f^n(k) = 4^n k$ for each $k \in \mathbf{Z}$ and each $n \in \mathbf{N}$.
(b) For this function f, describe the sets B', C, and D and functions f_1 and h given in Theorem 10.17.

10.28. Express each positive rational number as m/n, where $m, n \in \mathbf{N}$ and m/n is reduced to lowest terms. Let d_a denote the number of digits in $a \in \mathbf{N}$. Thus $d_2 = 1$, $d_{13} = 2$, and $d_{100} = 3$. Define the function $f : \mathbf{Q}^+ \to \mathbf{N}$ so that $f(m/n)$ is the positive integer with $2(d_m + d_n)$ digits whose first d_m digits is the integer m, whose final d_n digits is the integer n, and all of whose remaining $d_m + d_n$ digits are 0. Thus $f(2/3) = 2003$ and $f(10/271) = 1000000271$.

(a) Prove that f is one-to-one.
(b) Use the Schröder–Bernstein Theorem to prove that $\mathbf{Q}^+$ is denumerable.

ADDITIONAL EXERCISES FOR CHAPTER 10

10.29. Evaluate the proposed proof of the following result.

Result Let A and B be two sets with $|A| = |B|$. If $a \in A$ and $b \in B$, then $|A - \{a\}| = |B - \{b\}|$.

Proof Since A and B have the same number of elements and one element is removed from each of A and B, it follows that $|A - \{a\}| = |B - \{b\}|$. ∎

10.30. Evaluate the proposed proof of the following result.

Result The sets $(0, \infty)$ and $[0, \infty)$ are numerically equivalent.

Proof Define the function $f : (0, \infty) \to [0, \infty)$ by $f(x) = x$. First, we show that f is one-to-one. Assume that $f(a) = f(b)$. Then $a = b$ and so f is one-to-one.
Next, we show that f is onto. Let $r \in [0, \infty)$. Since $f(r) = r$, the function f is onto. Since f is bijective, $|(0, \infty)| = |[0, \infty)|$. ∎

10.31. For a real number x, the **floor** $\lfloor x \rfloor$ of x is the greatest integer less than or equal to x. Therefore, $\lfloor 5.5 \rfloor = 5$, $\lfloor 3 \rfloor = 3$, and $\lfloor -5.5 \rfloor = -6$. Let $f : \mathbf{N} \to \mathbf{Z}$ be defined by $f(n) = (-1)^n \lfloor n/2 \rfloor$.

(a) Prove that f is bijective.
(b) What does (a) tell us about $\mathbf{Z}$? (See Result 10.2.)

12 Proofs in Calculus

The proofs that occur in calculus are considerably different than any of those we have seen thus far. Calculus is the study of functions and limits. The functions encountered in calculus are real-valued functions defined on sets of real numbers. That is, each function that we study in calculus is of the type $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$. In the study of limits, we are often interested in such functions having the property that either (1) $X = \mathbf{N}$ and increasing values in the domain $\mathbf{N}$ result in functional values approaching some real number L, or (2) the function is defined for all real numbers near some specified real number a and values approaching a result in functional values approaching some real number L. We begin with (1), where $X = \mathbf{N}$.

12.1 Limits of Sequences

A **sequence** (of real numbers) is a real-valued function defined on the set of natural numbers; that is, a **sequence** is a function $f : \mathbf{N} \to \mathbf{R}$. If $f(n) = a_n$ for each $n \in \mathbf{N}$, then $f = \{(1, a_1), (2, a_2), (3, a_3), \cdots\}$. Since only the numbers $a_1, a_2, a_3, \cdots$ are relevant in f, we also refer to $a_1, a_2, a_3, \cdots$ as the sequence, which we often denote by $\{a_n\}$. The numbers a_1, a_2, a_3, etc. are called the **terms** of $\{a_n\}$, with a_1 being the first term, a_2 the second term, etc. Thus a_n is the nth term of the sequence. Hence $\{\frac{1}{n}\}$ is the sequence $1, 1/2, 1/3, \cdots$; while $\{\frac{n}{2n+1}\}$ is the sequence $1/3, 2/5, 3/7, \cdots$. In these two examples, the nth term of a sequence is given and, from this, we can easily find the first few terms and, in fact, any particular term. On the other hand, finding the nth term of a sequence whose first few terms are given can be challenging. For example, the nth term of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots$$

is $1/2n$; the nth term of the sequence

$$1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{8}, \cdots$$

is $1 + 1/2^n$; the nth term of the sequence

$$1, \frac{3}{5}, \frac{1}{2}, \frac{5}{11}, \frac{3}{7}, \frac{7}{17}, \cdots$$

is $(n + 1)/(3n - 1)$; the nth term of the sequence

$$1, -1, 1, -1, 1, -1, \cdots$$

is $(-1)^{n+1}$; while the nth term of the sequence $1, 4, 9, 16, \cdots$ is n^2.

For the sequence $\{\frac{1}{n}\}$, the larger the integer n, the closer $1/n$ is to 0; and for the sequence $\{\frac{n}{2n+1}\}$, the larger the integer n, the closer $n/(2n + 1)$ is to $1/2$. On the other hand, for the sequence $\{n^2\}$, as the integer n becomes larger, n^2 becomes increasingly large and does not approach any real number.

When we discuss how close two numbers are to each other, we are actually considering the distance between them. The **distance** between two real numbers a and b is defined as $|a - b|$. Recall that the absolute value of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Hence the distance between $a = 3$ and $b = 5$ is $|3 - 5| = |5 - 3| = 2$; while the distance between 0 and $1/n$, where $n \in \mathbf{N}$, is $|0 - \frac{1}{n}| = |\frac{1}{n} - 0| = \frac{1}{n}$.

For a fixed positive real number r, the inequality $|x| < r$ is equivalent to the inequalities $-r < x < r$. Hence $|x| < 3$ is equivalent to $-3 < x < 3$; while $|x - 2| < 4$ is equivalent to $-4 < x - 2 < 4$. Adding 2 throughout these inequalities, we obtain $-4 + 2 < (x - 2) + 2 < 4 + 2$ and so $-2 < x < 6$. We have seen in Exercise 4.23 and Theorem 4.17 that for real numbers x and y,

$$|xy| = |x||y| \quad \text{and} \quad |x + y| \leq |x| + |y|.$$

Both of these properties will be useful in our discussion of calculus.

We mentioned that for some sequences $\{a_n\}$, there is a real number L (or at least there appears to be a real number L) such that the larger the integer n becomes, the closer a_n is to L. We have now arrived at an important and fundamental idea in the study of sequences and are prepared to introduce a new concept to describe this situation.

A sequence $\{a_n\}$ of real numbers is said to converge to a real number L if the larger the integer n, the closer a_n is to L. Since the words "larger" and "closer" are vague and consequently are open to interpretation, we need to make these words considerably more precise.

What we want to say then is that we can make a_n as close to L as we wish (that is, we can make $|a_n - L|$ as small as we wish) provided that n is large enough. Let ϵ (the Greek letter epsilon) denote how small we want $|a_n - L|$ to be; that is, we want $|a_n - L| < \epsilon$ by choosing n large enough. This is equivalent to $-\epsilon < a_n - L < \epsilon$, that is, $L - \epsilon < a_n < L + \epsilon$. Hence we require that a_n be a number in the open interval $(L - \epsilon, L + \epsilon)$ when n is large enough. Now we need to know what we mean by "large enough". What we mean by this is that there is some positive integer N such that if n is an integer greater than N, then $a_n \in (L - \epsilon, L + \epsilon)$. If such a positive integer N can be found for every positive number ϵ, regardless of how small ϵ might be, then we say that $\{a_n\}$ converges to L. This is illustrated in Figure 12.1.

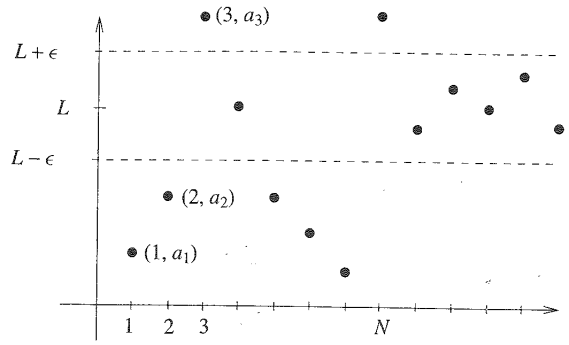

Figure 12.1 A sequence $\{a_n\}$ that converges to L

Formally then, a sequence $\{a_n\}$ of real numbers is said to **converge** to the real number L if for every real number $\epsilon > 0$, there exists a positive integer N such that if n is an integer with $n > N$, then $|a_n - L| < \epsilon$. As we indicated, the number ϵ is a measure of how close the terms a_n are required to be to the number L, and N indicates a position in the sequence beyond which the required condition is satisfied. If a sequence $\{a_n\}$ converges to L, then L is referred to as the **limit** of $\{a_n\}$, and we write $\lim_{n\to\infty} a_n = L$. If a sequence does not converge, it is said to **diverge**. Consequently, if a sequence $\{a_n\}$ diverges, then there is *no* real number L such that $\lim_{n\to\infty} a_n = L$.

Before looking at a few examples, we introduce some useful notation. For a real number x, recall that $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The integer $\lceil x \rceil$ is often called the **ceiling** of x. Consequently, $\lceil 8/3 \rceil = 3$, $\lceil \sqrt{2} \rceil = 2$, $\lceil -1.6 \rceil = -1$, and $\lceil 5 \rceil = 5$. By the definition of $\lceil x \rceil$, it follows that if x is an integer, then $\lceil x \rceil = x$; while if x is not an integer, then $\lceil x \rceil > x$. In particular, if n is an integer such that $n > \lceil x \rceil$, then $n > x$.

We now show how the definition of convergent sequence is used to prove that a sequence converges to some number.

Result to Prove The sequence $\{\frac{1}{n}\}$ converges to 0.

PROOF STRATEGY Here we are required to show, for a given real number $\epsilon > 0$, that there is a positive integer N such that if $n > N$, then $|\frac{1}{n} - 0| = |\frac{1}{n}| = \frac{1}{n} < \epsilon$. The inequality $\frac{1}{n} < \epsilon$ is equivalent to $n > 1/\epsilon$. Hence if we let $N = \lceil 1/\epsilon \rceil$ and take n to be an integer greater than N, then $n > \frac{1}{\epsilon}$. We can now present a formal proof. ♦

Result 12.1 *The sequence* $\{\frac{1}{n}\}$ *converges to* 0.

Proof Let $\epsilon > 0$. Choose $N = \lceil 1/\epsilon \rceil$ and let n be any integer such that $n > N$. Thus $n > 1/\epsilon$ and so $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$. ■

PROOF ANALYSIS Although the proof of Result 12.1 is quite short, the real work in constructing the proof occurred in the proof strategy (our "scratch paper" work) that preceded the proof, but which is not part of the proof. This explains why we chose N as we did and why this choice of N was successful. In the proof of Result 12.1, we chose $N = \lceil 1/\epsilon \rceil$ and showed that with this value of N, every integer n with $n > N$ yields $|\frac{1}{n} - 0| < \epsilon$, which, of course, was our goal. There is nothing unique about this choice of N, however. Indeed, we could have chosen N to be *any* integer greater than $\lceil 1/\epsilon \rceil$ or, equivalently, any integer greater than $1/\epsilon$, and reached the desired conclusion as well. We could not, however, choose N to be an integer smaller than $\lceil 1/\epsilon \rceil$. We cannot in general choose $N = 1/\epsilon$ since there is no guarantee that $1/\epsilon$ is an integer. ♦

We now consider another illustration of a convergent sequence.

Result to Prove The sequence $\{3 + \frac{2}{n^2}\}$ converges to 3.

PROOF STRATEGY Here we are required to show, for a given $\epsilon > 0$, that there exists a positive integer N such that if $n > N$, then

$$\left|\left(3 + \frac{2}{n^2}\right) - 3\right| = \left|\frac{2}{n^2}\right| = \frac{2}{n^2} < \epsilon.$$

The inequality $\frac{2}{n^2} < \epsilon$ is equivalent to $\frac{n^2}{2} > \frac{1}{\epsilon}$ and $n > \sqrt{2/\epsilon}$. Therefore, if we let $N = \lceil \sqrt{2/\epsilon} \rceil$ and choose n to be an integer greater than N, then $n > \sqrt{2/\epsilon}$. We can now give a proof. ♦

Result 12.2 *The sequence* $\{3 + \frac{2}{n^2}\}$ *converges to* 3.

Proof Let $\epsilon > 0$. Choose $N = \lceil \sqrt{2/\epsilon} \rceil$ and let n be any integer such that $n > N$. Thus $n > \sqrt{2/\epsilon}$ and $n^2 > 2/\epsilon$. So $\frac{1}{n^2} < \frac{\epsilon}{2}$ and $\frac{2}{n^2} < \epsilon$. Therefore,

$$\left|\left(3 + \frac{2}{n^2}\right) - 3\right| = \left|\frac{2}{n^2}\right| = \frac{2}{n^2} < \epsilon.$$

■

We now consider a somewhat more complicated example.

Result to Prove The sequence $\{\frac{n}{2n+1}\}$ converges to $\frac{1}{2}$.

PROOF STRATEGY Observe that

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n - 2n - 1}{2(2n+1)}\right| = \left|-\frac{1}{4n+2}\right| = \frac{1}{4n+2}.$$

The inequality $\frac{1}{4n+2} < \epsilon$ is equivalent to $4n + 2 > 1/\epsilon$, which, in turn, is equivalent to $n > \frac{1}{4\epsilon} - \frac{1}{2}$. It may appear that the proper choice for N is $\lceil \frac{1}{4\epsilon} - \frac{1}{2} \rceil$; but if $\epsilon \geq 1/2$, then $N = 0$, which is not acceptable since N is required to be a positive integer. However, notice that $\frac{1}{4\epsilon} > \frac{1}{4\epsilon} - \frac{1}{2}$. So if $n > \frac{1}{4\epsilon}$, then $n > \frac{1}{4\epsilon} - \frac{1}{2}$ as well. Hence if we choose $N = \lceil 1/4\epsilon \rceil$, then we can obtain the desired inequality. ♦

Result 12.3 *The sequence* $\{\frac{n}{2n+1}\}$ *converges to* $\frac{1}{2}$.

Proof Let $\epsilon > 0$ be given. Choose $N = \lceil 1/4\epsilon \rceil$ and let $n > N$. Then $n > \frac{1}{4\epsilon} > \frac{1}{4\epsilon} - \frac{1}{2}$, and so $4n > \frac{1}{\epsilon} - 2$ and $4n + 2 > 1/\epsilon$. Hence $\frac{1}{4n+2} < \epsilon$. Thus

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{2n - 2n - 1}{2(2n+1)}\right| = \left|-\frac{1}{4n+2}\right| = \frac{1}{4n+2} < \epsilon.$$

■

Again, the choice made for N in the proof of Result 12.3 is not unique. We could choose N to be any integer greater than $\frac{1}{4\epsilon}$.

We mentioned that a sequence $\{a_n\}$ is said to diverge if it does not converge. To prove that a sequence $\{a_n\}$ diverges, a proof by contradiction would be anticipated. We would begin such a proof by assuming, to the contrary, that $\{a_n\}$ converges, say to some real number L. We know that for every $\epsilon > 0$, there is a positive integer N such that if $n > N$, then $|a_n - L| < \epsilon$. If we could show for even one choice of $\epsilon > 0$ that no such positive integer N exists, then we would have produced a contradiction and proved the desired result. Let's see how this works in two examples.

Result to Prove The sequence $\{(-1)^{n+1}\}$ is divergent.

PROOF STRATEGY In a proof by contradiction, we begin by assuming that $\{(-1)^{n+1}\}$ converges, to the limit L say. Our goal is to show that there is some value of $\epsilon > 0$ for which there is no positive integer N that satisfies the requirement. We choose $\epsilon = 1$. According to the definition of what it means for $\{(-1)^{n+1}\}$ to converge to L, there must exist a positive integer N such that if n is an integer with $n > N$, then $|(-1)^{n+1} - L| < \epsilon = 1$. Let k be an odd integer such that $k > N$. Then

$$|(-1)^{k+1} - L| = |1 - L| = |L - 1| < 1.$$

Therefore, $-1 < L - 1 < 1$ and $0 < L < 2$. Now let ℓ be an even integer such that $\ell > N$. Then

$$|(-1)^{\ell+1} - L| = |-1 - L| = |L + 1| < 1.$$

Thus, $-1 < L + 1 < 1$ and $-2 < L < 0$. So $L < 0 < L$, which, of course, is impossible. We now repeat what we have just said in a formal proof. ♦

Result 12.4 *The sequence* $\{(-1)^{n+1}\}$ *is divergent.*

Proof Assume, to the contrary, that the sequence $\{(-1)^{n+1}\}$ converges. Then $\lim_{n\to\infty} (-1)^{n+1} = L$ for some real number L. Let $\epsilon = 1$. Then there exists a positive integer N such that if $n > N$, then $|(-1)^{n+1} - L| < \epsilon = 1$. Let k be an odd integer such that $k > N$. Then

$$|(-1)^{k+1} - L| = |1 - L| = |L - 1| < 1.$$

Therefore, $-1 < L - 1 < 1$ and $0 < L < 2$. Next, let ℓ be an even integer such that $\ell > N$. Then

$$|(-1)^{\ell+1} - L| = |-1 - L| = |L + 1| < 1.$$

So $-1 < L + 1 < 1$ and $-2 < L < 0$. Therefore, $L < 0 < L$, which is a contradiction. ■

PROOF ANALYSIS One question that now occurs is how we knew to choose $\epsilon = 1$. If ϵ denotes an arbitrary positive integer, then both inequalities $|L-1| < \epsilon$ and $|L+1| < \epsilon$ must be satisfied, but these result in the inequalities

$$1 - \epsilon < L < 1 + \epsilon \text{ and } -1 - \epsilon < L < -1 + \epsilon.$$

In particular, $1 - \epsilon < L < -1 + \epsilon$ and so $1 - \epsilon < -1 + \epsilon$. This is only possible if $2\epsilon > 2$ or $\epsilon > 1$. Hence if we choose ϵ to be any number such that $0 < \epsilon \leq 1$, a contradiction will be produced. We decided to choose $\epsilon = 1$. ♦

Result to Prove The sequence $\{(-1)^{n+1}\frac{n}{n+1}\}$ is divergent.

PROOF STRATEGY As expected, we will attempt a proof by contradiction and assume that $\{(-1)^{n+1}\frac{n}{n+1}\}$ is a convergent sequence, with limit L say. For $\epsilon > 0$, there is a positive integer N then such that

$$\left|(-1)^{n+1}\frac{n}{n+1} - L\right| < \epsilon$$

for each integer n such that $n > N$. There are some useful observations.

First, if $n > N$ and n is odd, then

$$\left|\frac{n}{n+1} - L\right| < \epsilon \text{ and so } -\epsilon < \frac{n}{n+1} - L < \epsilon.$$

Hence

$$L - \epsilon < \frac{n}{n+1} < L + \epsilon.$$

Second, if $n > N$ and n is even, then

$$\left|-\frac{n}{n+1} - L\right| < \epsilon \text{ and so } -\epsilon < -\frac{n}{n+1} - L < \epsilon.$$

Hence

$$L - \epsilon < -\frac{n}{n+1} < L + \epsilon.$$

Also, since $n > 1$, we have $n + n > n + 1$ and so $2n > n + 1$. Hence $\frac{n}{n+1} > \frac{1}{2}$.

Depending on whether $L = 0$, $L > 0$, or $L < 0$, we are faced with the decision as to how to choose ϵ in each case to produce a contradiction. ♦

Result 12.5 *The sequence* $\{(-1)^{n+1}\frac{n}{n+1}\}$ *is divergent.*

Proof Assume, to the contrary, that $\{(-1)^{n+1}\frac{n}{n+1}\}$ converges. Then $\lim_{n\to\infty}(-1)^{n+1}\frac{n}{n+1} = L$ for some real number L. We consider three cases, depending on whether $L = 0$, $L > 0$, or $L < 0$.

Case 1. $L = 0$. Let $\epsilon = \frac{1}{2}$. Then there exists a positive integer N such that if $n > N$, then $|(-1)^{n+1}\frac{n}{n+1} - 0| < \frac{1}{2}$ or $\frac{n}{n+1} < \frac{1}{2}$. Then $2n < n + 1$ and so $n < 1$, which is a contradiction.

Case 2. $L > 0$. Let $\epsilon = \frac{L}{2}$. Then there exists a positive integer N such that if $n > N$, then $|(-1)^{n+1}\frac{n}{n+1} - L| < \frac{L}{2}$. Let n be an even integer such that $n > N$. Then

$$-\frac{L}{2} < -\frac{n}{n+1} - L < \frac{L}{2}.$$

Hence $\frac{L}{2} < -\frac{n}{n+1} < \frac{3L}{2}$, which is a contradiction.

Case 3. $L < 0$. Let $\epsilon = -\frac{L}{2}$. Then there exists a positive integer N such that if $n > N$, then $|(-1)^{n+1}\frac{n}{n+1} - L| < -\frac{L}{2}$. Let n be an odd integer such that $n > N$. Then

$$\frac{L}{2} < \frac{n}{n+1} - L < -\frac{L}{2}$$

and so $\frac{3L}{2} < \frac{n}{n+1} < \frac{L}{2}$. This is a contradiction. ■

A sequence $\{a_n\}$ may diverge because as n becomes larger, a_n becomes larger and eventually exceeds any given real number. If a sequence has this property, then $\{a_n\}$ is said to diverge to infinity. More formally, a sequence $\{a_n\}$ **diverges to infinity**, written $\lim_{n\to\infty} a_n = \infty$, if for every positive number M, there exists a positive integer N such that if n is an integer such that $n > N$, then $a_n > M$. The sequence $\{(-1)^{n+1}\}$ encountered in Result 12.4, although divergent, does not diverge to infinity. However, the sequence $\{n^2 + \frac{1}{n}\}$ *does* diverge to infinity.

Result to Prove $\lim_{n\to\infty}\left(n^2 + \frac{1}{n}\right) = \infty.$

PROOF STRATEGY For a given positive number M, we are required to show the existence of a positive integer N such that if $n > N$, then $n^2 + \frac{1}{n} > M$. Notice that if $n^2 > M$, then $n^2 + \frac{1}{n} > n^2 > M$. Since $M > 0$, it follows that $n^2 > M$ is equivalent to $n > \sqrt{M}$. A formal proof can now be constructed. ♦

Result 12.6 $\lim_{n\to\infty}\left(n^2 + \frac{1}{n}\right) = \infty.$

Proof Let M be a positive number. Choose $N = \lceil\sqrt{M}\rceil$ and let n be any integer such that $n > N$. Hence $n > \sqrt{M}$ and so $n^2 > M$. Thus $n^2 + \frac{1}{n} > n^2 > M$. ■

12.2 Infinite Series

An important concept in calculus involving sequences is infinite series. For real numbers $a_1, a_2, a_3, \cdots$, we write $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$ to denote an **infinite series** (often simply called a **series**). For example,

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{ and } \sum_{k=1}^{\infty}\frac{k}{2k^2+1} = \frac{1}{3} + \frac{2}{9} + \frac{3}{19} + \cdots$$

are infinite series.

The numbers $a_1, a_2, a_3, \cdots$ are called the **terms** of the series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$. The notation certainly seems to suggest that we are adding the terms $a_1, a_2, a_3, \cdots$. But what does it mean to add infinitely many numbers? A meaning must be given to this. For this reason, we construct a sequence $\{s_n\}$, called the **sequence of partial sums** of the series. Here $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and, in general, for $n \in \mathbf{N}$,

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Because s_n is determined by adding a finite number of terms, there is no confusion in understanding the terms of the sequence $\{s_n\}$. If the sequence $\{s_n\}$ converges, say to the number L, then the series $\sum_{k=1}^{\infty} a_k$ is said to **converge** to L and we write $\sum_{k=1}^{\infty} a_k = L$. This number L is called the **sum** of $\sum_{k=1}^{\infty} a_k$. If $\{s_n\}$ diverges, then $\sum_{k=1}^{\infty} a_k$ is said to **diverge**.

The French mathematician Augustin-Louis Cauchy is one of the most productive mathematicians of the 19th century. Among his many accomplishments was his definition of convergence of infinite series, a definition which is still used today. In his work *Cours d'Analyse*, Cauchy considered the sequence $\{s_n\}$ of partial sums of a series. He stated the following:

> *If, for increasing values of n, the sum s_n approaches indefinitely a certain limit s, the series will be called convergent, and this limit in question will be called the sum of the series.*

We consider an example of a convergent series.

Result to Prove The infinite series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges to 1.

PROOF STRATEGY First, we consider the sequence $\{s_n\}$ of partial sums for this series. Since

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots,$$

it follows that $s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$, $s_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, and $s_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$.

Based on these three terms, it appears that $s_n = \frac{n}{n+1}$ for every positive integer n. We prove that this is indeed the case. ♦

Lemma 12.7 *For every positive integer n,*

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Proof of Lemma 12.7 We proceed by induction. For $n = 1$, we have $s_1 = \frac{1}{1\cdot 2} = \frac{1}{1+1}$ and the result holds. Assume that $s_k = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$, where k is a positive integer. We show that

$$s_{k+1} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Observe that

$$\begin{aligned} s_{k+1} &= \left[\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)}\right] + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}. \end{aligned}$$

By the Principle of Mathematical Induction, $s_n = \frac{n}{n+1}$ for every positive integer n. ■

There is another way that we might have been able to see that $s_n = \frac{n}{n+1}$. If we had observed that

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

then $a_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2}$, $a_2 = \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$, etc. In particular,

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}. \end{aligned}$$

In any case, since we now know that $s_n = \frac{n}{n+1}$, it remains only to prove that

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Lemma to Prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

PROOF STRATEGY For a given $\epsilon > 0$, we are required to find a positive integer N such that if $n > N$, then $|\frac{n}{n+1} - 1| < \epsilon$. Now

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{n-n-1}{n+1}\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1}.$$

The inequality $\frac{1}{n+1} < \epsilon$ is equivalent to $n + 1 > \frac{1}{\epsilon}$, which in turn is equivalent to $n > \frac{1}{\epsilon} - 1$. If $n > \frac{1}{\epsilon}$, then $n > \frac{1}{\epsilon} - 1$. We can now present a proof of this lemma. ♦

Lemma 12.8 $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Proof of Lemma 12.8 Let $\epsilon > 0$ be given. Choose $N = \lceil 1/\epsilon \rceil$ and let $n > N$. Then $n > \frac{1}{\epsilon} > \frac{1}{\epsilon} - 1$. So $n > \frac{1}{\epsilon} - 1$. Thus $n + 1 > \frac{1}{\epsilon}$ and $\frac{1}{n+1} < \epsilon$. Hence

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| = \frac{1}{n+1} < \epsilon.$$ ■

We are now prepared to give a proof of the result.

Result 12.9 *The infinite series* $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ *converges to* 1.

Proof The nth term of the sequence $\{s_n\}$ of partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

By Lemma 12.7,

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

and so $s_n = \frac{n}{n+1}$. By Lemma 12.8,

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

It follows that $\lim_{n\to\infty} s_n = 1$ and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. ∎

We now turn to a divergent series. The series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is famous and is called the **harmonic series**. Indeed, it is probably the best known divergent series.

Result 12.10 *The harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges.*

Proof Assume, to the contrary, that $\sum_{k=1}^{\infty} \frac{1}{k}$ converges, say to the number L. For each positive integer n, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Hence the sequence $\{s_n\}$ of partial sums converges to L. Therefore, for each $\epsilon > 0$, there exists a positive integer N such that if $n > N$, then $|s_n - L| < \epsilon$. Let's consider $\epsilon = 1/4$ and let n be an integer with $n > N$. Then

$$-\frac{1}{4} < s_n - L < \frac{1}{4}.$$

Since $2n > N$, it is also the case that $|s_{2n} - L| < \frac{1}{4}$ and so $-\frac{1}{4} < s_{2n} - L < \frac{1}{4}$. Observe that

$$s_{2n} = s_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > s_n + n\left(\frac{1}{2n}\right) = s_n + \frac{1}{2}.$$

Hence

$$\frac{1}{4} > s_{2n} - L > s_n + \frac{1}{2} - L = (s_n - L) + \frac{1}{2} > -\frac{1}{4} + \frac{1}{2} = \frac{1}{4},$$

which is impossible. ∎

PROOF ANALYSIS In Result 12.10, we showed that a certain series diverges, that is, it does not converge. Consequently, it is not surprising that we proved this by contradiction. By assuming that the sequence $\{s_n\}$ converges, this meant that the sequence has a limit L. This tells us that an inequality of the type $|s_n - L| < \epsilon$ exists for every positive number ϵ and for sufficiently large integers n (which depend on ϵ). The goal, of course, was to obtain a contradiction. We did this by making a choice of ϵ ($\epsilon = 1/4$ worked!) that eventually produced a mathematical impossibility. ♦

The harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ not only diverges, it diverges to infinity; that is, if $\{s_n\}$ is the sequence of partial sums for the harmonic series, then $\lim_{n\to\infty} s_n = \infty$. We also establish this fact. First, we verify a lemma, which shows once again that mathematical induction can be a useful proof technique in calculus.

Lemma 12.11 *Let* $s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, *where* $n \in \mathbf{N}$. *Then* $s_{2^n} \geq 1 + \frac{n}{2}$ *for every positive integer* n.

Proof We proceed by induction. For $n = 1$, $s_{2^1} = 1 + \frac{1}{2}$ and so the result holds for $n = 1$. Assume that $s_{2^k} \geq 1 + \frac{k}{2}$, where $k \in \mathbf{N}$. We show that $s_{2^{k+1}} \geq 1 + \frac{k+1}{2}$. Now observe that

$$\begin{aligned} s_{2^{k+1}} &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^{k+1}} \\ &= s_{2^k} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}} \\ &\geq s_{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}} \\ &= s_{2^k} + \frac{2^k}{2^{k+1}} = s_{2^k} + \frac{1}{2} \\ &\geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}. \end{aligned}$$

By the Principle of Mathematical Induction, $s_{2^n} \geq 1 + \frac{n}{2}$ for every positive integer n. ∎

Result 12.12 *The harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges to infinity.*

Proof For $n \in \mathbf{N}$, let $s_n = \sum_{k=1}^{n} \frac{1}{k}$. Thus $\{s_n\}$ is the sequence of partial sums for the harmonic series. We show that $\lim_{n\to\infty} s_n = \infty$. Let M be a positive integer and choose $N = 2^{2M}$. Let $n > N$. Then, using Lemma 12.11, we have

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{N} + \frac{1}{N+1} + \cdots + \frac{1}{n} \\ &= s_N + \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{n} \\ &> s_N = s_{2^{2M}} \geq 1 + \frac{2M}{2} > M. \end{aligned}$$

∎

12.3 Limits of Functions

We now turn to another common type of limit problem (perhaps the most common). Here we consider a function $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$, and study the behavior of f near some real number (point) a. For the present, we are not concerned whether $a \in X$, but since we *are* concerned about the numbers $f(x)$ for real numbers x near a, it is necessary that f is defined in some "deleted neighborhood" of a. By a **deleted neighborhood** of a, we mean a set of the type $(a - \delta, a) \cup (a, a + \delta) = (a - \delta, a + \delta) - \{a\} \subseteq X$ for some

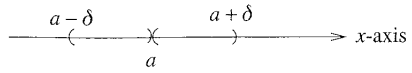

Figure 12.2 A deleted neighborhood of a

positive real number δ (the Greek letter delta). (See Figure 12.2.) It may actually be the case that $(a-\delta, a+\delta) \subseteq X$ for some $\delta > 0$. For example, if $f : X \to \mathbf{R}$ is defined by $f(x) = \frac{|x|}{x}$ and we are interested in the behavior of f near 0, then $0 \notin X$. In fact, it might very well be that $X = \mathbf{R} - \{0\}$, in which case, $(-\delta, 0) \cup (0, \delta) \subseteq X$ for every positive real number δ. On the other hand, if $f : X \to \mathbf{R}$ is defined by $f(x) = \frac{x}{x^2-1}$ and, once again, we are interested in the behavior of f near 0, then $1, -1 \notin X$. A natural choice for X is $\mathbf{R} - \{1, -1\}$, in which case $(-\delta, \delta) \subseteq X$ for every real number δ such that $0 < \delta \leq 1$.

We are now prepared to present the definition of the limit of a function. Let f be a real-valued function defined on a set X of real numbers. Also, let $a \in \mathbf{R}$ such that f is defined in some deleted neighborhood of a. Then we say that the real number L is the limit of $f(x)$ as x approaches a, written $\lim_{x\to a} f(x) = L$, if the closer x is to a, the closer $f(x)$ is to L. The vagueness of the word "closer" again requires a considerably more precise definition. Let the positive number ϵ indicate how close $f(x)$ is required to be to L; that is, we require that $|f(x) - L| < \epsilon$. Then the claim is that if x is sufficiently close to a, then $|f(x) - L| < \epsilon$. We use the positive number δ to represent how close x must be to a in order for the inequality $|f(x) - L| < \epsilon$ to be satisfied, recalling that we are not concerned about how, or even if, f is defined at a.

More precisely then, L is the **limit** of $f(x)$ as x approaches a, written $\lim_{x\to a} f(x) = L$, if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for every real number x with $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$. This implies that if $0 < |x - a| < \delta$, then certainly $f(x)$ is defined. If there exists a number L such that $\lim_{x\to a} f(x) = L$, then we say that the limit $\lim_{x\to a} f(x)$ exists and is equal to L; otherwise, this limit does not exist. Thus to show that $\lim_{x\to a} f(x) = L$, it is necessary to specify $\epsilon > 0$ first and then show the existence of a real number $\delta > 0$. Ordinarily, the smaller the value of ϵ, the smaller the value of δ. However, we must be certain that the number δ selected satisfies the requirement regardless of how small (or large) ϵ may be. Even though our choice of δ depends on ϵ, it should not depend on which real number x with $0 < |x - a| < \delta$ is being considered.

Accordingly, if $\lim_{x\to a} f(x) = L$, then for a given $\epsilon > 0$, there exists $\delta > 0$ such that if x is any number in the open interval $(a - \delta, a + \delta)$ that is different from a, then $f(x)$ is a number in the interval $(L - \epsilon, L + \epsilon)$. This geometric interpretation of the definition of limit is illustrated in Figure 12.3.

We illustrate these ideas with an example.

Result to Prove $\lim_{x\to 4} (3x - 7) = 5.$

PROOF STRATEGY Before giving a formal proof of this limit, let's discuss the procedure we will use. The proof begins by letting $\epsilon > 0$ be given. What we are required to do is to find a number $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(3x - 7) - 5| < \epsilon$ or, equivalently,

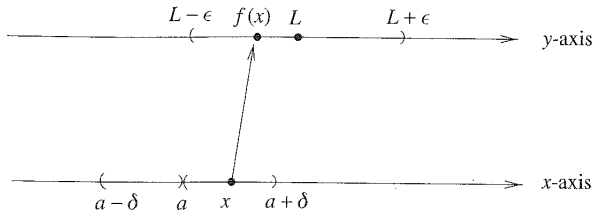

Figure 12.3 A geometric interpretation of $\lim_{x\to a} f(x) = L$

$|3(x - 4)| < \epsilon$. This is also equivalent to $|3| \cdot |x - 4| < \epsilon$ and to $|x - 4| < \epsilon/3$. This suggests our choice of δ. We can now give a proof. ♦

Result 12.13 $\lim_{x\to 4} (3x - 7) = 5.$

Proof Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/3$. Let $x \in \mathbf{R}$ such that $0 < |x - 4| < \delta = \epsilon/3$. Then

$$|(3x - 7) - 5| = |3x - 12| = |3(x - 4)| = 3|x - 4| < 3(\epsilon/3) = \epsilon.$$ ■

Let's consider another example.

Result to Prove $\lim_{x\to -3} (-2x + 1) = 7.$

PROOF STRATEGY First we do some preliminary algebra. The inequality $|(-2x + 1) - 7| < \epsilon$ is equivalent to $|-2x - 6| < \epsilon$ and to $2|x + 3| < \epsilon$. This suggests a desired value of δ. We can now give a proof. ♦

Result 12.14 $\lim_{x\to -3} (-2x + 1) = 7.$

Proof Let $\epsilon > 0$ be given and choose $\delta = \epsilon/2$. Let $x \in \mathbf{R}$ such that $0 < |x - (-3)| < \delta = \epsilon/2$, so $0 < |x + 3| < \epsilon/2$. Then

$$\begin{aligned} |(-2x + 1) - 7| &= |-2x - 6| = |-2(x + 3)| \\ &= |-2||x + 3| = 2|x + 3| < 2(\epsilon/2) = \epsilon. \end{aligned}$$ ■

The two examples that we have seen thus far should tell us how to proceed when the function is linear (that is, $f(x) = ax + b$, where $a, b \in \mathbf{R}$). We now present a slight variation of this.

Result to Prove $\lim_{x\to \frac{3}{2}} \dfrac{4x^2 - 9}{2x - 3} = 6.$

PROOF STRATEGY In this example, $|f(x) - L| < \epsilon$ becomes $|\frac{4x^2-9}{2x-3} - 6| < \epsilon$ or, after simplifying, $|\frac{(2x+3)(2x-3)}{2x-3} - 6| < \epsilon$. However, since the numbers x are in a deleted neighborhood of

$3/2$, it follows that $2x - 3 \neq 0$ and so $|\frac{(2x+3)(2x-3)}{2x-3} - 6| < \epsilon$ becomes $|(2x + 3) - 6| < \epsilon$, or $|2x - 3| < \epsilon$. Therefore, $2|x - 3/2| < \epsilon$ and $|x - 3/2| < \epsilon/2$. We are now prepared to give a proof. ♦

Result 12.15 $\lim_{x \to \frac{3}{2}} \dfrac{4x^2 - 9}{2x - 3} = 6.$

Proof Let $\epsilon > 0$ be given and choose $\delta = \epsilon/2$. Let $x \in \mathbf{R}$ such that $0 < |x - 3/2| < \delta = \epsilon/2$. So $2|x - 3/2| < \epsilon$ and $|2x - 3| < \epsilon$. Hence $|(2x + 3) - 6| < \epsilon$. Since $2x - 3 \neq 0$, it follows that $|\frac{(2x+3)(2x-3)}{2x-3} - 6| < \epsilon$ and so $|\frac{4x^2-9}{2x-3} - 6| < \epsilon$. ■

We now turn to a limit of a quadratic function.

Result to Prove $\lim_{x \to 3} x^2 = 9.$

PROOF STRATEGY Once again for a given $\epsilon > 0$, we are required to find $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $|x^2 - 9| < \epsilon$. To find an appropriate choice of δ in terms of ϵ, we begin with $|x^2 - 9| < \epsilon$. We wish to work the expression $|x - 3|$ into this inequality. Actually, this is quite easy since $|x^2 - 9| < \epsilon$ is equivalent to $|x - 3||x + 3| < \epsilon$. This might make us think of writing $|x - 3| < \frac{\epsilon}{|x+3|}$ and choosing $\delta = \frac{\epsilon}{|x+3|}$. However, δ is required to be a positive number (a constant) which depends on ϵ but not on x. The expression $|x + 3|$ can be eliminated, though, as we now show. Since it is our choice how to select δ, we can certainly require $\delta \leq 1$, which we do. Thus $|x - 3| < 1$ and so $-1 < x - 3 < 1$. Hence $2 < x < 4$. Thus $5 < x + 3 < 7$ and so $|x + 3| < 7$. So, under this restriction for δ, it follows that $|x - 3||x + 3| < 7|x - 3|$. Now if $7|x - 3| < \epsilon$, that is, if $|x - 3| < \epsilon/7$, then it will certainly follow that $|x - 3||x + 3| < \epsilon$. Arriving at this inequality required that both $|x - 3| < 1$ and $|x - 3| < \epsilon/7$. This suggests an appropriate choice of δ. ♦

Result 12.16 $\lim_{x \to 3} x^2 = 9.$

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1, \epsilon/7)$. Let $x \in \mathbf{R}$ such that $0 < |x - 3| < \delta = \min(1, \epsilon/7)$. Since $|x - 3| < 1$, it follows that $-1 < x - 3 < 1$ and so $5 < x + 3 < 7$. In particular, $|x + 3| < 7$. Because $|x - 3| < \epsilon/7$, it follows that

$$|x^2 - 9| = |x - 3||x + 3| < |x - 3| \cdot 7 < (\epsilon/7) \cdot 7 = \epsilon.$$

■

We have now seen four proofs of limits of type $\lim_{x \to a} f(x) = L$. In Result 12.13, we chose $\delta = \epsilon/3$ for the given $\epsilon > 0$ and in Result 12.14, we chose $\delta = \epsilon/2$. In each case, if we had considered a different value of a for the same function, then the same choice of δ would be successful. This is because the function is linear in each case. In Result 12.15, for a given $\epsilon > 0$, the selection of $\delta = \epsilon/2$ would also be successful if $a \neq 3/2$, provided $3/2 \notin (a - \delta, a + \delta)$. This is because the function f in Result 12.15 defined by $f(x) = (4x^2 - 9)/(2x - 3)$ is "nearly linear", that is, $f(x) = 2x + 3$ if $x \neq 3/2$ and $f(3/2)$ is not defined. However, our choice of $\delta = \epsilon/7$ in the proof of Result 12.16 depended on

$a = 3$; that is, if $a \neq 3$, a different choice of δ is needed. For example, if we were to prove that $\lim_{x \to 4} x^2 = 16$, then for a given $\epsilon > 0$, an appropriate choice for δ is $\min(1, \epsilon/9)$.

Next we consider a limit involving a polynomial function of a higher degree.

Result to Prove $\lim_{x \to 2} (x^5 - 2x^3 - 3x - 7) = 3.$

PROOF STRATEGY For a given $\epsilon > 0$, we are required to show that $|(x^5 - 2x^3 - 3x - 7) - 3| < \epsilon$ if $0 < |x - 2| < \delta$ for a suitable choice of $\delta > 0$. We then need to work $|x - 2|$ into the expression $|x^5 - 2x^3 - 3x - 10|$. Dividing $x^5 - 2x^3 - 3x - 10$ by $x - 2$, we obtain $x^5 - 2x^3 - 3x - 10 = (x - 2)(x^4 + 2x^3 + 2x^2 + 4x + 5)$. Hence we have

$$|x^5 - 2x^3 - 3x - 10| = |x - 2||x^4 + 2x^3 + 2x^2 + 4x + 5|.$$

Thus we seek an upper bound for $|x^4 + 2x^3 + 2x^2 + 4x + 5|$. To do this, we impose the restriction $\delta \leq 1$. Thus $|x - 2| < \delta \leq 1$. So $-1 < x - 2 < 1$ and $1 < x < 3$. Hence

$$|x^4 + 2x^3 + 2x^2 + 4x + 5| \leq |x^4| + |2x^3| + |2x^2| + |4x| + |5| < 170.$$

We are now prepared to prove Result 12.17. ♦

Result 12.17 $\lim_{x \to 2} (x^5 - 2x^3 - 3x - 7) = 3.$

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1, \epsilon/170)$. Let $x \in \mathbf{R}$ such that $0 < |x - 2| < \delta = \min(1, \epsilon/170)$. Since $|x - 2| < 1$, it follows that $1 < x < 3$ and so

$$|x^4 + 2x^3 + 2x^2 + 4x + 5| \leq |x^4| + |2x^3| + |2x^2| + |4x| + |5| < 170.$$

Since $|x - 2| < \epsilon/170$, we have

$$\begin{aligned} |(x^5 - 2x^3 - 3x - 7) - 3| &= |x^5 - 2x^3 - 3x - 10| \\ &= |x - 2| \cdot |x^4 + 2x^3 + 2x^2 + 4x + 5| \\ &< (\epsilon/170) \cdot 170 = \epsilon. \end{aligned}$$

■

Our next example involves a rational function (the ratio of two polynomials).

Result to Prove $\lim_{x \to 1} \dfrac{x^2 + 1}{x^2 + 4} = \dfrac{2}{5}.$

PROOF STRATEGY First observe that

$$\left|\frac{x^2 + 1}{x^2 + 4} - \frac{2}{5}\right| = \left|\frac{5(x^2 + 1) - 2(x^2 + 4)}{5(x^2 + 4)}\right| = \frac{|3x^2 - 3|}{5(x^2 + 4)} = \frac{3|x - 1||x + 1|}{5(x^2 + 4)}.$$

Hence it is necessary to find an upper bound for $\frac{3|x+1|}{5(x^2+4)}$. Once again we restrict δ so that $\delta \leq 1$. Then $|x - 1| < 1$ or $0 < x < 2$. Hence $1 < x + 1 < 3$; so $3|x + 1| < 9$. Also, since $x > 0$, it follows that $5(x^2 + 4) > 20$; so $\frac{1}{5(x^2+4)} < \frac{1}{20}$. Therefore,

$$\frac{3|x + 1|}{5(x^2 + 4)} < 9\left(\frac{1}{20}\right) = \frac{9}{20}.$$

We now present a proof of this result. ♦

Result 12.18 $\lim_{x\to 1} \dfrac{x^2+1}{x^2+4} = \dfrac{2}{5}$.

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1, 20\epsilon/9)$. Let $x \in \mathbf{R}$ such that $0 < |x-1| < \delta$. Since $|x-1| < 1$, we have $0 < x < 2$ and $1 < x+1 < 3$. Hence $3|x+1| < 3 \cdot 3 = 9$ and $5(x^2+4) > 20$, so $\frac{1}{5(x^2+4)} < \frac{1}{20}$. Therefore, $\frac{3|x+1|}{5(x^2+4)} < 9/20$. Since $|x-1| < 20\epsilon/9$, it follows that

$$\left|\frac{x^2+1}{x^2+4} - \frac{2}{5}\right| = \left|\frac{5(x^2+1)-2(x^2+4)}{5(x^2+4)}\right| = \frac{|3x^2-3|}{5(x^2+4)}$$
$$= \frac{3|x-1||x+1|}{5(x^2+4)} < \frac{20\epsilon}{9}\cdot\frac{9}{20} = \epsilon.$$

■

We now present one additional example on this topic.

Example 12.19 *Determine* $\lim_{x\to 1} \frac{x^2-1}{2x-1}$ *and verify your answer.*

Solution Since it appears that $\lim_{x\to 1}(x^2-1) = 0$ and $\lim_{x\to 1}(2x-1) = 1$, we would expect that $\lim_{x\to 1} \frac{x^2-1}{2x-1} = \frac{0}{1} = 0$. To verify this, we need to show that for a given $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x-1| < \delta$, then

$$\left|\frac{x^2-1}{2x-1} - 0\right| = \left|\frac{x^2-1}{2x-1}\right| < \epsilon.$$

Observe that

$$\left|\frac{x^2-1}{2x-1}\right| = \left|\frac{(x-1)(x+1)}{2x-1}\right| = \frac{|x+1|}{|2x-1|}|x-1|.$$

Proceeding as before, we find an upper bound for $\frac{|x+1|}{|2x-1|}$. Ordinarily, we might restrict $\delta \le 1$, as before, but in this situation, we have a problem. If $\delta \le 1$, then $0 < |x-1| < \delta$ and so $|x-1| < 1$. Thus $0 < x < 2$ or $x \in (0, 2)$. However, this interval of real numbers includes $1/2$ and $\frac{|x+1|}{|2x-1|}$ is not defined when $x = 1/2$. Thus we place a tighter restriction on δ. The restriction $\delta \le 1/2$ is not suitable either, for if $|x-1| < \delta \le 1/2$, then $1/2 < x < 3/2$. Even though $\frac{|x+1|}{|2x-1|}$ is defined for all real numbers x in this interval, this expression becomes arbitrarily large if x is arbitrarily close to $1/2$, allowing $|2x-1|$ to be arbitrarily close to 0. That is, we cannot find an upper bound for $\frac{|x+1|}{|2x-1|}$ if $\delta = 1/2$. Hence we require that $\delta \le 1/4$, say, and so $|x-1| < \delta \le 1/4$. Thus $3/4 < x < 5/4$. Hence $|x+1| < 9/4$. Also, $|2x-1| > 2(\frac{3}{4}) - 1 = 1/2$ and so $\frac{1}{|2x-1|} < 2$. Therefore, $\frac{|x+1|}{|2x-1|} < \frac{9}{4}\cdot 2 = \frac{9}{2}$. We now give a formal proof. ♦

Result 12.20 $\lim_{x\to 1} \dfrac{x^2-1}{2x-1} = 0.$

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1/4, 2\epsilon/9)$. Let $x \in \mathbf{R}$ such that $0 < |x-1| < \delta$. Since $\delta \le 1/4$, it follows that $|x-1| < 1/4$ and so $3/4 < x < 5/4$. Hence $|x+1| < 5/4 + 1 = 9/4$. Also, $|2x-1| > 2(\frac{3}{4}) - 1 = 1/2$ and so $\frac{1}{|2x-1|} < 2$.

Therefore, $\frac{|x+1|}{|2x-1|} < \frac{9}{4}\cdot 2 = \frac{9}{2}$. Since $|x-1| < \delta \le 2\epsilon/9$, it follows that

$$\left|\frac{x^2-1}{2x-1} - 0\right| = \left|\frac{x^2-1}{2x-1}\right| = \frac{|x+1|}{|2x-1|}|x-1| < \frac{2\epsilon}{9}\cdot\frac{9}{2} = \epsilon.$$

■

Next we consider a limit problem where the limit does not exist.

Result to Prove $\lim_{x\to 0} \dfrac{1}{x}$ does not exist.

PROOF STRATEGY As expected, we will give a proof by contradiction. If $\lim_{x\to 0} \frac{1}{x}$ does exist, then there exists a real number L such that $\lim_{x\to 0} \frac{1}{x} = L$. Hence for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x| < \delta$, then $|\frac{1}{x} - L| < \epsilon$. For numbers x "close" to 0, it certainly appears that $\frac{1}{x}$ is "large" (in absolute value). Hence, regardless of the value of ϵ, it seems that there should be a real number x with $0 < |x| < \delta$ such that $|\frac{1}{x} - L| \ge \epsilon$. It is our plan to show that this is indeed the case. Thus, we choose $\epsilon = 1$, for example, and show that no desired δ can be found. ♦

Result 12.21 $\lim_{x\to 0} \dfrac{1}{x}$ *does not exist.*

Proof Assume, to the contrary, that $\lim_{x\to 0} \frac{1}{x}$ exists. Then there exists a real number L such that $\lim_{x\to 0} \frac{1}{x} = L$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if x is a real number for which $0 < |x| < \delta$, then $|\frac{1}{x} - L| < \epsilon = 1$. Choose an integer n such that $n > \lceil 1/\delta \rceil \ge 1$. Since $n > 1/\delta$, it follows that $0 < 1/n < \delta$. We consider two cases.

Case 1. $L \le 0$. Let $x = 1/n$. So $0 < |x| < \delta$. Since $-L \ge 0$, it follows that

$$\left|\frac{1}{x} - L\right| = |n - L| = n - L \ge n > 1 = \epsilon,$$

which is a contradiction.

Case 2. $L > 0$. Let $x = -1/n$. So $0 < |x| < \delta$. Thus

$$\left|\frac{1}{x} - L\right| = |-n - L| = |-(n+L)| = n + L > n > 1 = \epsilon,$$

producing a contradiction in this case as well. ■

Result to Prove Let $f(x) = |x|/x$, where $x \in \mathbf{R}$ and $x \ne 0$. Then $\lim_{x\to 0} f(x)$ does not exist.

PROOF STRATEGY The graph of this function is shown in Figure 12.4. If $x > 0$, then $f(x) = |x|/x = x/x = 1$; while if $x < 0$, then $f(x) = |x|/x = -x/x = -1$. Hence there are numbers x that are "near" 0 such that $f(x) = 1$ and numbers x that are "near" 0 such that $f(x) = -1$. This suggests a proof. ♦

Result 12.22 *Let* $f(x) = |x|/x$, *where* $x \in \mathbf{R}$ *and* $x \ne 0$. *Then* $\lim_{x\to 0} f(x)$ *does not exist.*

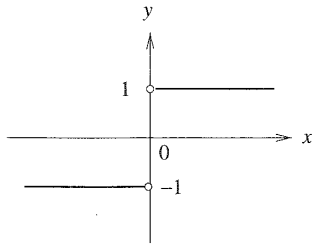

Figure 12.4 The graph of the function $f(x) = |x|/x$

Proof Assume, to the contrary, that $\lim_{x\to 0} f(x)$ exists. Then there exists a real number L such that $\lim_{x\to 0} f(x) = L$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if x is a real number satisfying $0 < |x - 0| = |x| < \delta$, then $|f(x) - L| < \epsilon = 1$. We consider two cases.

Case 1. $L \geq 0$. Consider $x = -\delta/2$. Then $|x| = \delta/2 < \delta$. However, $f(x) = f(-\delta/2) = (\delta/2)/(-\delta/2) = -1$. So $|f(x) - L| = |-1 - L| = 1 + L \geq 1$, a contradiction.

Case 2. $L < 0$. Let $x = \delta/2$. Then $|x| = \delta/2 < \delta$. Also, $f(x) = f(\delta/2) = (\delta/2)/(\delta/2) = 1$. So $|f(x) - L| = |1 - L| = 1 - L > 1$, a contradiction. ∎

12.4 Fundamental Properties of Limits of Functions

If we were to continue computing limits, then it would be essential to have some theorems at our disposal that would allow us to compute limits more rapidly. We now present some theorems that will allow us to determine limits more easily. We begin with a standard theorem on limits of sums of functions.

Theorem to Prove If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

$$\lim_{x\to a} (f(x) + g(x)) = L + M.$$

PROOF STRATEGY In this case, we are required to show, for a given $\epsilon > 0$, that $|(f(x) + g(x)) - (L + M)| < \epsilon$ if $0 < |x - a| < \delta$ for a suitable choice of $\delta > 0$. Now

$$|(f(x) + g(x)) - (L + M)| = |(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M|.$$

Hence if we can show that both $|f(x) - L| < \epsilon/2$ and $|g(x) - M| < \epsilon/2$, for example, then we will have obtained the desired inequality. However, because of the hypothesis, this can be accomplished. We now make all of this precise. ♦

Theorem 12.23 *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then*

$$\lim_{x\to a} (f(x) + g(x)) = L + M.$$

Proof Let $\epsilon > 0$. Since $\epsilon/2 > 0$, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \epsilon/2$. Also, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon/2$. Choose $\delta = \min(\delta_1, \delta_2)$ and let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Since $0 < |x - a| < \delta$, it follows that both $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$. Therefore,

$$\begin{aligned} |(f(x) + g(x)) - (L + M)| &= |(f(x) - L) + (g(x) - M)| \\ &\leq |f(x) - L| + |g(x) - M| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$ ∎

Theorem 12.23 states that the limit of the sum of two functions is the sum of their limits. Next we show that this is also true for products. Before getting to this theorem, let's see what would be involved to prove it. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. This means that we can make the expressions $|f(x) - L|$ and $|g(x) - M|$ as small as we wish. Our goal is to show that we can make $|f(x) \cdot g(x) - LM|$ as small as we wish, say less than ϵ for every given $\epsilon > 0$. The question then becomes how to use what we know about $|f(x) - L|$ and $|g(x) - M|$ as we consider $|f(x) \cdot g(x) - LM|$. A common way to do this is to add and subtract the same quantity to and from $f(x) \cdot g(x) - LM$. For example,

$$\begin{aligned} |f(x) \cdot g(x) - LM| &= |f(x) \cdot g(x) - f(x) \cdot M + f(x) \cdot M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

If we can make each of $|f(x)||g(x) - M|$ and $|f(x) - L||M|$ less than $\epsilon/2$, say, then we will have accomplished our goal. Since $|M|$ is a nonnegative constant and $|f(x) - L|$ and $|g(x) - M|$ can be made arbitrarily small, only $|f(x)|$ is in question. In fact, all that is required to show is that $f(x)$ can be bounded in a deleted neighborhood of a, that is, $|f(x)| \leq B$ for some constant $B > 0$.

Lemma 12.24 *Suppose that* $\lim_{x\to a} f(x) = L$. *Then there exists* $\delta > 0$ *such that if* $0 < |x - a| < \delta$, *then* $|f(x)| < 1 + |L|$.

Proof Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < 1$. Thus

$$|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + |L|.$$ ∎

We are now prepared to show that the limit of the product of two functions is the product of their limits.

Theorem to Prove If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x) \cdot g(x) = LM$.

PROOF STRATEGY As we discussed earlier,

$$\begin{aligned} |f(x) \cdot g(x) - LM| &= |f(x) \cdot g(x) - f(x) \cdot M + f(x) \cdot M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

For a given $\epsilon > 0$, we show that each of $|f(x)||g(x) - M|$ and $|f(x) - L||M|$ can be made less than $\epsilon/2$, which will give us a proof of the result. Of course, this follows immediately for $|f(x) - L||M|$ if $M = 0$. Otherwise, we can make $|f(x) - L|$ less than $\epsilon/(2|M|)$. By Lemma 12.24, we can make $|f(x)|$ less than $1 + |L|$. Thus we make $|g(x) - M| < \epsilon/(2(1 + |L|))$. Now, let's put all of the pieces together. ♦

Theorem 12.25 *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$, *then* $\lim_{x\to a} f(x) \cdot g(x) = LM$.

Proof Let $\epsilon > 0$ be given. By Lemma 12.24, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $|f(x)| < 1 + |L|$. Since $\lim_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \epsilon/(2(1 + |L|))$. We consider two cases.

Case 1. $M = 0$. Choose $\delta = \min(\delta_1, \delta_2)$. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Then

$$\begin{aligned} |f(x) \cdot g(x) - LM| &= |f(x) \cdot g(x) - f(x) \cdot M + f(x) \cdot M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\le |f(x)||g(x) - M| + |f(x) - L||M| \\ &< (1 + |L|)\epsilon/(2(1 + |L|)) + 0 = \epsilon/2 < \epsilon. \end{aligned}$$

Case 2. $M \neq 0$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then $|f(x) - L| < \epsilon/(2|M|)$. In this case, we choose $\delta = \min(\delta_1, \delta_2, \delta_3)$. Now let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Then

$$\begin{aligned} |f(x) \cdot g(x) - LM| &= |f(x) \cdot g(x) - f(x) \cdot M + f(x) \cdot M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\le |f(x)||g(x) - M| + |f(x) - L||M| \\ &< (1 + |L|)\epsilon/(2(1 + |L|)) + (\epsilon/2|M|)|M| \\ &= \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

■

Next we consider the limit of the quotient of two functions. As before, let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Our goal is to show that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$. Of course, this is not true if $M = 0$, so we will need to assume that $M \neq 0$. To prove that $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, we are required to show that $|\frac{f(x)}{g(x)} - \frac{L}{M}|$ can be made arbitrarily small. Observe that

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \left|\frac{f(x) \cdot M - L \cdot g(x)}{g(x) \cdot M}\right| = \left|\frac{f(x) \cdot M - LM + LM - L \cdot g(x)}{g(x) \cdot M}\right| \\ &= \left|\frac{(f(x) - L)M + L(M - g(x))}{g(x) \cdot M}\right| \le \frac{|f(x) - L||M| + |L||M - g(x)|}{|g(x)||M|} \\ &= \frac{|f(x) - L|}{|g(x)|} + \frac{|L||M - g(x)|}{|g(x)||M|}. \end{aligned}$$

Thus to show that $|\frac{f(x)}{g(x)} - \frac{L}{M}|$ can be made less than ϵ for any given positive number ϵ, it is sufficient to show that each of $\frac{|f(x)-L|}{|g(x)|}$ and $\frac{|L||M-g(x)|}{|g(x)||M|}$ can be made less than $\epsilon/2$. Only $1/|g(x)|$ requires study. In particular, we need to show that there is an upper bound for $1/|g(x)|$ in some deleted neighborhood of a.

Lemma 12.26 *If* $\lim_{x\to a} g(x) = M \neq 0$, *then* $1/|g(x)| < 2/|M|$ *for all* x *in some deleted neighborhood of* a.

Proof Let $\epsilon = |M|/2$. Then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < |M|/2$. Therefore,

$$|M| = |M - g(x) + g(x)| \le |M - g(x)| + |g(x)|$$

Hence $|g(x)| \ge |M| - |M - g(x)| > |M| - |M|/2 = |M|/2$. Thus $1/|g(x)| < 2/|M|$. ■

Theorem to Prove If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

PROOF STRATEGY Returning to our earlier discussion, we now have

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\le \frac{|f(x) - L|}{|g(x)|} + \frac{|L||M - g(x)|}{|g(x)||M|} \\ &< |f(x) - L| \cdot \frac{2}{|M|} + |L||M - g(x)| \cdot \frac{2}{|M|^2}. \end{aligned}$$

This suggests how small we must make $|f(x) - L|$ and $|g(x) - M| = |M - g(x)|$ to accomplish our goal. ♦

Theorem 12.27 *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M \neq 0$, *then* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof Let $\epsilon > 0$ be given. By Lemma 12.26, there exists $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$, then $1/|g(x)| < 2/|M|$. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$, then $|f(x) - L| < |M|\epsilon/4$. We consider two cases.

Case 1. $L = 0$. Define $\delta = \min(\delta_1, \delta_2)$. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Then

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\le \frac{|f(x) - L|}{|g(x)|} + \frac{|L||M - g(x)|}{|g(x)||M|} \\ &< \frac{|M|\epsilon}{4} \cdot \frac{2}{|M|} + 0 = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Case 2. $L \neq 0$. Since $\lim_{x\to a} g(x) = M$, there exists $\delta_3 > 0$ such that if $0 < |x - a| < \delta_3$, then $|g(x) - M| < |M|^2\epsilon/(4|L|)$. In this case, define $\delta = \min(\delta_1, \delta_2, \delta_3)$. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Then

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &\le \frac{|f(x) - L|}{|g(x)|} + \frac{|L||M - g(x)|}{|g(x)||M|} \\ &< \frac{|M|\epsilon}{4} \cdot \frac{2}{|M|} + \frac{|L|}{|M|} \cdot \frac{|M|^2\epsilon}{4|L|} \cdot \frac{2}{|M|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

■

Now, with the aid of Theorems 12.23, 12.25, 12.27, and a few other general results, it is possible to give simpler arguments for some of the limits we have discussed. First, we present some additional results, beginning with an observation concerning constant functions and followed by limits of polynomial functions defined by $f(x) = x^n$ for some $n \in \mathbf{N}$.

Theorem 12.28 *Let $a, c \in \mathbf{R}$. If $f(x) = c$ for all $x \in \mathbf{R}$, then $\lim_{x\to a} f(x) = c$.*

Proof Let $\epsilon > 0$ be given, and choose δ to be any positive number. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Since $f(x) = c$ for all $x \in \mathbf{R}$, it follows that $|f(x) - c| = |c - c| = 0 < \epsilon$. ∎

Theorem 12.29 *Let $f(x) = x$ for all $x \in \mathbf{R}$. For each $a \in \mathbf{R}$, $\lim_{x\to a} f(x) = a$.*

Proof Let $\epsilon > 0$ be given, and choose $\delta = \epsilon$. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. Then $|f(x) - a| = |x - a| < \delta = \epsilon$. ∎

We now extend the result in Theorem 12.29.

Theorem 12.30 *Let $n \in \mathbf{N}$ and let $f(x) = x^n$ for all $x \in \mathbf{R}$. Then for each $a \in \mathbf{R}$, $\lim_{x\to a} f(x) = a^n$.*

Proof We proceed by induction. The statement is true for $n = 1$ since if $f(x) = x$, then $\lim_{x\to a} f(x) = a$ by Theorem 12.29. Assume that $\lim_{x\to a} x^k = a^k$, where $k \in \mathbf{N}$. We show that $\lim_{x\to a} x^{k+1} = a^{k+1}$. Observe that $\lim_{x\to a} x^{k+1} = \lim_{x\to a} \left(x^k \cdot x\right)$. By Theorems 12.25 and 12.29 and the induction hypothesis,

$$\lim_{x\to a} x^{k+1} = \lim_{x\to a} \left(x^k \cdot x\right) = \left(\lim_{x\to a} x^k\right)\left(\lim_{x\to a} x\right) = \left(a^k\right)(a) = a^{k+1}.$$

By the Principle of Mathematical Induction, $\lim_{x\to a} x^n = a^n$ for every $n \in \mathbf{N}$. ∎

It is possible to prove the following theorem by induction as well. We leave its proof as an exercise (Exercise 12.21).

Theorem 12.31 *Let $f_1, f_2, \cdots, f_n$ be $n \geq 2$ functions such that $\lim_{x\to a} f_i(x) = L_i$ for $1 \leq i \leq n$. Then*

$$\lim_{x\to a} \left(f_1(x) + f_2(x) + \cdots + f_n(x)\right) = L_1 + L_2 + \cdots + L_n.$$

With the results we have now presented, it is possible to prove that if $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ is a polynomial, then for each $a \in \mathbf{R}$

$$\lim_{x\to a} p(x) = c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a). \tag{12.1}$$

For example, applying this to Result 12.13, we have

$$\lim_{x\to 4}(3x - 7) = 3 \cdot 4 - 7 = 5.$$

Similarly, Result 12.14 can be established. Result 12.15 cannot be established directly since $\lim_{x\to\frac{3}{2}}(2x - 3) = 0$. Applying what we now know to Result 12.16, we have $\lim_{x\to 3} x^2 = 3^2 = 9$ and in Result 12.17,

$$\lim_{x\to 2}(x^5 - 2x^3 - 3x - 7) = 2^5 - 2 \cdot 2^3 - 3 \cdot 2 - 7 = 3.$$

Also, if r is a rational function, that is, if $r(x)$ is the ratio $p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$ such that $q(a) \neq 0$ for $a \in \mathbf{R}$, then by Theorem 12.27,

$$\lim_{x\to a} r(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{p(a)}{q(a)} = r(a). \tag{12.2}$$

So, in Result 12.18, we have

$$\lim_{x\to 1} \frac{x^2 + 1}{x^2 + 4} = \frac{1^2 + 1}{1^2 + 4} = \frac{2}{5}.$$

Although it is simpler and certainly less time-consuming to verify certain limits with the aid of these theorems, we should also know how to verify limits by the $\epsilon - \delta$ definition.

12.5 Continuity

Once again, let $f : X \to \mathbf{R}$ be a function, where $X \subseteq \mathbf{R}$, and let a be a real number such that f is defined in some deleted neighborhood of a. Recall that $\lim_{x\to a} f(x) = L$ for some real number L if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in (a - \delta, a + \delta)$ and $x \neq a$, then $|f(x) - L| < \epsilon$. If f is defined at a and $f(a) = L$, then f is said to be **continuous** at a. That is, f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. Therefore, a function f is continuous at a if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. (Notice that in this instance, $0 < |x - a| < \delta$ is being replaced by $|x - a| < \delta$.) Thus for f to be continuous at a, three conditions must be satisfied:

(1) f is defined at a; (2) $\lim_{x\to a} f(x)$ exists; (3) $\lim_{x\to a} f(x) = f(a)$.

We now illustrate this.

Problem 12.32 *A function f is defined by $f(x) = (x^2 - 3x + 2)/(x^2 - 1)$ for all $x \in \mathbf{R} - \{-1, 1\}$. Is f continuous at 1 under any of the following circumstances:* (a) *f is not defined at 1;* (b) *$f(1) = 0$;* (c) *$f(1) = -1/2$?*

Solution For f to be continuous at 1, the function f must be defined at 1. So we can answer question (a) immediately. The answer is no. In order to answer questions (b) and (c), we must first determine whether $\lim_{x\to 1} f(x)$ exists. Observe that

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - 1} = \frac{(x - 1)(x - 2)}{(x - 1)(x + 1)} = \frac{x - 2}{x + 1}$$

since $x \neq 1$. Because $f(x) = \frac{x-2}{x+1}$ is a rational function, we can apply (12.2) to obtain

$$\lim_{x\to 1} \frac{x - 2}{x + 1} = \frac{\lim_{x\to 1}(x - 2)}{\lim_{x\to 1}(x + 1)} = \frac{-1}{2} = -\frac{1}{2}.$$

Hence if $f(1) = -1/2$, then f is continuous at 1. Therefore, the answer to question (b) is no and the answer to (c) is yes. ♦

For additional practice, we present an $\epsilon - \delta$ proof that $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1} = -\frac{1}{2}$.

Result to Prove $\lim_{x\to 1} \dfrac{x^2 - 3x + 2}{x^2 - 1} = -\dfrac{1}{2}$.

PROOF STRATEGY Observe that

$$\left|\frac{x^2-3x+2}{x^2-1}-\left(-\frac{1}{2}\right)\right| = \left|\frac{(x-1)(x-2)}{(x-1)(x+1)}+\frac{1}{2}\right| = \left|\frac{(x-2)}{(x+1)}+\frac{1}{2}\right|$$
$$= \left|\frac{2(x-2)+(x+1)}{2(x+1)}\right| = \left|\frac{3x-3}{2(x+1)}\right| = \frac{3}{2}\frac{|x-1|}{|x+1|}.$$

If $|x-1|<1$, then $0<x<2$ and $|x+1|>1$, so $1/|x+1|<1$. We are now prepared to prove that $\lim_{x\to 1}(x^2-3x+2)/(x^2-1) = -1/2$. ♦

Result 12.33 $\lim_{x\to 1}\dfrac{x^2-3x+2}{x^2-1} = -\dfrac{1}{2}$.

Proof Let $\epsilon>0$ and choose $\delta = \min(1, 2\epsilon/3)$. Let $x\in\mathbf{R}$ such that $|x-1|<\delta$. Since $|x-1|<1$, it follows that $0<x<2$. So $|x+1|>1$ and $1/|x+1|<1$. Hence

$$\left|\frac{x^2-3x+2}{x^2-1}-\left(-\frac{1}{2}\right)\right| = \left|\frac{(x-1)(x-2)}{(x-1)(x+1)}+\frac{1}{2}\right| = \left|\frac{(x-2)}{(x+1)}+\frac{1}{2}\right|$$
$$= \left|\frac{3x-3}{2(x+1)}\right| = \frac{3}{2}\frac{|x-1|}{|x+1|} < \frac{3}{2}\cdot\frac{2\epsilon}{3} = \epsilon.$$ ■

Indeed, (12.2) states that if a rational function r is defined by $r(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials such that $q(a)\neq 0$, then r is continuous at a. Also, (12.1) implies that if p is a polynomial function defined by $p(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$, then p is continuous at every real number a.

We now present some examples concerning continuity for functions that are neither polynomials nor rational functions.

Result to Prove The function f defined by $f(x) = \sqrt{x}$ for $x\geq 0$ is continuous at 4.

PROOF STRATEGY Because $f(4) = 2$, it suffices to show that $\lim_{x\to 4}\sqrt{x} = 2$. Thus $|f(x)-L| = |\sqrt{x}-2|$. To work $x-4$ into the expression $\sqrt{x}-2$, we multiply $\sqrt{x}-2$ by $(\sqrt{x}+2)/(\sqrt{x}+2)$, obtaining

$$|\sqrt{x}-2| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \frac{|x-4|}{\sqrt{x}+2}.$$

First we require that $\delta\leq 1$, that is, $|x-4|<1$, so $3<x<5$. Since $\sqrt{x}+2>3$, it follows that $1/(\sqrt{x}+2)<1/3$. Hence

$$|\sqrt{x}-2| = \frac{|x-4|}{\sqrt{x}+2} < \frac{|x-4|}{3}.$$

This suggests an appropriate choice for δ. ♦

Result 12.34 *The function f defined by $f(x) = \sqrt{x}$ for $x\geq 0$ is continuous at* 4.

Proof Let $\epsilon>0$ be given and choose $\delta = \min(1, 3\epsilon)$. Let $x\in\mathbf{R}$ such that $|x-4|<\delta$. Since $|x-4|<1$, it follows that $3<x<5$ and so $\sqrt{x}+2>3$. Therefore, $1/(\sqrt{x}+2)<1/3$.

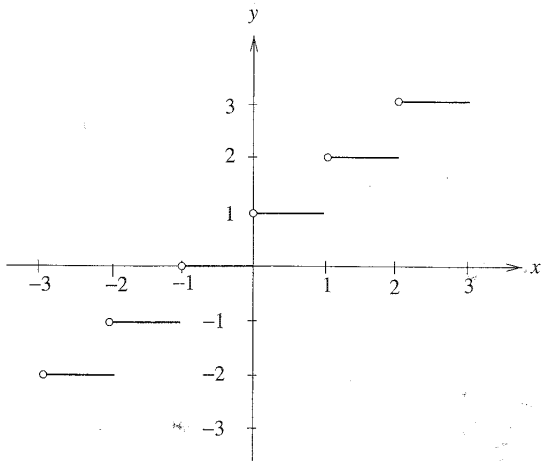

Figure 12.5 The graph of the ceiling function $f(x) = \lceil x\rceil$

Hence

$$|\sqrt{x}-2| = \left|\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2}\right| = \frac{|x-4|}{\sqrt{x}+2} < \frac{1}{3}(3\epsilon) = \epsilon.$$ ■

Figure 12.5 gives the graph of the ceiling function $f:\mathbf{R}\to\mathbf{Z}$ defined by $f(x) = \lceil x\rceil$. This function is not continuous at any integer but is continuous at all other real numbers. We verify the first of these remarks and leave the proof of the second remark as an exercise (Exercise 12.24).

Result 12.35 *The ceiling function $f:\mathbf{R}\to\mathbf{Z}$ defined by $f(x) = \lceil x\rceil$ is not continuous at any integer.*

Proof Assume, to the contrary, that there is some integer k such that f is continuous at k. Therefore, $\lim_{x\to k} f(x) = f(k) = \lceil k\rceil = k$. Hence for $\epsilon = 1$, there exists $\delta>0$ such that if $|x-k|<\delta$, then $|f(x)-f(k)| = |f(x)-k|<\epsilon = 1$. Let $\delta_1 = \min(\delta, 1)$ and let $x_1\in(k, k+\delta_1)$. Thus $k<x_1<k+\delta$ *and* $k<x_1<k+1$. Hence $f(x_1) = \lceil x_1\rceil = k+1$ and $|f(x_1)-k| = |(k+1)-k| = 1 < 1$, a contradiction. ■

12.6 Differentiability

We have discussed the existence and nonexistence of limits $\lim_{x\to a} f(x)$ for functions $f:X\to\mathbf{R}$ with $X\subseteq\mathbf{R}$, where f is defined in a deleted neighborhood of the real number a and, in the case of continuity at a, investigated whether $\lim_{x\to a} f(x) = f(a)$ if f is defined in a neighborhood of a. If f is defined in a neighborhood of a, then there is an important limit that concerns the ratio of the differences $f(x)-f(a)$ and $x-a$.

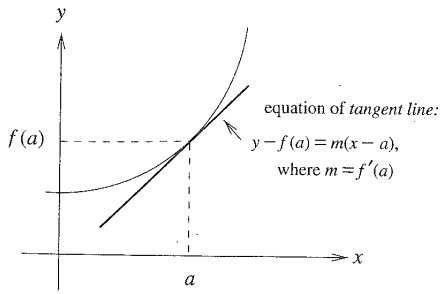

Figure 12.6 Derivatives and slopes of tangent lines

A function $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$, that is defined in a neighborhood of a real number a, is said to be **differentiable** at a if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. This limit is called the **derivative** of f at a and is denoted by $f'(a)$. Therefore,

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

You probably already know that $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. Indeed, if $f'(a) = m$, then the equation of this line is $y - f(a) = m(x - a)$. (See Figure 12.6.)

We illustrate derivatives with an example.

Example 12.36 *Show that the function f defined by $f(x) = 1/x^2$ for $x \neq 0$ is differentiable at 1 and determine $f'(1)$.*

Solution Thus we need to show that $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x^2}-1}{x-1}$ exists. In a deleted neighborhood of 1,

$$\frac{\frac{1}{x^2} - 1}{x - 1} = \frac{\frac{1-x^2}{x^2}}{x - 1} = \frac{1 - x^2}{x^2(x - 1)} = \frac{(1 - x)(1 + x)}{x^2(x - 1)} = -\frac{1 + x}{x^2}. \tag{12.3}$$

Since $\frac{1+x}{-x^2}$ is a rational function, we can once again use (12.2) to see that

$$\lim_{x\to 1} \frac{1 + x}{-x^2} = \frac{\lim_{x\to 1}(1 + x)}{\lim_{x\to 1}(-x^2)} = \frac{2}{-1} = -2$$

and so $f'(1) = -2$.

We present an $\epsilon - \delta$ proof of this limit as well. For a given $\epsilon > 0$, we are required to find $\delta > 0$ such that if $x \in \mathbf{R}$ with $0 < |x - 1| < \delta$, then $|\frac{\frac{1}{x^2}-1}{x-1} - (-2)| < \epsilon$. Observe that

$$\left|\frac{\frac{1}{x^2} - 1}{x - 1} - (-2)\right| = \left|-\frac{1 + x}{x^2} + 2\right| = \left|\frac{2x^2 - x - 1}{x^2}\right| = \frac{|x - 1||2x + 1|}{x^2}.$$

If we restrict δ so that $\delta \leq 1/2$, then $|x - 1| < 1/2$ and so $1/2 < x < 3/2$. Since $x > 1/2$, it follows that $x^2 > 1/4$ and $1/x^2 < 4$. Also, since $x < 3/2$, it follows that $|2x + 1| < 4$.

Hence $|x - 1||2x + 1|/x^2 < 16|x - 1|$. This shows us how to select δ. We now prove that $f'(1) = -2$. ♦

Result 12.37 *Let f be the function defined by $f(x) = 1/x^2$ for $x \neq 0$. Then $f'(1) = -2$.*

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1/2, \epsilon/16)$. Let $x \in \mathbf{R}$ such that $0 < |x - 1| < \delta$. Since $|x - 1| < 1/2$, it follows that $1/2 < x < 3/2$. Thus $x^2 > 1/4$ and so $1/x^2 < 4$. Also, $|2x + 1| < 4$. Since $|x - 1| < \epsilon/16$, it follows that

$$\left|\frac{f(x) - f(1)}{x - 1} - (-2)\right| = \left|\frac{\frac{1}{x^2} - 1}{x - 1} - (-2)\right| = \left|-\frac{1 + x}{x^2} + 2\right|$$

$$= \left|\frac{2x^2 - x - 1}{x^2}\right| = \frac{|2x + 1|}{x^2} \cdot |x - 1| < 4 \cdot 4 \cdot \frac{\epsilon}{16} = \epsilon. \quad \blacksquare$$

From Result 12.37, it now follows that the slope of the tangent line to the graph of $y = 1/x^2$ at the point $(1, 1)$ is -2 and, consequently, that the equation of this tangent line is $y - 1 = -2(x - 1)$. Differentiability of a function at some number a implies continuity at a, as we now show.

Theorem 12.38 *If a function f is differentiable at a, then f is continuous at a.*

Proof Since f is differentiable at a, it follows that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and equals the real number $f'(a)$. To show that f is continuous at a, we need to show that $\lim_{x\to a} f(x) = f(a)$. We write $f(x)$ as

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a).$$

Now, using properties of limits, we have

$$\lim_{x\to a} f(x) = \left[\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right] \lim_{x\to a}(x - a) + \lim_{x\to a} f(a)$$

$$= f'(a) \cdot 0 + f(a) = f(a). \quad \blacksquare$$

The converse of Theorem 12.38 is not true. For example, the functions f and g defined by $f(x) = |x|$ and $g(x) = \sqrt[3]{x}$ are continuous at 0 but neither is differentiable at 0. That f is not differentiable at 0 is actually established in Result 12.22.

EXERCISES FOR CHAPTER 12

Section 12.1: Limits of Sequences

12.1. Prove that the sequence $\{\frac{1}{2n}\}$ converges to 0.

12.2. Prove that the sequence $\{\frac{1}{n^2+1}\}$ converges to 0.

12.3. Prove that the sequence $\{1 + \frac{1}{2^n}\}$ converges to 1.

12.4. Prove that the sequence $\{\frac{n+2}{2n+3}\}$ converges to $\frac{1}{2}$.

12.5. By definition, $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$, there exists a positive integer N such that if n is an integer with $n > N$, then $|a_n - L| < \epsilon$. By taking the negation of this definition, write out the meaning of $\lim_{n\to\infty} a_n \neq L$ using quantifiers. Then write out the meaning of $\{a_n\}$ diverges using quantifiers.

12.6. Show that the sequence $\{n^4\}$ diverges to infinity.

12.7. Show that the sequence $\{\frac{n^5+2n}{n^2}\}$ diverges to infinity.

Section 12.2: Infinite Series

12.8. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$ converges and determine its sum by

(a) computing the first few terms of the sequence $\{s_n\}$ of partial sums and conjecturing a formula for s_n;

(b) using mathematical induction to verify that your conjecture in (a) is correct;

(c) completing the proof.

12.9. Prove that the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges and determine its sum by

(a) computing the first few terms of the sequence $\{s_n\}$ of partial sums and conjecturing a formula for s_n;

(b) using mathematical induction to verify that your conjecture in (a) is correct;

(c) completing the proof.

12.10. The terms $a_1, a_2, a_3, \cdots$ of the series $\sum_{k=1}^{\infty} a_k$ are defined recursively by $a_1 = \frac{1}{6}$ and

$$a_n = a_{n-1} - \frac{2}{n(n+1)(n+2)}$$

for $n \geq 2$. Prove that $\sum_{k=1}^{\infty} a_k$ converges and determine its value.

Section 12.3: Limits of Functions

12.11. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2}(\frac{3}{2}x + 1) = 4$.

12.12. Give an $\epsilon - \delta$ proof that $\lim_{x\to -1}(3x - 5) = -8$.

12.13. Determine $\lim_{x\to 3} \frac{x^2-2x-3}{x^2-8x+15}$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.14. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2}(2x^2 - x - 5) = 1$.

12.15. Give an $\epsilon - \delta$ proof that $\lim_{x\to 2} x^3 = 8$.

12.16. Give an $\epsilon - \delta$ proof that $\lim_{x\to 3} \frac{3x+1}{4x+3} = \frac{2}{3}$.

12.17. Determine $\lim_{x\to 1} \frac{1}{5x-4}$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.18. Show that $\lim_{x\to 0} \frac{1}{x^2}$ does not exist.

12.19. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} 1 & x < 3 \\ 1.5 & x = 3 \\ 2 & x > 3. \end{cases}$$

(a) Determine whether $\lim_{x\to 3} f(x)$ exists and verify your answer.

(b) Determine whether $\lim_{x\to \pi} f(x)$ exists and verify your answer.

Section 12.4: Fundamental Properties of Limits of Functions

12.20. Use induction to prove that $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ for all $n \in \mathbf{N}$.

12.21. Use induction to prove that for every integer $n \geq 2$ and every n functions $f_1, f_2, \cdots, f_n$ such that $\lim_{x\to a} f_i(x) = L_i$ for $1 \leq i \leq n$,

$$\lim_{x\to a}(f_1(x) + f_2(x) + \cdots + f_n(x)) = L_1 + L_2 + \cdots + L_n.$$

12.22. Use limit theorems to determine the following:

(a) $\lim_{x\to 1}(x^3 - 2x^2 - 5x + 8)$

(b) $\lim_{x\to 1}(4x + 7)(3x^2 - 2)$

(c) $\lim_{x\to 2} \frac{2x^2-1}{3x^3+1}$

Section 12.5: Continuity

12.23. Use Theorem 12.23 to prove that every polynomial is continuous at every real number.

12.24. Let $f : \mathbf{R} \to \mathbf{Z}$ be the ceiling function defined by $f(x) = \lceil x \rceil$. Give an $\epsilon - \delta$ proof that if a is a real number that is not an integer, then f is continuous at a.

12.25. The function f defined by $f(x) = \frac{x^2-9}{x^2-3x}$ is not defined at 3. Is it possible to define f at 3 such that f is continuous there? Verify your answer with an $\epsilon - \delta$ proof.

12.26. The function $f : \mathbf{R} - \{0, 2\} \to \mathbf{R}$ is defined by $f(x) = \frac{x^2-4}{x^3-2x^2}$. Use limit theorems to determine whether f can be defined at 2 such that f is continuous at 2.

12.27. Prove that the function $f : [1, \infty) \to [0, \infty)$ defined by $f(x) = \sqrt{x-1}$ is continuous at $x = 10$.

Section 12.6: Differentiability

12.28. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2$. Determine $f'(3)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.29. The function $f : \mathbf{R} - \{-2\} \to \mathbf{R}$ is defined by $f(x) = \frac{1}{x+2}$. Determine $f'(1)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

12.30. The function $f : \mathbf{R} \to \mathbf{R}$ is defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine $f'(0)$ and verify that your answer is correct with an $\epsilon - \delta$ proof.

ADDITIONAL EXERCISES FOR CHAPTER 12

12.31. Prove that the sequence $\{\frac{n+1}{3n-1}\}$ converges to $\frac{1}{3}$.

12.32. Prove that $\lim_{n\to\infty} \frac{2n^2}{4n^2+1} = \frac{1}{2}$.

12.33. Prove that the sequence $\{1 + (-2)^n\}$ diverges.

12.34. Prove that $\lim_{n\to\infty}(\sqrt{n^2+1} - n) = 0$.

12.35. Let $a, c_0, c_1 \in \mathbf{R}$ such that $c_1 \neq 0$. Give an $\epsilon - \delta$ proof that $\lim_{x\to a}(c_1 x + c_0) = c_1 a + c_0$.

12.36. Evaluate the proposed solution of the following problem.

13.43. Evaluate the proposed proof of the following statement.

Result There exists no abelian group containing exactly three elements x such that $x^2 = e$.

Proof Assume, to the contrary, that there exists an abelian group G such that $x^2 = e$ for exactly three distinct elements x of G. Certainly, $e^2 = e$, so there are two non-identity elements a and b such that $a^2 = b^2 = e$. Observe that $(ab)^2 = a^2b^2 = ee = e$. Hence either $ab = a$, $ab = b$, or $ab = e$, which implies, respectively, that $b = e$, $a = e$, or $a = b$, producing a contradiction. ∎

13.44. Prove or disprove the following: For each odd integer $k \geq 3$, there exists no abelian group containing exactly k elements x such that $x^2 = e$.

Solutions to Odd-Numbered Section Exercises

EXERCISES FOR CHAPTER 1

Section 1.1: Describing a Set

1.1. Only (d) and (e) are sets.

1.3. **(a)** $|A| = 5$, **(b)** $|B| = 11$, **(c)** $|C| = 51$, **(d)** $|D| = 2$, **(e)** $|E| = 1$, **(f)** $|F| = 2$

1.5. **(a)** $A = \{-1, -2, -3, \ldots\} = \{x \in \mathbf{Z} : x \leq -1\}$
(b) $B = \{-3, -2, \ldots, 3\} = \{x \in \mathbf{Z} : -3 \leq x \leq 3\} = \{x \in \mathbf{Z} : |x| \leq 3\}$
(c) $C = \{-2, -1, 1, 2\} = \{x \in \mathbf{Z} : -2 \leq x \leq 2, x \neq 0\} = \{x \in \mathbf{Z} : 0 < |x| \leq 2\}$

1.7. **(a)** $A = \{\cdots, -4, -1, 2, 5, 8, \cdots\} = \{3x + 2 : x \in \mathbf{Z}\}$
(b) $B = \{\cdots, -10, -5, 0, 5, 10, \cdots\} = \{5x : x \in \mathbf{Z}\}$
(c) $C = \{1, 8, 27, 64, 125, \cdots\} = \{x^3 : x \in \mathbf{N}\}$

Section 1.2: Subsets

1.9. Let $r = \min(c - a, b - c)$ and let $I = (c - r, c + r)$. Then I is centered at c and $I \subseteq (a, b)$.

1.11. See Figure 1.

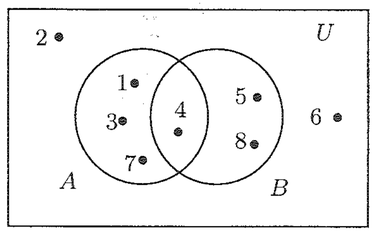

Figure 1 Answer for Exercise 1.11

1.13. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\{0\}\}, A\}$

1.15. $\mathcal{P}(A) = \{\emptyset, \{0\}, \{\emptyset\}, \{\{\emptyset\}\}, \{0, \emptyset\}, \{0, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}, A\}$; $|\mathcal{P}(A)| = 8$

Section 1.3: Set Operations

1.17. **(a)** $A \cup B = \{1, 3, 5, 9, 13, 15\}$ **(b)** $A \cap B = \{9\}$ **(c)** $A - B = \{1, 5, 13\}$
(d) $B - A = \{3, 15\}$ **(e)** $\overline{A} = \{3, 7, 11, 15\}$ **(f)** $A \cap \overline{B} = \{1, 5, 13\}$

1.19. Let $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{2, 3\}$. Then $B \neq C$ but $B - A = C - A = \{3\}$.

1.21. **(a)** and **(b)** are the same, as are **(c)** and **(d)**

1.23. See Figures 2(a) and (b) below.

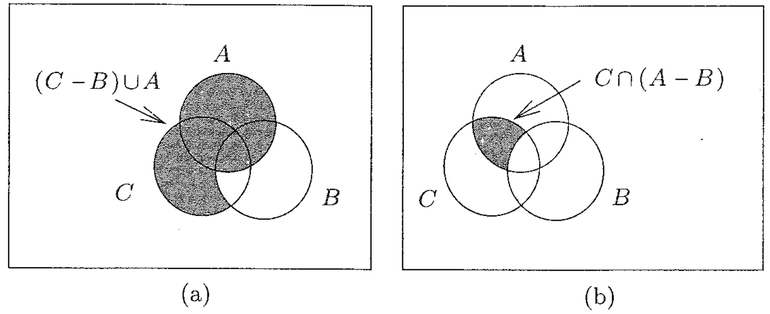

Figure 2 Answers for Exercise 1.23

Section 1.4: Indexed Collections of Sets

1.25. Let $U = \{1, 2, \ldots, 8\}$, $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 4, 6\}$, and $C = \{1, 3, 4, 7\}$.

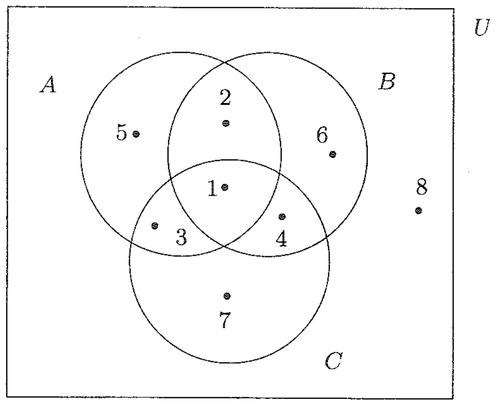

Figure 3 Answer for Exercise 1.25

1.27. $\bigcup_{X \in S} X = A \cup B \cup C = \{0, 1, 2, \ldots, 5\}$ and $\bigcap_{X \in S} X = A \cap B \cap C = \{2\}$.

1.29. Since $|A| = 26$ and $|A_\alpha| = 3$ for each $\alpha \in A$, we need to have at least nine sets of cardinality 3 for their union to be A; that is, in order for $\bigcup_{\alpha \in S} A_\alpha = A$, we must have $|S| \geq 9$. However, if we let $S = \{a, d, g, j, m, p, s, v, y\}$, then $\bigcup_{\alpha \in S} A_\alpha = A$. Hence the smallest cardinality of a set S with $\bigcup_{\alpha \in S} A_\alpha = A$ is 9.

1.31. **(a)** $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{x \in \mathbf{R} : 0 \leq x \leq 1/n\} = [0, 1/n]$.
(b) $\{A_n\}_{n \in \mathbf{N}}$, where $A_n = \{a \in \mathbf{Z} : |a| \leq n\} = \{-n, -(n-1), \ldots, (n-1), n\}$.

Section 1.5: Partitions of Sets

1.33. **(a)** S_1 is not a partition of A since 4 belongs to no element of S_1.
(b) S_2 is a partition of A. S_2 can be written as $\{\{1, 2\}, \{3, 4, 5\}\}$.
(c) S_3 is not a partition of A because 2 belongs to two elements of S_3.
(d) S_4 is not a partition of A since S_4 is not a set of subsets of A.

1.35. $A = \{1, 2, 3, 4\}$. $S_1 = \{\{1\}, \{2\}, \{3, 4\}\}$ and $S_2 = \{\{1, 2\}, \{3\}, \{4\}\}$.

1.37. Let $S = \{A_1, A_2, A_3\}$, where $A_1 = \{x \in \mathbf{Q} : x > 1\}$, $A_2 = \{x \in \mathbf{Q} : x < 1\}$, and $A_3 = \{1\}$.

1.39. Let $S = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{x \in \mathbf{Z} : x$ is odd and x is positive$\}$, $A_2 = \{x \in \mathbf{Z} : x$ is odd and x is negative, $A_3 = \{x \in \mathbf{Z} : x$ is even and x is nonnegative$\}$, $A_4 = \{x \in \mathbf{Z} : x$ is even and x is negative$\}$.

Section 1.6: Cartesian Products of Sets

1.41. $A \times B = \{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}$.

1.43. $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$, $A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, A), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, A)\}$.

1.45. $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$, $\mathcal{P}(B) = \{\emptyset, B\}$, $A \times B = \{(1, \emptyset), (2, \emptyset)\}$,
$\mathcal{P}(A) \times \mathcal{P}(B) = \{(\emptyset, \emptyset), (\emptyset, B), (\{1\}, \emptyset), (\{1\}, B), (\{2\}, \emptyset), (\{2\}, B), (A, \emptyset), (A, B)\}$.

1.47. $S = \{(3, 0), (2, 1), (1, 2), (0, 3), (-3, 0), (-2, 1), (-1, 2), (2, -1), (1, -2), (0, -3), (-2, -1), (-1, -2)\}$.

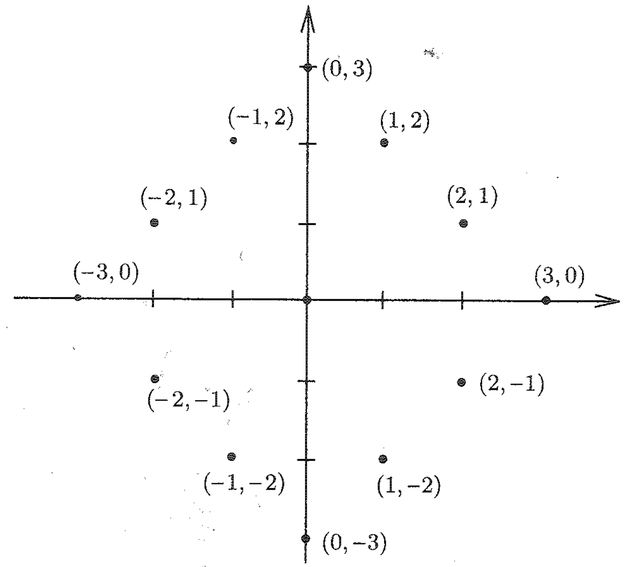

Figure 4 Answer for Exercise 1.47

EXERCISES FOR CHAPTER 2

Section 2.1: Statements

2.1. **(a)** A false statement **(b)** A true statement **(c)** Not a statement **(d)** Not a statement (an open sentence)
(e) Not a statement **(f)** Not a statement (an open sentence) **(g)** Not a statement

2.3. **(a)** False. $\emptyset$ has no elements. **(b)** True **(c)** True
(d) False. $\{\emptyset\}$ has $\emptyset$ as its only element. **(e)** True **(f)** False. 1 is not a set.

2.5. **(a)** $\{x \in \mathbf{Z} : x > 2\}$ **(b)** $\{x \in \mathbf{Z} : x \leq 2\}$

2.7. 3, 5, 11, 17, 41, 59

Section 2.2: The Negation of a Statement

2.9. See Figure 5.

P	Q	$\sim P$	$\sim Q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

Figure 5 Answer for Exercise 2.9

Section 2.3: The Disjunction and Conjunction of Statements

2.11. **(a)** True, **(b)** False, **(c)** False, **(d)** True, **(e)** True.

2.13. **(a)** All nonempty subsets of $\{1, 3, 5\}$. **(b)** All subsets of $\{1, 3, 5\}$.
(c) There are no subsets A of S for which $(\sim P(A)) \wedge (\sim Q(A))$ is true.

Section 2.4: The Implication

2.15. See Figure 6.

P	Q	$\sim P$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow (\sim P)$
T	T	F	T	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Figure 6 Answer for Exercise 2.15

2.17. **(a)** $(P \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (True)
(b) $(P \wedge Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)
(c) $((\sim P) \wedge Q) \Rightarrow R$: If $\sqrt{2}$ is not rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational. (False)
(d) $(P \vee Q) \Rightarrow (\sim R)$: If $\sqrt{2}$ is rational or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is not rational. (True)

Section 2.5: More On Implications

2.19. **(a)** $P(x) \Rightarrow Q(x)$: If $|x| = 4$, then $x = 4$. $P(-4) \Rightarrow Q(-4)$ is false. $P(-3) \Rightarrow Q(-3)$ is true. $P(1) \Rightarrow Q(1)$ is true. $P(4) \Rightarrow Q(4)$ is true. $P(5) \Rightarrow Q(5)$ is true.
(b) $P(x) \Rightarrow Q(x)$: If $x^2 = 16$, then $|x| = 4$. True for all $x \in S$.
(c) $P(x) \Rightarrow Q(x)$: If $x > 3$, then $4x - 1 > 12$. True for all $x \in S$.

2.21. **(a)** True for $(x, y) = (3, 4)$ and $(x, y) = (5, 5)$, false for $(x, y) = (1, -1)$.
(b) True for $(x, y) = (1, 2)$ and $(x, y) = (6, 6)$, false for $(x, y) = (2, -2)$.
(c) True for $(x, y) \in \{(1, -1), (-3, 4), (1, 0)\}$ and false for $(x, y) = (0, -1)$.

Section 2.6: The Biconditional

2.23. **(a)** $\sim P(x) : x \neq -2$. True if $x = 0, 2$.
(b) $P(x) \vee Q(x) : x = -2$ or $x^2 = 4$. True if $x = -2, 2$.
(c) $P(x) \wedge Q(x) : x = -2$ and $x^2 = 4$. True if $x = -2$.

(d) $P(x) \Rightarrow Q(x)$: If $x = -2$, then $x^2 = 4$. True for all x.
(e) $Q(x) \Rightarrow P(x)$: If $x^2 = 4$, then $x = -2$. True if $x = 0, -2$.
(f) $P(x) \Leftrightarrow Q(x) : x = -2$ if and only if $x^2 = 4$. True if $x = 0, -2$.

2.25. x is odd if and only if x^2 is odd.
That x is odd is a necessary and sufficient condition for x^2 to be odd.

2.27. **(a)** True for $(x, y) \in \{(3, 4), (5, 5)\}$. **(b)** True for $(x, y) \in \{(1, 2), (6, 6)\}$. **(c)** True for $(x, y) \in \{(1, -1), (1, 0)\}$.

2.29. **(i)** $P(1) \Rightarrow Q(1)$ is false; **(ii)** $Q(4) \Rightarrow P(4)$ is true;
(iii) $P(2) \Leftrightarrow R(2)$ is true; **(iv)** $Q(3) \Leftrightarrow R(3)$ is false.

Section 2.7: Tautologies and Contradictions

2.31. The compound statement $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ is a contradiction since it is false for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$\sim Q$	$P \wedge Q$	$P \wedge (\sim Q)$	$(P \wedge (\sim Q)) \wedge (P \wedge Q)$
T	T	F	T	F	**F**
T	F	T	F	T	**F**
F	T	F	F	F	**F**
F	F	T	F	F	**F**

2.33. The compound statement $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$ is a tautology since it is true for all combinations of truth values for the component statements P, Q, and R. See the truth table below.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$(P \Rightarrow Q) \wedge (Q \Rightarrow R)$	$P \Rightarrow R$	$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
T	T	T	T	T	T	T	**T**
T	F	T	F	T	F	T	**T**
F	T	T	T	T	T	T	**T**
F	F	T	T	T	T	T	**T**
T	T	F	T	F	F	F	**T**
T	F	F	F	T	F	F	**T**
F	T	F	T	F	F	T	**T**
F	F	F	T	T	T	T	**T**

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$: If P implies Q and Q implies R, then P implies R.

Section 2.8: Logical Equivalence

2.35. **(a)** See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$(\sim P) \vee (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**T**
F	T	T	F	T	**F**	**T**
F	F	T	T	F	**T**	**T**

Since $\sim(P \vee Q)$ and $(\sim P) \vee (\sim Q)$ do not have the same truth values for all combinations of truth values for the component statements P and Q, the compound statements $\sim(P \vee Q)$ and $(\sim P) \vee (\sim Q)$ are not logically equivalent.

(b) The biconditional $\sim(P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$ is not a tautology, and so there are instances when this biconditional is false.

2.37. The statements Q and $(\sim Q) \Rightarrow (P \wedge (\sim P))$ are logically equivalent since they have the same truth values for all combinations of truth values for the component statements P and Q. See the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \wedge (\sim P)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	**T**	F	F	F	**T**
T	**F**	F	T	F	**F**
F	**T**	T	F	F	**T**
F	**F**	T	T	F	**F**

Section 2.9: Some Fundamental Properties of Logical Equivalence

2.39. **(a)** The statement $P \vee (Q \wedge R)$ is equivalent to $(P \vee Q) \wedge (P \vee R)$ since the last two columns in the truth table of Figure 7 are the same.

P	Q	R	$P \vee Q$	$P \vee R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	**T**	**T**
T	F	T	T	T	F	**T**	**T**
F	T	T	T	T	T	**T**	**T**
F	F	T	F	T	F	**F**	**F**
T	T	F	T	T	F	**T**	**T**
T	F	F	T	T	F	**T**	**T**
F	T	F	T	F	F	**F**	**F**
F	F	F	F	F	F	**F**	**F**

Figure 7 Answer for Exercise 2.39(a)

(b) The statement $\sim(P \vee Q)$ is equivalent to $(\sim P) \wedge (\sim Q)$ since the last two columns in the truth table of Figure 8 are the same.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$(\sim P) \wedge (\sim Q)$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Figure 8 Answer for Exercise 2.39(b)

2.41. **(a)** x and y are even only if xy is even.
(b) If xy is even, then x and y are even.
(c) Either at least one of x and y is odd or xy is even.
(d) x and y are even and xy is odd.

Section 2.10: Quantified Statements

2.43. $\forall x \in S$, $P(x)$: For every odd integer x, the integer $x^2 + 1$ is even.
$\exists x \in S$, $Q(x)$: There exists an odd integer x such that x^2 is even.

2.45. **(a)** There exists a set A such that $A \cap \overline{A} \neq \emptyset$.
(b) For every set A, we have $\overline{A} \not\subseteq A$.

2.47. **(a)** False, since $P(1)$ is false. **(b)** True, for example, $P(3)$ is true.

2.49. **(a)** $\exists a, b \in \mathbf{Z}$, $ab < 0$ and $a + b > 0$.
(b) $\forall x, y \in \mathbf{R}$, $x \neq y$ implies that $x^2 + y^2 > 0$.
(c) For all integers a and b either $ab \geq 0$ or $a + b \leq 0$.
There exist real numbers x and y such that $x \neq y$ and $x^2 + y^2 \leq 0$.
(d) $\forall a, b \in \mathbf{Z}$, $ab \geq 0$ or $a + b \leq 0$.
$\exists x, y \in \mathbf{R}$, $x \neq y$ and $x^2 + y^2 \leq 0$.

2.51. Let $S = \{3, 5, 11\}$ and $P(s, t) : st - 2$ is prime.
(a) $\forall s, t \in S$, $P(s, t)$.
(b) True since $P(s, t)$ is true for all $s, t \in S$.
(c) $\exists s, t \in S$, $\sim P(s, t)$.
(d) There exist $s, t \in S$ such that $st - 2$ is not prime.
(e) False since the statement in (a) is true.

Section 2.11: Characterizations of Statements

2.53. An integer n is odd if and only if n^2 is odd.

2.55. **(a)** a characterization. **(b)** a characterization. **(c)** a characterization.
(d) a characterization. (Pythagorean theorem) **(e)** not a characterization. (Every positive number is the area of some rectangle.)

EXERCISES FOR CHAPTER 3

Section 3.1: Trivial and Vacuous Proofs

3.1. ***Proof*** Since $x^2 - 2x + 2 = (x - 1)^2 + 1 \geq 1$, it follows that $x^2 - 2x + 2 \neq 0$ for all $x \in \mathbf{R}$. Hence the statement is true trivially. ∎

3.3. ***Proof*** Note that $\frac{r^2+1}{r} = r + \frac{1}{r}$. If $r \geq 1$, then $r + \frac{1}{r} > 1$; while if $0 < r < 1$, then $\frac{1}{r} > 1$ and so $r + \frac{1}{r} > 1$. Thus $\frac{r^2+1}{r} \leq 1$ is false for all $r \in \mathbf{Q}^+$ and so the statement is true vacuously. ∎

3.5. ***Proof*** Since $n^2 - 2n + 1 = (n - 1)^2 \geq 0$, it follows that $n^2 + 1 \geq 2n$ and so $n + \frac{1}{n} \geq 2$. Thus the statement is true vacuously. ∎

Section 3.2: Direct Proofs

3.7. ***Proof*** Let x be an even integer. Then $x = 2a$ for some integer a. Thus

$$5x - 3 = 5(2a) - 3 = 10a - 4 + 1 = 2(5a - 2) + 1.$$

Since $5a - 2$ is an integer, $5x - 3$ is odd. ∎

3.9. ***Proof*** Let $1 - n^2 > 0$. Then $n = 0$. Thus $3n - 2 = 3 \cdot 0 - 2 = -2$ is an even integer. ∎

3.11. ***Proof*** Assume that $(n + 1)^2(n + 2)^2/4$ is even, where $n \in S$. Then $n = 2$. For $n = 2$, $(n + 2)^2(n + 3)^2/4 = 100$, which is even. ∎

Section 3.3: Proof by Contrapositive

3.13. First, we prove a lemma. **Lemma** Let $n \in \mathbf{Z}$. If $15n$ is even, then n is even.
(Use a proof by contrapositive to verify this lemma.) Then use this lemma to prove the result.
Proof of Result Assume that $15n$ is even. By the lemma, n is even and so $n = 2a$ for some integer a. Hence

$$9n = 9(2a) = 2(9a).$$

Since $9a$ is an integer, $9n$ is even. ∎
[Note: This result could also be proved by assuming that $15n$ is even (and so $15n = 2a$ for some integer a) and observing that $9n = 15n - 6n = 2a - 6n$.]

3.15. **Lemma** Let $x \in \mathbf{Z}$. If $7x + 4$ is even, then x is even. (Use a proof by contrapositive to verify this lemma.)
Proof of Result Assume that $7x + 4$ is even. Then by the lemma, x is even and so $x = 2a$ for some integer a. Hence

$$3x - 11 = 3(2a) - 11 = 6a - 12 + 1 = 2(3a - 6) + 1.$$

Since $3a - 6$ is an integer, $3x - 11$ is odd. ∎

3.17. The proof would begin by assuming that $n^2(n + 1)^2/4$ is odd, where $n \in S$. Then $n = 2$ and so $n^2(n - 1)^2/4 = 1$ is odd.

Section 3.4: Proof by Cases

3.19. ***Proof*** Let $n \in \mathbf{Z}$. We consider two cases.
Case 1. *n is even.* Then $n = 2a$ for some integer a. Thus

$$n^2 - 3n + 9 = 4a^2 - 3(2a) + 9 = 2(2a^2 - 3a + 4) + 1.$$

Since $2a^2 - 3a + 4$ is an integer, $n^2 - 3n + 9$ is odd.
Case 2. *n is odd.* Then $n = 2b + 1$ for some integer b. Observe that

$$\begin{aligned} n^2 - 3n + 9 &= (2b+1)^2 - 3(2b+1) + 9 \\ &= 4b^2 + 4b + 1 - 6b - 3 + 9 = 4b^2 - 2b + 7 \\ &= 2(2b^2 - b + 3) + 1. \end{aligned}$$

Since $2b^2 - b + 3$ is an integer, $n^2 - 3n + 9$ is odd. ∎

3.21. ***Proof*** Assume that x or y is even, say x is even. Then $x = 2a$ for some integer a. Thus $xy = (2a)y = 2(ay)$. Since ay is an integer, xy is even. ∎

3.23. One possibility is to begin by proving the implication "If x and y are of the same parity, then $x - y$ is even." Use a direct proof and consider two cases, according to whether x and y are both even or x and y are both odd.
For the converse of this implication, use a proof by contrapositive and consider two cases, where say *Case 1. x is even and y is odd.* and *Case* 2. *x is odd and y is even.*

3.25. **(a)** Use the following facts:
(1) Let $x, y \in \mathbf{Z}$. Then $x + y$ is even if and only if x and y are of the same parity.
(2) Let $x \in \mathbf{Z}$. Then x^2 is even if and only if x even.
(b) Let x and y be integers. Then $(x + y)^2$ is odd if and only if x and y are of opposite parity.

Section 3.5: Proof Evaluations

3.27. (3) is proved.
3.29. The converse of the result has been proved. No proof has been given of the result itself.

EXERCISES FOR CHAPTER 4

Section 4.1: Proofs Involving Divisibility of Integers

4.1. ***Proof*** Assume that $a \mid b$. Then $b = ac$ for some integer c. Then $b^2 = (ac)^2 = a^2c^2$. Since c^2 is an integer, $a^2 \mid b^2$. ∎

4.3. **(a)** ***Proof*** Assume that $3 \mid m$. Then $m = 3q$ for some integer q. Hence $m^2 = (3q)^2 = 9q^2 = 3(3q^2)$. Since $3q^2$ is an integer, $3 \mid m^2$. ∎
(b) Let $m \in \mathbf{Z}$. If $3 \nmid m^2$, then $3 \nmid m$.
(c) Start with the following: Assume that $3 \nmid m$. Then $m = 3q + 1$ or $m = 3q + 2$, where $q \in \mathbf{Z}$. Consider these two cases.
(d) Let $m \in \mathbf{Z}$. If $3 \mid m^2$, then $3 \mid m$.
(e) Let $m \in \mathbf{Z}$. Then $3 \mid m$ if and only if $3 \mid m^2$.

4.5. ***Proof*** Assume that $a \mid b$ or $a \mid c$, say the latter. Then $c = ak$ for some integer k. Thus $bc = b(ak) = a(bk)$. Since bk is an integer, $a \mid bc$. ∎

4.7. For the implication "If $3 \nmid n$, then $3 \mid (2n^2 + 1)$.", use a direct proof. Assume that $3 \nmid n$. Then $n = 3q + 1$ or $n = 3q + 2$ for some integer q. Then consider these two cases.
For the converse "If $3 \mid (2n^2 + 1)$, then $3 \nmid n$." use a proof by contrapositive.

4.9. ***Proof*** Let $n \in \mathbf{Z}$ with $n \geq 8$. Then $n = 3q$, where $q \geq 3$, or $n = 3q + 1$, where $q \geq 3$, or $n = 3q + 2$, where $q \geq 2$. We consider these three cases.
Case 1. $n = 3q$, *where* $q \geq 3$. Then $n = 3a + 5b$, where $a \geq 3$ and $b = 0$.

Case 2. $n = 3q + 1$, *where* $q \geq 3$. Then $n = 3(q - 3) + 10$, where $q - 3 \geq 0$. Thus $n = 3a + 5b$, where $a = q - 3 \geq 0$ and $b = 2$.
Case 3. $n = 3q + 2$, *where* $q \geq 2$. Then $n = 3(q - 1) + 5$, where $q - 1 \geq 1$. Thus $n = 3a + 5b$, where $a = q - 1 \geq 1$ and $b = 1$. ∎

Section 4.2: Proofs Involving Congruence of Integers

4.11. ***Proof*** Assume that $a \equiv b \pmod{n}$ and $a \equiv c \pmod{n}$. Then $n \mid (a - b)$ and $n \mid (a - c)$. Hence $a - b = nx$ and $a - c = ny$, where $x, y \in \mathbf{Z}$. Thus $b = a - nx$ and $c = a - ny$. Therefore, $b - c = (a - nx) - (a - ny) = ny - nx = n(y - x)$. Since $y - x$ is an integer, $n \mid (b - c)$ and so $b \equiv c \pmod{n}$. ∎

4.13. **(a)** ***Proof*** Assume that $a \equiv 1 \pmod{5}$. Then $5 \mid (a - 1)$. So $a - 1 = 5k$ for some integer k. Thus $a = 5k + 1$ and so

$$a^2 = (5k+1)^2 = 25a^2 + 10a + 1 = 5(5a^2 + 2a) + 1.$$

Thus

$$a^2 - 1 = 5(5a^2 + 2a).$$

Since $5a^2 + 2a$ is an integer, $5 \mid (a^2 - 1)$ and so $a^2 \equiv 1 \pmod{5}$. ∎
(b) We can conclude that $b^2 \equiv 1 \pmod{5}$.

4.15. ***Proof*** Assume that $a \equiv 5 \pmod{6}$ and $b \equiv 3 \pmod{4}$. Then $6 \mid (a - 5)$ and $4 \mid (b - 3)$. Thus $a - 5 = 6x$ and $b - 3 = 4y$, where $x, y \in \mathbf{Z}$. So $a = 6x + 5$ and $b = 4y + 3$. Observe that

$$4a + 6b = 4(6x + 5) + 6(4y + 3) = 24x + 20 + 24y + 18 = 24x + 24y + 38 = 8(3x + 3y + 4) + 6.$$

Since $3x + 3y + 4$ is an integer, $8 \mid (4a + 6b - 6)$ and so $4a + 6b \equiv 6 \pmod{8}$. ∎

4.17. ***Proof*** Either $a = 3q$, $a = 3q + 1$ or $a = 3q + 2$ for some integer q. We consider these three cases.
Case 1. $a = 3q$. Then

$$a^3 - a = (3q)^3 - (3q) = 27q^3 - 3q = 3(9q^3 - q).$$

Since $9q^3 - q$ is an integer, $3 \mid (a^3 - a)$ and so $a^3 \equiv a \pmod{3}$.
Case 2. $a = 3q + 1$. Then

$$\begin{aligned} a^3 - a &= (3q+1)^3 - (3q+1) = 27q^3 + 27q^2 + 9q + 1 - 3q - 1 \\ &= 27q^3 + 27q^2 + 6q = 3(9q^3 + 9q^2 + 2q). \end{aligned}$$

Since $9q^3 + 9q^2 + 2q$ is an integer, $3 \mid (a^3 - a)$ and so $a^3 \equiv a \pmod{3}$.
Case 3. $a = 3q + 2$. Then

$$\begin{aligned} a^3 - a &= (3q+2)^3 - (3q+2) = (27q^3 + 54q^2 + 36q + 8) - 3q - 2 \\ &= 27q^3 + 54q^2 + 33q + 6 = 3(9q^3 + 18q^2 + 11q + 2). \end{aligned}$$

Since $9q^3 + 18q^2 + 11q + 2$ is an integer, $3 \mid (a^3 - a)$ and so $a^3 \equiv a \pmod{3}$. ∎

Section 4.3: Proofs Involving Real Numbers

4.19. ***Proof*** Assume that $a < 3m + 1$ and $b < 2m + 1$. Since a and b are integers, $a \leq 3m$ and $b \leq 2m$. Therefore,

$$2a + 3b \leq 2(3m) + 3(2m) = 12m < 12m + 1,$$

as desired. ∎

4.21. This exercise states that the arithmetic mean of two positive numbers is at least as large as their geometric mean.
(a) ***Proof*** Since $(a - b)^2 \geq 0$, it follows that $a^2 - 2ab + b^2 \geq 0$. Adding $4ab$ to both sides, we obtain $a^2 + 2ab + b^2 \geq 4ab$ or $(a + b)^2 \geq 4ab$. Taking square roots of both sides, we have $a + b \geq 2\sqrt{ab}$ and so $\sqrt{ab} \leq (a + b)/2$, as desired. ∎

(b) Assume that $\sqrt{ab} = (a+b)/2$. Taking the steps in part (a) in reverse order, we obtain $(a-b)^2 = 0$ and so $a = b$.

4.23. Observe that if $x = 0$ or $y = 0$, then the result holds. Thus we may assume that $x \neq 0$ and $y \neq 0$. There are three cases.

Case 1. $x > 0$ *and* $y > 0$.

Case 2. $x < 0$ *and* $y < 0$.

Case 3. *One of* x *and* y *is positive and the other is negative, say* $x > 0$ *and* $y < 0$.

4.25. ***Proof*** Since $|x - z| = |(x - y) + (y - z)|$, it follows that $|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|$. ∎

Section 4.4: Proofs Involving Sets

4.27. We first show that $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Assume, without loss of generality, that $x \in A$. We consider two cases.

Case 1. $x \in B$. Since $x \in A$ and $x \in B$, it follows that $x \in A \cap B$. Thus $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

Case 2. $x \notin B$. Since $x \in A$ and $x \notin B$, it follows that $x \in A - B$. Again, $x \in (A - B) \cup (B - A) \cup (A \cap B)$.

Next, we verify that $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$. Let $y \in (A - B) \cup (B - A) \cup (A \cap B)$. Then $y \in A - B$, $y \in B - A$, or $y \in A \cap B$. In each case, either $y \in A$ or $y \in B$. Therefore, $y \in A \cup B$.

4.29. ***Proof*** Assume that $A \cap B = A$. We show that $A \subseteq B$. Let $x \in A$. Since $A = A \cap B$, it follows that $x \in A \cap B$ and so $x \in B$. Hence $A \subseteq B$.

For the converse, assume that $A \subseteq B$. We show that $A \cap B = A$. Since $A \cap B \subseteq A$, it suffices to show that $A \subseteq A \cap B$. Let $x \in A$. Since $A \subseteq B$, it follows that $x \in B$. Thus $x \in A$ and $x \in B$, implying that $x \in A \cap B$. Therefore, $A \subseteq A \cap B$. ∎

4.31. ***Proof*** Assume that $A = \emptyset$ and $B = \emptyset$. Then $A \cup B = \emptyset \cup \emptyset = \emptyset$. ∎

4.33. ***Proof*** Assume that $A = B$. Then $A \cup B = A \cap B = A$. It remains to verify the converse. Assume that $A \neq B$. Thus $A \not\subseteq B$ or $B \not\subseteq A$, say the former. Thus there exists $a \in A$ such that $a \notin B$. Since $a \notin B$, it follows that $a \notin A \cap B$. On the other hand, $a \in A$ implies that $a \in A \cup B$. Therefore, $A \cup B \neq A \cap B$. ∎

Section 4.5: Fundamental Properties of Set Operations

4.35. ***Proof*** First, we show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, it follows that $x \in B$ or $x \in C$, say $x \in B$. Because $x \in A$ and $x \in B$, it follows that $x \in A \cap B$. Hence $x \in (A \cap B) \cup (A \cap C)$.

Next, we show that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $y \in (A \cap B) \cup (A \cap C)$. Then $y \in A \cap B$ or $y \in A \cap C$, say the former. Since $y \in A \cap B$, it follows that $y \in A$ and $y \in B$ and so $y \in A$ and $y \in B \cup C$. Thus $y \in A \cap (B \cup C)$. ∎

4.37. ***Proof*** We first show that $(A - B) \cap (A - C) \subseteq A - (B \cup C)$. Let $x \in (A - B) \cap (A - C)$. Then $x \in A - B$ and $x \in A - C$. Since $x \in A - B$, it follows that $x \in A$ and $x \notin B$. Because $x \in A - C$, we have $x \in A$ and $x \notin C$. Since $x \notin B$ and $x \notin C$, we have $x \notin B \cup C$. Thus $x \in A - (B \cup C)$.

Next, we show that $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. Let $y \in A - (B \cup C)$. Thus $y \in A$ and $y \notin B \cup C$. Since $y \notin B \cup C$, it follows that $y \notin B$ and $y \notin C$. Thus $y \in A - B$ and $y \in A - C$. Therefore, $y \in (A - B) \cap (A - C)$. ∎

4.39. ***Proof*** By Theorem 4.21,

$$\begin{aligned}\overline{\overline{A} \cup (\overline{B} \cap C)} &= \overline{\overline{A}} \cap (\overline{\overline{B} \cap C}) = A \cap (\overline{\overline{B}} \cup \overline{C}) \\ &= A \cap (B \cup \overline{C}) = (A \cap B) \cup (A \cap \overline{C}) \\ &= (A \cap B) \cup (A - C),\end{aligned}$$

as desired. ∎

Section 4.6: Proofs Involving Cartesian Products of Sets

4.41. Let A and B be sets. Then $A \times B = B \times A$ if and only if $A = B$ or one of A and B is empty.

Proof First, we show that if $A = B$ or one of A and B is empty, then $A \times B = B \times A$. If $A = B$, then certainly $A \times B = B \times A$; while if one of A and B is empty, say $A = \emptyset$, then $A \times B = \emptyset \times B = \emptyset = B \times \emptyset = B \times A$.

For the converse, assume that A and B are nonemptysets with $A \neq B$. Since $A \neq B$, at least one of A and B is not a subset of the other, say $A \not\subseteq B$. Then there is an element $a \in A$ such that $a \notin B$. Since $B \neq \emptyset$, there exists an element $b \in B$. Then $(a, b) \in A \times B$ but $(a, b) \notin B \times A$. Hence $A \times B \neq B \times A$. ∎

4.43. ***Proof*** First, assume that $A \times C \subseteq B \times C$. We show that $A \subseteq B$. Let $a \in A$. Since $C \neq \emptyset$, there exists $c \in C$ and so $(a, c) \in A \times C$. Since $A \times C \subseteq B \times C$, it follows that $(a, c) \in B \times C$ and so $a \in B$.

For the converse, assume that $A \subseteq B$. We show that $A \times C \subseteq B \times C$. Let $(a, c) \in A \times C$. Then $a \in A$ and $c \in C$. Since $A \subseteq B$, it follows that $a \in B$. Thus $(a, c) \in B \times C$, as desired. ∎

4.45. ***Proof*** We first show that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and $y \in B \cap C$. Thus $y \in B$ and $y \in C$. Thus $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Therefore, $(x, y) \in (A \times B) \cap (A \times C)$.

It remains to show that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. Let $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ and $(x, y) \in A \times C$. So $x \in A$, $y \in B$, and $y \in C$. Hence $y \in B \cap C$ and so $(x, y) \in A \times (B \cap C)$. ∎

4.47. ***Proof*** Let $(x, y) \in (A \times B) \cup (C \times D)$. Then $(x, y) \in A \times B$ or $(x, y) \in C \times D$. Assume, without loss of generality, that $(x, y) \in A \times B$. Thus $x \in A$ and $y \in B$. This implies that $x \in A \cup C$ and $y \in B \cup D$. Therefore, $(x, y) \in (A \cup C) \times (B \cup D)$. ∎

EXERCISES FOR CHAPTER 5

Section 5.1: Counterexamples

5.1. Let $a = b = -1$. Then $\log(ab) = \log 1 = 0$ but $\log(a)$ and $\log(b)$ are not defined. Thus $a = b = -1$ is a counterexample.

5.3. If $n = 3$, then $(2n^2 + 1) = 19$. Since $3 \nmid 19$, it follows that $n = 3$ is a counterexample.

5.5. If $a = 1$ and $b = 2$, then $(a + b)^3 = 3^3 = 27$, but $a^3 + 2a^2b + 2ab + 2ab^2 + b^3 = 1 + 4 + 4 + 8 + 8 = 25$. Thus $a = 1$ and $b = 2$ form a counterexample.

Section 5.2: Proof by Contradiction

5.7. ***Proof*** Assume, to the contrary, that there exists a largest negative rational number r. Thus $r = a/b$, where $a, b \in \mathbf{Z}$ and $b \neq 0$. Consider $r/2 = a/2b$. Since $a, 2b \in \mathbf{Z}$ and $2b \neq 0$, the number $r/2$ is rational. Because $r < r/2 < 0$, this contradicts r being the largest negative rational number. ∎

(Note: The fact that $r/2$ is a rational number may be sufficiently clear that this does not have to be verified.)

5.9. ***Proof*** Assume, to the contrary, that 200 can be written as the sum of an odd integer a and two even integers b and c. Then $a = 2x + 1$, $b = 2y$, and $c = 2z$, where $x, y, z \in \mathbf{Z}$. Thus

$$200 = a + b + c = (2x + 1) + 2y + 2z = 2(x + y + z) + 1.$$

Since $x + y + z \in \mathbf{Z}$, it follows that 200 is odd, which is a contradiction. ∎

5.11. ***Proof*** Let $a \geq 2$ and b be integers and assume, to the contrary, that $a \mid b$ and $a \mid (b + 1)$. So $b = ax$ and $b + 1 = ay$, where $x, y \in \mathbf{Z}$. Then $b + 1 = ax + 1 = ay$ and so $1 = ay - ax = a(y - x)$. Since $y - x$ is an integer, $a \mid 1$, which is a contradiction since $a \geq 2$. ∎

5.13. ***Proof*** Assume, to the contrary, that there exist an irrational number a and a nonzero rational number b such that ab is rational. Since b is a nonzero rational number, $b = r/s$, where $r, s \in \mathbf{Z}$ and $r, s \neq 0$. Then $ab = p/q$, where $p, q \in \mathbf{Z}$ and $q \neq 0$. Then $a = p/(bq) = (sp)/(rq)$. Since $sp, rq \in \mathbf{Z}$ and $rq \neq 0$, it follows that a is a rational number, which is a contradiction. ∎

5.15. Assume, to the contrary, that $ar + s$ and $ar - s$ are both rational. Then $(ar + s) + (ar - s) = 2ar$ is rational. Thus $2ar = p/q$, where $p, q \in \mathbf{Z}$ and $p, q \neq 0$. Then show that $a = p/(2qr)$ is rational, producing a contradiction.

5.17. Consider beginning as follows: Assume, to the contrary, that $a = \sqrt{2} + \sqrt{3}$ is a rational number. Then $a - \sqrt{2} = \sqrt{3}$. Squaring both sides, we obtain $a^2 - 2a\sqrt{2} + 2 = 3$ and so $\sqrt{2} = (a^2 - 1)/(2a)$. This will lead to $\sqrt{2}$ being rational, producing a contradiction.

5.19. ***Proof*** Let $t \in \mathbf{Q}$. Then $t = t + 0 \cdot \sqrt{2} = t + 0 \cdot \sqrt{3} \in S \cap T$. Hence $\mathbf{Q} \subseteq S \cap T$. We now show that $S \cap T \subseteq \mathbf{Q}$. Let x be an arbitrary element of $S \cap T$. Then there exist $p, q, r, s \in \mathbf{Q}$ such that $x = p + q\sqrt{2}$ and $x = r + s\sqrt{3}$. Thus $p + q\sqrt{2} = r + s\sqrt{3}$. Hence $p - r = s\sqrt{3} - q\sqrt{2}$. Squaring both sides, we obtain

$$(p - r)^2 = 3s^2 - 2sq\sqrt{6} + 2q^2.$$

If $sq \neq 0$, then

$$\sqrt{6} = \frac{(p - r)^2 - 3s^2 - 2q^2}{-2sq}$$

is a rational number. However, we saw in Exercise 5.18(a) that $\sqrt{6}$ is irrational. Thus $sq = 0$, implying that $s = 0$ or $q = 0$. In either case, $x \in \mathbf{Q}$. Thus $S \cap T \subseteq \mathbf{Q}$ and so $S \cap T = \mathbf{Q}$. ∎

5.21. ***Proof*** Assume to the contrary, that there exists a positive integer x such that $2x < x^2 < 3x$. Dividing these inequalities by (the positive integer) x, we obtain $2 < x < 3$. This is impossible since there is no integer between 2 and 3. ∎

5.23. Assume, to the contrary, that there exist odd integers x and y such that $x^2 + y^2 = z^2$, where $z \in \mathbf{Z}$. Then $x = 2a + 1$ and $y = 2b + 1$, where $a, b \in \mathbf{Z}$. Thus

$$\begin{aligned} x^2 + y^2 &= (2a + 1)^2 + (2b + 1)^2 = 4a^2 + 4a + 1 + 4b^2 + 4b + 1 \\ &= 4(a^2 + a + b^2 + b) + 2 = 2[2(a^2 + a + b^2 + b) + 1] = 2s, \end{aligned}$$

where $s = 2(a^2 + a + b^2 + b) + 1$ is an odd integer. If z is even, then $z = 2c$ for some integer c and so $z = 2(2c^2)$, where $2c^2$ is an even integer; while if z is odd, then z^2 is odd. Produce a contradiction in each case.

Section 5.3: A Review of Three Proof Techniques

5.25. (a) ***Proof*** Let n be an odd integer. Then $n = 2x + 1$ for some integer x. Thus

$$7n - 5 = 7(2x + 1) - 5 = 14x + 2 = 2(7x + 1).$$

Since $7x + 1$ is an integer, $7n - 5$ is even. ∎

(b) ***Proof*** Assume that $7n - 5$ is odd. Then $7n - 5 = 2x + 1$ for some integer x. Hence

$$\begin{aligned} n &= (8n - 5) - (7n - 5) = (8n - 5) - (2x + 1) \\ &= 8n - 2x - 6 = 2(4n - x - 3). \end{aligned}$$

Since $4n - x - 3$ is an integer, n is even. ∎

(c) ***Proof*** Assume, to the contrary, that there exists an odd integer n such that $7n - 5$ is odd. Thus $n = 2x + 1$ for some integer x. Thus

$$7n - 5 = 7(2x + 1) - 5 = 14x + 2 = 2(7x + 1).$$

Since $7x + 1$ is an integer, $7n - 5$ is even, producing a contradiction. ∎

5.27. This result can be proved using either a proof by contrapositive or a proof by contradiction.

Section 5.4: Existence Proofs

5.29. ***Proof*** For the rational number $a = 1$ and the irrational number $b = \sqrt{2}$, the number $1^{\sqrt{2}} = 1$ is rational. ∎

5.31. ***Proof*** Consider the irrational numbers $\sqrt{3}$ and $\sqrt{2}$. If $\sqrt{3}^{\sqrt{2}}$ is rational, then $a = \sqrt{3}$ and $b = \sqrt{2}$ have the desired properties. On the other hand, if $\sqrt{3}^{\sqrt{2}}$ is irrational, then

$$(\sqrt{3}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{3}^{\sqrt{2}\sqrt{2}} = \sqrt{3}^2 = 3$$

is rational. Thus $a = \sqrt{3}^{\sqrt{2}}$ and $b = \sqrt{2}$ have the desired properties. ∎

5.33. ***Proof*** Let $f(x) = x^3 + x^2 - 1$. Since f is a polynomial function, it is continuous on the set of all real numbers and so f is continuous on the interval $[2/3, 1]$. Because $f(2/3) = -7/27 < 0$ and $f(1) = 1 > 0$, it follows by the Intermediate Value Theorem of Calculus that there is a number c between $x = 2/3$ and $x = 1$ such that $f(c) = 0$. Hence c is a solution.

We now show that c is the unique solution of $f(x) = 0$ between $2/3$ and 1. Let c_1 and c_2 be solutions of $f(x) = 0$ between $2/3$ and 1. Then $c_1^3 + c_1^2 - 1 = 0$ and $c_2^3 + c_2^2 - 1 = 0$. Hence $c_1^3 + c_1^2 - 1 = c_2^3 + c_2^2 - 1$, implying that $c_1^3 + c_1^2 = c_2^3 + c_2^2$ and so

$$\begin{aligned} c_1^3 - c_2^3 + c_1^2 - c_2^2 &= (c_1 - c_2)(c_1^2 + c_1c_2 + c_2^2) + (c_1 - c_2)(c_1 + c_2) \\ &= (c_1 - c_2)(c_1^2 + c_1c_2 + c_2^2 + c_1 + c_2) = 0. \end{aligned}$$

Dividing by the positive number $c_1^2 + c_1c_2 + c_2^2 + c_1 + c_2$, we obtain $c_1 - c_2 = 0$ and so $c_1 = c_2$. ∎

Section 5.5: Disproving Existence Statements

5.35. We show that if a and b are odd integers, then $4 \nmid (3a^2 + 7b^2)$. Let a and b be odd integers. Then $a = 2x + 1$ and $b = 2y + 1$ for integers x and y. Then

$$\begin{aligned} 3a^2 + 7b^2 &= 3(2x + 1)^2 + 7(2y + 1)^2 = 3(4x^2 + 4x + 1) + 7(4y^2 + 4y + 1) \\ &= 12x^2 + 12x + 3 + 28y^2 + 28y + 7 = 4(3x^2 + 3x + 7y^2 + 7y + 2) + 2. \end{aligned}$$

Since 2 is the remainder when $3a^2 + 7b^2$ is divided by 4, it follows that $4 \nmid (3a^2 + 7b^2)$.

5.37. We show that if n is an integer, then

$$\begin{aligned} n^4 + n^3 + n^2 + n &= (n^4 + n^2) + (n^3 + n) = n^2(n^2 + 1) + n(n^2 + 1) \\ &= n(n + 1)(n^2 + 1) \end{aligned}$$

is even. Let $n \in \mathbf{Z}$. Then n is even or n is odd. We consider these two cases.

Case 1. *n is even.* Then $n = 2a$ for some integer a. Then

$$n^4 + n^3 + n^2 + n = n(n + 1)(n^2 + 1) = 2a(n + 1)(n^2 + 1) = 2[a(n + 1)(n^2 + 1)].$$

Since $a(n + 1)(n^2 + 1)$ is an integer, $n^4 + n^3 + n^2 + n$ is even.

Case 2. *n is odd.* Then $n = 2b + 1$ for some integer b and so $n + 1 = 2b + 2 = 2(b + 1)$. Thus

$$n^4 + n^3 + n^2 + n = n(n + 1)(n^2 + 1) = 2n(b + 1)(n^2 + 1) = 2[n(b + 1)(n^2 + 1)].$$

Since $n(b + 1)(n^2 + 1)$ is an integer, $n^4 + n^3 + n^2 + n$ is even.

EXERCISES FOR CHAPTER 6

Section 6.1: The Principle of Mathematical Induction

6.1. The sets in (b) and (d) are well-ordered.

6.3. ***Proof*** Let S be a nonempty set of negative integers. Let $T = \{n : -n \in S\}$. Hence T is a nonempty set of positive integers. By the Well-Ordering Principle, T has a least element m. Hence $m \leq n$ for all $n \in T$. Therefore, $-m \in S$ and $-m \geq -n$ for all $-n \in S$. Thus $-m$ is the largest element of S. ∎

6.5. ***Proof*** We use induction. Since $1 = 2 \cdot 1^2 - 1$, the formula holds for $n = 1$. Assume that the formula holds for some integer $k \geq 1$, that is,

$$1 + 5 + 9 + \cdots + (4k - 3) = 2k^2 - k.$$

We show that

$$1 + 5 + 9 + \cdots + [4(k + 1) - 3] = 2(k + 1)^2 - (k + 1).$$

Observe that

$$\begin{aligned}1+5+9+\cdots+[4(k+1)-3] &= [1+5+9+\cdots+(4k-3)]+4(k+1)-3\\ &= (2k^2-k)+(4k+1) = 2k^2+3k+1\\ &= 2(k+1)^2-(k+1).\end{aligned}$$

The result then follows by the Principle of Mathematical Induction. ∎

6.7. One possibility: $1+7+13+\cdots+(6n-5) = 3n^2-2n$.

6.9. ***Proof*** We proceed by induction. For $n=1$, we have $1\cdot 3 = 3 = \frac{1\cdot(1+1)(2\cdot 1+7)}{6}$, which is true. Assume that $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+k(k+2) = \frac{k(k+1)(2k+7)}{6}$, where $k\in\mathbf{N}$. We then show that

$$\begin{aligned}1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(k+1)(k+3) &= \frac{(k+1)(k+2)[2(k+1)+7]}{6}\\ &= \frac{(k+1)(k+2)(2k+9)}{6}.\end{aligned}$$

Observe that

$$\begin{aligned}&1\cdot 3+2\cdot 4+3\cdot 5+\cdots+(k+1)(k+3)\\ &= [1\cdot 3+2\cdot 4+3\cdot 5+\cdots+k(k+2)]+(k+1)(k+3)\\ &= \frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)\\ &= \frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6}\\ &= \frac{(k+1)(2k^2+7k+6k+18)}{6} = \frac{(k+1)(2k^2+13k+18)}{6}\\ &= \frac{(k+1)(k+2)(2k+9)}{6}.\end{aligned}$$

By the Principle of Mathematical Induction,

$$1\cdot 3+2\cdot 4+3\cdot 5+\cdots+n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

for every positive integer n. ∎

6.11. ***Proof*** We proceed by induction. Since $\frac{1}{3\cdot 4} = \frac{1}{3+9}$, the formula holds for $n=1$. Assume that

$$\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(k+2)(k+3)} = \frac{k}{3k+9},$$

where k is a positive integer. We show that

$$\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(k+3)(k+4)} = \frac{k+1}{3(k+1)+9} = \frac{k+1}{3(k+4)}.$$

Observe that

$$\begin{aligned}&\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(k+3)(k+4)}\\ &= \left[\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(k+2)(k+3)}\right]+\frac{1}{(k+3)(k+4)}\\ &= \frac{k}{3k+9}+\frac{1}{(k+3)(k+4)} = \frac{k(k+4)+3}{3(k+3)(k+4)}\\ &= \frac{k^2+4k+3}{3(k+3)(k+4)} = \frac{(k+1)(k+3)}{3(k+3)(k+4)}\\ &= \frac{k+1}{3(k+4)}.\end{aligned}$$

By the Principle of Mathematical Induction, $\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+2)(n+3)} = \frac{n}{3n+9}$ for every positive integer n. ∎

Section 6.2: A More General Principle of Mathematical Induction

6.13. ***Proof*** Since $1024 = 2^{10} > 10^3 = 1000$, the inequality holds when $n=10$. Assume that $2^k > k^3$, where $k\geq 10$ is an arbitrary integer. We show that $2^{k+1} > (k+1)^3$. Observe that

$$\begin{aligned}2^{k+1} &= 2\cdot 2^k > 2k^3 = k^3+k^3 \geq k^3+10k^2 = k^3+3k^2+7k^2\\ &> k^3+3k^2+7k = k^3+3k^2+3k+4k\\ &> k^3+3k^2+3k+1 = (k+1)^3.\end{aligned}$$

By the Principle of Mathematical Induction, $2^n > n^3$ for every integer $n\geq 10$. ∎

6.15. ***Proof*** We proceed by induction. Since $3^1 > 1^2$, the inequality holds for $n=1$. Assume that $3^k > k^2$, where k is a positive integer. We show that $3^{k+1} > (k+1)^2$. If $k=1$, then $3^{k+1} = 3^2 = 9 > 4 = (1+1)^2$. Thus we may assume $k\geq 2$. Observe that

$$\begin{aligned}3^{k+1} &= 3\cdot 3^k > 3k^2 = k^2+2k^2 = k^2+2k\cdot k \geq k^2+2k\cdot 2\\ &= k^2+4k = k^2+2k+2k \geq k^2+2k+4 > k^2+2k+1 = (k+1)^2.\end{aligned}$$

By the Principle of Mathematical Induction, $3^n > n^2$ for every positive integer n. ∎

6.17. ***Proof*** We proceed by induction. Since $(1+x)^1 = 1+1x$, the inequality holds when $n=1$. Assume that $(1+x)^k \geq 1+kx$, where k is an arbitrary positive integer. We show that

$$(1+x)^{k+1} \geq 1+(k+1)x.$$

Observe that

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx)$$

since $1+x>0$. Thus

$$(1+x)^{k+1} \geq (1+x)(1+kx) = 1+(k+1)x+kx^2 \geq 1+(k+1)x$$

since $kx^2\geq 0$. By the Principle of Mathematical Induction, $(1+x)^n \geq 1+nx$ for every positive integer n. ∎

6.19. ***Proof*** We proceed by induction. Since $81 \mid (10-10)$, the statement is true for $n=0$. Assume that $81 \mid (10^{k+1}-9k-10)$, where k is a nonnegative integer. We show that $81 \mid (10^{k+2}-9(k+1)-10)$. Since $81 \mid (10^{k+1}-9k-10)$, it follows that $10^{k+1}-9k-10 = 81x$, where $x\in\mathbf{Z}$. Thus $10^{k+1} = 9k+10+81x$. Therefore,

$$\begin{aligned}10^{k+2}-9(k+1)-10 &= 10\cdot 10^{k+1}-9k-19\\ &= 10(9k+10+81x)-9k-19\\ &= 81k+81+810x = 81(k+1+10x).\end{aligned}$$

Since $(k+1+10x)\in\mathbf{Z}$, it follows that $81 \mid (10^{k+2}-9(k+1)-10)$. By the Principle of Mathematical Induction, $81 \mid (10^{n+1}-9n-10)$ for every nonnegative integer n. ∎

6.21. **Lemma** Let $a\in\mathbf{Z}$. If $3 \mid 2a$, where $a\in\mathbf{Z}$, then $3 \mid a$.

Proof of Result We employ mathematical induction. By the lemma, the result holds for $n=1$. Assume for some positive integer k that if $3 \mid 2^k a$, then $3 \mid a$. We show that if $3 \mid 2^{k+1}a$, then $3 \mid a$. Assume that $3 \mid 2^{k+1}a$. Then $2^{k+1}a = 3x$ for some integer x. Observe that

$$2^{k+1}a = 2(2^k a) = 3x.$$

Since $3 \mid 2(2^k a)$, it follows by the lemma that $3 \mid 2^k a$. By the induction hypothesis, $3 \mid a$.

By the Principle of Mathematical Induction, it follows that for every positive integer n, if $3 \mid 2^n a$, then $3 \mid a$. ∎

6.23. (a) ***Proof*** We proceed by induction. Certainly, the statement is true for $m = 1$. Assume that for some positive integer k and any $2k$ integers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ for which $a_i \equiv b_i \pmod{n}$ for $1 \le i \le k$, we have $a_1 + a_2 + \cdots + a_k \equiv b_1 + b_2 + \cdots + b_k \pmod{n}$. Now let $c_1, c_2, \ldots, c_{k+1}$ and $d_1, d_2, \ldots, d_{k+1}$ be $2(k+1)$ integers such that $c_i \equiv d_i \pmod{n}$ for $1 \le i \le k+1$. Let $c = c_1 + c_2 + \cdots + c_k$ and $d = d_1 + d_2 + \cdots + d_k$. By the induction hypothesis, $c \equiv d \pmod{n}$. By Result 4.10, $c + c_{k+1} \equiv d + d_{k+1} \pmod{n}$. Thus $c_1 + c_2 + \cdots + c_{k+1} \equiv d_1 + d_2 + \cdots + d_{k+1} \pmod{n}$. The result then follows by the Principle of Mathematical Induction. ∎

(b) The proof of (b) is similar to the one in (a).

6.25. (a) ***Proof*** We use induction to prove that every set with n real numbers, where $n \in \mathbf{N}$, has a largest element. Certainly, the only element of a set with one element is the largest element of this set. Thus the statement is true for $n = 1$. Assume that every set with k real numbers, where $k \in \mathbf{N}$, has a largest element. We show that every set with $k+1$ real numbers has a largest element. Let $S = \{a_1, a_2, \ldots, a_{k+1}\}$ be a set with $k+1$ real numbers. Then the subset $T = \{a_1, a_2, \ldots, a_k\}$ of S has k real numbers. By the induction hypothesis, T has a largest element, say a. If $a \ge a_{k+1}$, then a is the largest element of S; otherwise, a_{k+1} is the largest element of S. In either case, S has a largest element.

By the Principle of Mathematical Induction, every finite nonempty set of real numbers has a largest element. ∎

(b) ***Proof*** Let S be a finite nonempty set of real numbers. Define $S' = \{x : -x \in S\}$. Since S' is also a finite nonempty set of real numbers, it follows by (a) that S' has a largest element y. Thus $y \ge x$ for all $x \in S'$. Therefore, $-y \in S$ and $-y \le -x$ for all $-x \in S$. So $-y$ is a smallest element of S. ∎

Section 6.3: Proof by Minimum Counterexample

6.27. ***Proof*** Assume, to the contrary, that there is a positive integer n such that $3 \nmid (2^{2n} - 1)$. Then there is a smallest positive integer n such that $3 \nmid (2^{2n} - 1)$. Let m be this integer. Since $3 \mid (2^2 - 1)$, it follows that $3 \mid (2^{2n} - 1)$ when $n = 1$ and so $m \ge 2$. Thus $m = k + 1$, where $1 \le k < m$. So $3 \mid (2^{2k} - 1)$. Hence $2^{2k} - 1 = 3x$ for some integer x and so $2^{2k} = 3x + 1$. Now

$$2^{2m} - 1 = 2^{2(k+1)} - 1 = 4 \cdot 2^{2k} - 1 = 4(3x + 1) - 1 = 3(4x + 1).$$

Since $4x + 1 \in \mathbf{Z}$, it follows that $3 \mid (2^{2m} - 1)$, producing a contradiction. ∎

6.29. ***Proof*** Certainly $5 \mid (n^5 - n)$ for $n = 0$. We now show that $5 \mid (n^5 - n)$ if n is a positive integer. Assume, to the contrary, that there is some positive integer n such that $5 \nmid (n^5 - n)$. Then there is a smallest positive integer n such that $5 \nmid (n^5 - n)$. Let m be this integer. Since $5 \mid (1^5 - 1)$, it follows that $m \ge 2$. So we can write $m = k + 1$, where $1 \le k < m$. Thus $5 \mid (k^5 - k)$ and so $k^5 - k = 5x$ for some integer x. Then

$$\begin{aligned} m^5 - m &= (k+1)^5 - (k+1) = k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\ &= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k = 5x + 5k^4 + 10k^3 + 10k^2 + 5k \\ &= 5(x + k^4 + 2k^3 + 2k^2 + k). \end{aligned}$$

Since $x + k^4 + 2k^3 + 2k^2 + k \in \mathbf{Z}$, it follows that $5 \mid (m^5 - m)$, which is a contradiction.

Suppose next that $n < 0$. Then $n = -p$, where $p \in \mathbf{N}$ and so $5 \mid (p^5 - p)$. Thus $p^5 - p = 5y$ for some integer y. Since

$$n^5 - n = (-p)^5 - (-p) = -(p^5 - p) = -(5y) = 5(-y)$$

and $-y \in \mathbf{Z}$, it follows that $5 \mid (n^5 - n)$. ∎

6.31. ***Proof*** Assume, to the contrary, that there is a positive integer n for which there is no subset S_n of S such that $\sum_{i \in S_n} i = n$. Let m be the smallest such integer. If we let $S_1 = \{1\}$, then $\sum_{i \in S_1} i = 1$. So $m \ge 2$. Thus m can be expressed as $m = k + 1$, where $1 \le k < m$. Consequently, there exists a subset S_k of S such that $\sum_{i \in S_k} i = k$. If $1 \notin S_k$, then $S_{k+1} = S_k \cup \{1\}$ has the desired property. Otherwise, there is a smallest positive integer t such that

$2^t \notin S_k$. Thus $2^0, 2^1, \ldots, 2^{t-1} \in S_k$. Since $2^0 + 2^1 + \cdots + 2^{t-1} = 2^t - 1$, it follows that if we let

$$S_{k+1} = (S_k \cup \{2^t\}) - \{2^0, 2^1, \ldots, 2^{t-1}\},$$

then $\sum_{i \in S_{k+1}} i = k + 1 = m$, producing a contradiction. ∎

Section 6.4: The Strong Principle of Mathematical Induction

6.33. **Conjecture** A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \ge 3$. Then $a_n = 2^{n-1}$ for every positive integer n.

Proof We proceed by the Strong Principle of Mathematical Induction. Since $a_1 = 1$, the conjecture is true for $n = 1$. Assume that $a_i = 2^{i-1}$ for every integer i with $1 \le i \le k$, where $k \in \mathbf{N}$. We show that $a_{k+1} = 2^k$. Since $a_{1+1} = a_2 = 2 = 2^1$, it follows that $a_{k+1} = 2^k$ for $k = 1$. Hence we may assume that $k \ge 2$. Thus

$$\begin{aligned} a_{k+1} &= a_k + 2a_{k-1} = 2^{k-1} + 2 \cdot 2^{k-2} = 2^{k-1} + 2^{k-1} \\ &= 2 \cdot 2^{k-1} = 2^k. \end{aligned}$$

The result then follows by the Strong Principle of Mathematical Induction. ∎

6.35. (a) The sequence $\{F_n\}$ is defined recursively by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$.

(b) ***Proof*** We proceed by the Strong Principle of Mathematical Induction. Since $F_1 = 1$ is odd and $3 \nmid 1$, it follows that $2 \mid F_1$ if and only if $3 \mid 1$ and the statement is true for $n = 1$. Assume that $2 \mid F_i$ if and only if $3 \mid i$ for every integer i with $1 \le i \le k$ and $k \in \mathbf{N}$. We show that $2 \mid F_{k+1}$ if and only if $3 \mid (k+1)$. Since $F_2 = F_{1+1} = 1$ and $3 \nmid 2$, the statement is true for $k = 1$. Hence we may assume that $k \ge 2$. We now consider three cases, according to whether $k + 1 = 3q$, $k + 1 = 3q + 1$, or $k + 1 = 3q + 2$ for some integer q.

Case 1. $k + 1 = 3q$. Thus $3 \nmid k$ and $3 \nmid (k - 1)$. By the inductive hypothesis, F_k and F_{k-1} are odd. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is even.

Case 2. $k + 1 = 3q + 1$. Thus $3 \mid k$ and $3 \nmid (k - 1)$. By the inductive hypothesis, F_k is even and F_{k-1} is odd. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is odd.

Case 3. $k + 1 = 3q + 2$. Thus $3 \nmid k$ and $3 \mid (k - 1)$. By the inductive hypothesis, F_k is odd and F_{k-1} is even. Since $F_{k+1} = F_k + F_{k-1}$, it follows that F_{k+1} is odd.

By the Strong Principle of Mathematical Induction, $2 \mid F_n$ if and only if $3 \mid n$ for every positive integer n. ∎

6.37. ***Proof*** We use the Strong Principle of Mathematical Induction. Since $12 = 3 \cdot 4 + 7 \cdot 0$, the statement is true when $n = 12$. Assume for an integer $k \ge 12$ that for every integer i with $12 \le i \le k$, there exist nonnegative integers a and b such that $i = 3a + 7b$. We show that there exist nonnegative integers x and y such that $k + 1 = 3x + 7y$. Since $13 = 3 \cdot 2 + 7 \cdot 1$ and $14 = 3 \cdot 0 + 7 \cdot 2$, we may assume that $k \ge 14$. Since $k - 2 \ge 12$, there exist nonnegative integers c and d such that $k - 2 = 3c + 7d$. Hence $k + 1 = 3(c + 1) + 7d$. By the Strong Principle of Mathematical Induction, for each integer $n \ge 12$, there are nonnegative integers a and b such that $n = 3a + 7b$. ∎

EXERCISES FOR CHAPTER 7

Section 7.2: Revisiting Quantified Statements

7.1. (a) Let S be the set of all odd integers and let $P(n) : 3n + 1$ is even. $\forall n \in S$, $P(n)$.

(b) ***Proof*** Let $n \in S$. Then $n = 2k + 1$ for some integer k. Thus $3n + 1 = 3(2k + 1) + 1 = 6k + 4 = 2(3k + 2)$. Since $3k + 2$ is an integer, $3n + 1$ is even. ∎

7.3. (a) Let $P(n) : n^{n-1}$ is even. $\forall n \in \mathbf{N}$, $P(n)$.

(b) Note that $P(1)$ is false and so the statement in (a) is false.

7.5. (a) Let $P(m, n) : n < m < 2n$. $\forall n \in \mathbf{N} - \{1\}$, $\exists m \in \mathbf{Z}$, $P(m, n)$.

(b) ***Proof*** Let $n \ge 2$ be an integer and let $m = n + 1$. Since $n \ge 2$, it follows that $n < n + 1 = m < n + 2 \le n + n = 2n$. ∎

7.7. **(a)** Let $P(m, n)$: $(n-2)(m-2) > 0$. $\forall n \in \mathbf{Z}, \exists m \in \mathbf{Z}, P(m, n)$.
(b) $\exists n \in \mathbf{Z}, \forall m \in \mathbf{Z}, \sim P(m, n)$.
(c) Let $n = 2$. Then $(n-2)(m-2) = 0 \cdot (m-2) = 0$ for all $m \in \mathbf{N}$.

7.9. **(a)** Let $P(a, b, x)$: $|bx| < a$ and $Q(a, b) : |b| < a$. $\forall a \in \mathbf{N}, \exists b \in \mathbf{Z}, (Q(a, b) \wedge (\forall x \in \mathbf{R}, P(a, b, x)))$.
(b) ***Proof*** Let $a \in \mathbf{N}$ and let $b = 0$. Then $|bx| = 0 < a$ for every real number x. ∎

7.11. **(a)** Let $P(x, y, n)$: $x^2 + y^2 \geq n$. $\exists n \in \mathbf{Z}, \forall x, y \in \mathbf{R}, P(x, y, n)$.
(b) ***Proof*** Let $n = 0$. Then for every two real numbers x and y, $x^2 + y^2 \geq 0 = n$. ∎

7.13. **(a)** Let $P(a, b, n)$: $a < \frac{1}{n} < b$. $\exists a, b \in \mathbf{Z}, \forall n \in \mathbf{N}, P(a, b, n)$.
(b) ***Proof*** Let $a = 0$ and $b = 2$. Then for every $n \in \mathbf{N}$, $a = 0 < \frac{1}{n} < 2 = b$. ∎

7.15. **(a)** Let S be the set of odd integers and $P(a, b, c)$: abc is odd. $\forall a, b, c \in S, P(a, b, c)$.
(b) Let a, b, and c be odd integers. Then $a = 2x + 1$, $b = 2y + 1$, and $c = 2z + 1$, where $x, y, z \in \mathbf{Z}$. Then show that $abc = (2x + 1)(2y + 1)(2z + 1)$ is odd.

Section 7.3: Testing Statements

7.17. The statement is true.
Proof Since each of the following statements
$P(1) \Rightarrow Q(1)$: If 7 is prime, then 5 is prime.
$P(2) \Rightarrow Q(2)$: If 2 is prime, then 7 is prime.
$P(3) \Rightarrow Q(3)$: If 28 is prime, then 9 is prime.
$P(4) \Rightarrow Q(4)$: If 8 is prime, then 11 is prime.
is true, $\forall n \in S$, $P(n) \Rightarrow Q(n)$ is true. ∎

7.19. This statement is false. Let $x = 1$. Then $4x + 7 = 11$ is odd and $x = 1$ is odd. Thus $x = 1$ is a counterexample.

7.21. This statement is true.
Proof Let x be an even integer. Then $x = 2n$ for some integer n. Observe that $x = (2n + 1) + (-1)$. Since n is an integer, $2n + 1$ is odd. Since -1 is odd as well, both $2n + 1$ and -1 are odd. ∎

7.23. This statement is false. Let $A = \{1, 2, 3\}$ and $B = \{2, 3\}$. Then $A \cup B = \{1, 2, 3\}$ and $(A \cup B) - B = \{1\} \neq A$. Consequently, $A = \{1, 2, 3\}$ and $B = \{2, 3\}$ constitute a counterexample.

7.25. The statement is true.
Proof Consider the integer 35. Then $3 + 5 = 8$ is even and $3 \cdot 5 = 15$ is odd. ∎

7.27. The statement is false. Let $x = 3$ and $y = -1$. Then $|x + y| = |3 + (-1)| = |2| = 2$ and $|x| + |y| = |3| + |-1| = 3 + 1 = 4$. Thus $|x + y| \neq |x| + |y|$. So $x = 3$ and $y = -1$ is a counterexample.

7.29. The statement is false. We show that there is no real number x such that $x^2 < x < x^3$.
Suppose that there is a real number x such that $x^2 < x < x^3$. Since $x^2 \geq 0$, it follows that $x > 0$. Dividing $x^2 < x < x^3$ by x, we have $x < 1 < x^2$. Thus $0 < x < 1$ and $x^2 > 1$, which is impossible. ∎

7.31. The statement is true. For $a = 0$, any two real numbers b and $c \neq 0$ satisfy the equality.

7.33. The statement is false. Note that $x^4 + x^2 + 1 \geq 1 > 0$ for every $x \in \mathbf{R}$.

7.35. The statement is false. Neither of the expressions $\frac{x^3+x}{x^4-1}$ or $\frac{x}{x^2-1}$ is defined when $x = 1$ or $x = -1$.

7.37. The statement is false. Let $x = 6$ and $y = 4$. Then $z = 2$.

7.39. The statement is true.
Proof Assume that $A - B = \emptyset$ for every set B. Let $B = \emptyset$. Then $A - B = A - \emptyset = A = \emptyset$. ∎

7.41. The statement is true.
Proof Let A be a nonempty set. Let $B = A$. Then $A - B = B - A = \emptyset$. So $|A - B| = |B - A| = 0$. ∎

7.43. The statement is false. Observe that $4 = 1 + 3$.

7.45. The statement is true. Consider $c = 1$ and $d = 2b + 1$.

7.47. The statement is true. For each even integer n, $n = n + 0$.

7.49. The statement is false. Consider $A = \{1\}$, $B = \{2\}$, and $C = D = \{1, 2\}$.

7.51. The statement is true. Let $a = \sqrt{2}$ and $b = 1$.

7.53. The statement is true. Consider the set $B = S - A$.

7.55. The statement is false. Let $A = \{1\}$ and $B = \{2\}$. Then $\{1, 2\} \in \mathcal{P}(A \cup B)$ but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

7.57. The statement is false. Consider $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1\}$.

7.59. The statement is true. Observe that at least two of a, b, and c are of the same parity, say a and b are of the same parity. Then $a + b$ is even.

7.61. The statement is false. Consider $a = 2$ and $c = 1$.

7.63. The statement is false. Consider $n = 1$.

7.65. The statement is true. Let $x = 51$ and $y = 50$. Then $x^2 = (51)^2 = (50 + 1)^2 = (50)^2 + 2 \cdot 50 + 1$.

7.67. The statement is true.
Proof Let p be an odd prime. Then $p = 2k + 1$ for some $k \in \mathbf{N}$. For $a = k + 1$ and $b = k$, $a^2 - b^2 = (k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = p$. ∎

EXERCISES FOR CHAPTER 8

Section 8.1: Relations

8.1. $\text{dom } R = \{a, b\}$ and $\text{ran } R = \{s, t\}$.

8.3. Since $A \times A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ and $|A \times A| = 4$, the number of subsets of $A \times A$ and hence the number of relations on A is $2^4 = 16$. Four of these 16 relations are $\emptyset$, $A \times A$, $\{(0, 0)\}$, and $\{(0, 0), (0, 1), (1, 0)\}$.

Section 8.2: Properties of Relations

8.5. The relation R is reflexive and transitive. Since $(a, d) \in R$ and $(d, a) \notin R$, it follows that R is not symmetric.

8.7. The relation R is transitive but neither reflexive nor symmetric.

8.9. The relation R is reflexive and symmetric. Observe that $3 \; R \; 1$ and $1 \; R \; 0$ but $3 \not R \; 0$. Thus R is not transitive.

8.11. The relation R is symmetric and transitive but not reflexive.

8.13. The relation R is reflexive and symmetric. Observe that $-1 \; R \; 0$ and $0 \; R \; 2$ but $-1 \not R \; 2$. Thus R is not transitive.

Section 8.3: Equivalence Relations

8.15. ***Proof*** Since $a^3 = a^3$ for each $a \in \mathbf{Z}$, it follows that $a \; R \; a$ and R is reflexive. Let $a, b \in \mathbf{Z}$ such that $a \; R \; b$. Then $a^3 = b^3$ and so $b^3 = a^3$. Thus $b \; R \; a$ and R is symmetric. Let $a, b, c \in \mathbf{Z}$ such that $a \; R \; b$ and $b \; R \; c$. Thus $a^3 = b^3$ and $b^3 = c^3$. Hence $a^3 = c^3$ and so $a \; R \; c$ and R is transitive. ∎
Let $a, b \in \mathbf{Z}$. Note that $a^3 = b^3$ if and only if $a = b$. Thus $[a] = \{a\}$ for every $a \in \mathbf{Z}$.

8.17. There are three distinct equivalence classes, namely $[1] = \{1, 5\}$, $[2] = \{2, 3, 6\}$, and $[4] = \{4\}$.

8.19. ***Proof*** Assume that $a \; R \; b$, $c \; R \; d$, and $a \; R \; d$. Since $a \; R \; b$ and R is symmetric, $b \; R \; a$. Similarly, $d \; R \; c$. Because $b \; R \; a$, $a \; R \; d$, and R is transitive, $b \; R \; d$. Finally, since $b \; R \; d$ and $d \; R \; c$, it follows that $b \; R \; c$, as desired. ∎

Section 8.4: Properties of Equivalence Classes

8.21. Let $R = \{(v, v), (w, w), (x, x), (y, y), (z, z), (v, w), (w, v), (x, y), (y, x)\}$. Then $[v] = \{v, w\}$, $[x] = \{x, y\}$, and $[z] = \{z\}$ are three distinct equivalence classes.

8.23. Observe that $2 \; R \; 6$ and $6 \; R \; 3$, but $2 \not R \; 3$. Thus R is not transitive, and so R is not an equivalence relation.

8.25. ***Proof*** Let $x \in \mathbf{Z}$. Since $3x - 7x = -4x = 2(-2x)$ and $-2x$ is an integer, $3x - 7x$ is even. Thus $x \; R \; x$ and R is reflexive.
Next, we show that R is symmetric. Let $x \; R \; y$, where $x, y \in \mathbf{Z}$. Thus $3x - 7y$ is even and so $3x - 7y = 2a$ for some integer a. Observe that

$$3y - 7x = (3x - 7y) - 10x + 10y = 2a - 10x + 10y = 2(a - 5x + 5y).$$

Since $a - 5x + 5y$ is an integer, $3y - 7x$ is even. So $y \; R \; x$ and R is symmetric.
Finally, we show that R is transitive. Assume that $x \; R \; y$ and $y \; R \; z$, where $x, y, z \in \mathbf{Z}$. Then $3x - 7y$ and $3y - 7z$ are even. So $3x - 7y = 2a$ and $3y - 7z = 2b$, where $a, b \in \mathbf{Z}$. Adding these two equations, we obtain

$$(3x - 7y) + (3y - 7z) = 3x - 4y - 7z = 2a + 2b$$

and so $3x - 7z = 2a + 2b + 4y = 2(a + b + 2y)$. Since $a + b + 2y$ is an integer, $3x - 7z$ is even. Therefore, $x \; R \; z$ and R is transitive. ∎
There are two distinct equivalence classes, namely, $[0] = \{0, \pm 2, \pm 4, \ldots\}$ and $[1] = \{\pm 1, \pm 3, \pm 5, \ldots\}$.

8.27. For the set $S = \{1, 2, 3\}$, let

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\} \text{ and } R_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}.$$

Then R_1 and R_2 are equivalence relations on S, but

$$R = R_1 \cup R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

is not an equivalence relation on S. For example, $(1, 2), (2, 3) \in R$ but $(1, 3) \notin R$, so R is not transitive.

Section 8.5: Congruence Modulo n

8.29. ***Proof*** Let $a \in \mathbf{Z}$. Since $3a + 5a = 8a$, it follows that $8 \mid (3a + 5a)$ and so $3a + 5a \equiv 0 \pmod 8$. Hence $a \; R \; a$ and R is reflexive.

Next, we show that R is symmetric. Assume that $a \; R \; b$, where $a, b \in \mathbf{Z}$. Then $3a + 5b \equiv 0 \pmod 8$, that is, $3a + 5b = 8k$ for some integer k. Observe that $(3a + 5b) + (3b + 5a) = 8a + 8b$. Thus

$$3b + 5a = 8a + 8b - (3a + 5b) = 8a + 8b - 8k = 8(a + b - k).$$

Since $a + b - k$ is an integer, $8 \mid (3b + 5a)$ and so $3b + 5a \equiv 0 \pmod 8$. Hence $b \; R \; a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a \; R \; b$ and $b \; R \; c$, where $a, b, c \in \mathbf{Z}$. Thus $3a + 5b \equiv 0 \pmod 8$ and $3b + 5c \equiv 0 \pmod 8$. So $3a + 5b = 8x$ and $3b + 5c = 8y$, where $x, y \in \mathbf{Z}$. Observe that

$$(3a + 5b) + (3b + 5c) = 3a + 8b + 5c = 8x + 8y.$$

Thus $3a + 5c = 8x + 8y - 8b = 8(x + y - b)$. Since $x + y - b$ is an integer, $8 \mid (3a + 5c)$ and $3a + 5c \equiv 0 \pmod 8$. Therefore, $a \; R \; c$ and R is transitive. ∎

8.31. There are two distinct equivalence classes, namely $[0] = \{0, \pm 2, \pm 4, \ldots\}$ and $[1] = \{\pm 1, \pm 3, \pm 5, \ldots\}$.

8.33. ***Proof*** Let $a \in \mathbf{Z}$. Since $5a - 2a = 3a$, it follows that $3 \mid (5a - 2a)$ and so $5a \equiv 2a \pmod 3$. Hence $a \; R \; a$ and R is reflexive.

Next, we show that R is symmetric. Assume that $a \; R \; b$, where $a, b \in \mathbf{Z}$. Then $5a \equiv 2b \pmod 3$, that is, $5a - 2b = 3k$ for some integer k. Observe that $(5a - 2b) + (5b - 2a) = 3a + 3b$. Thus

$$5b - 2a = 3a + 3b - (5a - 2b) = 3a + 3b - 3k = 3(a + b - k).$$

Since $a + b - k$ is an integer, $3 \mid (5b - 2a)$ and so $5b \equiv 2a \pmod 3$. Hence $b \; R \; a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a \; R \; b$ and $b \; R \; c$, where $a, b, c \in \mathbf{Z}$. Thus $5a \equiv 2b \pmod 3$ and $5b \equiv 2c \pmod 3$. So $5a - 2b = 3x$ and $5b - 2c = 3y$, where $x, y \in \mathbf{Z}$. Observe that

$$(5a - 2b) + (5b - 2c) = (5a - 2c) + 3b = 3x + 3y.$$

Thus $5a - 2c = 3x + 3y - 3b = 3(x + y - b)$. Since $x + y - b$ is an integer, $3 \mid (5a - 2c)$ and $5a \equiv 2c \pmod 3$. Therefore, $a \; R \; c$ and R is transitive. ∎

There are three distinct equivalence classes, namely

$[0] = \{0, \pm 3, \pm 6, \ldots\}$,

$[1] = \{\ldots, -5, -2, 1, 4, \ldots\}$, and

$[2] = \{\ldots, -4, -1, 2, 5, \ldots\}$.

8.35. ***Proof*** First, we show that R is reflexive. Let $a \in \mathbf{Z}$. Since $2a + 3a = 5a$, it follows that $5 \mid (2a + 3a)$ and so $a \; R \; a$. Hence R is reflexive.

Next, we show that R is symmetric. Assume that $a \; R \; b$, where $a, b \in \mathbf{Z}$. Then $2a + 3b \equiv 0 \pmod 5$. Hence $2a + 3b = 5k$ for some integer k. Observe that $(2a + 3b) + (2b + 3a) = 5a + 5b$. Thus

$$2b + 3a = 5a + 5b - (2a + 3b) = 5a + 5b - 5k = 5(a + b - k).$$

Since $a + b - k$ is an integer, $5 \mid (2b + 3a)$ and so $2b + 3a \equiv 0 \pmod 5$. Hence $b \; R \; a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a \; R \; b$ and $b \; R \; c$, where $a, b, c \in \mathbf{Z}$. Thus $2a + 3b \equiv 0 \pmod 5$ and $2b + 3c \equiv 0 \pmod 5$. So $2a + 3b = 5x$ and $2b + 3c = 5y$, where $x, y \in \mathbf{Z}$.

Observe that

$$(2a + 3b) + (2b + 3c) = 2a + 5b + 3c = 5x + 5y.$$

Thus $2a + 3c = 5x + 5y - 5b = 5(x + y - b)$. Since $x + y - b$ is an integer, $5 \mid (2a + 3c)$ and $2a + 3c \equiv 0 \pmod 5$. Therefore, $a \; R \; c$ and R is transitive. ∎

The distinct equivalence classes are [0], [1], [2], [3], and [4]. In fact, the set of distinct equivalence classes is $\mathbf{Z}_5$.

Section 8.6: The Integers Modulo n

8.37. The addition and multiplication tables in $\mathbf{Z}_4$ are shown below.

+	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

·	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

8.39. **(a)** $[7] + [5] = [12] = [1]$.
(b) $[7] \cdot [5] = [35] = [2]$.
(c) $[-82] + [207] = [6] + [9] = [4]$.
(d) $[-82] \cdot [207] = [6] \cdot [9] = [10]$.

8.41. ***Proof*** Let $[a], [b], [c], [d] \in \mathbf{Z}_n$, where $[a] = [b]$ and $[c] = [d]$. We prove that $[ac] = [bd]$. Since $[a] = [b]$, it follows that $a \; R \; b$, where R is the relation defined in Theorem 8.6. Similarly, $c \; R \; d$. Therefore, $a \equiv b \pmod n$ and $c \equiv d \pmod n$. Thus, $n \mid (a - b)$ and $n \mid (c - d)$. Hence, there exist integers x and y so that

$$a - b = nx \text{ and } c - d = ny.$$

Thus $a = nx + b$ and $c = ny + d$ and so $ac = (nx + b)(ny + d) = nxny + nxd + bny + bd$. Hence

$$ac - bd = nxny + nxd + bny = n(nxy + xd + by).$$

This implies that $n \mid (ac - bd)$. Thus, $ac \equiv bd \pmod n$. From this, we conclude that $ac \; R \; bd$, which implies that $[ac] = [bd]$. ∎

EXERCISES FOR CHAPTER 9

Section 9.1: The Definition of Function

9.1. $\text{dom } f = \{a, b, c, d\}$ and $\text{ran } f = \{y, z\}$.

9.3. Since R is an equivalence relation, R is reflexive. So $(a, a) \in R$ for every $a \in A$. Since R is also a function from A to A, we must have $R = \{(a, a) : a \in A\}$ and so R is the identity function on A.

9.5. Let $A' = \{a \in A : (a, b) \in R \text{ for some } b \in B\}$. Furthermore, for each element $a' \in A'$, select exactly one element $b' \in \{b \in B : (a', b) \in R\}$. Then $f = \{(a', b') : a' \in A'\}$ is a function from A' to B.

Section 9.2: The Set of All Functions From A to B

9.7. $B^A = \{f_1, f_2, \ldots, f_8\}$, where $f_1 = \{(1, x), (2, x), (3, x)\}$, $f_2 = \{(1, x), (2, x), (3, y)\}$, $f_3 = \{(1, x), (2, y), (3, x)\}$, $f_4 = \{(1, y), (2, x), (3, x)\}$. By interchanging x and y in f_1, f_2, f_3, f_4, we obtain f_5, f_6, f_7, f_8.

9.9. For $A = \{a, b, c\}$ and $B = \{0, 1\}$, there are 8 different functions from A to B, namely

$f_1 = \{(a, 0), (b, 0), (c, 0)\}$, $f_2 = \{(a, 0), (b, 0), (c, 1)\}$,
$f_3 = \{(a, 0), (b, 1), (c, 0)\}$, $f_4 = \{(a, 0), (b, 1), (c, 1)\}$,
$f_5 = \{(a, 1), (b, 0), (c, 0)\}$, $f_6 = \{(a, 1), (b, 0), (c, 1)\}$,
$f_7 = \{(a, 1), (b, 1), (c, 0)\}$, $f_8 = \{(a, 1), (b, 1), (c, 1)\}$.

Section 9.3: One-to-one and Onto Functions

9.11. Let $f = \{(w, r), (x, r), (y, r), (z, s)\}$. Since $f(w) = f(x) = r$ and t is not an image of any element of A, it follows that f is neither one-to-one nor onto.

9.13. The function f is injective, but not surjective. There is no $n \in \mathbf{Z}$ such that $f(n) = 2$.

9.15. The function f is injective but not surjective. There is no $n \in \mathbf{Z}$ such that $f(n) = 5$.

9.17. **(a)** Since $f(0) = f(-4)$, it follows that f is not one-to-one.

(b) Note that $f(x) = (x + 2)^2 + 5 \geq 5$, so f is not onto. For example, there is no $x \in \mathbf{R}$ such that $f(x) = 4$.

9.19. **(a)** Define $f(n) = n$ for all $n \in \mathbf{N}$.

(b) Define $f(n) = 2n$ for all $n \in \mathbf{N}$.

(c) Define $f(1) = 1$ and $f(n) = n - 1$ for each integer $n \geq 2$.

(d) Define $f(n) = 1$ for all $n \in \mathbf{N}$.

Section 9.4: Bijective Functions

9.21. ***Proof*** We first show that f is one-to-one. Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R} - \{2\}$. Then $\frac{5a+1}{a-2} = \frac{5b+1}{b-2}$. Multiplying both sides by $(a - 2)(b - 2)$, we obtain $(5a + 1)(b - 2) = (5b + 1)(a - 2)$. Simplifying, we have $5ab - 10a + b - 2 = 5ab - 10b + a - 2$. Subtracting $5ab - 2$ from both sides, we have $-10a + b = -10b + a$. Thus $11a = 11b$ and so $a = b$. Therefore, f is one-to-one.

To show that f is onto, let $r \in \mathbf{R} - \{5\}$. We show that there exists $x \in \mathbf{R} - \{2\}$ such that $f(x) = r$. Choose $x = \frac{2r+1}{r-5}$. Then $x \in \mathbf{R} - \{2\}$ and

$$f(x) = f\left(\frac{2r+1}{r-5}\right) = \frac{5\left(\frac{2r+1}{r-5}\right) + 1}{\frac{2r+1}{r-5} - 2} = \frac{5(2r+1) + (r-5)}{(2r+1) - 2(r-5)} = \frac{11r}{11} = r,$$

implying that f is onto. Therefore f is bijective. ∎

9.23. **(a)** Consider $S = \{2, 5, 6\}$. Observe that for each $y \in B$, there exists $x \in S$ such that x is related to y. This says that $\gamma(R) \leq 3$. On the other hand, let $S' \subseteq A$ such that for every element y of B, there is an element $x \in S'$ such that x is related to y. Observe that S' must contain 6, at least one of 2 and 3, and at least one of 4, 5 and 7. Thus $|S'| \geq 3$. Therefore, $\gamma(R) = 3$.

(b) If R is an equivalence relation defined on a finite nonempty set A, then $\gamma(R)$ is the number of distinct equivalence classes of R.

(c) If f is a bijective function from A to B, then $\gamma(f) = |A|$.

9.25. ***Proof*** We first show that f is one-to-one. Let $a, b \in A$ such that $f(a) = f(b)$. Now

$$a = i_A(a) = (f \circ f)(a) = f(f(a)) = f(f(b))$$
$$= (f \circ f)(b) = i_A(b) = b.$$

Thus f is one-to-one.

Next, we show that f is onto. Let $c \in A$. Suppose that $f(c) = d \in A$. Observe that

$$f(d) = f(f(c)) = (f \circ f)(c) = i_A(c) = c.$$

Thus f is onto. ∎

Section 9.5: Composition of Functions

9.27. $(g \circ f)(1) = g(f(1)) = g(4) = 17$ and $(f \circ g)(1) = f(g(1)) = f(2) = 13$.

9.29. **(a)** The statement is true. This is Corollary 9.8.

(b) The statement is false. Let $A = \{1, 2\}$, $B = \{a, b\}$, and $C = \{x, y\}$; and let $f : A \to B$ and $g : B \to C$ be defined by $f = \{(1, a), (2, a)\}$ and $g = \{(a, x), (b, y)\}$. Then $g \circ f = \{(1, x), (2, x)\}$. Thus g is onto but $g \circ f$ is not.

(c) The statement is false. Consider the functions f and g in (b).

(d) The statement is true.

Proof Let $A = \{1, 2\}$, $B = \{a, b, c\}$, and $C = \{x, y\}$; and let $f : A \to B$ and $g : B \to C$ be defined by $f = \{(1, a), (2, b)\}$ and $g = \{(a, x), (b, y), (c, y)\}$. Then $g \circ f = \{(1, x), (2, y)\}$ is onto but f is not onto. ∎

(e) The statement is false. We show that for functions $f : A \to B$ and $g : B \to C$, if f is not one-to-one, then $g \circ f : A \to C$ is not one-to-one.

Since f is not one-to-one, there exist $a, b \in A$ such that $a \neq b$ and $f(a) = f(b)$. Thus $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ f)(b)$ and so $g \circ f$ is not one-to-one.

9.31. **(a)** ***Proof*** Let $(x, y) \in A \times A$. Then $x = 4a$ and $y = 4b$, where $a, b \in \mathbf{Z}$. Since $f(x, y) = xy = (4a)(4b) = 2(8ab)$ and $8ab \in \mathbf{Z}$, it follows that $f(x, y) \in B'$ and so $g \circ f$ is defined. ∎

(b) $(g \circ f)(4k, 4\ell) = g(f(4k, 4\ell)) = g(16k\ell) = 8k\ell$.

Section 9.6: Inverse Functions

9.33. ***Proof*** First, we show that f is one-to-one. Assume that $f(a) = f(b)$, where $a, b \in \mathbf{R}$. Then $4a - 3 = 4b - 3$. Adding 3 to both sides and dividing by 4, we obtain $a = b$. Next we show that f is onto. Let $r \in \mathbf{R}$. Then $(r + 3)/4 \in \mathbf{R}$. Therefore, $f(\frac{r+3}{4}) = 4(\frac{r+3}{4}) - 3 = r$.

Note that $f^{-1}(x) = (x + 3)/4$ for $x \in \mathbf{R}$. ∎

9.35. **(a)** ***Proof*** Let $f(a) = f(b)$, where $a, b \in \mathbf{R}$. Then $2a + 3 = 2b + 3$. Adding -3 to both side and dividing by 2, we have $a = b$ and so f is one-to-one. Let $r \in \mathbf{R}$. Letting $x = (r - 3)/2$, we have

$$f(x) = 2\left(\frac{r-3}{2}\right) + 3 = (r - 3) + 3 = r$$

and so f is onto. ∎

(b) The proof is similar to that in (a).

(c) $(g \circ f)(x) = -6x - 4$.

(d) $f^{-1}(x) = \frac{x-3}{2}$ and $g^{-1}(x) = \frac{5-x}{3}$.

(e) $(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x) = -(x + 4)/6$.

9.37. **(a)** The statement is false. Let $A = \{1, 2\}$, $B = \{x, y\}$, and $C = \{r, s\}$. Define $f = \{(1, x), (2, x)\}$, $g = \{(x, r), (y, r)\}$, and $h = \{(x, r), (y, s)\}$. Then $g \circ f = \{(1, r), (2, r)\} = h \circ f$ but $g \neq h$.

(b) The statement is false. Let $A = \{1\}$, $B = \{x, y\}$, and $C = \{r, s\}$. Define $f = \{(1, x)\}$, $g = \{(x, r), (y, r)\}$, and $h = \{(x, r), (y, s)\}$. Then f is one-to-one, $g \circ f = \{(1, r)\} = h \circ f$ but $g \neq h$.

Section 9.7: Permutations

9.39. **(a)** $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 3 & 5 & 2 \end{pmatrix}$ and $\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 2 & 6 & 1 & 3 \end{pmatrix}$.

(b) $\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 6 & 2 & 1 \end{pmatrix}$ and $\beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 5 & 1 & 6 \end{pmatrix}$.

EXERCISES FOR CHAPTER 10

Section 10.2: Denumerable Sets

10.1. ***Proof*** Since A and B are denumerable, the sets A and B can be expressed as

$$A = \{a_1, a_2, a_3, \ldots\} \text{ and } B = \{b_1, b_2, b_3, \ldots\}.$$

The function $f : \mathbf{N} \to A \cup B$ defined by

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \cdots \end{array}$$

is bijective. Therefore, $A \cup B$ is denumerable. ∎

10.3. **(a)** $1 + \sqrt{2}, (4 + \sqrt{2})/2, (9 + \sqrt{2})/3$.

(b) ***Proof*** Assume that $f(a) = f(b)$, where $a, b \in \mathbf{N}$. Then $\frac{a^2+\sqrt{2}}{a} = \frac{b^2+\sqrt{2}}{b}$. Multiplying by ab, we obtain $a^2b + \sqrt{2}b = ab^2 + \sqrt{2}a$. Thus $a^2b - ab^2 + \sqrt{2}b - \sqrt{2}a = ab(a - b) - \sqrt{2}(a - b) = (a - b)(ab - \sqrt{2}) = 0$. Thus $a = b$ or $ab = \sqrt{2}$. Since $ab \in \mathbf{N}$ and $\sqrt{2}$ is irrational, $ab \neq \sqrt{2}$. Therefore, $a = b$ and f is one-to-one. ∎

(c) ***Proof*** Let $x \in S$. Then $x = (n^2 + \sqrt{2})/n$ for some $n \in \mathbf{N}$. Then $f(n) = x$. ∎

(d) Yes, since $\mathbf{N}$ is denumerable and $f : \mathbf{N} \to S$ is a bijection by (b) and (c).

10.5. Since A is denumerable, $A = \{a_1, a_2, \ldots\}$. Observe that

$$A \times B = \{(a_1, x), (a_1, y), (a_2, x), (a_2, y), \ldots\}.$$

10.7. Note that S is an infinite subset of the set $\mathbf{N} \times \mathbf{N}$. The result follows by Theorem 10.3 and Result 10.5.

10.9. Construct a table (as shown below), where the set $\{i, j\}$ with $i < j$ occurs in row j, column i.

	1	2	3	4	⋯
1					
2	$\{1, 2\}$				
3	$\{1, 3\}$	$\{2, 3\}$			
4	$\{1, 4\}$	$\{2, 4\}$	$\{3, 4\}$		
⋮	⋮	⋮	⋮		

10.11. Since the sets $A_1, A_2, A_3, \ldots$ are denumerable sets, we can write $A_i = \{a_{i1}, a_{i2}, a_{i3}, \ldots\}$ for each $i \in \mathbf{N}$. Construct a table where a_{ij} is in row i, column j.

10.13. Since $\mathbf{Z} - \{2\}$ is an infinite subset of the denumerable set $\mathbf{Z}$, it follows by Theorem 10.3 that $\mathbf{Z} - \{2\}$ is denumerable and so $|\mathbf{Z}| = |\mathbf{Z} - \{2\}|$.

Section 10.3: Uncountable Sets

10.15. ***Proof*** Denote the set of irrational numbers by $\mathbf{I}$. Assume, to the contrary, that $\mathbf{I}$ is denumerable. Since $\mathbf{Q}$ and $\mathbf{I}$ are disjoint denumerable sets, $\mathbf{Q} \cup \mathbf{I}$ is denumerable by Exercise 10.1. Since $\mathbf{Q} \cup \mathbf{I} = \mathbf{R}$, it follows that $\mathbf{R}$ is denumerable, which is a contradiction. ∎

10.17. ***Proof*** Consider the function $f : (0, 2) \to \mathbf{R}$ defined by

$$f(x) = \frac{1 - x}{x(x - 2)}$$

for all $x \in (0, 2)$. First, we show that f is one-to-one. Let $f(a) = f(b)$, where $a, b \in (0, 2)$. Then

$$\frac{1 - a}{a^2 - 2a} = \frac{1 - b}{b^2 - 2b}.$$

Multiplying both sides by $(a^2 - 2a)(b^2 - 2b)$ and simplifying, we obtain

$$(a - b)(a + b - ab - 2) = 0.$$

We claim that $a = b$. Assume, to the contrary, that $a \neq b$. We may assume that $a > b$. Then $a + b - ab - 2 = 0$. Since $a + b - ab - 2 = (a - 1)(1 - b) - 1 = 0$, it follows that $(a - 1)(1 - b) = 1$. Thus $a \neq 1$. If $a < 1$, then $b < a < 1$ and so $(a - 1)(1 - b) < 0$, which is impossible. Thus $a > 1$ and $b < 1$. Since $1 < a < 2$ and $0 < b < 1$, it follows that $0 < a - 1 < 1$ and $0 < 1 - b < 1$. However then, $(a - 1)(1 - b) < 1$, producing a contradiction. Thus $a = b$, as claimed, and f is one-to-one.

Next we show that f is onto. Let $r \in \mathbf{R}$. Since $f(1) = 0$, we may assume that $r \neq 0$. For $r \neq 0$, let $x = \frac{2r - 1 + \sqrt{4r^2 + 1}}{2r}$ (obtained from the quadratic formula). Then $0 < x < 1$ if $r < 0$ and $1 < x < 2$ if $r > 0$. It follows that $f(x) = r$ and so f is onto. ∎

Section 10.4: Comparing Cardinalities of Sets

10.19. **(a)** False. For example, $|\mathcal{P}(\mathbf{R})| > |\mathbf{R}|$.

(b) False. $|\mathbf{Q}| \neq |\mathbf{R}|$.

(c) True.

Proof Since A is denumerable and $A \subseteq B$, the set B is infinite. Since B is an infinite subset of the denumerable set C, it follows that B is denumerable. ∎

(d) True. Consider the function $f : \mathbf{N} \to S$ defined by $f(n) = \sqrt{2}/n$. The function f is bijective.

(e) True. (See (d).)

(f) False. Consider $\mathbf{R}$.

(g) False. The function $f : \mathbf{N} \to \mathbf{R}$ defined by $f(n) = n$ is injective but $|\mathbf{N}| \neq |\mathbf{R}|$.

10.21. The cardinalities of these sets are the same. Consider $f : [0, 1] \to [1, 3]$ defined by $f(x) = 2x + 1$ for all $x \in [0, 1]$.

10.23. Let $b \in B$. Then the function $f : A \to A \times B$ defined by $f(a) = (a, b)$ for each $a \in A$ is one-to-one. Thus $|A| \leq |A \times B|$.

Section 10.5: The Schröder–Bernstein Theorem

10.25. ***Proof*** Since $(0, 1) \subseteq [0, 1]$, the identity function $i : (0, 1) \to [0, 1]$ defined by $i(x) = x$ is an injective function. The function $f : [0, 1] \to (0, 1)$ defined by $f(x) = \frac{1}{2}x + \frac{1}{4}$ is also injective. It then follows by the Schröder–Bernstein Theorem that $|(0, 1)| = |[0, 1]|$. ∎

10.27. **(a)** ***Proof*** We use induction on n. Since $f(k) = 4k = 4^1k$ for all $k \in \mathbf{Z}$, the result holds for $n = 1$. Assume that $f^m(k) = 4^m k$ for all $k \in \mathbf{Z}$, where m is a positive integer. We show that $f^{m+1}(k) = 4^{m+1}k$. Observe that

$$f^{m+1}(k) = f(f^m(k)) = f(4^m k) = 4(4^m k) = 4^{m+1}k.$$

The result then follows by the Principle of Mathematical Induction. ∎

(b) $B' = \{f^n(x) : x \text{ is odd}, n \in \mathbf{N}\} = \{4^n x : x \text{ is odd}, n \in \mathbf{N}\}$.
$C = \{x : x \text{ is odd}\} \cup B' = \{x : x \text{ is odd}\} \cup \{4^n x : x \text{ is odd}, n \in \mathbf{N}\} = \{4^n x : x \text{ is odd}, n \in \mathbf{N} \cup \{0\}\}$.
$D = 2\mathbf{Z} - B' = 2\mathbf{Z} - \{4^n x : x \text{ is odd}, n \in \mathbf{N}\} = \{2^{2t-1}x : x \text{ is odd}, t \in \mathbf{N}\}$.
The function f_1 is the restriction of f to C. Thus $f_1 : C \to B'$ is defined by $f_1(x) = 4x$ for $x \in \{4^n y : y \text{ is odd}, n \in \mathbf{N} \cup \{0\}\}$.
The function $h : C \cup D \to B' \cup D$ is defined by

$$h(x) = \begin{cases} f_1(x) & \text{if } x \in C \\ i_D(x) & \text{if } x \in D \end{cases} = \begin{cases} 4x & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

EXERCISES FOR CHAPTER 11

Section 11.1: Divisibility Properties of Integers

11.1. ***Proof*** Assume that $a \mid b$ and $c \mid d$. Then $b = ax$ and $d = cy$ for integers x and y. Then $ad + bc = a(cy) + (ax)c = ac(y + x)$. Since $y + x$ is an integer, $ac \mid (ad + bc)$. ∎

11.3. ***Proof*** Assume that $ac \mid bc$. Then $bc = (ac)x = c(ax)$ for some integer x. Since $c \neq 0$, we can divide by c, obtaining $b = ax$. So $a \mid b$. ∎

11.5. ***Proof*** Assume, to the contrary, that there exists a prime $n \geq 3$ that can be expressed as $k^3 + 1 \geq 3$ for some integer k. Since $n = k^3 + 1 = (k + 1)(k^2 - k + 1)$, it follows that $k + 1 = 1$ or $k^2 - k + 1 = 1$, which implies that $k = 0$ or $k = 1$. Thus $n = 1$ or $n = 2$, which is a contradiction. ∎

11.7. ***Proof*** We employ induction. For $n = 1$, we have $5^{2 \cdot 1} + 7 = 32$ and $8 \mid 32$. Thus the result is true for $n = 1$. Assume that

$$8 \mid (5^{2k} + 7)$$

for some positive integer k. We show that

$$8 \mid (5^{2(k+1)} + 7).$$

Since $8 \mid (5^{2k} + 7)$, it follows that $5^{2k} + 7 = 8a$ for some integer a and so $5^{2k} = 8a - 7$. Thus

$$\begin{aligned} 5^{2(k+1)} + 7 &= 5^2 \cdot 5^{2k} + 7 = 25(8a - 7) + 7 \\ &= 200a - 175 + 7 = 200a - 168 = 8(25a - 21). \end{aligned}$$

Since $25a - 21$ is an integer, $8 \mid (5^{2(k+1)} + 7)$. The result then follows by the Principle of Mathematical Induction. ∎

11.9. Consider the n numbers $2 + (n+1)!, 3 + (n+1)!, \ldots, n + (n+1)!, (n+1) + (n+1)!$. Observe for $2 \le k \le n+1$ that k divides $k + (n+1)!$. Thus these n numbers are composite.

11.11. ***Proof*** We employ induction. By Theorem 11.2, if a and x are integers such that $d \mid a$, then $d \mid ax$. Thus the statement is true for $n = 1$. Assume for some positive integer k, that if $a_1, a_2, \ldots, a_k$ and $x_1, x_2, \ldots, x_k$ are $2k \ge 2$ integers such that $d \mid a_i$ for all i $(1 \le i \le k)$, then $d \mid \sum_{i=1}^{k} a_i x_i$. Let $b_1, b_2, \ldots, b_{k+1}$ and $y_1, y_2, \ldots, y_{k+1}$ be $2(k+1)$ integers such that $d \mid b_i$ for all i $(1 \le i \le k+1)$. Let $b = \sum_{i=1}^{k} b_i y_i$. By the induction hypothesis, $d \mid b$. By Theorem 11.2, $d \mid b_{k+1} y_{k+1}$. Again by Theorem 11.2, $d \mid (b + b_{k+1} y_{k+1})$. Thus $d \mid \sum_{i=1}^{k+1} b_i y_i$. The result then follows by the Principle of Mathematical Induction. ∎

Section 11.2: The Division Algorithm

11.13. **(a)** $125 = 17 \cdot 7 + 6$ $(q = 7, r = 6)$
(b) $125 = (-17) \cdot (-7) + 6$ $(q = -7, r = 6)$
(c) $96 = 8 \cdot 12 + 0$ $(q = 12, r = 0)$
(d) $96 = (-8) \cdot (-12) + 0$ $(q = -12, r = 0)$
(e) $-17 = 22 \cdot (-1) + 5$ $(q = -1, r = 5)$
(f) $-17 = (-22) \cdot 1 + 5$ $(q = 1, r = 5)$
(g) $0 = 15 \cdot 0 + 0$ $(q = 0, r = 0)$
(h) $0 = (-15) \cdot 0 + 0$ $(q = 0, r = 0)$

11.15. ***Proof*** Let a be an odd integer. Then $a = 2b + 1$ for some integer b. Thus

$$a^2 = (2b+1)^2 = 4b^2 + 4b + 1 = 4(b^2 + b) + 1.$$

Since $k = b^2 + b$ is an integer, $a = 4k + 1$. ∎

11.17. **Result** The square of an integer that is not a multiple of 5 is either of the form $5k + 1$ or $5k + 4$ for some integer k.
Proof Let n be an integer that is not a multiple of 5. Then $a = 5q + r$ for some integers q and r with $1 \le r \le 4$. We consider these four cases.
Case 1. $n = 5q + 1$. Then

$$n^2 = (5q+1)^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1,$$

where $5q^2 + 2q \in \mathbf{Z}$.
(The other three cases are handled similarly.) ∎

11.19. **(a)** ***Proof*** Let p be an odd prime. Then $p = 2a + 1$ for some integer a. We consider two cases, depending on whether a is even or a is odd.
Case 1. *a is even.* Then $a = 2k$, where $k \in \mathbf{Z}$. Thus $p = 2a + 1 = 2(2k) + 1 = 4k + 1$.
Case 2. *a is odd.* Then $a = 2k + 1$, where $k \in \mathbf{Z}$. Thus $p = 2a + 1 = 2(2k + 1) + 1 = 4k + 3$. ∎
(b) ***Proof*** Let $p \ge 5$ be an odd prime. Then $p = 2a + 1$ for some integer a. We consider three cases, depending on whether $a = 3k$, $a = 3k + 1$, $a = 3k + 2$ or some integer k.
Case 1. $a = 3k$. Then $p = 2a + 1 = 2(3k) + 1 = 6k + 1$.
Case 2. $a = 3k + 1$. Then $p = 2a + 1 = 2(3k + 1) + 1 = 6k + 3 = 3(2k + 1)$. Since $2k + 1$ is an integer, $3 \mid p$, which is impossible as $p \ge 5$ is a prime. Thus this case cannot occur.
Case 3. $a = 3k + 2$. Then $p = 2a + 1 = 2(3k + 2) + 1 = 6k + 5$. ∎

11.21. **(a)** Observe that $n = 6q + 5 = 3(2q) + 3 + 2 = 3(2q + 1) + 2$. Letting $k = 2q + 1$, we see that $n = 3k + 2$.
(b) The converse is false. The integer $2 = 3 \cdot 0 + 2$ is of the form $3k + 2$, but 2 is not of the form $6q + 5$ since $6q + 5 = 2(3q + 2) + 1$ is always odd.

11.23. ***Proof*** Assume that an even number of a, b, and c are congruent to 1 modulo 3. We consider two cases.
Case 1. *None of a, b, and c is congruent to* 1 *modulo* 3. We consider two subcases.
Subcase 1.1. *At least one of a, b, and c is congruent to* 0 *modulo* 3, *say* $a \equiv 0 \pmod 3$. Then $a = 3q$ for some integer q. Thus $abc = 3qbc$. Since $qbc \in \mathbf{Z}$, it follows that $3 \mid abc$ and $abc \equiv 0 \pmod 3$. Hence $abc \not\equiv 1 \pmod 3$.

Subcase 1.2. *None of a, b, and c is congruent to* 0 *modulo* 3. Then all of a, b, and c are congruent to 2 modulo 3. By Result 4.11, $ab \equiv 1 \pmod 3$. Applying Result 4.11 again, we have $abc \equiv 2 \pmod 3$ and so $abc \not\equiv 1 \pmod 3$.
Case 2. *Exactly two of a, b, and c are congruent to* 1 *modulo* 3, *say a and b are congruent to* 1 *modulo* 3 *and c is not congruent to* 1 *modulo* 3. (The proof is similar to that of Case 1.) ∎

11.25. **(a)** ***Proof*** Let $S_k = \{a_1, a_2, \ldots, a_k\}$ for each integer k with $1 \le k \le n$. For each integer k $(1 \le k \le n)$, $\sum_{i=1}^{k} a_i \equiv r \pmod n$ for some integer r, where $0 \le r \le n - 1$. We consider two cases.
Case 1. $\sum_{i=1}^{k} a_i \equiv 0 \pmod n$ *for some integer k*. Then $n \mid \sum_{i=1}^{k} a_i$, that is, n divides the sum of the elements of S_k.
Case 2. $\sum_{i=1}^{k} a_i \not\equiv 0 \pmod n$ *for all integers k* $(1 \le k \le n)$. Hence there exist integers s and t with $1 \le s < t \le n$ such that $\sum_{i=1}^{s} a_i \equiv r \pmod n$ and $\sum_{i=1}^{t} a_i \equiv r \pmod n$ for an integer r with $1 \le r \le n - 1$. Therefore,

$$\sum_{i=1}^{s} a_i \equiv \sum_{i=1}^{t} a_i \pmod n$$

and so

$$n \mid \left(\sum_{i=1}^{t} a_i - \sum_{i=1}^{s} a_i \right).$$

Hence

$$n \mid \sum_{i=s+1}^{t} a_i,$$

that is, n divides the sum of the elements of the set $T = \{a_{s+1}, a_{s+2}, \ldots, a_t\}$. ∎
(b) No, except it would be better not to use the word "set". Show, for every n integers $a_1, a_2, \ldots, a_n$, distinct or not, that n divides the sum of some k of them $(1 \le k \le n)$.

Section 11.4: The Euclidean Algorithm

11.27. **(a)** $\gcd(51, 288) = 3 = 51 \cdot (17) + 288 \cdot (-3)$
(b) $\gcd(357, 629) = 17 = 357 \cdot (-7) + 629 \cdot 4$
(c) $\gcd(180, 252) = 36 = 180 \cdot 3 + 252 \cdot (-2)$

11.29. ***Proof*** Assume first that n is a linear combination of a and b. Thus $n = as + bt$ for some integers s and t. Since $d = \gcd(a, b)$, it follows that $d \mid a$ and $d \mid b$. By Result 11.2, $d \mid (as + bt)$ and so $d \mid n$.
For the converse, assume that $d \mid n$. Then $n = dc$ for some integer c. Since $d = \gcd(a, b)$, it follows by Theorem 11.7 that $d = ax + by$ for some integers x and y. Therefore,

$$n = dc = (ax + by)c = a(xc) + b(yc).$$

Since xc and yc are integers, n is a linear combination of a and b. ∎

11.31. ***Proof*** Since $d = \gcd(a, b)$, it follows by Theorem 11.7 that $d = as + bt$ for some integers s and t. Thus

$$d = as + bt = (a_1 d)s + (b_1 d)t = d(a_1 s + b_1 t).$$

Dividing both sides by d, we obtain $a_1 s + b_1 t = 1$. It then follows by Theorem 11.12 that $\gcd(a_1, b_1) = 1$. ∎

Section 11.5: Relatively Prime Integers

11.33. **(a)** Consider $a = 4$ and $b = c = 2$. **(b)** Consider $a = b = c = 2$.

11.35. ***Proof*** Assume, to the contrary, that $\sqrt{6}$ is rational. The $\sqrt{6} = a/b$, where $a, b \in \mathbf{Z}$ and $b \ne 0$. Furthermore, we may assume that $\gcd(a, b) = 1$. Hence $6 = a^2/b^2$ and $a^2 = 6b^2 = 2(3b^2)$. Since $3b^2$ is an integer, a^2 is even. By Theorem 3.12, a is even. Thus $a = 2c$ for some integer c. Hence $a^2 = (2c)^2 = 4c^2 = 6b^2$ and so

$2c^2 = 3b^2$. Since c^2 is an integer, $2 \mid 3b^2$. Since $\gcd(2, 3) = 1$, it follows by Theorem 11.13 that $2 \mid b^2$. By Theorem 3.12, b is even. This contradicts our assumption that a/b has been reduced to lowest terms. ∎

11.37. ***Proof*** We give a proof by contrapositive. Hence we show that if $p \geq 2$ is an integer that is not a prime, then there exist two integers a and b such that $p \mid ab$ but $p \nmid a$ and $p \nmid b$. Assume that p is not a prime. Then there exist two integers a and b such that $1 < a < p$, $1 < b < p$, and $p = ab$. Thus $p \mid ab$. Since $a < p$ and $b < p$, it follows that $p \nmid a$ and $p \nmid b$. ∎

11.39. **(a)** False. Consider $n = 3$.

(b) True since $(-3)(2n + 1) + 2(3n + 2) = 1$. ∎

11.41. **(a)** ***Proof*** Let (a, b, c) be a Pythagorean triple. Then $a^2 + b^2 = c^2$. Therefore, $(an)^2 + (bn)^2 = a^2n^2 + b^2n^2 = (a^2 + b^2)n^2 = c^2n^2 = (cn)^2$. Thus (an, bn, cn) is a Pythagorean triple. ∎

(b) ***Proof*** Assume, to the contrary, that ab is odd. So a and b are both odd. Then $a = 2x + 1$ and $b = 2y + 1$, where $x, y \in \mathbf{Z}$. Observe that

$$a^2 + b^2 = (2x + 1)^2 + (2y + 1)^2 = 4x^2 + 4x + 1 + 4y^2 + 4y + 1.$$

Thus $c^2 = 4x^2 + 4x + 4y^2 + 4y + 2 = 2(2x^2 + 2x + 2y^2 + 2y + 1)$. Since $2x^2 + 2x + 2y^2 + 2y + 1 \in \mathbf{Z}$, it follows that c^2 is even and so c is even. Let $c = 2z$, where $z \in \mathbf{Z}$. Thus

$$2 = (2z)^2 - (4x^2 + 4x + 4y^2 + 4y) = 4z^2 - (4x^2 + 4x + 4y^2 + 4y) = 4(z^2 - x^2 - x - y^2 - y).$$

This implies that $4 \mid 2$, which is a contradiction. ∎

(c) ***Proof*** Assume, to the contrary, that a and b are of the same parity. By (b), ab is even and so at least one of a and b is even. By our assumption then, a and b are both even. Thus $\gcd(a, b) \geq 2$, which is a contradiction. ∎

11.43. ***Proof*** Assume that $ac \equiv bc \pmod{n}$ and $\gcd(c, n) = 1$. Thus $n \mid (ac - bc)$ and so $n \mid c(a - b)$. Since $\gcd(c, n) = 1$, it follows by Theorem 11.13 that $n \mid (a - b)$. Hence $a \equiv b \pmod{n}$. ∎

Section 11.6: The Fundamental Theorem of Arithmetic

11.45. **(a)** $4725 = 3^3 \cdot 5^2 \cdot 7$ **(b)** $9702 = 2 \cdot 3^2 \cdot 7^2 \cdot 11$ **(c)** $180625 = 5^4 \cdot 17^2$.

11.47. **(a)** $4278 = 2 \cdot 3 \cdot 23 \cdot 31$ and $71929 = 11 \cdot 13 \cdot 503$.

(b) $\gcd(4278, 71929) = 1$

EXERCISES FOR CHAPTER 12

Section 12.1: Limits of Sequences

12.1. ***Proof*** Let $\epsilon > 0$ be given. Choose $N = \lceil 1/2\epsilon \rceil$ and let $n > N$. Thus $n > 1/2\epsilon$ and so $|\frac{1}{2n} - 0| = \frac{1}{2n} < \epsilon$. ∎

12.3. ***Proof*** Let $\epsilon > 0$ be given. Choose $N = \max(1, \lceil \log_2(\frac{1}{\epsilon}) \rceil)$ and let $n > N$. Thus $n > \log_2(\frac{1}{\epsilon})$, and so $2^n > 1/\epsilon$ and $1/2^n < \epsilon$. Therefore, $|(1 + \frac{1}{2^n}) - 1| = \frac{1}{2^n} < \epsilon$. ∎

12.5. There exists a real number $\epsilon > 0$ such that for each positive integer N, there exists an integer $n > N$ such that $|a_n - L| \geq \epsilon$.

Let $P(L, \epsilon, n) : |a_n - L| \geq \epsilon$. $\forall L \in \mathbf{R}, \exists \epsilon \in \mathbf{R}^+, \forall N \in \mathbf{N}, \exists n \in \mathbf{N}, n > N$, $P(L, \epsilon, n)$.

12.7. ***Proof*** Let M be a positive number. Choose $N = \lceil \sqrt[3]{M} \rceil$ and let n be any integer such that $n > N$. Hence $n > \sqrt[3]{M}$ and so $n^3 > M$. Thus $\frac{n^5 + 2n}{n^2} = n^3 + \frac{2}{n} > n^3 > M$. ∎

Section 12.2: Infinite Series

12.9. Let $s_n = \sum_{i=1}^{n} \frac{1}{2^i}$ for each integer $n \geq 1$.

(a) $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, $s_3 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$.

Conjecture $s_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbf{N}$.

(b) ***Proof*** We proceed by induction. Since $s_1 = \frac{1}{2} = 1 - \frac{1}{2^1}$, the formula s_n holds for $n = 1$. Thus the statement is true for $n = 1$. Assume that $s_k = 1 - \frac{1}{2^k}$ for a positive integer k. We show that $s_{k+1} = 1 - \frac{1}{2^{k+1}}$.

Observe that

$$\sum_{i=1}^{k+1} \frac{1}{2^i} = \left(\sum_{i=1}^{k} \frac{1}{2^i}\right) + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$
$$= 1 - \left(\frac{1}{2^k} - \frac{1}{2^{k+1}}\right) = 1 - \frac{2 - 1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}.$$

By the Principle of Mathematical Induction, $s_n = 1 - \frac{1}{2^n}$ for all $n \in \mathbf{N}$. ∎

(c) The proof that $\lim_{n \to \infty}(1 - \frac{1}{2^n}) = 1$ is similar to the one in Exercise 12.3. ∎

Section 12.3: Limits of Functions

12.11. ***Proof*** Let $\epsilon > 0$ be given and choose $\delta = 2\epsilon/3$. Let $x \in \mathbf{R}$ such that $0 < |x - 2| < \delta = 2\epsilon/3$. Thus $|(\frac{3}{2}x + 1) - 4| = |\frac{3}{2}x - 3| = \frac{3}{2}|x - 2| < \frac{3}{2} \cdot \frac{2\epsilon}{3} = \epsilon$. ∎

12.13. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 8x + 15} = -2$.

Proof For a given $\epsilon > 0$, choose $\delta = \min(1, \epsilon/3)$. Let $x \in \mathbf{R}$ such that $0 < |x - 3| < \delta \leq 1$. Thus $2 < x < 4$ and so $|x - 5| > 1$. Hence $\frac{1}{|x-5|} < 1$. Observe that

$$\frac{x^2 - 2x - 3}{x^2 - 8x + 15} - (-2) = \frac{x^2 - 2x - 3}{x^2 - 8x + 15} + 2 = \frac{(x^2 - 2x - 3) + 2(x^2 - 8x + 15)}{x^2 - 8x + 15}$$
$$= \frac{3x^2 - 18x + 27}{x^2 - 8x + 15} = \frac{3(x^2 - 6x + 9)}{x^2 - 8x + 15}$$
$$= \frac{3(x - 3)^2}{(x - 3)(x - 5)} = \frac{3(x - 3)}{(x - 5)}.$$

Thus $|(\frac{x^2-2x-3}{x^2-8x+15}) - (-2)| = \frac{3|x-3|}{|x-5|} < 3|x - 3| < 3(\epsilon/3) = \epsilon$. ∎

12.15. ***Proof*** Let $\epsilon > 0$ be given and choose $\delta = \min(1, \epsilon/19)$. Let $x \in \mathbf{R}$ such that $0 < |x - 2| < \delta = \min(1, \epsilon/19)$. Since $|x - 2| < 1$, it follows that $-1 < x - 2 < 1$ and so $1 < x < 3$. Thus $|x^2 + 2x + 4| < 19$. Because $|x - 2| < \epsilon/19$, it follows that $|x^3 - 8| = |x - 2||x^2 + 2x + 4| < |x - 2| \cdot 19 < (\epsilon/19) \cdot 19 = \epsilon$. ∎

12.17. $\lim_{x \to 1} \frac{1}{5x-4} = 1$.

Proof For a given $\epsilon > 0$, choose $\delta = \min(1/10, \epsilon/10)$. Let $x \in \mathbf{R}$ such that $0 < |x - 1| < \delta$. Since $|x - 1| < \delta \leq \frac{1}{10}$, it follow that $\frac{9}{10} < x < \frac{11}{10}$ and so $\frac{1}{2} < 5x - 4 < \frac{3}{2}$. Hence $|5x - 4| > \frac{1}{2}$ and $\frac{1}{|5x-4|} < 2$. Therefore,

$$\left|\frac{1}{5x - 4} - 1\right| = \left|\frac{-5x + 5}{5x - 4}\right| = \frac{5|x - 1|}{|5x - 4|} < 10|x - 1| < 10\frac{\epsilon}{10} = \epsilon.$$

∎

12.19. **(a)** $\lim_{x \to 3} f(x)$ does not exist.

Proof Assume, to the contrary, that $\lim_{x \to 3} f(x)$ exists. Then $\lim_{x \to 3} f(x) = L$ for some real number L. Let $\epsilon = 1/2$. Then there exists $\delta > 0$ such that if $x \in \mathbf{R}$ and $0 < |x - 3| < \delta$, then $|f(x) - L| < \epsilon = \frac{1}{2}$. If $0 < x - 3 < \delta$, then $f(x) = 2$. So $|2 - L| < \frac{1}{2}$. Thus $L > 1.5$. If $-\delta < x - 3 < 0$, then $f(x) = 1$ and $|1 - L| < \frac{1}{2}$. So $L < 1.5$. Since $1.5 < L < 1.5$, this is a contradiction. ∎

(b) $\lim_{x \to \pi} f(x) = 2$.

Proof Let $\epsilon > 0$ be given. Choose $\delta = .1$. Let $x \in \mathbf{R}$ such that $0 < |x - \pi| < \delta$. Then $x > \pi - .1 > 3$. Thus $f(x) = 2$ and so $|f(x) - 2| = 0 < \epsilon$. ∎

Section 12.4: Fundamental Properties of Limits of Functions

12.21. By Theorem 12.23,

$$\lim_{x \to a}(f_1(x) + f_2(x)) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) = L_1 + L_2$$

and so the result is true for $n = 2$. Assume that if $g_1, g_2, \ldots, g_k$ are k functions, where $k \geq 2$, such that $\lim_{x\to a} g_i(x) = L_i$ for $1 \leq i \leq k$, then

$$\lim_{x\to a}(g_1(x) + g_2(x) + \cdots + g_k(x)) = L_1 + L_2 + \cdots + L_k.$$

Let $f_1, f_2, \ldots, f_{k+1}$ be $k+1$ functions such that $\lim_{x\to a} f_i(x) = M_i$ for $1 \leq i \leq k+1$. We show that

$$\lim_{x\to a}(f_1(x) + f_2(x) + \cdots + f_{k+1}(x)) = M_1 + M_2 + \cdots + M_{k+1}.$$

Observe that

$$f_1(x) + f_2(x) + \cdots + f_{k+1}(x) = [f_1(x) + f_2(x) + \cdots + f_k(x)] + f_{k+1}(x).$$

We can use Theorem 12.23 and the induction hypothesis to obtain the desired result.

Section 12.5: Continuity

12.23. ***Proof*** We prove by induction on the degree n of a polynomial p that for every real number a, $\lim_{x\to a} p(x) = p(a)$. Suppose first that $n = 0$ and that p is a polynomial c of degree 0. Then p is a constant polynomial and $\lim_{x\to a} p(x) = \lim_{x\to a} c = c = p(a)$. Assume that the result is true for all polynomials of degree $k \geq 0$, and let p be a polynomial of degree $k+1$. Hence

$$p(x) = c_{k+1}x^{k+1} + c_k x^k + \cdots + c_1 x + c_0,$$

where $c_i \in \mathbf{R}$ for $0 \leq i \leq k+1$. Let $q(x) = c_k x^k + c_{k-1}x^{k-1} \cdots + c_1 x + c_0$. By the induction hypothesis, $\lim_{x\to a} q(x) = q(a)$. Also, $\lim_{x\to a} c_{k+1}x^{k+1} = c_{k+1}a^{k+1}$. By Theorem 12.23,

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}(c_{k+1}x^{k+1} + c_k x^k + \cdots + c_1 x + c_0)\\ &= \lim_{x\to a}(c_{k+1}x^{k+1} + q(x)) = \lim_{x\to a} c_{k+1}x^{k+1} + \lim_{x\to a} q(x)\\ &= c_{k+1}a^{k+1} + q(a) = p(a).\end{aligned}$$

The result then follows by the Principle of Mathematical Induction. ∎

12.25. Yes, define $f(3) = 2$. Then $\lim_{x\to 3} \frac{x^2-9}{x^2-3x} = 2$. (Use an argument similar to that in Result 12.15.)

12.27. We show that $\lim_{x\to 10} \sqrt{x-1} = f(10) = 3$.

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1, 5\epsilon)$. Let $x \in \mathbf{R}$ such that $0 < |x - 10| < \delta$. Since $|x - 10| < 1$, it follows that $9 < x < 11$ and so $\sqrt{x-1} + 3 > 5$. Therefore, $1/(\sqrt{x-1}+3) < 1/5$. Hence

$$\left|\sqrt{x-1} - 3\right| = \left|\frac{(\sqrt{x-1}-3)(\sqrt{x-1}+3)}{\sqrt{x-1}+3}\right| = \frac{|x-10|}{\sqrt{x-1}+3} < \frac{1}{5}(5\epsilon) = \epsilon.$$ ∎

Section 12.6: Differentiability

12.29. $f'(1) = -\frac{1}{9}$.

Proof Let $\epsilon > 0$ be given and choose $\delta = \min(1, 18\epsilon)$. Let $x \in \mathbf{R}$ such that $0 < |x - 1| < \delta$. Since $|x - 1| < 1$, it follows that $2 < x + 2 < 4$ and so $\frac{1}{x+2} < \frac{1}{2}$. Then

$$\begin{aligned}\left|\frac{f(x)-f(1)}{x-1} - \left(-\frac{1}{9}\right)\right| &= \left|\frac{\frac{1}{x+2} - \frac{1}{3}}{x-1} + \frac{1}{9}\right| = \left|\frac{-1+x}{9(x+2)}\right|\\ &= \frac{|x-1|}{9(x+2)} < \frac{|x-1|}{18} < \frac{18\epsilon}{18} = \epsilon.\end{aligned}$$

Thus $f'(1) = -\frac{1}{9}$. ∎

EXERCISES FOR CHAPTER 13

Section 13.1: Binary Operations

13.1. **(a)** $x * (y * z) = x * x = y$ and $(x * y) * z = z * z = y$. So $x * (y * z) = (x * y) * z$.
(b) $x * (x * x) = x * y = z$ and $(x * x) * x = y * x = y$.
(c) $y * (y * y) = y * x = y$ and $(y * y) * y = x * y = z$.
(d) The binary operation $*$ is neither associative nor commutative.

13.3. **(a)** Let $A_1, A_2 \in T$. Then $A_1 = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix}$ for some $a_1, b_1, a_2, b_2 \in \mathbf{R}$. Then $A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & -(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix}$. Since $A_1 + A_2 \in T$, it follows that T is closed under addition.

(b) Since $A_1 A_2 = \begin{bmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{bmatrix}\begin{bmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 - b_1b_2 & -(a_1b_2 + b_1a_2) \\ a_1b_2 + b_1a_2 & a_1a_2 - b_1b_2 \end{bmatrix} \in T$, it follows that T is closed under matrix multiplication.

13.5. ***Proof*** Let $a, b \in T$. Thus $a * a = a$ and $b * b = b$. Hence

$$\begin{aligned}(a * b) * (a * b) &= (a * b) * (b * a) = a * (b * (b * a)) = a * ((b * b) * a)\\ &= a * (b * a) = a * (a * b) = (a * a) * b = a * b.\end{aligned}$$ ∎

Section 13.2: Groups

13.7. See the table below.

$*$	a	b	c	d
a	d	c	b	a
b	c	d	a	b
c	b	a	d	c
d	a	b	c	d

Section 13.3: Permutation Groups

13.9. The table for $(F, \circ)$ is shown below. Composition of functions is always associative. All other properties can be obtained from the table.

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_5	f_1	f_6	f_2	f_4
f_4	f_4	f_6	f_2	f_5	f_1	f_3
f_5	f_5	f_3	f_6	f_1	f_4	f_2
f_6	f_6	f_4	f_5	f_2	f_3	f_1

13.11. **(a)** S_2 **(b)** S_3 **(c)** $(\mathbf{Z}, +)$ **(d)** $(M_2^*(\mathbf{R}), \cdot)$.

13.13. The table for $(G, \circ)$ is shown below. That G is an abelian group can be seen from the table.

$\circ$	γ_1	γ_2	γ_3	γ_4
γ_1	γ_1	γ_2	γ_3	γ_4
γ_2	γ_2	γ_3	γ_4	γ_1
γ_3	γ_3	γ_4	γ_1	γ_2
γ_4	γ_4	γ_1	γ_2	γ_3

Section 13.4: Fundamental Properties of Groups

13.15. ***Proof*** Let s be an inverse for a and let $x = b * s$. Then

$$x * a = (b * s) * a = b * (s * a) = b * e = b.$$

Hence $x = b * s$ is a solution of the equation $x * a = b$.

Next we show that $x * a = b$ has a unique solution x in G. Suppose that x_1 and x_2 are both solutions of $x * a = b$. Then $x_1 * a = b$ and $x_2 * a = b$. Hence $x_1 * a = x_2 * a$. Applying the Right Cancellation Law, we have $x_1 = x_2$. ∎

13.17. Since G has even order, $G - \{e\}$ has an odd number of elements. Consider those elements $g \in G$ for which $g \neq g^{-1}$ and let $S_g = \{g, g^{-1}\}$. Hence $S_g = S_{g^{-1}}$. If we take the union of all such sets S_g for which $g \neq g^{-1}$, then $\cup S_g \subset G - \{e\}$. Hence there exists an element $h \in G - \{e\}$ such that $h \notin \cup S_g$ and so $h = h^{-1}$. Thus $h^2 = e$.

13.19. ***Proof*** Assume that $ab = ba$. Applying Theorem 13.11, we obtain

$$a^{-1}b^{-1} = (ba)^{-1} = (ab)^{-1} = b^{-1}a^{-1},$$

giving the desired result. ∎

13.21. See the table below.

+	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
[3]	[3]	[4]	[5]	[6]	[7]	[8]	[0]	[1]	[2]
[6]	[6]	[7]	[8]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[0]
[4]	[4]	[5]	[6]	[7]	[8]	[0]	[1]	[2]	[3]
[7]	[7]	[8]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[0]	[1]
[5]	[5]	[6]	[7]	[8]	[0]	[1]	[2]	[3]	[4]
[8]	[8]	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]

Section 13.5: Subgroups

13.23. ***Proof*** First assume that H is a subgroup of G and let $a, b \in H$. Since $b \in H$, it follows by the Subgroup Test that $b^{-1} \in H$. Since $a, b^{-1} \in H$, we have, again by the Subgroup Test, that $ab^{-1} \in H$.

We now verify the converse. Assume, for a nonempty subset H of a group G, that $ab^{-1} \in H$ whenever $a, b \in H$. Since $H \neq \emptyset$, the set H contains an element h. Thus $hh^{-1} = e \in H$. Let $a \in H$. Then $e, a \in H$ and so $ea^{-1} = a^{-1} \in H$. Now let $a, b \in H$. Then $b^{-1} \in H$ and so $a, b^{-1} \in H$. Therefore, $a(b^{-1})^{-1} = ab \in H$. By the Subgroup Test, H is a subgroup of G. ∎

13.25. **(a)** The statement is true.

Proof Since H and K are subgroups of G, it follows that $e \in H$ and $e \in K$. So $e \in H \cap K$ and $H \cap K \neq \emptyset$. Let $a, b \in H \cap K$. Then $a, b \in H$ and $a, b \in K$. Since H and K are subgroups of G, it follows that $ab \in H$ and $ab \in K$. So $ab \in H \cap K$. Let $a \in H \cap K$. It remains to show that $a^{-1} \in H \cap K$. Since $a \in H$, $a \in K$, and H and K are subgroups of G, it follows that $a^{-1} \in H$ and $a^{-1} \in K$. So $a^{-1} \in H \cap K$. By the Subgroup Test, $H \cap K$ is a subgroup of G. ∎

(b) The statement is false. For example, $H = \{[0], [3]\}$ and $K = \{[0], [2], [4]\}$ are subgroups of $(\mathbf{Z}_6, +)$, but $H \cup K$ is not a subgroup of $(\mathbf{Z}_6, +)$.

13.27. ***Proof*** Since $\sqrt{3} \in H$, it follows that $H \neq \emptyset$. First, we show that H is closed under multiplication. Let $r = a + b\sqrt{3}$ and $s = c + d\sqrt{3}$ be elements of H, where at least one of a and b is nonzero and at least one of c and d is nonzero. Therefore, $r \neq 0$ and $s \neq 0$. Hence

$$rs = (ac + 3bd) + (ad + bc)\sqrt{3} \neq 0.$$

Thus at least one of $ac + 3bd$ and $ad + bc$ is nonzero. Since $ac + 3bd, ad + bc \in \mathbf{Q}$, it follows that $rs \in H$, and so H is closed under multiplication.

Next, we show that every element of H has an inverse in H. Let $r = a + b\sqrt{3} \in H$, where at least one of a and b is nonzero. Then

$$\frac{1}{r} = \frac{1}{a + b\sqrt{3}} = \frac{1}{a + b\sqrt{3}} \cdot \frac{a - b\sqrt{3}}{a - b\sqrt{3}}$$
$$= -\frac{a}{3b^2 - a^2} + \frac{b}{3b^2 - a^2}\sqrt{3}.$$

Observe that $3b^2 - a^2 \neq 0$; for if $3b^2 - a^2 = 0$, then $a/b = \pm\sqrt{3}$, which is impossible since $a/b \in \mathbf{Q}$ and $\sqrt{3} \in \mathbf{I}$. Hence $1/r \in H$.

By the Subgroup Test, H is a subgroup. ∎

13.29. ***Proof*** Let e be the identity in G. Since $e^2 = e \in H$, it follows that $H \neq \emptyset$. Let $a^2, b^2 \in H$, where $a, b \in G$. Since G is abelian, $a^2b^2 = (ab)^2 \in H$. Also, if $a^2 \in H$, then $(a^2)^{-1} = (a^{-1})^2 \in H$. By the Subgroup Test, H is a subgroup of G. ∎

Section 13.6: Isomorphic Groups

13.31. **(a)** ***Proof*** Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$, it follows that $H \neq \emptyset$. Let $A, B \in H$. Then $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, where $a, b \in \mathbf{Z}$. Then $AB = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \in H$. Also, if $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in H$, then $A^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \in H$. By the Subgroup Test, H is a subgroup of $(M_2^*(\mathbf{R}), \cdot)$. ∎

(b) ***Proof*** First, we show that f is one-to-one. Suppose that $f(a) = f(b)$, where $a, b \in \mathbf{Z}$. Then $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Hence $a = b$. Next, we show that f is onto. Let $A = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in H$. Then $f(n) = A$ and so f is onto. Finally, we show that f is operation-preserving. Let $a, b \in \mathbf{Z}$. Then

$$f(a+b) = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = f(a) \cdot f(b)$$

and so f is operation-preserving. Therefore, f is an isomorphism. ∎

(c) It suggests that

$$H_1 = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbf{Q} \right\} \text{ and } H_2 = \left\{ \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} : r \in \mathbf{R} \right\}$$

are also subgroups of $(M_2^*(\mathbf{R}), \cdot)$, where $(\mathbf{Q}, +)$ is isomorphic to $(H_1, \cdot)$ and $(\mathbf{R}, +)$ is isomorphic to $(H_2, \cdot)$.

13.33. The function ϕ is an isomorphism. ***Proof*** First we show that ϕ is one-to-one. Let $\phi(r) = \phi(s)$, where $r, s \in R^+$. Then $r^2 = s^2$. Since $r, s \in R^+$, it follows that $r = s$ and so ϕ is one-to-one. Given $r \in \mathbf{R}^+$, let $x = \sqrt{r} \in \mathbf{R}^+$. Then $\phi(x) = r$ and so ϕ is onto. Moreover, $\phi(rs) = (rs)^2 = r^2s^2 = \phi(r)\phi(s)$. Therefore, ϕ is operation-preserving and so ϕ is an isomorphism. ∎

13.35. ***Proof*** By Corollary 9.8, the composition $\phi_2 \circ \phi_1$ of two bijections ϕ_1 and ϕ_2 is also a bijection. Since $\phi_1 : G \to H$ and $\phi_2 : H \to K$ are isomorphisms, $\phi_1(st) = \phi_1(s)\phi_1(t)$ for $s, t \in G$ and $\phi_2(ab) = \phi_2(a)\phi_2(b)$ for $a, b \in H$. Therefore, if $s, t \in G$, then

$$(\phi_2 \circ \phi_1)(st) = \phi_2(\phi_1(st)) = \phi_2(\phi_1(s)\phi_1(t))$$
$$= \phi_2(\phi_1(s))\phi_2(\phi_1(t)) = (\phi_2 \circ \phi_1)(s)(\phi_2 \circ \phi_1)(t),$$

implying that $\phi_2 \circ \phi_1$ is an isomorphism. ∎

Chapter 14

Proofs in Ring Theory

We have noted that many of the proofs we have seen thus far involve integers and their properties. This was certainly the case in Chapter 11, where we were primarily concerned with additive and multiplicative properties of integers. Many important properties of integers follow from just a very few familiar additive and multiplicative properties of integers. In particular, every three integers a, b, and c satisfy the following:

$$\begin{array}{llll} (1) & a+b=b+a & (2) & (a+b)+c=a+(b+c) \\ (3) & a+0=a & (4) & a+(-a)=0 \\ (5) & a(bc)=(ab)c & (6) & a(b+c)=ab+ac \end{array} \tag{14.1}$$

Properties (1) – (4) tell us that the integers form an abelian group under addition, a fact we observed in Chapter 13. You can probably think of other familiar properties of integers (such as $ab = ba$), but let's concentrate on the six properties listed above. We saw in Chapter 13 that some of these properties have names. For example, (1) is called the commutative law of addition; while (2) and (5) are the associative laws of addition and multiplication, respectively. Property (6) is called the distributive law. Property (3) states that the integer 0 is the identity under addition; while property (4) tells us that for an integer a, the integer $-a$ is its inverse under addition. Properties (3) and (4) in particular may seem as if they are such basic properties of the integers that they should not even be mentioned. However, it is precisely that these six properties are so basic and natural that makes them important and draws our attention to them.

A question now arises: Just what facts about the integers are consequences only of these six properties? An even more basic question is: If we have a nonempty set S of objects (not necessarily integers) for which it is possible to add and multiply every two elements of S (and in each case obtain an element of S) such that properties (1) - (6) are satisfied, then what additional properties must S possess? Of course, whatever properties that can be deduced about the elements of S will be properties of the integers as well.

In fact, this is the essence of the area of abstract algebra that we are about to encounter (and often of all mathematics). While studying a familiar set of objects, we may discover an interesting fact about this set. But what features of this set led us to this conclusion? And if any other set had these same features, does this interesting fact hold for these sets as well? We are now prepared to explore nonempty sets on which addition and multiplication have been defined that satisfy properties (1) - (6).

14.1 Rings

Addition and multiplication of integers are binary operations since each associates an integer with each (ordered) pair of integers. Binary operations in general are discussed in Chapter 13.

In the current context, a nonempty set with one or more binary operations that are required to satisfy certain prescribed properties is referred to as an **algebraic structure**. Hence we have already seen examples of algebraic structures. Indeed, every group (see Chapter 13) is an algebraic structure. Studying algebraic structures is fundamental to abstract algebra.

We mentioned that the familiar operations of addition and multiplication defined on the integers satisfy the six properties listed in (14.1). Other familiar sets of numbers with these operations also satisfy these six properties, including the rational numbers, the real numbers, and the complex numbers. The situation is different for the irrational numbers, however, since addition and multiplication are not even binary operations. For example, $\sqrt{2}$ and $-\sqrt{2}$ are irrational numbers while $\sqrt{2} \cdot \sqrt{2} = 2$ and $\sqrt{2} + (-\sqrt{2}) = 0$ are not.

These and other examples suggest a general concept. A set R (this is *not* the symbol used for the set of real numbers) with two binary operations, one of which is *called* **addition** and denoted by $+$ and the other *called* **multiplication** and denoted by $\cdot$ (where we often write ab rather than $a \cdot b$ for $a, b \in R$), is called a **ring** if it satisfies the following six properties:

R1 **Commutative Law of Addition:** $a + b = b + a$ for all $a, b \in R$;

R2 **Associative Law of Addition:** $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$;

R3 **Existence of Additive Identity:** There exists an element $0 \in R$ such that $a + 0 = a$ for all $a \in R$;

R4 **Existence of Additive Inverse:** For each $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0$;

R5 **Associative Law of Multiplication:** $a(bc) = (ab)c$ for all $a, b, c \in R$;

R6 **Distributive Laws:** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Notice that property R3 requires the existence of at least one element in R, which implies that every ring is nonempty. Recall that if S is a set with a binary operation $*$ and $e \in S$ is an identity for S under $*$, then $e * a = a * e = a$ for all $a \in S$. Since a ring R has two binary operations and an identity element is only required for the operation of addition, we refer to an element 0 specified in property R3 as an **additive identity**. The notation 0 for an additive identity is chosen because the integer 0 is an additive identity in $\mathbf{Z}$. In other words, an additive identity in a ring R has the same characteristic as the integer 0 under addition in $\mathbf{Z}$. It is important to realize that when we refer to an additive identity 0 in a ring R, we are referring only to an element in R that we are denoting by 0 and that satisfies property R3, namely, $a + 0 = a$ for all $a \in R$. Since property R1 holds in every ring, we also have $0 + a = a$.

Also, if an algebraic structure $(S, *)$ has an identity e, then an element $a \in S$ has an inverse $b \in S$ if $a * b = b * a = e$. Each element of a ring R is only required to have this property for the operation of addition. Thus, an inverse of an element $a \in R$ with respect to addition is called an **additive inverse** of a. In $\mathbf{Z}$, an additive inverse of an integer m is its negative $-m$. For this reason, we use $-a$ to denote an additive inverse of an element a in a ring R. We must keep in mind that an element $-a$ in R stands only for some element in R that satisfies property R4, namely, $a + (-a) = 0$. By property R1, we also know that $(-a) + a = 0$. Since properties R1 – R4 are required of every ring R, it follows that $(R, +)$ is an abelian group.

A ring with binary operations $+$ and $\cdot$ is commonly denoted by $(R, +, \cdot)$. However, if the two operations involved are clear, then we simply write R. In particular, if we are dealing with a familiar set with standard operations of addition and multiplication (*and* these are the

operations we are using), then we write only the symbol for that set. Thus $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ are rings.

We now look at some other common examples of rings.

Result 14.1 *The set $2\mathbf{Z}$ of even integers is a ring under ordinary addition and multiplication.*

Proof. First we show that ordinary addition and multiplication are binary operations on $2\mathbf{Z}$. Let $a, b \in 2\mathbf{Z}$. Then $a = 2x$ and $b = 2y$ for $x, y \in \mathbf{Z}$. Then $a + b = 2x + 2y = 2(x + y)$ and $ab = (2x)(2y) = 2(2xy)$. Since $x + y$ and $2xy$ are integers, $a + b$ and ab belong to $2\mathbf{Z}$.

Since $2\mathbf{Z} \subseteq \mathbf{Z}$ and the binary operations in $2\mathbf{Z}$ are the same as those in $\mathbf{Z}$, properties R1, R2, R5, and R6 are automatically satisfied. Moreover, since the integer 0 is even, $0 \in 2\mathbf{Z}$ and so $2\mathbf{Z}$ has an additive identity. To show that property R4 is also satisfied, let $a \in 2\mathbf{Z}$. So $a = 2x$, where $x \in \mathbf{Z}$. Then $-a = -(2x) = 2(-x)$. Since $-x \in \mathbf{Z}$, it follows that $-a \in 2\mathbf{Z}$. ∎

Result 14.2 *The set $\mathbf{Z}_n = \{[0], [1], [2], \cdots, [n-1]\}$, $n \geq 2$, of residue classes is a ring under residue class addition and residue class multiplication.*

Proof. It was indicated in Chapter 7 that both residue class addition and multiplication defined by $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [ab]$ are well-defined and so are binary operations in $\mathbf{Z}_n$. That properties R1, R2, R5, and R6 are satisfied depends only on the corresponding properties in the ring $\mathbf{Z}$. For example, to see that R1 and R2 are satisfied, let $[a], [b], [c] \in \mathbf{Z}_n$. Then

$$[a] + [b] = [a + b] = [b + a] = [b] + [a]$$

and

$$\begin{aligned} ([a] + [b]) + [c] &= [a + b] + [c] = [(a + b) + c] = [a + (b + c)] \\ &= [a] + [b + c] = [a] + ([b] + [c]). \end{aligned}$$

The proofs of properties R5 and R6 are similar. The residue class [0] is an additive identity in $\mathbf{Z}_n$ and an additive inverse for $[a]$ is $[-a]$ since $[a] + [-a] = [a + (-a)] = [0]$. ∎

The ring $(\mathbf{Z}_n, +, \cdot)$ described in Result 14.2 is commonly called the **ring of residue classes modulo** n.

Result 14.3 *The set $M_2(\mathbf{R})$ of 2×2 matrices over $\mathbf{R}$ is a ring under matrix addition and matrix multiplication.*

Proof. Recall that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ in $M_2(\mathbf{R})$, addition and multiplication are defined by

$$A + B = \begin{bmatrix} a + e & b + f \\ c + g & d + h \end{bmatrix} \text{ and } AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}.$$

An additive identity for $M_2(\mathbf{R})$ is the zero matrix $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and an additive inverse for the matrix A given above is the matrix $-A = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. The verification of properties R1, R2, R5, and R6 depends only on the properties of the ring $\mathbf{R}$. ∎

Not only is $M_2(\mathbf{R})$ a ring under matrix addition and matrix multiplication, so too is $M_n(\mathbf{R})$ for each integer $n \geq 2$.

Result 14.4 *The set $\mathcal{F}_{\mathbf{R}} = \{f \ : \ f : \mathbf{R} \to \mathbf{R}\}$ of real-valued functions with domain $\mathbf{R}$ is a ring under function addition and function multiplication.*

Proof. Recall that for $f, g \in \mathcal{F}_{\mathbf{R}}$, addition and multiplication are defined by

$$(f+g)(x) = f(x) + g(x) \ \text{ and } \ (f \cdot g)(x) = f(x) \cdot g(x)$$

for all $x \in \mathbf{R}$. The proofs of properties R1, R2, R5, and R6 depend only on properties of the ring $\mathbf{R}$. For example, property R1 follows because

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

for all $x \in \mathbf{R}$ and so $f + g = g + f$; while property R5 follows because

$$\begin{aligned} ((f \cdot g) \cdot h)(x) &= (f \cdot g)(x) \cdot h(x) = (f(x) \cdot g(x)) \cdot h(x) \\ &= f(x) \cdot (g(x) \cdot h(x)) = f(x) \cdot (g \cdot h)(x) = (f \cdot (g \cdot h))(x) \end{aligned}$$

for all $x \in \mathbf{R}$ and so $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

The zero function $f_0 : \mathbf{R} \to \mathbf{R}$ defined by $f_0(x) = 0$ for all $x \in \mathbf{R}$ is an additive identity since for each $f \in \mathcal{F}_{\mathbf{R}}$ and all $x \in \mathbf{R}$,

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x)$$

and so $f + f_0 = f$.

For $f \in \mathcal{F}_{\mathbf{R}}$, the function $-f \in \mathcal{F}_{\mathbf{R}}$ defined by $(-f)(x) = -(f(x))$ for all $x \in \mathbf{R}$ is an additive inverse for f since for all $x \in \mathbf{R}$,

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0 = f_0(x)$$

and so $f + (-f) = f_0$. ∎

A less common, though useful, example of a ring is given next.

Result 14.5 *The set $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$ is a ring under the addition $(a,b) + (c,d) = (a+c, b+d)$ and multiplication $(a,b) \cdot (c,d) = (ac, bd)$.*

Before proving this result, it is important to know that we are defining a new sum $(a,b)+(c,d)$ in terms of the familiar sums $a + c$ and $b + c$ of two real numbers. Hence $+$ has two different meanings here. A similar distinction exists between the product in $\mathbf{R}^2$ and the standard product of real numbers.

Proof of Result 14.5. Certainly the addition and multiplication defined here are binary operations on $\mathbf{R}^2$. That $\mathbf{R}^2$ satisfies property R1 follows because addition in $\mathbf{R}$ is commutative. Let $(a,b), (c,d) \in \mathbf{R}^2$. Then

$$(a,b) + (c,d) = (a+c, b+d) = (c+a, d+b) = (c,d) + (a,b).$$

Let $(a,b) \in \mathbf{R}^2$. Observe that $(0,0) \in \mathbf{R}^2$ and that

$$(a,b) + (0,0) = (a+0, b+0) = (a,b).$$

Thus $(0,0)$ is an additive identity in $\mathbf{R}^2$. Moreover, $(-a,-b) \in \mathbf{R}^2$ and

$$(a,b) + (-a,-b) = (a + (-a), b + (-b)) = (0,0).$$

Hence $(-a,-b)$ is an additive inverse of (a,b) and properties R3 and R4 hold.

We only verify one of the distributive laws for a ring as the argument for the remaining law is similar. Again, let $(a,b),(c,d),(e,f) \in \mathbf{R}^2$. Applying the distributive law for addition and multiplication in $\mathbf{R}$, we have

$$\begin{aligned}(a,b)[(c,d)+(e,f)] &= (a,b)(c+e,d+f) = (a(c+e),b(d+f)) \\ &= (ac+ae,bd+bf) = (ac,bd)+(ae,bf) \\ &= (a,b)(c,d)+(a,b)(e,f),\end{aligned}$$

establishing this distributive law in $\mathbf{R}^2$.

The associative properties R2 and R5 can be established in $\mathbf{R}^2$ in a similar manner. ∎

Next we show that familiar sets, under unfamiliar binary operations, need not be rings.

Example 14.6 *For $a,b \in \mathbf{R}$, define addition $\bigoplus$ and multiplication $\bigodot$ by*

$$a \bigoplus b = a+b-1 \text{ and } a \bigodot b = ab,$$

where the operations indicated in $a+b-1$ and ab are ordinary addition, subtraction, and multiplication. Then $(\mathbf{R}, \bigoplus, \bigodot)$ is not a ring.

Solution. It was shown in Example 13.4 that the binary operation $\bigoplus$ satisfies properties R1-R4, that is, $(\mathbf{R}, \bigoplus)$ is an abelian group. Because $\bigodot$ is ordinary multiplication, property R5 holds as well. However, property R6 is not satisfied since for $a=b=c=0$,

$$a \bigodot (b \bigoplus c) = 0 \bigodot (-1) = 0 \quad \text{and} \quad (a \bigodot b) \bigoplus (a \bigodot c) = 0 \bigoplus 0 = -1.$$

Therefore, $(\mathbf{R}, \bigoplus, \bigodot)$ is not a ring. ♢

Let's see what happens when ordinary addition and multiplication of real numbers are reversed.

Example 14.7 *The set $\mathbf{R}$ of real numbers is not a ring when addition $*$ is defined as ordinary multiplication and multiplication $\circ$ is defined as ordinary addition.*

Solution. We denote ordinary addition of real numbers by $+$ and ordinary multiplication by $\cdot$ (though we write $a \cdot b$ as ab, as usual). We show that a distributive law fails in $(\mathbf{R}, *, \circ)$. Let $a=b=c=-1$. Then

$$a \circ (b * c) = a + (bc) = (-1) + (-1)(-1) = (-1) + 1 = 0,$$

while

$$(a \circ b) * (a \circ c) = (a+b)(a+c) = [(-1)+(-1)][(-1)+(-1)] = (-2)(-2) = 4.$$

Therefore, $(\mathbf{R}, *, \circ)$ is not a ring. ♢

Some rings satisfy properties beyond the six properties required of all rings. We have already mentioned that the integers satisfy the familiar property: $ab = ba$ for all $a,b \in \mathbf{Z}$. This, of course, is the commutative law of multiplication. Rings are not required to have this property. However, when they do, we give these rings a special name. A ring $(R,+,\cdot)$ is called a **commutative ring** if it satisfies

R7 **Commutative Law of Multiplication:** $ab = ba$ for all $a, b \in R$.

A ring $(R, +, \cdot)$ that does not satisfy the Commutative Law of Multiplication is called a **noncommutative ring**. While the rings $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, 2\mathbf{Z}, \mathbf{Z}_n, \mathcal{F}_{\mathbf{R}}$, and $\mathbf{R}^2$ are commutative, the ring $M_2(\mathbf{R})$ is noncommutative. For example, if we let

$$A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = BA.$$

Another property that $\mathbf{Z}$ possesses, which is basic yet important, is that it contains an integer e with the property that $a \cdot e = e \cdot a = a$ for every integer a. Of course, 1 has this property in $\mathbf{Z}$. In general, a ring $(R, +, \cdot)$ is called a **ring with unity** (or a **ring with multiplicative identity**) if it satisfies

R8 **Existence of Multiplicative Identity:** There exists an element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$.

If $(R, +, \cdot)$ has an element 1 satisfying property R8, then 1 is called a **unity** for R. Again, we stress that much care is needed here. When we write 1, we mean only an element of R that satisfies property R8, namely, $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. It does not imply that 1 is the *integer* 1. Indeed, R itself could be a ring with unity that contains no integers whatsoever. Also, if R is a commutative ring, then to show that some element $1 \in R$ is a unity requires only to show that $a \cdot 1 = a$ for all $a \in R$ since $a \cdot 1 = 1 \cdot a$ for all $a \in R$.

The rings $\mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}, \mathbf{Z}_n, \mathcal{F}_{\mathbf{R}}$, and $\mathbf{R}^2$ are rings with unity. The number 1 is a unity for $\mathbf{Z}$, $\mathbf{Q}$, and $\mathbf{R}$, as is $1 = 1 + 0i$ a unity for $\mathbf{C}$. The residue class [1] is a unity for $\mathbf{Z}_n$; while the constant function $f_1 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = 1$ for all $x \in \mathbf{R}$ is a unity for $\mathcal{F}_{\mathbf{R}}$. Furthermore, the ordered pair $(1, 1)$ is a unity for $\mathbf{R}^2$. The noncommutative ring $M_2(\mathbf{R})$ has a unity as well. Indeed, the 2×2 **identity matrix**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a unity for $M_2(\mathbf{R})$ since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for all $a, b, c, d \in \mathbf{R}$. On the other hand, not all rings have a unity. In particular, the ring $2\mathbf{Z}$ of even integers does not have a unity since the only integer e such that $e \cdot a = a$ for every integer a is $e = 1$ but $1 \notin 2\mathbf{Z}$.

14.2 Elementary Properties of Rings

Despite the fact that there are many different kinds of rings, there are properties that all rings have in common. Necessarily, of course, any such properties are consequences of the six defining properties of a ring. We now present some properties that all rings share, beginning with the uniqueness of certain types of elements in rings.

The definition of a ring R guarantees that it contains an additive identity, that is, an element 0 such that $a + 0 = a$ for all $a \in R$. Although the definition does not specify that there is only one such element, there is, in fact, only one. Also, the definition of R states that for each $a \in R$, there is an element $-a \in R$ such that $a + (-a) = 0$. Again, there is no indication that each element of R has only one additive inverse, but, in fact, this is the case. Actually, these are consequences of the fact that R is a group under addition (see Theorem 13.9), but we verify these facts here.

Theorem 14.8 *Let R be a ring. Then*

(*i*) *R has a unique additive identity, and*

(*ii*) *each element in R has a unique additive inverse.*

Proof. We first verify (*i*). Suppose that both 0 and $0'$ are additive identities for R. Since 0 is an additive identity, $0' + 0 = 0'$. Also, since $0'$ is an additive identity, $0 + 0' = 0$. It then follows by the commutative law that $0' = 0' + 0 = 0 + 0' = 0$ and so $0' = 0$. Therefore, there is only one additive identity in R and (*i*) holds.

We now verify (*ii*). Suppose that $-x$ and x' are both additive inverses for the element $x \in R$. Then $x + (-x) = 0$ and $x + x' = 0$. Hence

$$-x = -x + 0 = -x + (x + x') = (-x + x) + x' = 0 + x' = x'.$$

So each element in R has a unique additive inverse. ∎

Proof Analysis Let's revisit the proof of the uniqueness of additive inverses in Theorem 14.8 to see how this proof may have been constructed. We know that $x + (-x) = 0$ and $x + x' = 0$. Since $x + (-x) = 0 = x + x'$, it follows, by adding $-x$ to the equal elements $x + (-x)$ and $x + x'$, that

$$-x + (x + (-x)) = -x + (x + x'). \tag{14.2}$$

The left side of (14.2) is $-x + 0 = -x$. Our goal was to show that $-x = x'$, so this suggests starting with $-x = -x + 0 = -x + (x + x')$ and the remainder of the proof follows quite naturally. The resulting proof given in Theorem 14.8 is certainly much clearer than giving a list of equalities, with no accompanying explanations:

$$\begin{aligned} x + (-x) &= x + x' \\ -x + (x + (-x)) &= -x + (x + x') \\ (-x + x) + (-x) &= (-x + x) + x' \\ 0 + (-x) &= 0 + x' \\ -x &= x'. \quad \diamondsuit \end{aligned}$$

In view of Theorem 14.8, we can now refer to *the* additive identity of a ring and *the* additive inverse of an element in a ring. The additive identity of a ring R is called the **zero element** of R.

Not only are the additive identity and the additive inverse of every element in a ring R unique, but if R has a unity, then this element is unique as well.

Theorem 14.9 *If R is a ring with unity, then R has a unique unity.*

Proof. Let 1 and $1'$ be unities in R. Since 1 is a unity, $1 \cdot 1' = 1' \cdot 1 = 1'$; while since $1'$ is a unity, $1 \cdot 1' = 1' \cdot 1 = 1$. Therefore, $1 = 1 \cdot 1' = 1'$. ∎

A basic fact concerning rings allows us to simplify certain algebraic expressions. Although the next theorem is a consequence of the fact that R is an abelian group under addition (see Theorem 13.7), we provide a proof of this theorem.

Theorem 14.10 (**Cancellation Law of Addition**) *If a, b, and c are elements in a ring $(R, +, \cdot)$ such that $a + b = a + c$, then $b = c$.*

Proof. Observe that

$$\begin{aligned} b &= 0 + b = [(-a) + a] + b = (-a) + (a + b) \\ &= (-a) + (a + c) = [(-a) + a] + c = 0 + c = c. \end{aligned}$$

Therefore, the Cancellation Law of Addition holds in $(R, +, \cdot)$. ∎

Proof Analysis Another version of the preceding proof begins with $a + b = a + c$ (that is, $a + b$ and $a + c$ represent the same element in R). If the additive inverse $-a$ of a is now added to this element, we obtain

$$-a + (a + b) = -a + (a + c).$$

By the associative law,

$$(-a + a) + b = (-a + a) = c;$$

so $0 + b = 0 + c$ and thus $b = c$. ♢

We have seen that the zero element 0 in a ring R has the property that $0 + 0 = 0$. Hence R contains an element c such that $c + c = c$, namely, $c = 0$. However, as an immediate consequence of the Cancellation Law of Addition, no other element of R has this property.

Corollary 14.11 *Let $(R, +, \cdot)$ be a ring. If c is an element of R such that $c + c = c$, then $c = 0$.*

Proof. Since $c+c = c$, we also have $c+c = c+0$. Now, canceling c, it follows by Theorem 14.10 that $c = 0$. ∎

Although the defining property of the zero element of a ring $(R, +, \cdot)$ concerns only one of the two operations, namely addition, it has a property involving multiplication that is probably not unexpected.

Theorem 14.12 *For every element a in a ring $(R, +, \cdot)$,*

$$a \cdot 0 = 0 \cdot a = 0.$$

Proof. Since the proofs that $a \cdot 0 = 0$ and $0 \cdot a = 0$ are similar, we only verify the first of these. Observe that

$$a \cdot 0 + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$$

The result now follows from Corollary 14.11 (where $c = a \cdot 0$). ∎

We now turn our attention to properties of rings involving additive inverses. At times, a very simple argument for some fact can be given by recognizing that $-a$ represents the unique element which when added to a results in 0. Two examples of this appear in the following theorem.

Theorem 14.13 *Let $(R, +, \cdot)$ be a ring and let $a, b \in R$. Then*

(i) $-(-a) = a$

(ii) *if $a = -b$, then $b = -a$.*

Proof. Since $a + (-a) = 0$, it follows that a is the additive inverse of $-a$, that is, $a = -(-a)$. This verifies (i).

To establish (ii), let $a = -b$. Hence a is the additive inverse of b and so $a + b = 0$. This, however, implies that b is the additive inverse of a and so $b = -a$. ∎

We now consider some results concerning the product of two elements in a ring, at least one of which is an additive inverse. Since the additive inverse is an element that is defined only in terms of addition, it would seem natural that any property concerning such an element that involves multiplication must be a consequence of the distributive laws. (This is exactly what occurred in Theorem 14.12.)

Theorem 14.14 *Let $(R, +, \cdot)$ be a ring and let $a, b \in R$. Then*

$$(-a) \cdot b = a \cdot (-b) = -(ab).$$

Proof. To show that $(-a) \cdot b = -(ab)$, it suffices to verify that $(-a) \cdot b$ is the additive inverse of $a \cdot b$. This can be accomplished by showing that $a \cdot b + (-a) \cdot b = 0$. Observe that

$$a \cdot b + (-a) \cdot b = [a + (-a)] \cdot b = 0 \cdot b = 0.$$

A proof that $a \cdot (-b) = -(a \cdot b)$ is similar. ∎

Corollary 14.15 *Let $(R, +, \cdot)$ be a ring and let $a, b \in R$. Then*

$$(-a) \cdot (-b) = ab.$$

Proof. By Theorem 14.14, $(-a) \cdot (-b) = a \cdot [-(-b)]$, and by Theorem 14.13, $-(-b) = b$. Thus $(-a) \cdot (-b) = a \cdot b$. ∎

In the ring of integers we know that if $a, b \in \mathbf{Z}$, then $a+(-b) = a-b$. We follow this convention in an arbitrary ring. If $(R, +, \cdot)$ is a ring and $a, b \in R$, then we define the **subtraction** of b from a as $a - b = a + (-b)$. In particular, if $a = b$ in R, then we arrive at the seemingly obvious fact that $a - b = b - b = b + (-b) = 0$.

We present a basic fact concerning subtraction.

Result 14.16 *Let $(R, +, \cdot)$ be a ring and let $a, b, c \in R$. Then $a(b - c) = ab - ac$.*

Proof. Observe that $a(b-c) = a[b+(-c)] = ab+a \cdot (-c)$. By Theorem 14.14, $a(-c) = -(ac)$, so $a(b - c) = ab + [-(ac)] = ab - ac$. ∎

14.3 Subrings

We have seen that the subset $2\mathbf{Z}$ of $\mathbf{Z}$ is a ring when the operations of addition and multiplication used in $2\mathbf{Z}$ are the same as those of $\mathbf{Z}$. Since $\mathbf{Z}$ is already a ring, we found that it was relatively easy to prove that $2\mathbf{Z}$ is a ring. We saw that $2\mathbf{Z}$ inherits the properties R1, R2, R5, and R6 of a ring from $\mathbf{Z}$. What we didn't know automatically, and therefore had to verify, was

that $2\mathbf{Z}$ is closed under addition and multiplication, that the zero element of $\mathbf{Z}$ is also in $2\mathbf{Z}$, and that each element of $2\mathbf{Z}$ has an additive inverse in $2\mathbf{Z}$. In general, then, it is much easier to prove that a subset S of a known ring R is a ring under the same operations defined on R. This observation leads us to an important concept in the study of rings.

Let R be a ring. If S is a subset of R such that S is a ring under the same operations defined on R, then S is called a **subring** of R. If R contains at least two elements, then R contains at least two subrings, namely R itself and the "zero subring" $\{0\}$. We now state exactly what properties need to be verified to show that a subset of a known ring R is a subring of R.

Theorem 14.17 (**The Subring Test**) *A nonempty subset S of a ring R is a subring of R if and only if S is closed under subtraction and multiplication.*

Proof. If S is a subring of R, then certainly S is closed under subtraction and multiplication. For the converse, let R be a ring and S a nonempty subset of R that is closed under subtraction and multiplication. We show that S itself is a ring. Since $S \neq \emptyset$, there is some element $s \in S$. Because S is closed under subtraction, $s - s = 0 \in S$, that is, the zero element of R belongs to S and so property R3 holds. Now let $a \in S$. Again, since S is closed under subtraction, $0 - a = 0 + (-a) = -a \in S$ and so property R4 holds. This implies that the additive inverse of an element of S also belongs to S. For $a, b \in S$, we know that $-b \in S$ and so $a - (-b) = a + [-(-b)] = a + b \in S$. Hence S is closed under addition as well.

Now it remains to show that addition is commutative, that addition and multiplication are associative, and that distributive laws hold, namely, properties R1, R2, R5, and R6 hold in S. But all these properties are inherited from R and so hold in S as well. ∎

Consequently, to show that a subset S of a ring R is a subring, we need only show that S is nonempty and that S is closed under subtraction and multiplication. We now illustrate how the Subring Test is used by presenting several examples, beginning with a new proof that $2\mathbf{Z}$ is a ring under ordinary addition and multiplication.

Result 14.18 *The subset $2\mathbf{Z}$ of even integers is a subring of $\mathbf{Z}$.*

Proof. Since 0 is an even integer, $2\mathbf{Z}$ is nonempty. Let $a, b \in 2\mathbf{Z}$. Then $a = 2x$ and $b = 2y$, where $x, y \in \mathbf{Z}$. Observe that $a - b = 2x - 2y = 2(x - y)$ and $ab = (2x)(2y) = 2(2xy)$. Since $x - y$ and $2xy$ are integers, $a - b$ and ab belong to $2\mathbf{Z}$. By the Subring Test, $2\mathbf{Z}$ is a subring of $\mathbf{Z}$. ∎

Result 14.19 *The subset $\mathbf{R} \times \{0\} = \{(x, 0) : x \in \mathbf{R}\}$ of the ring $\mathbf{R} \times \mathbf{R}$ is a subring of $\mathbf{R} \times \mathbf{R}$.*

Proof. Since $(0, 0) \in \mathbf{R} \times \{0\}$, the set $\mathbf{R} \times \{0\}$ is nonempty. Let $a, b \in \mathbf{R} \times \{0\}$. Then $a = (x, 0)$ and $b = (y, 0)$ for some $x, y \in \mathbf{R}$. Thus $a - b = (x, 0) - (y, 0) = (x - y, 0 - 0) = (x - y, 0) \in \mathbf{R} \times \{0\}$ and $a \cdot b = (x, 0) \cdot (y, 0) = (xy, 0) \in \mathbf{R} \times \{0\}$. By the Subring Test, $\mathbf{R} \times \{0\}$ is a subring of $\mathbf{R} \times \mathbf{R}$. ∎

The next example concerns a subring of the ring of complex numbers. A complex number of the form $a + bi$, where $a, b \in \mathbf{Z}$ and $i = \sqrt{-1}$, is called a **Gaussian integer**.

Result 14.20 *The set $G = \{a + bi : a, b \in \mathbf{Z}\}$ of Gaussian integers is a subring of the ring $\mathbf{C}$ of complex numbers.*

Proof. Since $0 = 0 + 0i \in G$, the set G is nonempty. Let $x, y \in G$. Then $x = a + bi$ and $y = c + di$, where $a, b, c, d \in \mathbf{Z}$. Observe that

$$x - y = (a + bi) - (c + di) = (a - c) + (b - d)i$$

and

$$xy = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$

Since $a - c, b - d, ac - bd$, and $ad + bc$ are integers, $x - y$ and xy are Gaussian integers. By the Subring Test, G is a subring of $\mathbf{C}$. ∎

The elements that belong to two subrings of a ring R also produce a subring of R.

Result 14.21 *If S_1 and S_2 are subrings of a ring R, then $S_1 \cap S_2$ is also a subring of R.*

Proof. Since $0 \in S_1$ and $0 \in S_2$, it follows that $0 \in S_1 \cap S_2$ and so $S_1 \cap S_2$ is nonempty. Let $a, b \in S_1 \cap S_2$. Then $a, b \in S_i$ for $i = 1, 2$. Since S_1 and S_2 are subrings of R, it follows that $a - b \in S_i$ and $ab \in S_i$ for $i = 1, 2$. Hence $a - b \in S_1 \cap S_2$ and $ab \in S_1 \cap S_2$. Therefore, by the Subring Test, $S_1 \cap S_2$ is a subring of R. ∎

14.4 Integral Domains

Properties possessed by the integers have led us to the concept of a ring as well as two special kinds of rings, namely commutative rings and rings with unity. We have seen that if R is a ring, then $a \cdot 0 = 0 \cdot a = 0$ for every $a \in R$. This property can be stated in another way:

$$\text{Let } a, b \in R. \text{ If } a = 0 \text{ or } b = 0, \text{ then } a \cdot b = 0. \tag{14.3}$$

Of course, the converse of (14.3) also holds in the ring $\mathbf{Z}$:

$$\text{Let } a, b \in \mathbf{Z}. \text{ If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0. \tag{14.4}$$

The implication (14.4) also holds in the ring of real numbers. Indeed, (14.4) is the critical property of real numbers needed for solving many equations. For example, if $(x-3)(x+2) = 0$, where $x \in \mathbf{R}$, then $x = 3$ or $x = -2$. This leads us to another important concept.

A nonzero element a in a ring R is called a **zero divisor** of R if there exists a nonzero element b in R such that either $ab = 0$ or $ba = 0$. Of course, in this case, b is a zero divisor of R as well.

Certainly then, the rings $\mathbf{Z}$ and $\mathbf{R}$ have no zero divisors. Furthermore, $2\mathbf{Z}, \mathbf{Q}$, and $\mathbf{C}$ are rings possessing no zero divisors. There are, however, some well-known rings that do have zero divisors. In $\mathbf{Z}_6$, we have seen that $[2][3] = [6] = [0]$. Since $[2] \neq [0]$ and $[3] \neq [0]$, it follows that $[2]$ and $[3]$ are zero divisors in $\mathbf{Z}_6$. The residue class $[4]$ is also a zero divisor in $\mathbf{Z}_6$ since $[4][3] = [0]$.

Consider the functions f and g in $\mathcal{F}_{\mathbf{R}}$ defined as:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathcal{I} \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Q} \\ 1 & \text{if } x \in \mathcal{I} \end{cases}$$

Then $(f \cdot g)(x) = f(x) \cdot g(x) = 0 = f_0(x)$ for all $x \in \mathbf{R}$. Hence $f \cdot g = f_0$, the zero element of $\mathcal{F}_{\mathbf{R}}$, but $f \neq f_0$ and $g \neq f_0$. So f and g are zero divisors in $\mathcal{F}_{\mathbf{R}}$.

In $M_2(\mathbf{R})$, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ while } BA = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Hence A and B are zero divisors in the noncommutative ring $M_2(\mathbf{R})$.

From what we have seen, it is not all that uncommon for a ring R to contain nonzero elements whose product is the zero element of R. Thus, it is useful to distinguish those rings that contain zero divisors from those that do not. Before proceeding further though, we need to address one special kind of ring. A ring R is called **trivial** if it contains only one element – necessarily, then, the zero element. That is, R is trivial if $R = \{0\}$. If R is **nontrivial**, then it contains at least two elements, and consequently at least one nonzero element. If R is a trivial ring, then certainly $a \cdot 0 = 0 \cdot a = a$ for all $a \in R$ since $a = 0$ is the only element of R. Therefore, if R is trivial, then it contains a unity (namely 0). Obviously, a trivial ring is commutative as well. On the other hand, if R is a nontrivial ring with unity, then it cannot occur that the unity and zero elements are the same.

Theorem 14.22 *If R is a nontrivial ring with unity 1, then $1 \neq 0$.*

Proof. Assume, to the contrary, that $1 = 0$. Since R is a nontrivial ring, there is an element $a \in R$ such that $a \neq 0$. However, then

$$a = a \cdot 1 = a \cdot 0 = 0,$$

which is a contradiction. ∎

A nontrivial commutative ring with unity that contains no zero divisors is called an **integral domain**. Therefore, all of the rings $\mathbf{Z}$, $\mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ are integral domains.

Not all commutative rings with unity are integral domains, however. For example, we saw that [2] and [3] are zero divisors in $\mathbf{Z}_6$. We also saw that $\mathcal{F}_{\mathbf{R}}$ possesses zero divisors. Moreover, since $(0,1)\cdot(1,0) = (0,0)$ in $\mathbf{R}^2$, it follows that $(0,1)$ and $(1,0)$ are zero divisors in $\mathbf{R}^2$. Therefore, although all of $\mathbf{Z}_6$, $\mathcal{F}_{\mathbf{R}}$, and $\mathbf{R}^2$ are commutative rings with unity, none is an integral domain.

Since an integral domain is required to be a commutative ring with a unity, $2\mathbf{Z}$ is not an integral domain, despite the fact that it is both commutative and contains no zero divisors, as it does not contain a unity.

We have seen that every ring satisfies the Cancellation Law of Addition. For multiplication, the situation can be quite different. There are two possible cancellation laws in this case.

Cancellation Laws of Multiplication: *Let R be a ring and let $a, b, c \in R$.*

(1) *If $ab = ac$, where $a \neq 0$, then $b = c$.*

(2) *If $ac = bc$, where $c \neq 0$, then $a = b$.*

Of course, if R is a commutative ring, then (1) and (2) say the same thing. In a noncommutative ring, (1) is referred to as the *Left Cancellation Law of Multiplication* and (2) as the *Right Cancellation Law of Multiplication*. In the ring $\mathbf{Z}_6$, $[3]\cdot[2] = [3]\cdot[4]$ but $[2] \neq [4]$. So the Cancellation Laws of Multiplication fail to hold in $\mathbf{Z}_6$. The Cancellation Laws of Multiplication never fail to hold in rings without zero divisors, however.

Theorem 14.23 *Let R be a ring. Then the Cancellation Laws of Multiplication hold in R if and only if R contains no zero divisors.*

Proof. Assume first that R is a ring without zero divisors. We only verify the Left Cancellation Law (1) since the proof of (2) is similar. Let $a, b, c \in R$, where $a \neq 0$ and $ab = ac$. Since $ab = ac$, it follows that $ab + (-(ac)) = ac + (-(ac))$ and so $ab - ac = 0$. Thus $a(b - c) = 0$. Since R contains no zero divisors and $a \neq 0$, it follows that $b - c = 0$ and so $b = c$.

For the converse, assume that R is a ring in which the Cancellation Laws of Multiplication hold. We show that R contains no zero divisors. Let $a, b \in R$ such that $ab = 0$. We show that $a = 0$ or $b = 0$. If $a = 0$, then we have the desired result. So we may assume that $a \neq 0$. Hence $a \cdot b = 0 = a \cdot 0$ and so $a \cdot b = a \cdot 0$. By the (Left) Cancellation Law of Multiplication, the element a can be canceled in $a \cdot b = a \cdot 0$, arriving at $b = 0$. Thus R has no zero divisors. ∎

Since a ring R satisfying the Cancellation Laws of Multiplication is equivalent to R containing no zero divisors, we have an immediate consequence of Theorem 14.23.

Corollary 14.24 *Let R be a nontrivial commutative ring with unity. Then R is an integral domain if and only if the Cancellation Law of Multiplication holds in R.*

While $\mathbf{Z}_6$ is not an integral domain, it is not difficult to show that $\mathbf{Z}_5$ is (by constructing a multiplication table for $\mathbf{Z}_5$ as in the same manner done for $\mathbf{Z}_6$ in Figure 7.1 of Chapter 7). Consequently, some rings $\mathbf{Z}_n$ are integral domains while others are not. You might have seen a difference between $\mathbf{Z}_6$ and $\mathbf{Z}_5$ already, namely, 5 is prime and 6 is not. We are about to see that this is the key observation. In the proof of the next theorem, we will use the fact that if a and b are integers and p is a prime such that $p \mid ab$, then $p \mid a$ or $p \mid b$. (This theorem is discussed in detail in Chapter 11. In particular, see Corollary 11.14.)

Theorem 14.25 *For an integer $n \geq 2$, the ring $\mathbf{Z}_n$ is an integral domain if and only if n is a prime.*

Proof. First, we show that if $\mathbf{Z}_n$ is an integral domain, then n is a prime. Assume that n is not a prime. Then $n = ab$ for some integers a and b with $1 < a < n$ and $1 < b < n$. Thus $[a] \neq [0]$ and $[b] \neq [0]$ in $\mathbf{Z}_n$. On the other hand, $[a][b] = [ab] = [n] = [0]$ in $\mathbf{Z}_n$. Hence $[a]$ and $[b]$ are zero divisors in $\mathbf{Z}_n$ and so $\mathbf{Z}_n$ is not an integral domain.

For the converse, assume that n is a prime. We show that $\mathbf{Z}_n$ is an integral domain. Certainly, $\mathbf{Z}_n$ is a nontrivial commutative ring with unity. It remains only to show that $\mathbf{Z}_n$ has no zero divisors. Let $[a], [b] \in \mathbf{Z}_n$ such that $[a] \cdot [b] = [0]$. Then $[a] \cdot [b] = [ab] = [0]$, which implies that $ab \equiv 0 \pmod{n}$. Therefore, $n \mid ab$. Since n is a prime, it follows by Corollary 11.14 that $n \mid a$ or $n \mid b$; so $[a] = [0]$ or $[b] = [0]$. Thus $\mathbf{Z}_n$ contains no zero divisors. ∎

14.6 Fields

Initially, we saw that many fundamental properties of integers are shared by other algebraic structures. This led us to the concept of rings. Among the many rings we encountered are $\mathbf{Z}$, $2\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}_n$, $\mathcal{F}_{\mathbf{R}}$, and $M_2(\mathbf{R})$. However, only some of these are commutative rings with unity, namely, $\mathbf{Z}$,$\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}_n$, and $\mathcal{F}_{\mathbf{R}}$; and only some of these are integral domains, namely, $\mathbf{Z}$,$\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Z}_p$, where p is a prime. There is a property that $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{Z}_p$ possess that $\mathbf{Z}$ does not, however, which will finally allow us to distinguish $\mathbf{Z}$ from these rings.

Let a be a nonzero integer. Unless a is 1 or -1, there is no integer b such that $ab = 1$. On the other hand, if a is a nonzero rational number, then there is always a rational number b such that $ab = 1$. Indeed, $b = 1/a \in \mathbf{Q}$ has this property.

This discussion leads us to another concept. Let R be a ring with unity 1. A nonzero element a of R is called a **unit** if there is some element b in R such that $ab = ba = 1$. In this case, b is called a **multiplicative inverse** of a. (Of course, b is also a unit with multiplicative inverse a.) We must take care to distinguish between the terms "unit" and "unity" in a ring R. A unity in R is an element $1 \in R$ such $a \cdot 1 = 1 \cdot a = a$ for all $a \in R$. On the other hand, if R is a nontrivial ring with unity 1, then a nonzero element $a \in R$ is a unit if $a \cdot b = b \cdot a = 1$ for some $b \in R$. The unity 1 is always a unit since $1 \cdot 1 = 1$. As with additive inverses, multiplicative inverses in a ring are unique.

Theorem 14.26 *Let R be a nontrivial ring with unity. Then each unit in R has a unique multiplicative inverse.*

Proof. Let a be a unit in R and suppose that b and c are multiplicative inverses of a. Hence $ab = ba = 1$ and $ac = ca = 1$. It then follows that

$$b = b \cdot 1 = b(ac) = (ba)c = 1 \cdot c = c.$$

Therefore, a has a unique multiplicative inverse. ∎

For a unit a in a nontrivial ring with unity, we write a^{-1} for the (unique) multiplicative inverse of a. The only units in $\mathbf{Z}$ are 1 and -1 since these are the only integers a for which there is an integer b such that $ab = 1$. In $\mathbf{Q}$ and $\mathbf{R}$, however, all nonzero elements are units. In $\mathbf{Z}_6$, $[5] \cdot [5] = [25] = [1]$, so both [1] and [5] are units. Furthermore, there are no other units in $\mathbf{Z}_6$, as can be seen from the multiplication table (Figure 7.1) in Chapter 7.

A nontrivial commutative ring with unity in which every nonzero element is a unit is called a **field**. In addition to $\mathbf{Q}$ and $\mathbf{R}$, the ring $\mathbf{C}$ of complex numbers is a field.

Result 14.27 *The ring $\mathbf{C}$ of complex numbers is a field.*

Proof. We have already noted that $\mathbf{C}$ is a commutative ring with a unity, so we are only required to show that every nonzero complex number is a unit. Let x be a nonzero complex number. Hence $x = a + bi$, where $a, b \in \mathbf{R}$ and either $a \neq 0$ or $b \neq 0$. Thus $a^2 + b^2 \neq 0$. We show that there exists a complex number $y = c + di$, where $c, d \in \mathbf{R}$, such that $xy = 1 = 1 + 0i$. Let $c = \dfrac{a}{a^2 + b^2}$ and $d = \dfrac{-b}{a^2 + b^2}$ and observe that

$$\begin{aligned} xy &= (a + bi)(c + di) = (a + bi)\left(\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i\right) \\ &= \frac{1}{a^2 + b^2}(a + bi)(a - bi) = \frac{1}{a^2 + b^2}\left(a^2 - b^2 i^2\right) \\ &= \frac{a^2 + b^2}{a^2 + b^2} = 1 = 1 + 0i. \end{aligned}$$

Hence x has a multiplicative inverse, namely, $x^{-1} = \dfrac{a}{a^2 + b^2} + \dfrac{-b}{a^2 + b^2}i$. ∎

Proof Analysis In the proof of the preceding result, for the nonzero complex number $x = a + bi$, how did we know to choose $y = c + di$ so that $xy = 1 = 1 + 0i$? That is, how did we know what the multiplicative inverse of x was? Actually, that was not so difficult.

Since $xy = (a + bi)(c + di) = 1 + 0i$, it follows that $(ac - bd) + (ad + bc)i = 1 + 0i$. Hence

$$ac - bd = 1 \tag{14.5}$$

and

$$ad + bc = 0. \tag{14.6}$$

Multiplying equation (14.5) by a, equation (14.6) by b, and adding, we obtain

$$(a^2 + b^2)c = a; \tag{14.7}$$

while multiplying equation (14.5) by $-b$, equation (14.6) by a, and adding, we obtain

$$(a^2 + b^2)d = -b. \tag{14.8}$$

Solving (14.7) for c and (14.8) for d, we find that

$$c = \frac{a}{a^2 + b^2} \quad \text{and} \quad d = \frac{-b}{a^2 + b^2}.$$

Hence $\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$ is the logical choice for x^{-1}. That $c + di$ is actually x^{-1} was, of course, verified in the proof of Result 14.27. $\diamondsuit$

Fields are actually special kinds of integral domains, as we now show.

Theorem 14.28 *Every field is an integral domain.*

Proof. Let F be a field. To verify that F is also an integral domain, we need only show that F contains no zero divisors. Let a be a nonzero element of F and let $b \in F$ such that $ab = 0$. Then $0 = a^{-1} \cdot 0 = a^{-1}(ab) = (a^{-1}a)\, b = 1b = b$. Since $b = 0$, it follows that a is not a zero divisor. ∎

Certainly, the converse of Theorem 14.28 is not true since $\mathbf{Z}$ is an integral domain that is not a field. However, under a certain restriction, an integral domain is a field as well.

Theorem 14.29 *Every finite integral domain is a field.*

Proof. Let D be a finite integral domain, say $D = \{a_1, a_2, \cdots, a_n\}$. To show that D is a field, we need only show that every nonzero element of D has a multiplicative inverse. Let $a \in D$, where $a \neq 0$, and consider the elements $aa_1, aa_2, \cdots, aa_n$. If $aa_i = aa_j$, where $1 \leq i \leq n$ and $1 \leq j \leq n$, then $a_i = a_j$ by the Cancellation Law of Multiplication. This implies that the elements $aa_1, aa_2, \cdots, aa_n$ are distinct and are, in fact, all n elements of D. Thus one of these elements is 1 and so $aa_k = 1$ for some integer k with $1 \leq k \leq n$. Hence $a_k = a^{-1}$ and a has a multiplicative inverse. ∎

We have seen in Theorem 14.25 that $\mathbf{Z}_n$ is an integral domain if and only if n is a prime. Theorem 14.29 now gives us the following result.

Corollary 14.30 *The ring $\mathbf{Z}_n$ is a field if and only if n is prime.*

Exercises for Chapter 14

14.1 Verify that each of the following is a ring by showing that (1) the indicated addition and multiplication are binary operations and (2) the required six properties are satisfied. (You may assume that both $\mathbf{Z}$ and $\mathbf{R}$ are rings under ordinary addition and multiplication.)

(**a**) The set $k\mathbf{Z}$, where $k \in \mathbf{Z}$ and $k \geq 2$, under ordinary addition and multiplication.

(*b*) The set $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} \ : \ a, b \in \mathbf{Z}\}$ under ordinary addition and multiplication.

14.2 Verify that each of the following is not a ring.

(*a*) The set $\mathcal{F}_{\mathbf{R}}$ under function addition and function composition.

(*b*) The set $\mathbf{Z}$ under the addition defined by $a * b = a$ and ordinary multiplication.

(*c*) The set $\mathbf{Z}$ under ordinary addition and the multiplication defined by $a * b = a$.

(*d*) The set $\mathbf{Z}$ under the addition defined by $a*b = \min\{a, b\}$ and ordinary multiplication.

(*e*) The set $\mathbf{Z}$ under ordinary addition and the multiplication defined by $a*b = \min\{a, b\}$.

14.3 For a given set S and binary operations $*$ and $\circ$, determine whether $(S, *, \circ)$ is a ring.

(a) $S = \mathbf{R}$, $a * b = a + b + 1$, $a \circ b = ab$.

(b) $S = \mathbf{R}^+$, the set of positive real numbers, $a * b = ab$ and $a \circ b = a^b$.

14.4 Let a be an element in a ring $(R, +, \cdot)$. Complete the proof of Theorem 14.12 by proving that $0 \cdot a = 0$.

14.5 Let a and b be elements in a ring $(R, +, \cdot)$. Complete the proof of Theorem 14.14 by proving that $a \cdot (-b) = -(a \cdot b)$.

14.6 Let R be a ring with unity 1. Use Theorem 14.14 to prove that $(-1)a = -a$ for all $a \in R$.

14.7 Let $(R, +, \cdot)$ be a ring with the property that $a^2 = a \cdot a = a$ for every $a \in R$.

(a) Prove that every element in R is its own additive inverse, that is, prove that $-a = a$ for every $a \in R$. [Hint: Consider $(a + a)^2$.]

(b) Prove that R is a commutative ring. [Hint: Consider $(a + b)^2$.]

14.8 Does there exist an example of a nontrivial ring $(R, +, \cdot)$, that is, R has at least two elements, such that addition and multiplication in R are the same, namely, $a + b = ab$ for all $a, b \in R$? Justify your answer.

14.9 Verify that each of the following subsets is a subring of the given ring.

(**a**) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \ : \ a, b \in \mathbf{R} \right\}$ in the ring $M_2(\mathbf{R})$.

(b) $S = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \ : \ a, b, c \in \mathbf{Q}\}$ in the ring $\mathbf{R}$.

14.10 Prove that the subset $S = \{[0], [2], [4]\}$ is a subring of $\mathbf{Z}_6$.

14.11 Recall that a Gaussian integer is a complex number of the type $a+bi$, where $a, b \in \mathbf{Z}$ and $i = \sqrt{-1}$, and that the set G of Gaussian integers is a subring of the ring $\mathbf{C}$ of complex numbers. Define an *even Gaussian integer* to be a complex number of the type $a+bi$, where $a, b \in 2\mathbf{Z}$. Is the set $2G$ of even Gaussian integers a subring of G? Justify your answer.

14.12 By Result 14.21, if S_1 and S_2 are subrings of a ring R, then $S_1 \cap S_2$ is a subring of R. Both $2\mathbf{Z}$ and $3\mathbf{Z}$ are subrings of the ring $\mathbf{Z}$. Give a simple description of the subring $2\mathbf{Z} \cap 3\mathbf{Z}$ in $\mathbf{Z}$. Justify your answer.

14.13 Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \ : \ a, b \in \mathbf{R} \right\}$.

(a) Prove that S is a subring of $M_2(\mathbf{R})$.

(b) Prove that there is an element $E \in S$ such that $EA = A$ for all $A \in S$, but there is an element $C \in S$ such that $CE \neq C$.

(c) Prove that S does not possess a unity.

14.14 Use Theorem 14.23 to prove Corollary 14.24.

14.15 Define multiplication $\circ$ on $2\mathbf{Z}$ by $a \circ b = ab/2$. Prove that $(2\mathbf{Z}, +, \circ)$ is an integral domain, where $+$ is ordinary addition.

14.16 Let R be a commutative ring with unity.

(a) Prove that a unit of R is not a zero divisor in R.

(b) Determine whether the converse of (a) is true.

(c) Prove that if R is a finite ring and a is not a zero divisor of R, then a has a multiplicative inverse in R.

14.17 Define addition $*$ and multiplication $\circ$ on $\mathbf{Z}$ as follows:

$$a * b = a + b - 1 \quad \text{and} \quad a \circ b = a + b + ab.$$

Prove that $(\mathbf{Z}, *, \circ)$ is a ring with unity and answer the following questions.

(a) Is this ring commutative?

(b) Is this ring an integral domain?

(c) Is this ring a field?

14.18 Show that $\mathbf{Z}[\sqrt{2}] = \{a + b\sqrt{2} \ : \ a, b \in \mathbf{Z}\}$ is not a field.

14.19 Give an example of a ring that is not a field but has a subring that is a field.

14.20 Let R be a nontrivial commutative ring with unity. Prove that R is a field if and only if for every $a, b \in R$ with $a \neq 0$, the equation $ax = b$ has a solution $x \in R$.

14.21 Prove that $\mathbf{Q}[i] = \{a + bi \ : \ a, b \in \mathbf{Q}\}$ is a field.

14.22 Let $(F, +, \cdot)$ be a field and let $a, b \in F$ with $a \neq 0$. Show that the equation $a \cdot x = b$ has a unique solution $x \in F$.

14.23 Give examples of the following (if they exist):

(a) a finite ring

(b) an infinite ring

(c) a noncommutative finite ring

(d) a noncommutative infinite ring

(e) a ring with unity

(f) a ring without unity

(g) a noncommutative ring with unity

(h) a noncommutative ring without unity

(i) a ring that is not an integral domain

(j) a finite integral domain

(k) an infinite integral domain

(ℓ) an integral domain that is not a field

(m) a finite field

(n) an infinite field

14.24 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let $A = \{n \in \mathbf{N} : n = 0\}$. Then A is a subring of $(\mathbf{Z}, +, \cdot)$.

Proof. Let $a, b \in S$. Then $a = 0$ and $b = 0$. Since $a - b = 0 - 0 = 0 \in A$ and $a \cdot b = 0 \cdot 0 = 0 \in A$, it follows that A is closed under subtraction and multiplication. By the Subring Test, $(A, +, \cdot)$ is a subring of $(\mathbf{Z}, +, \cdot)$. ∎

14.25 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let R be a ring with unity containing at least two elements and let

$$R' = \{a \in R \ : \ a - r \text{ is a unit for each } r \in R\}.$$

Then R' is a subring of R.

Proof. Let $a, b \in R'$. First, consider $a - b$ and $r \in R$. Then $(a - b) - r = a - (b + r)$. Since $a \in R'$ and $b + r \in R$, it follows that $(a - b) - r$ is a unit and so $a - b \in R'$. Next, consider ab and $r' \in R$. Then $ab - r' = a - (a - ab + r')$. Since $a \in R'$ and $a - ab + r' \in R$, it follows that $ab - r'$ is a unit. Thus $ab \in R'$. By the Subring Test, R' is a subring of R.∎

Chapter 15

Proofs in Linear Algebra

A topic you may very well have studied in geometry, calculus, or physics is vectors. You might recall vectors both in the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ and in 3-space $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. Often one thinks of a vector as a directed line segment from the origin to some other point. Examples of these (both in the plane and in 3-space) are shown in Figure 15.1.

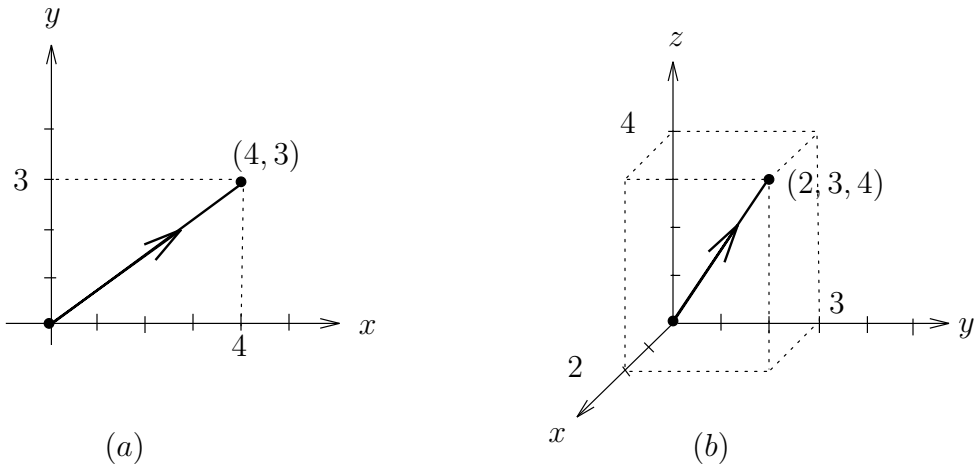

Figure 15.1: Vectors in the plane and 3-space

The vector $\mathbf{u}$ in the plane (it is customary to print vectors in bold) shown in Figure 15.1(a) can be expressed as $\mathbf{u} = (4, 3)$; while the vector $\mathbf{v}$ in 3-space shown in Figure 15.1(b) can be expressed as $\mathbf{v} = (2, 3, 4)$. The vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ in the plane and $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ in 3-space will be of special interest to us.

15.1 Properties of Vectors in 3-Space

One important feature of vectors is that they can be added (to produce another vector); while another is that a vector can be multiplied by an element of some set, usually a real number (again to produce another vector). In this context, these elements are called **scalars**. Let's focus on vectors in 3-space for the present. Let $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2)$, where a_i, b_i, c_i $(i = 1, 2)$ are real numbers. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

and the **scalar multiple** of $\mathbf{u}$ by a scalar (real number) α is defined by

$$\alpha\mathbf{u} = (\alpha a_1, \alpha b_1, \alpha c_1).$$

From this definition, it follows that

$$\begin{aligned} \mathbf{u} &= (a_1, b_1, c_1) = (a_1, 0, 0) + (0, b_1, 0) + (0, 0, c_1) \\ &= a_1(1, 0, 0) + b_1(0, 1, 0) + c_1(0, 0, 1) = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}. \end{aligned}$$

That is, it is possible to express a vector $\mathbf{u}$ in 3-space in terms of (and to be called a linear combination of) the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in 3-space. Listed below are eight simple, yet fundamental, properties that follow from these definitions of vector addition and scalar multiplication in $\mathbf{R}^3$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$.
3. For $\mathbf{z} = (0, 0, 0)$, $\mathbf{u} + \mathbf{z} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^3$.
4. For each $\mathbf{u} \in \mathbf{R}^3$, there exists a vector in $\mathbf{R}^3$ which we denote by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{z} = (0, 0, 0)$.
5. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all $\alpha \in \mathbf{R}$ and all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$.
6. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ for all $\alpha, \beta \in \mathbf{R}$ and all $\mathbf{u} \in \mathbf{R}^3$.
7. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ for all $\alpha, \beta \in \mathbf{R}$ and all $\mathbf{u} \in \mathbf{R}^3$.
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^3$.

These properties are rather straightforward to verify, as we illustrate with properties 1, 4, and 6. To verify property 1, observe that

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (b_1 + a_1, b_2 + a_2, b_3 + a_3) = \mathbf{v} + \mathbf{u}. \end{aligned}$$

Here, we used only the definition of addition of vectors in $\mathbf{R}^3$ and the fact that addition of real numbers is commutative.

To verify property 4, we begin with a vector $\mathbf{v} = (b_1, b_2, b_3) \in \mathbf{R}^3$ and show that there is some vector in $\mathbf{R}^3$, which we denote by $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{z} = (0, 0, 0)$. There is an obvious choice for $-\mathbf{v}$, however, namely $(-b_1, -b_2, -b_3)$. Observe that

$$\begin{aligned} \mathbf{v} + (-b_1, -b_2, -b_3) &= (b_1, b_2, b_3) + (-b_1, -b_2, -b_3) \\ &= (b_1 + (-b_1),\ b_2 + (-b_2),\ b_3 + (-b_3)) = (0, 0, 0). \end{aligned}$$

Hence, $-\mathbf{v} = (-b_1, -b_2, -b_3)$ has the desired property. We note also that, according to the definition of scalar multiplication in $\mathbf{R}^3$,

$$(-1)\mathbf{v} = ((-1)b_1, (-1)b_2, (-1)b_3) = (-b_1, -b_2, -b_3) = -\mathbf{v}.$$

We will revisit this observation later.

To establish property 6, observe that

$$\begin{aligned}(\alpha+\beta)\mathbf{u} &= (\alpha+\beta)(a_1,b_1,c_1)\\ &= ((\alpha+\beta)a_1,(\alpha+\beta)b_1,(\alpha+\beta)c_1))\\ &= (\alpha a_1+\beta a_1,\alpha b_1+\beta b_1,\alpha c_1+\beta c_1)\\ &= (\alpha a_1,\alpha b_1,\alpha c_1)+(\beta a_1,\beta b_1,\beta c_1)\\ &= \alpha(a_1,b_1,c_1)+\beta(a_1,b_1,c_1)\\ &= \alpha\mathbf{u}+\beta\mathbf{u}.\end{aligned}$$

Thus, showing that $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$ also depends only on some familiar properties of addition and multiplication of real numbers. Vectors in the plane can be added and multiplied by scalars in the expected manner and, in fact, satisfy properties 1-8 as well.

15.2 Vector Spaces

In addition to vectors in the plane and 3-space, there are other mathematical objects that can be added and multiplied by scalars so that properties 1-8 are satisfied. Indeed, these objects provide a generalization of vectors in the plane and 3-space. For this reason, we will refer to these more abstract objects as vectors as well. The study of vectors is a major topic in the area of mathematics called linear algebra.

A nonempty set V, every two elements of which can be added (that is, if $\mathbf{u},\mathbf{v}\in V$, then $\mathbf{u}+\mathbf{v}$ is a unique vector of V) and each element of which can be multiplied by any real number (that is, if $\alpha\in\mathbf{R}$ and $\mathbf{v}\in V$, then $\alpha\mathbf{v}$ is a unique element in V) is called a **vector space** (in fact, a **vector space over R**) if it satisfies the following eight properties:

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$. (Commutative Property)
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$. (Associative Property)
3. There exists an element $\mathbf{z}\in V$ such that $\mathbf{v}+\mathbf{z}=\mathbf{v}$ for all $\mathbf{v}\in V$.
4. For each $\mathbf{v}\in V$, there exists an element $-\mathbf{v}\in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{z}$.
5. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$ for all $\alpha\in\mathbf{R}$ and all $\mathbf{u},\mathbf{v}\in V$.
6. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$ for all $\alpha,\beta\in\mathbf{R}$ and all $\mathbf{v}\in V$.
7. $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$ for $\alpha,\beta\in\mathbf{R}$ and all $\mathbf{v}\in V$.
8. $1\mathbf{v}=\mathbf{v}$ for all $\mathbf{v}\in V$.

The elements of V are called **vectors** and the real numbers in this definition are called **scalars**. Hence if $\mathbf{u},\mathbf{v}\in V$ and $\alpha,\beta\in\mathbf{R}$, then both $\alpha\mathbf{u}$ and $\beta\mathbf{v}$ belong to V. Therefore, $\alpha\mathbf{u}+\beta\mathbf{v}\in V$. The vector $\alpha\mathbf{u}+\beta\mathbf{v}$ is called a **linear combination** of $\mathbf{u}$ and $\mathbf{v}$. We can also discuss linear combinations of more than two vectors. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three vectors in V and let α,β,γ be three scalars (real numbers). Therefore, $\alpha\mathbf{u}$, $\beta\mathbf{v}$, and $\gamma\mathbf{w}$ are three vectors in V and $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. We've now encountered a familiar situation in mathematics. Since addition in V is only defined for two vectors, what exactly is meant by $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$? There are two obvious interpretations of $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$, namely, $(\alpha\mathbf{u}+\beta\mathbf{v})+\gamma\mathbf{w}$ (where $\alpha\mathbf{u}$ and $\beta\mathbf{v}$ are added first, producing the vector $\alpha\mathbf{u}+\beta\mathbf{v}$, which is then added to $\gamma\mathbf{w}$) and $\alpha\mathbf{u}+(\beta\mathbf{v}+\gamma\mathbf{w})$. However, property 2 (the associative law

of addition of vectors) guarantees that both interpretations give us the same vector and consequently, there is nothing ambiguous about writing $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ without parentheses. In fact, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$, then $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The element $\mathbf{z} \in V$ described in property 3 (and used in property 4) is called a **zero vector** and an element $-\mathbf{v}$ in property 4 is called a **negative** of $\mathbf{v}$. By the commutative property, we also know that $\mathbf{z} + \mathbf{v} = \mathbf{v}$ and $(-\mathbf{v}) + \mathbf{v} = \mathbf{z}$ for every vector $\mathbf{v} \in V$. Since V satisfies properties 1–4, the set V forms an abelian group under addition (see Chapter 13).

Although we have only defined a vector space over the set $\mathbf{R}$ of real numbers (and this is all we will deal with), it is not always required that the scalars be real numbers. Indeed, there are certain situations when complex numbers are not only suitable scalars but in fact, the preferred scalars. Other possibilities exist as well.

Of course, we have seen two examples of vector spaces, namely, $\mathbf{R}^2$ and $\mathbf{R}^3$ (with addition and scalar multiplication defined above). More generally, n-space $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}$ (n factors) is a vector space where addition of two vectors $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ is defined by

$$\mathbf{u} + \mathbf{v} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication $\alpha\mathbf{u}$, where $\alpha \in \mathbf{R}$, is defined by

$$\alpha\mathbf{u} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

We now describe two vector spaces of a very different nature. Recall that $\mathcal{F}_{\mathbf{R}}$ is the set of all functions from $\mathbf{R}$ to $\mathbf{R}$, that is,

$$\mathcal{F}_{\mathbf{R}} = \{f \ : \ f : \mathbf{R} \to \mathbf{R}\}.$$

Therefore, the well-known trigonometric function $f_1 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = \sin x$ for all $x \in \mathbf{R}$ belongs to $\mathcal{F}_{\mathbf{R}}$. The function $f_2 : \mathbf{R} \to \mathbf{R}$ defined by $f_2(x) = 3x + x/(x^2 + 1)$ for all $x \in \mathbf{R}$ also belongs to $\mathcal{F}_{\mathbf{R}}$.

For $f, g \in \mathcal{F}_{\mathbf{R}}$ and a scalar (real number) α, addition and scalar multiplication are defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \quad \text{for all } x \in \mathbf{R}, \\ (\alpha f)(x) &= \alpha(f(x)) \quad \text{for all } x \in \mathbf{R}. \end{aligned}$$

For the functions f_1 and f_2 defined above,

$$(f_1 + f_2)(x) = \sin x + 3x + \frac{x}{x^2+1} \quad \text{and} \quad (5f_2)(x) = 15x + \frac{5x}{x^2+1}.$$

Under these definitions of addition and scalar multiplication, $\mathcal{F}_{\mathbf{R}}$ is a vector space, the verification of which depends only on ordinary addition and multiplication of real numbers. As an illustration, we verify that $\mathcal{F}_{\mathbf{R}}$ satisfies properties 2-5 of a vector space.

First we verify property 2. Let $f, g, h \in \mathcal{F}_{\mathbf{R}}$. Then

$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x) + h(x) = (f(x) + g(x)) + h(x) \\ &= f(x) + (g(x) + h(x)) = f(x) + (g+h)(x) \\ &= (f + (g+h))(x) \end{aligned}$$

for all $x \in \mathbf{R}$. Therefore, $(f+g)+h = f+(g+h)$.

Second we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 3 of a vector space. Define the (constant) function $f_0 : \mathbf{R} \to \mathbf{R}$ by $f_0(x) = 0$ for all $x \in \mathbf{R}$. We show that f_0 is a zero vector for $\mathcal{F}_{\mathbf{R}}$. For $f \in \mathcal{F}_{\mathbf{R}}$,

$$(f+f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x)$$

for all $x \in \mathbf{R}$. Therefore, $f + f_0 = f$. The function f_0 is called the **zero function** in $\mathcal{F}_{\mathbf{R}}$.

Next we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 4 of a vector space. For each function $f \in \mathcal{F}_{\mathbf{R}}$, define the function $-f : \mathbf{R} \to \mathbf{R}$ by $(-f)(x) = -(f(x))$ for all $x \in \mathbf{R}$. Since

$$(f+(-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0 = f_0(x)$$

for all $x \in \mathbf{R}$, it follows that $f + (-f) = f_0$ and so $-f$ is a negative of f.

Finally, we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 5 of a vector space. Let $f, g \in \mathcal{F}_{\mathbf{R}}$ and $\alpha \in \mathbf{R}$. Then, for each $x \in \mathbf{R}$,

$$\begin{aligned} (\alpha(f+g))(x) &= \alpha\left((f+g)(x)\right) = \alpha\left(f(x)+g(x)\right) \\ &= \alpha f(x) + \alpha g(x) = (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x) \end{aligned}$$

and so $\alpha(f+g) = \alpha f + \alpha g$.

We now consider a special class of real-valued functions defined on $\mathbf{R}$. These functions are important in many areas of mathematics, not only linear algebra. A function $p : \mathbf{R} \to \mathbf{R}$ is called a **polynomial function** (actually a **polynomial function over R**) if

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

for all $x \in \mathbf{R}$, where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers. The expression $p(x)$ itself is called a **polynomial** in x. You may recall that if $a_n \neq 0$, then n is the **degree** of $p(x)$. The zero function f_0 is a polynomial function. It is assigned no degree, however. We denote the set of all polynomial functions over $\mathbf{R}$ by $\mathbf{R}[x]$. Hence $\mathbf{R}[x] \subseteq \mathcal{F}_{\mathbf{R}}$.

Let $f, g \in \mathbf{R}[x]$ and let $\alpha \in \mathbf{R}$. Then

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \ldots + b_m x^m,$$

where n and m are nonnegative integers and $a_i, b_j \in \mathbf{R}$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. If we assume, say, that $m \geq n$, then the sum $f + g$ is the polynomial function defined by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ &= (a_0+b_0) + (a_1+b_1)x + \ldots + (a_n+b_n)x^n + b_{n+1}x^{n+1} + \ldots + b_m x^m; \end{aligned}$$

while the scalar multiple αf of f by α is the polynomial function defined by

$$(\alpha f)(x) = \alpha(f(x)) = (\alpha a_0) + (\alpha a_1)x + \ldots + (\alpha a_n)x^n.$$

These definitions are, of course, exactly the same as the sum of two elements of $\mathcal{F}_{\mathbf{R}}$ and the scalar product of an element of $\mathcal{F}_{\mathbf{R}}$ by a real number.

Actually, $\mathbf{R}[x]$ is itself a vector space over $\mathbf{R}$ under the addition and scalar multiplication we have just defined. For example, let $f, g \in \mathbf{R}[x]$. Since $\mathbf{R}[x] \subseteq \mathcal{F}_{\mathbf{R}}$ and addition in $\mathbf{R}[x]$ is defined exactly the same as in $\mathcal{F}_{\mathbf{R}}$, it follows that $f + g = g + f$; that is, property 1 of a vector space is satisfied. By the same reasoning, property 2 and properties 5-8 are satisfied as well. The zero function f_0 is in $\mathbf{R}[x]$ and we know that $f + f_0 = f$ for all $f \in \mathcal{F}_{\mathbf{R}}$. Hence $p + f_0 = p$ for all $p \in \mathbf{R}[x]$. So f_0 is a zero vector for $\mathbf{R}[x]$. For $f \in \mathbf{R}[x]$ defined by $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, we know that $-f$ is given by $(-f)(x) = -(f(x)) = (-a_0) + (-a_1)x + \ldots + (-a_n)x^n$. Thus $-f \in \mathbf{R}[x]$ is a negative of f. Thus properties 3 and 4 are satisfied as well, and so $\mathbf{R}[x]$ is a vector space over $\mathbf{R}$.

15.3 Matrices

Among the best known and most important examples of vector spaces are those concerning matrices. A rectangular array of real numbers is called a **matrix**. The plural of "matrix" is "matrices". (In general, a matrix need not be an array of real numbers — it can be a rectangular array of elements from any prescribed set. However, we will deal only with real numbers.) Thus a matrix has m rows and n columns for some pair m, n of positive integers and contains mn real numbers, each of which is located in some row i and column j for integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. A matrix with m rows and n columns is said to have **size** $m \times n$ and is called an $m \times n$ **matrix** (read as "m by n matrix"). Hence

$$B = \begin{bmatrix} 1 & \sqrt{2} & -3/2 \\ 0 & -.8 & 4 \end{bmatrix}$$

is a 2×3 matrix, while

$$C = \begin{bmatrix} 4 & 1 & 9 \\ 0 & 3 & 2 \\ 7 & -1 & 1 \end{bmatrix}$$

is a 3×3 matrix. A general $m \times n$ matrix A is commonly written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Therefore, a_{ij} represents the element located in row i and column j of A. This is referred to as the (i, j)**-entry** of A. In fact, it is convenient shorthand notation to represent the matrix A by $[a_{ij}]$ and to write $A = [a_{ij}]$. The ith **row** of A is $[a_{i1}a_{i2}\dots a_{in}]$ and the jth **column** is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

For two matrices to be equal, they must have the same size. Furthermore, two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal**, written as $A = B$, if $a_{ij} = b_{ij}$ for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. That is, $A = B$ if A and B have the same size and corresponding entries are equal. Hence, in order for

$$A = \begin{bmatrix} 2 & x & -3 \\ 1/2 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4/5 & -3 \\ y & 4 & 0 \end{bmatrix}$$

to be equal, we must have $x = 4/5$ and $y = 1/2$.

For positive integers m and n, let $M_{mn}[\mathbf{R}]$ denote the set of all $m \times n$ matrices whose entries are real numbers. If $m = n$, then the matrices are called **square matrices**. The set of all $m \times m$ (square) matrices whose entries are real numbers is also denoted by $M_m[\mathbf{R}]$.

We now define addition and scalar multiplication in $M_{mn}[\mathbf{R}]$. Let $A, B \in M_{mn}[\mathbf{R}]$, where $A = [a_{ij}]$ and $B = [b_{ij}]$. The **sum** $A + B$ of A and B is defined as that $m \times n$ matrix $[c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$ for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. For $\alpha \in \mathbf{R}$, the **scalar multiple** αA of A by α is defined as $\alpha A = [d_{ij}]$, where $d_{ij} = \alpha a_{ij}$ for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$. For example, if

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -9 & 2 \\ -2 & 5 & 0 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 5 & -10 & -1 \\ -2 & 9 & 0 \end{bmatrix} \quad \text{and} \quad (-2)A = \begin{bmatrix} -4 & 2 & 6 \\ 0 & -8 & 0 \end{bmatrix}.$$

Under this addition and scalar multiplication, $M_{mn}[\mathbf{R}]$ is a vector space. As an illustration, we verify that properties 1 and 3-5 of a vector space are satisfied in $M_2[\mathbf{R}]$. Let $\alpha \in \mathbf{R}$ and let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = B + A. \end{aligned}$$

This verifies property 1 of a vector space. We see here that verifying property 1 depended only on the definition of addition of matrices and the fact that real numbers are commutative under addition.

Let $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, often called the 2×2 **zero matrix**. Then

$$\begin{aligned} A + Z &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ a_{21} + 0 & a_{22} + 0 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A \end{aligned}$$

and so Z is a zero element of $M_2[\mathbf{R}]$, thereby verifying property 3.

Next, let $-A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$. Consequently,

$$A + (-A) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Z,$$

and so $-A$ is a negative of A. Therefore, property 4 is satisfied. We note also that if A is multiplied by the scalar -1, then we obtain

$$(-1)A = (-1)\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = -A.$$

Finally,

$$\alpha(A + B) = \alpha \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} \alpha(a_{11} + b_{11}) & \alpha(a_{12} + b_{12}) \\ \alpha(a_{21} + b_{21}) & \alpha(a_{22} + b_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} + \alpha b_{11} & \alpha a_{12} + \alpha b_{12} \\ \alpha a_{21} + \alpha b_{21} & \alpha a_{22} + \alpha b_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \alpha b_{12} \\ \alpha b_{21} & \alpha b_{22} \end{bmatrix}$$

$$= \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \alpha \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \alpha A + \alpha B.$$

Under the right set of circumstances, matrices can also be multiplied — although this is, of course, not a requirement for a vector space.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times r$ matrix, that is, let A and B be two matrices, where the number of columns in A equals the number of rows in B. In this case, we define the **product** AB of A and B as that $m \times r$ matrix $[c_{ij}]$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \tag{15.1}$$

for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq r$. Hence the (i, j)-entry of AB is obtained from the ith row of A and jth column of B, that is,

$$[a_{i1}\ a_{i2}\ \ldots\ a_{in}] \quad \text{and} \quad \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

by multiplying corresponding terms of this row and column and then adding all n products. The expression (15.1) is referred to as the **inner product** of the ith row of A and the jth column of B. For example, let

$$A = \begin{bmatrix} 1 & -3 & 5 & 0 \\ -1 & 0 & 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -6 & 5 \\ 2 & 0 & 1 \\ 3 & 3 & 2 \\ -6 & 9 & 0 \end{bmatrix}.$$

Since A is a 2×4 matrix and B is a 4×3 matrix, the product AB is defined and, in fact, $AB = [c_{ij}]$ is the 2×3 matrix, where the six inner products are

$$\begin{aligned} c_{11} &= 1 \cdot 1 + (-3) \cdot 2 + 5 \cdot 3 + 0 \cdot (-6) = 10 \\ c_{12} &= 1 \cdot (-6) + (-3) \cdot 0 + 5 \cdot 3 + 0 \cdot 9 = 9 \\ c_{13} &= 1 \cdot 5 + (-3) \cdot 1 + 5 \cdot 2 + 0 \cdot 0 = 12 \\ c_{21} &= (-1) \cdot 1 + 0 \cdot 2 + 6 \cdot 3 + 2 \cdot (-6) = 5 \\ c_{22} &= (-1) \cdot (-6) + 0 \cdot 0 + 6 \cdot 3 + 2 \cdot 9 = 42 \\ c_{23} &= (-1) \cdot 5 + 0 \cdot 1 + 6 \cdot 2 + 2 \cdot 0 = 7. \end{aligned}$$

Hence

$$AB = \begin{bmatrix} 10 & 9 & 12 \\ 5 & 42 & 7 \end{bmatrix}.$$

On the other hand, since the matrix B above is a 4×3 matrix and A is a 2×4 matrix, the product BA is not defined. Certainly, however, if A and B are any two square matrices of the same size, then AB and BA are both defined though they need not be equal. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \text{ while } \quad BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

15.4 Some Properties of Vector Spaces

Although we have now seen several different vector spaces, there are a number of properties that these vector spaces have in common (in addition to the eight defining properties). Indeed, there are a number of additional properties that *all* vector spaces have in common. Since vector spaces are defined by eight properties, one might expect, and rightfully so, that any other properties they have in common are consequences of these eight properties.

According to property 3, every vector space contains at least one zero vector and by property 4, every vector has at least one negative. We show that "at least one" can be replaced by "exactly one" in both instances. Actually, these are consequences of the fact that every vector space is a group under addition (Chapter 13). We verify these nevertheless.

Theorem 15.1 *Every vector space has a unique zero vector.*

Proof. Let V be a vector space and assume that $\mathbf{z}$ and $\mathbf{z}'$ are both zero vectors in V. Since $\mathbf{z}$ is a zero vector, $\mathbf{z}' + \mathbf{z} = \mathbf{z}'$. Moreover, since $\mathbf{z}'$ is a zero vector, $\mathbf{z} + \mathbf{z}' = \mathbf{z}$. Therefore, $\mathbf{z} = \mathbf{z} + \mathbf{z}' = \mathbf{z}' + \mathbf{z} = \mathbf{z}'$. ∎

As a consequence of Theorem 15.1, we now know that a vector space V possesses only one zero vector $\mathbf{z}$ that satisfies property 3 of a vector space. Hence we can now refer to $\mathbf{z}$ as *the* zero vector of V.

Theorem 15.2 *Let V be a vector space. Then every vector in V has a unique negative.*

Proof. Let $\mathbf{v} \in V$ and assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both negatives of $\mathbf{v}$. Thus $\mathbf{v} + \mathbf{v}_1 = \mathbf{z}$ and $\mathbf{v} + \mathbf{v}_2 = \mathbf{z}$. Hence

$$\mathbf{v}_1 = \mathbf{v}_1 + \mathbf{z} = \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 = \mathbf{z} + \mathbf{v}_2 = \mathbf{v}_2. \qquad ∎$$

Proof Analysis Let's revisit the proof of Theorem 15.2. We wanted to show that each vector $\mathbf{v}$ has only one negative. We assumed that there were two negatives of $\mathbf{v}$, namely $\mathbf{v}_1$ and $\mathbf{v}_2$. Our goal then was to show that $\mathbf{v}_1 = \mathbf{v}_2$. We started with $\mathbf{v}_1$. Our idea was to add $\mathbf{z}$ to $\mathbf{v}_1$, as this sum is the vector $\mathbf{v}_1$ again. Since $\mathbf{z}$ can also be expressed as $\mathbf{v} + \mathbf{v}_2$, we made this substitution, bringing the vector $\mathbf{v}_2$ into the discussion. Eventually, we showed that this expression for $\mathbf{v}_1$ was also equal to $\mathbf{v}_2$. There is another approach we could have tried.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are both negatives of $\mathbf{v}$, it follows that $\mathbf{v} + \mathbf{v}_1 = \mathbf{z}$ and $\mathbf{v} + \mathbf{v}_2 = \mathbf{z}$, that is, $\mathbf{v} + \mathbf{v}_1 = \mathbf{v} + \mathbf{v}_2$. If we add the same vector to both $\mathbf{v} + \mathbf{v}_1$ and $\mathbf{v} + \mathbf{v}_2$, we obtain equal vectors (since $\mathbf{v} + \mathbf{v}_1 = \mathbf{v} + \mathbf{v}_2$). A good choice of a vector to add to both $\mathbf{v} + \mathbf{v}_1$ and $\mathbf{v} + \mathbf{v}_2$ is a negative of $\mathbf{v}$ (either one!). This gives us the following list of equalities:

$$\begin{aligned} \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_1) &= \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) \\ (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_1 &= (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 \\ \mathbf{z} + \mathbf{v}_1 &= \mathbf{z} + \mathbf{v}_2 \\ \mathbf{v}_1 &= \mathbf{v}_2. \end{aligned}$$

Although this string of equalities results in $\mathbf{v}_1 = \mathbf{v}_2$, this is not a particularly well-written proof. However, since our goal is to show that $\mathbf{v}_1 = \mathbf{v}_2$, this suggests a way to arrive at our goal. We start with $\mathbf{v}_1$ (at the bottom of the left column), proceed upward, then to the right, and downward, producing

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{z} + \mathbf{v}_1 = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_1 = \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_1) \\ &= \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 = \mathbf{z} + \mathbf{v}_2 = \mathbf{v}_2, \end{aligned}$$

which is similar to the proof given in Theorem 15.2 (though a bit longer). ♢

As a consequence of Theorem 15.2, we can now refer to $-\mathbf{v}$ as *the* negative of $\mathbf{v}$. Of course, the zero vector $\mathbf{z}$ has the property that $\mathbf{z}+\mathbf{z} = \mathbf{z}$. However, no other vector has this property.

Theorem 15.3 *Let V be a vector space. If $\mathbf{v}$ is a vector such that $\mathbf{v}+\mathbf{v} = \mathbf{v}$, then $\mathbf{v} = \mathbf{z}$.*

Proof. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$, it follows that

$$\mathbf{z} = \mathbf{v} + (-\mathbf{v}) = (\mathbf{v} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (\mathbf{v} + (-\mathbf{v})) = \mathbf{v} + \mathbf{z} = \mathbf{v}. \qquad \blacksquare$$

A proof like that given for Theorem 15.3 can be obtained by adding $-\mathbf{v}$ to the equal vectors $\mathbf{v} + \mathbf{v}$ and $\mathbf{v}$ and proceeding as we did in the discussion following the proof of Theorem 15.2. Also, see Exercise 15.6(b).

We now describe two other properties concerning the zero vector that are consequences of Theorem 15.3.

Corollary 15.4 *Let V be a vector space. Then*

(*i*) $0\mathbf{v} = \mathbf{z}$ *for every vector* $\mathbf{v}$ *in* V *and*

(*ii*) $\alpha\mathbf{z} = \mathbf{z}$ *for every scalar* $\alpha \in \mathbf{R}$.

Proof. First, we prove (*i*). Observe that

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}.$$

By Theorem 15.3, $0\mathbf{v} = \mathbf{z}$.

Next we verify (*ii*). Observe that

$$\alpha\mathbf{z} = \alpha(\mathbf{z} + \mathbf{z}) = \alpha\mathbf{z} + \alpha\mathbf{z}.$$

Again, by Theorem 15.3, $\alpha\mathbf{z} = \mathbf{z}$. ■

Hence, by Corollary 15.4, $0\mathbf{v} = \mathbf{z}$ for every vector $\mathbf{v}$ in a vector space and $\alpha\mathbf{z} = \mathbf{z}$ for every scalar α. That is, if either $\alpha = 0$ or $\mathbf{v} = \mathbf{z}$, then $\alpha\mathbf{v} = \mathbf{z}$. We now show that the converse of this statement is true as well.

Theorem 15.5 *Let V be a vector space. If $\alpha\mathbf{v} = \mathbf{z}$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{z}$.*

Proof. If $\alpha = 0$, then, of course, the statement is true. So we may assume that $\alpha \neq 0$. In this case,

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{\alpha}\ \alpha\right)\mathbf{v} = \left(\frac{1}{\alpha}\right)(\alpha\mathbf{v}) = \left(\frac{1}{\alpha}\right)\mathbf{z} = \mathbf{z}. \qquad \blacksquare$$

Another useful property is that the scalar multiple of a vector by -1 is the negative of that vector. Actually, we have observed this earlier with two particular vector spaces but this is true in general.

Theorem to Prove If $\mathbf{v}$ is a vector in a vector space, then $(-1)\mathbf{v} = -\mathbf{v}$.

Proof Strategy Since $\mathbf{v}$ has a unique negative, to show that $(-1)\mathbf{v} = -\mathbf{v}$, we need only verify that the sum of $\mathbf{v}$ and $(-1)\mathbf{v}$ is $\mathbf{z}$. $\diamondsuit$

Theorem 15.6 *If $\mathbf{v}$ is a vector in a vector space, then $(-1)\mathbf{v} = -\mathbf{v}$.*

Proof. Observe that

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = 0\mathbf{v} = \mathbf{z}.$$

Hence $(-1)\mathbf{v} = -\mathbf{v}$. $\blacksquare$

15.5 Subspaces

Earlier we saw that $\mathcal{F}_{\mathbf{R}} = \{f\ :\ f : \mathbf{R} \to \mathbf{R}\}$ is a vector space (under function addition and scalar multiplication). Since the set $\mathbf{R}[x]$ of all polynomial functions over $\mathbf{R}$ is a subset of $\mathcal{F}_{\mathbf{R}}$ and the addition and scalar multiplication defined in $\mathbf{R}[x]$ are exactly the same as those defined in $\mathcal{F}_{\mathbf{R}}$, it was considerably easier to show that $\mathbf{R}[x]$ is a vector space. This idea can be made more general.

For a vector space V, a subset W of V is called a **subspace** of V if W is vector space under the same addition and scalar multiplication defined on V. Hence if W is a subspace of a known vector space V, then W itself is a vector space. Since every subspace contains a zero vector, W must be nonempty.

As we study vector spaces further, we will see that certain subspaces appear regularly and consequently it is beneficial to have an understanding of subspaces. Furthermore, some sets having an addition and scalar multiplication defined on them are subsets of known vector spaces and can be shown to be vector spaces more easily by verifying that they are subspaces.

What is required to show that a subset W of a vector space V is a subspace of V? Of course, W must satisfy the eight properties required of all vector spaces. In addition, if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v}$ must belong to W. This property is expressed by saying that W is **closed under addition**. Also, if α is a scalar (a real number) and $\mathbf{v} \in W$, then $\alpha\mathbf{v}$ must belong to W. We express this property by saying that W is **closed under scalar multiplication**.

Property 1 (the commutative property) requires that $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for every two vectors $\mathbf{u}$ and $\mathbf{v}$ in W. However, V is a vector space and satisfies property 1. Thus $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ and W satisfies property 1. By the same reasoning, property 2 and properties 5-8 are satisfied by W. These properties of W are said to be **inherited** from V. Hence for a nonempty subset W of a vector space V to be a subspace of V, it is necessary that W be closed under addition and scalar multiplication. Perhaps surprisingly, these requirements are sufficient as well for a nonempty subset W of V to be subspace of V.

Theorem 15.7 (**The Subspace Test**) *A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under addition and scalar multiplication.*

Proof. First, let W be a subspace of V. Certainly, W is closed under addition and scalar multiplication. For the converse, let W be a nonempty subset of V that is closed under addition and scalar multiplication. As we noted earlier, W inherits properties 1, 2 and 5-8 of a vector space from V. Since W is nonempty and is closed under addition and scalar multiplication, only properties 3 and 4 remain to be verified. Since $W \neq \emptyset$, there is some vector $\mathbf{v}$ in W. Since W is closed under scalar multiplication, it follows by Corollary 15.4(i) that $0\mathbf{v} = \mathbf{z} \in W$. Hence W contains a zero vector (namely the zero vector of V) and property 3 is satisfied. Now let $\mathbf{w}$ be any vector of W. Again, $(-1)\mathbf{w} \in W$. However, by Theorem 15.6, $(-1)\mathbf{w} = -\mathbf{w} \in W$, and so $\mathbf{w}$ has a negative in W (namely the negative of $\mathbf{w}$ in V). Thus property 4 is satisfied in W as well. ∎

The proof of Theorem 15.7 brought out two important facts. Namely, if W is a subspace of a vector space V, then W contains a zero vector (namely, the zero vector of V) and for every vector $\mathbf{w} \in W$, its negative $-\mathbf{w}$ belongs to W as well.

Every vector space V (containing at least two elements) always contains two subspaces, namely V itself and the subspace consisting only of the zero vector of V. We now present several examples to illustrate how the Subspace Test (Theorem 15.7) can be applied to show that certain subsets of a vector space are (or are not) subspaces of that vector space. The first two examples concern the vector space $\mathbf{R}^3$.

Result 15.8 *The set*

$$W = \{(a, b, 2a - b) \ : \ a, b \in \mathbf{R}\}$$

is a subspace of $\mathbf{R}^3$.

First observe that W contains all vectors of $\mathbf{R}^3$ whose 3rd coordinate is twice the first coordinate minus the second coordinate. So for example, W contains $(3, 2, 4)$, taking $a = 3$ and $b = 2$, and $(0, 0, 0)$, taking $a = b = 0$. Of course, if W is to be a subspace of $\mathbf{R}^3$, then it is essential that W contains the zero vector of $\mathbf{R}^3$.

Proof of Result 15.8. Since W contains the zero vector of $\mathbf{R}^3$, it follows that $W \neq \emptyset$. To show that W is a subspace of V, we need only show that W is closed under addition (that is, if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u}+\mathbf{v} \in W$) and that W is closed under scalar multiplication (that is, if $\mathbf{u} \in W$ and $\alpha \in \mathbf{R}$, then $\alpha\mathbf{u} \in W$). Let $\mathbf{u}, \mathbf{v} \in W$ and $\alpha \in \mathbf{R}$. Then $\mathbf{u} = (a, b, 2a - b)$ and $\mathbf{v} = (c, d, 2c - d)$, where $a, b, c, d \in \mathbf{R}$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (a + c, b + d, 2(a + c) - (b + d)) \in W \quad \text{and} \\ \alpha\mathbf{u} &= (\alpha a, \alpha b, 2(\alpha a) - (\alpha b)) \in W. \end{aligned}$$

By the Subspace Test, W is a subspace of $\mathbf{R}^3$. ∎

Example 15.9 *Determine whether*

$$W = \{(a, b, a^2 + b) \ : \ a, b \in \mathbf{R}\}$$

is a subspace of $\mathbf{R}^3$.

Solution. Taking $a = b = 1$, we see that $\mathbf{u} = (1, 1, 2) \in W$. Then $2\mathbf{u} = (2, 2, 4)$. Since $4 \neq 2^2 + 2$, it follows that $2\mathbf{u} \notin W$. Since W is not closed under scalar multiplication, W is not a subspace of $\mathbf{R}^3$. (The subset W of $\mathbf{R}$ is not closed under addition either since $\mathbf{u} + \mathbf{u} \notin W$.) $\diamondsuit$

We next consider the vector space $\mathcal{F}_{\mathbf{R}}$. We have already mentioned that $\mathbf{R}[x]$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. Also, the set $\mathcal{C}_{\mathbf{R}} = \{f \in \mathcal{F}_{\mathbf{R}} \ : \ f \text{ is continuous}\}$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. Indeed, $\mathbf{R}[x]$ is a subspace of $\mathcal{C}_{\mathbf{R}}$ as well.

Result 15.10 *Let $\mathcal{F}_0 = \{f \in \mathcal{F}_{\mathbf{R}} \ : \ f(1) = 0\}$. Then $\mathcal{F}_0$ is a subspace of $\mathcal{F}_{\mathbf{R}}$.*

Hence the function $f_1 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = x - 1$ belongs to $\mathcal{F}_0$, as does the zero function $f_0 : \mathbf{R} \to \mathbf{R}$ defined by $f_0(x) = 0$ for all x.

Proof of Result 15.10. Since $\mathcal{F}_0$ contains the zero function, $\mathcal{F}_0 \neq \emptyset$. Let $f, g \in \mathcal{F}_0$ and $\alpha \in \mathbf{R}$. Then

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0 \quad \text{and} \quad (\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0.$$

Thus $f + g \in \mathcal{F}_0$ and $\alpha f \in \mathcal{F}_0$. By the Subspace Test, $\mathcal{F}_0$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. ∎

Example 15.11 *Determine whether*

$$\mathcal{F}_1 = \{f \in \mathcal{F}_{\mathbf{R}} \ : \ f(0) = 1\}$$

is a subspace of $\mathcal{F}_{\mathbf{R}}$.

Solution. Observe that the functions $g, h \in \mathcal{F}_{\mathbf{R}}$ defined by $g(x) = x+1$ and $h(x) = x^2+1$ belong to $\mathcal{F}_1$. However, $(g+h)(x) = g(x)+h(x) = x^2+x+2$ and $(g+h)(0) = 2$, so $g+h \notin \mathcal{F}_1$. Therefore, $\mathcal{F}_1$ is not a subspace of $\mathcal{F}_{\mathbf{R}}$. $\diamondsuit$

The next example concerns the vector space $M_2(\mathbf{R})$ of 2×2 matrices with real entries.

Result 15.12 *The set*

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \ a, b, c \in \mathbf{R} \right\}$$

is a subspace of $M_2(\mathbf{R})$.

Hence W consists of all these 2×2 matrices whose $(1, 2)$-entry is 0. Thus the zero matrix, all of whose entries are 0, belongs to W.

Proof of Result 15.12. Since W contains the zero matrix, $W \neq \emptyset$. Let $A, B \in W$ and $\alpha \in \mathbf{R}$. So

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & 0 \\ e & f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbf{R}$. Then

$$A + B = \begin{bmatrix} a + d & 0 \\ b + e & c + f \end{bmatrix} \quad \text{and} \quad \alpha A = \begin{bmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{bmatrix}.$$

Therefore, $A + B$ and αA belong to W and by the Subspace Test, W is a subspace of $M_2(\mathbf{R})$. ∎

15.6 Spans of Vectors

In Result 15.12 we showed that the set

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \ a, b, c \in \mathbf{R} \right\}$$

is a subspace of $M_2(\mathbf{R})$. Thus if $A \in W$, then $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbf{R}$. Observe, also, that

$$\begin{aligned} A &= \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

In other words, A (and, consequently, every matrix in W) is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, W is the set of all linear combinations of these three matrices. This observation illustrates a more general situation.

Recall that if V is a vector space, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$, then every vector of the form $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Thus, by taking $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, we see that the zero vector is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Also, by taking $\alpha_i = 1$ for a fixed integer i $(1 \leq i \leq n)$ and all other scalars 0, we see that each vector $\mathbf{v_i}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We have noted that every linear combination of vectors in V is a vector in V and, of course, the set of all such linear combinations is a subset of V. In fact, more can be said of this subset.

Theorem 15.13 *Let V be a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then the set W of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a subspace of V.*

Proof. Since W contains the zero vector of V, it follows that $W \neq \emptyset$. Let $\mathbf{u}, \mathbf{w} \in W$ and let $\alpha \in \mathbf{R}$. Then $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n$, where $\alpha_i, \beta_i \in \mathbf{R}$ for $1 \leq i \leq n$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \ldots + (\alpha_n + \beta_n)\mathbf{v}_n \text{ and} \\ \alpha\mathbf{u} &= (\alpha\alpha_1)\mathbf{v}_1 + (\alpha\alpha_2)\mathbf{v}_2 + \ldots + (\alpha\alpha_n)\mathbf{v}_n. \end{aligned}$$

So both $\mathbf{u} + \mathbf{w}$ and $\alpha\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and hence belong to W. Thus by the Subspace Test, W is a subspace of V. ∎

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V, the subspace W of V consisting of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and is denoted by $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. Also, W is referred to as the subspace of V **spanned** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

By Result 15.12,

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \ a, b, c \in \mathbf{R} \right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

We saw in Result 15.8 that $W = \{(a, b, 2a - b) \ : \ a, b \in \mathbf{R}\}$ is a subspace of of $\mathbf{R}^3$. Since $(a, b, 2a - b) = a(1, 0, 2) + b(0, 1, -1)$, it follows that W is spanned by the vectors $(1, 0, 2)$ and $(0, 1, -1)$, that is, $W = \langle (1, 0, 2), (0, 1, -1) \rangle$.

We consider another illustration of spans of vectors.

Result 15.14 *Let f_1, f_2, f_3, g_2 and g_3 be five functions in $\mathbf{R}[x]$ defined by $f_1(x) = 1$, $f_2(x) = 1 + x^2$, $f_3(x) = 1 + x^2 + x^4$, $g_2(x) = x^2$, and $g_3(x) = x^4$ for all $x \in \mathbf{R}$, and let $W = \langle f_1, f_2, f_3 \rangle$ and $W' = \langle f_1, g_2, g_3 \rangle$. Then $W = W'$.*

Since W and W' are sets of vectors (polynomial functions) and our goal is to show that $W = W'$, we proceed in the standard manner by showing that each of W and W' is a subset of the other.

Proof of Result 15.14. First, we show that $W \subseteq W'$. Let $f \in W$. Then $f = af_1 + bf_2 + cf_3$ for some $a, b, c \in \mathbf{R}$. Hence, for each $x \in \mathbf{R}$,

$$\begin{aligned} f(x) &= a \cdot 1 + b \cdot \left(1 + x^2\right) + c \cdot \left(1 + x^2 + x^4\right) \\ &= (a + b + c) + (b + c) \cdot x^2 + c \cdot x^4. \end{aligned}$$

Thus, f is also a linear combination of f_1, g_2, and g_3. Consequently, $W \subseteq W'$. It remains to show that $W' \subseteq W$. Let $g \in W'$. Then

$$g = af_1 + bg_2 + cg_3 \text{ for some } a, b, c \in \mathbf{R}.$$

So, for each $x \in \mathbf{R}$,

$$\begin{aligned} g(x) &= a \cdot 1 + b \cdot x^2 + c \cdot x^4 = (a - b) \cdot 1 + b \cdot \left(1 + x^2\right) + c \cdot x^4 \\ &= (a - b) \cdot 1 + (b - c) \cdot \left(1 + x^2\right) + c \cdot \left(1 + x^2 + x^4\right). \end{aligned}$$

Hence g is also a linear combination of f_1, f_2, f_3 as well and so $W' \subseteq W$. ∎

From what we have seen, if V is a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ is a subspace of V (that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$). Quite possibly other subspaces of V contain $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as well. Of course, V itself is a subspace of V containing $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In a certain sense though, W is the smallest subspace of V containing $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Theorem 15.15 *Let V be a vector space containing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ and let $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. If W' is a subspace of V containing $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$, then W is a subspace of W'.*

Proof. Since W and W' are subspaces of V, we need only show that $W \subseteq W'$. Let $\mathbf{v} \in W$. Thus $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$, where $\alpha_i \in \mathbf{R}$ for $1 \leq i \leq n$. Since $\mathbf{v}_i \in W'$ for $1 \leq i \leq n$ and W' is a subspace of V, it follows that $\mathbf{v} \in W'$. Hence $W \subseteq W'$. ∎

There is a consequence of Theorem 15.15 that is especially useful.

Corollary 15.16 *Let V be a vector space spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$. If W is a subspace of V containing $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$, then $W = V$.*

Proof. Since W is a subspace of V, certainly $W \subseteq V$. By Theorem 15.15, $V \subseteq W$. Thus $W = V$. ∎

To illustrate a number of the concepts and results introduced thus far, we consider an example concerning 3-space.

Result 15.17

(*i*) *For the vectors* $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, *and* $\mathbf{k} = (0,0,1)$, $\mathbf{R}^3 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$.

(*ii*) *If* $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, *and* $\mathbf{w}_3 = (1,1,1)$, *then* $\mathbf{R}^3 = \langle \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \rangle$.

(*iii*) *Let* $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (1,1,0)$, *and* $\mathbf{u}_3 = (0,0,1)$. *Then* $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$.

Proof. Let $W_1 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$. Since W_1 is a subspace of $\mathbf{R}^3$, it follows that $W_1 \subseteq \mathbf{R}^3$. We now show that $\mathbf{R}^3 \subseteq W_1$. Let $\mathbf{v} \in \mathbf{R}^3$. So $\mathbf{v} = (a,b,c)$, where $a, b, c \in \mathbf{R}$. Then $\mathbf{v} = (a,0,0) + (0,b,0) + (0,0,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Hence $\mathbf{v}$ is a linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and so $\mathbf{v} \in W_1$. Hence $\mathbf{R}^3 \subseteq W_1$. This implies that $\mathbf{R}^3 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ and (*i*) is verified.

Next, we verify (*ii*). Let $W_2 = \langle \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \rangle$. To verify that $\mathbf{R}^3 = W_2$, it suffices to show by Corollary 15.16 and part (*i*) of this result that each of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ belongs to W_2. To show that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ belong to W_2, we are then required to show that each of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is a linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. Since $\mathbf{i} = (1,0,0) = (1,1,1) + (-1)(0,1,1)$, it follows that $\mathbf{i} = 0 \cdot \mathbf{w}_1 + (-1)\mathbf{w}_2 + 1 \cdot \mathbf{w}_3$. Now $\mathbf{j} = (0,1,0) = (1,1,0) + (0,1,1) + (-1)(1,1,1)$; so $\mathbf{j} = 1 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + (-1)\mathbf{w}_3$. Finally, $\mathbf{k} = (0,0,1) = (1,1,1) + (-1)(1,1,0)$ and so $\mathbf{k} = (-1)\mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 1 \cdot \mathbf{w}_3$. Hence $\mathbf{R}^3 = W_2$ and (ii) is established.

Finally, we verify (*iii*). Let $W = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ and $W' = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$. Since W' contains the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, it follows by Theorem 15.15 that $W \subseteq W'$.

By Corollary 15.16, to prove that $W' \subseteq W$, we need only show that each of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ belongs to W, that is, each of these three vectors is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. This is obvious for $\mathbf{u}_1$ and $\mathbf{u}_2$ as $\mathbf{u}_1 = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2$ and $\mathbf{u}_2 = 0 \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2$. Thus it remains only to show that $\mathbf{u}_3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. However, $\mathbf{u}_3 = (0,0,1) = (1,1,1) + (-1)(1,1,0) = 1 \cdot \mathbf{u}_1 + (-1)\mathbf{u}_2$, completing the proof. ∎

15.7 Linear Dependence and Independence

For the vectors $\mathbf{u}_1 = (1,1,0)$ and $\mathbf{u}_2 = (0,1,1)$ in $\mathbf{R}^3$, the vector $\mathbf{u}_3 = (-1,1,2) \in \mathbf{R}^3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ since

$$\mathbf{u}_3 = (-1,1,2) = (-1) \cdot \mathbf{u}_1 + 2 \cdot \mathbf{u}_2 = (-1) \cdot (1,1,0) + 2 \cdot (0,1,1).$$

Therefore, in a certain sense, the vector $\mathbf{u}_3$ *depends on* $\mathbf{u}_1$ and $\mathbf{u}_2$ *in a linear manner*. This linear dependence can be restated as

$$(-1) \cdot \mathbf{u}_1 + 2 \cdot \mathbf{u}_2 + (-1) \cdot \mathbf{u}_3 = (0,0,0).$$

This kind of dependence plays an important role in linear algebra.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a nonempty set of vectors in a vector space V. The set S is called **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_m$, not all 0, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{z}$. If S is not linearly dependent, then S is said to be **linearly independent**. For $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, we also say that the *vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are

linearly dependent or linearly independent according to whether the *set* S is linearly dependent or linearly independent, respectively. Consequently, the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are **linearly independent** if whenever $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{z}$, then $c_i = 0$ for *each* i $(1 \leq i \leq m)$.

We now consider some examples.

Example 15.18 *Determine whether* $S = \{(1,1,1), (1,1,0), (0,1,1)\}$ *is a linearly independent set of vectors in* $\mathbf{R}^3$.

Solution. Let a, b, and c be scalars such that

$$a \cdot (1,1,1) + b \cdot (1,1,0) + c \cdot (0,1,1) = (0,0,0).$$

By scalar multiplication and vector addition, we have $(a+b, a+b+c, a+c) = (0,0,0)$, arriving at the following system of equations:

$$\begin{aligned} a+b &= 0 \\ a+b+c &= 0 \\ a+c &= 0. \end{aligned}$$

Subtracting the first equation from the second, we obtain $c = 0$. Substituting $c = 0$ into the third equation, we obtain $a = 0$. Substituting $a = 0$ and $c = 0$ into the second equation, we obtain $b = 0$. Hence $a = b = c = 0$ and S is linearly independent. $\diamondsuit$

Example 15.19 *Determine whether*

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

is a linearly independent set of vectors in $M_2(\mathbf{R})$.

Solution. Again, let a, b, and c be scalars such that

$$a \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By scalar multiplication and matrix addition, we have

$$\begin{bmatrix} 2a+c & a+b+c \\ a+b+c & 2b+c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This results in the system of equations:

$$\begin{aligned} 2a+c &= 0 \\ a+b+c &= 0 \\ 2b+c &= 0 \end{aligned}$$

where the second equation actually occurs twice. From the first and third equations, it follows that $c = -2a$ and $c = -2b$ and so $a = b = -c/2$. Substituting these values for a

and b in the second equation gives $(-c/2) + (-c/2) + c = -c + c = 0$, that is, the second equation is satisfied for every value of c. Hence, if we let $c = -2$, say, then $a = b = 1$ and

$$1 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, S is a linearly dependent set of vectors. $\diamondsuit$

We now show that a familiar set of polynomial functions is linearly independent.

Theorem to Prove For every nonnegative integer n, the set $S_n = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathbf{R}[x]$.

Proof Strategy The elements of S_n are actually functions, say $S_n = \{f_0, f_1, f_2, \ldots, f_n\}$, where $f_i : \mathbf{R} \to \mathbf{R}$ is defined by $f_i(x) = x^i$ for $0 \leq i \leq n$ and for all $x \in \mathbf{R}$. To show that S_n is linearly independent, we are required to show that if $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$, where $c_i \in \mathbf{R}$ for $0 \leq i \leq n$, then $c_i = 0$ for all i. Of course, the question is how to do this. By choosing various values of x, we could arrive at a system of equations to solve. For example, we could begin by letting $x = 0$, obtaining $c_0 \cdot 1 + c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0$, and so $c_0 = 0$. Therefore, $c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$. Letting $x = 1$ and $x = 2$, we have $c_1 + c_2 + \ldots + c_n = 0$ and $2c_1 + 2^2 c_2 + \ldots + 2^n c_n = 0$. We could actually arrive at a system of n equations and n unknowns, but perhaps this is sounding complicated.

On the other hand, from the statement of the theorem, another approach is suggested. Quite often when we see a theorem stated as "for every nonnegative integer n", we think of applying induction. The main challenge to such a proof would be to show that if $\{1, x, x^2, \ldots, x^k\}$ is linearly independent, where $k \geq 0$, then $\{1, x, x^2, \ldots, x^{k+1}\}$ is linearly independent. Hence we would be dealing with the equation $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1} = 0$ for $c_i \in \mathbf{R}$, $0 \leq i \leq k+1$, attempting to show that $c_i = 0$ for all i $(0 \leq i \leq k+1)$. We already mentioned that showing $c_0 = 0$ is not difficult. In order to make use of the induction hypothesis, we need a linear combination of the polynomials $1, x, x^2, \ldots, x^k$. One idea for doing this is to take the derivative of $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1}$. $\diamondsuit$

Theorem 15.20 *For every nonnegative integer n, the set $S_n = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathbf{R}[x]$.*

Proof. We proceed by induction. For $n = 0$, we are required to show that $S_0 = \{1\}$ is linearly independent in $\mathbf{R}[x]$. Let c be a scalar such that $c \cdot 1 = 0$. Then surely $c = 0$ and so S_0 is linearly independent.

Assume that $S_k = \{1, x, x^2, \ldots, x^k\}$ is linearly independent in $\mathbf{R}[x]$, where k is a nonnegative integer. We show that $S_{k+1} = \{1, x, x^2, \ldots, x^{k+1}\}$ is linearly independent in $\mathbf{R}[x]$. Let $c_0, c_1, \ldots, c_{k+1}$ be scalars such that

$$c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1} = 0, \tag{15.2}$$

for all $x \in \mathbf{R}$. Letting $x = 0$ in (15.2), we see that $c_0 = 0$. Now taking the derivatives of both sides of (15.2), we see that

$$c_1 \cdot 1 + 2c_2 x + 3c_3 x^2 + \ldots + (k+1) c_{k+1} x^k = 0$$

for all $x \in \mathbf{R}$. By the induction hypothesis, S_k is a linearly independent set of vectors in $\mathbf{R}[x]$ and so $c_1 = 2c_2 = 3c_3 = \ldots = (k+1)c_{k+1} = 0$, which implies that $c_1 = c_2 = c_3 = \ldots = c_{k+1} = 0$. Since $c_0 = 0$ as well, it follows that S_{k+1} is linearly independent. ∎

Proof Analysis Before proceeding further, it is important that we understand the proof we have just given. The proof began by showing that $S_0 = \{1\}$ is linearly independent. What this means is that S_0 consists of the single constant polynomial function f defined by $f(x) = 1$ for all $x \in \mathbf{R}$. Let c be a scalar (real number) such that $c \cdot f = f_0$, where f_0 is the zero polynomial function defined by $f_0(x) = 0$ for all $x \in \mathbf{R}$. Thus, for each $x \in \mathbf{R}$, $(cf)(x) = f_0(x) = 0$, that is,

$$(cf)(x) = c \cdot f(x) = c \cdot 1 = 0 = f_0(x)$$

and so $c = 0$. $\diamondsuit$

We now consider a result for a general vector space.

Result 15.21 *If $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent vectors in a vector space V, then $\mathbf{v}_1$, $\mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are also linearly independent in V.*

Proof. Let a, b, and c be scalars such that

$$a \cdot \mathbf{v}_1 + b \cdot (\mathbf{v}_1 + \mathbf{v}_2) + c \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{z}.$$

From this, we have

$$(a + b + c) \cdot \mathbf{v}_1 + (b + c) \cdot \mathbf{v}_2 + c \cdot \mathbf{v}_3 = \mathbf{z}.$$

Since $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent, $a + b + c = b + c = c = 0$, from which it follows that $a = b = c = 0$ and so $\mathbf{v}_1$, $\mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent. ∎

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors, where $n \in \mathbf{N}$, and let S' be a nonempty subset of S. Then $|S'| = m$ for some integer m with $1 \leq m \leq n$. Since the order in which the elements of S are listed is irrelevant, these elements can be rearranged and relabeled if necessary so that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. This fact is quite useful at times.

Theorem 15.22 *Let S be a finite nonempty set of vectors in a vector space V. If S is linearly independent in V and S' is a nonempty subset of S, then S' is also linearly independent in V.*

Proof. We may assume that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $S = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$, where then $1 \leq m \leq n$. If $m = n$, then $S' = S$ and surely S' is linearly independent. Thus we can assume that $m < n$. Let $c_1, c_2, \ldots, c_m$ be scalars such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{z}.$$

However, then,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + 0\mathbf{v}_{m+2} + \ldots + 0\mathbf{v}_n = \mathbf{z}. \tag{15.3}$$

Since S is linearly independent, all scalars in (15.3) are 0. In particular, $c_1 = c_2 = \ldots = c_m = 0$, which implies that S' is linearly independent. ∎

We can restate Theorem 15.22 as follows: Let V be a vector space, and let S and S' be finite nonempty subsets of V such that $S' \subseteq S$. If S is linearly independent, then S' is linearly independent. The contrapositive of this implication gives us: If S' is linearly dependent, then S is linearly dependent.

Although we have only discussed linear independence and linear dependence in connection with finite sets of vectors, these concepts exist for infinite sets of vectors as well. An infinite set of vectors in a vector space V is **linearly independent** if *every* finite nonempty subset of S is linearly independent. Equivalently, an infinite set S of vectors in a vector space V is **linearly dependent** if some finite nonempty subset of S is linearly dependent. Every example we have seen of a (finite) set S of linearly dependent vectors in some vector space V gives rise to an infinite set T of linearly dependent vectors; namely, any infinite subset T of V such that $S \subseteq T$ is linearly dependent. But what is an example of a vector space that contains infinitely many linearly independent vectors? We provide such an example now.

Result 15.23 *The set $T = \{1, x, x^2, \ldots\}$ is linearly independent in $\mathbf{R}[x]$.*

Proof. Let S be a finite nonempty subset of T. Then there is a largest nonnegative integer m such that $x^m \in S$. Therefore, $S \subseteq S_m = \{1, x, x^2, \ldots, x^m\}$. By Theorem 15.20, S_m is linearly independent in $\mathbf{R}[x]$ and by Theorem 15.22, S is linearly independent. Consequently, T is linearly independent in $\mathbf{R}[x]$. ∎

15.8 Linear Transformations

We have seen that many properties of a vector space V, subspaces of V, the span of a set of vectors in V, and linear independence and linear dependence of vectors in V deal with a common concept: linear combinations of vectors. Perhaps this is not unexpected in an area of mathematics called linear algebra. There are occasions when two vectors spaces V and V' are so closely linked that with each vector $\mathbf{w} \in V$, there is an associated vector $\mathbf{w}' \in V'$ such that the vector associated with $\alpha\mathbf{u} + \beta\mathbf{v}$ in V is $\alpha\mathbf{u}' + \beta\mathbf{v}'$ in V'. Such an association describes a function from V to V'. In particular, a function $f : V \to V'$ is said to **preserve linear combinations** of vectors if $f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$ and every two scalars α and β. If $f : V \to V'$ has the property that $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$, then f is said to **preserve addition**; while if $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$ for all $\mathbf{u} \in V$ and every scalar α, then f is said to **preserve scalar multiplication**.

Let $\mathbf{z}'$ be the zero vector of V'. If $f : V \to V'$ preserves linear combinations and $\mathbf{u}, \mathbf{v} \in V$, then

$$f(\mathbf{u} + \mathbf{v}) = f(1 \cdot \mathbf{u} + 1 \cdot \mathbf{v}) = 1 \cdot f(\mathbf{u}) + 1 \cdot f(\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$

and $f(\alpha\mathbf{u}) = f(\alpha\mathbf{u} + 0\mathbf{v}) = \alpha f(\mathbf{u}) + 0f(\mathbf{v}) = \alpha f(\mathbf{u}) + \mathbf{z}' = \alpha f(\mathbf{u})$. Hence if $f : V \to V'$ is a function that preserves linear combinations, then f preserves addition and scalar multiplication as well.

Conversely, suppose that $f : V \to V'$ is a function that preserves both addition and scalar multiplication. Then for $\mathbf{u}, \mathbf{v} \in V$ and scalars α and β,

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) = f(\alpha\mathbf{u}) + f(\beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}),$$

that is, f preserves linear combinations. Because functions that preserve linear combinations are so important in linear algebra, they are given a special name.

Let V and V' be vector spaces. A function $T : V \to V'$ is called a **linear transformation** if it preserves both addition and scalar multiplication, that is, if it satisfies the following conditions:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$

for all $\mathbf{u}, \mathbf{v} \in V$ and all $\alpha \in \mathbf{R}$. There are some points in connection with these conditions that need to be addressed and that may not be self-evident. Condition 1 states that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every two vectors $\mathbf{u}$ and $\mathbf{v}$ of V. Hence the addition indicated in $T(\mathbf{u}+\mathbf{v})$ takes place in V; while, on the other hand, since $T(\mathbf{u})$ and $T(\mathbf{v})$ are vectors in V', the addition indicated in $T(\mathbf{u}) + T(\mathbf{v})$ takes place in V'. Also, condition 2 states that $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$ for every vector $\mathbf{v}$ in V and every scalar α. By the same reasoning, the scalar multiplication indicated in $T(\alpha\mathbf{v})$ takes place in V, while the scalar multiplication in $\alpha T(\mathbf{v})$ takes place in V'. From what we have already seen, every linear transformation preserves linear combinations of vectors (hence the name).

Let's consider an example of a linear transformation.

Result 15.24 *The function $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by*

$$T((a,b,c)) = T(a,b,c) = (2a+c, 3c-b)$$

is a linear transformation.

Before we prove Result 15.24, let's be certain that we understand what this function does. For example, $T(1,2,3) = (5,7)$, $T(1,-6,-2) = (0,0)$, while $T(0,0,0) = (0,0)$ as well. We now show that T is a linear transformation.

Proof of Result 15.24. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$. Then $\mathbf{u} = (a,b,c)$ and $\mathbf{v} = (d,e,f)$ for $a,b,c,d,e,f \in \mathbf{R}$. Then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(a+d, b+e, c+f) = (2(a+d)+c+f, 3(c+f)-(b+e)) \\ &= (2a+c, 3c-b) + (2d+f, 3f-e) \\ &= T(a,b,c) + T(d,e,f) = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} T(\alpha\mathbf{u}) &= T(\alpha(a,b,c)) = T(\alpha a, \alpha b, \alpha c) \\ &= (2\alpha a + \alpha c, 3\alpha c - \alpha b) = \alpha(2a+c, 3c-b) = \alpha T(\mathbf{u}), \end{aligned}$$

as desired. ∎

Sometimes the vectors in $\mathbf{R}^3$ are written as "column vectors", that is, as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ rather than (a,b,c) or the "row vector" $[a\ b\ c]$. In this case, notice that the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(a,b,c) = (2a+c, 3c-b)$ can be described as

$$T(a,b,c) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+c \\ -b+3c \end{bmatrix},$$

that is, if we let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}$, then this linear transformation can be defined in terms of the matrix A, namely,

$$T(\mathbf{v}) = A\mathbf{v}.$$

In general, if A is an $m \times n$ matrix, then the function $T : \mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\mathbf{u}) = A\mathbf{u}$ for an $n \times 1$ column vector $\mathbf{u} \in \mathbf{R}^n$ is a linear transformation. For example, consider the 3×2 matrix $A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}$. For $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, and $\alpha \in \mathbf{R}$,

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} a+c-2b-2d \\ 3a+3c-b-d \\ 2a+2c+5b+5d \end{bmatrix} \\ &= \begin{bmatrix} a-2b \\ 3a-b \\ 2a+5b \end{bmatrix} + \begin{bmatrix} c-2d \\ 3c-d \\ 2c+5d \end{bmatrix} = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} T(\alpha\mathbf{u}) &= T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix} = \begin{bmatrix} \alpha a - 2\alpha b \\ 3\alpha a - \alpha b \\ 2\alpha a + 5\alpha b \end{bmatrix} \\ &= \alpha\begin{bmatrix} a-2b \\ 3a-b \\ 2a+5b \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \alpha T(\mathbf{u}). \end{aligned}$$

Thus, $T : \mathbf{R}_2 \to \mathbf{R}_3$ is a linear transformation. The proof for a general $m \times n$ matrix is similar. As another illustration of a linear transformation, we consider a well-known function from $\mathbf{R}[x]$ to itself.

Result 15.25 *The function D (for differentiation) from $\mathbf{R}[x]$ to $\mathbf{R}[x]$ defined by*

$$D(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n) = c_1 + 2c_2x + \ldots + nc_nx^{n-1}$$

is a linear transformation.

Proof. Let $f, g \in \mathbf{R}[x]$, where $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_sx^s$ and, say, $r \leq s$. Then

$$\begin{aligned} D(f(x)+g(x)) &= D\left((a_0 + a_1x + \ldots + a_rx^r) + (b_0 + b_1x + \ldots + b_sx^s)\right) \\ &= D\left((a_0+b_0) + (a_1+b_1)x + \ldots + (a_r+b_r)x^r + b_{r+1}x^{r+1} + \ldots + b_sx^s\right) \\ &= (a_1+b_1) + \ldots + r(a_r+b_r)x^{r-1} + (r+1)b_{r+1}x^r + \ldots + sb_sx^{s-1} \\ &= \left(a_1 + 2a_2x + \ldots + ra_rx^{r-1}\right) + \left(b_1 + 2b_2x + \ldots + sb_sx^{s-1}\right) \\ &= D(f(x)) + D(g(x)) \end{aligned}$$

and

$$\begin{aligned} D(\alpha f(x)) &= D\left(\alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \ldots + \alpha a_rx^r\right) \\ &= \alpha a_1 + 2\alpha a_2x + \ldots + r\alpha a_rx^{r-1} \\ &= \alpha(a_1 + 2a_2x + \ldots + ra_rx^{r-1}) = \alpha D(f(x)). \end{aligned}$$

Since D preserves both addition and scalar multiplication, it is a linear transformation. ∎

There is a special kind a function from a vector space to itself that is always a linear transformation.

Result 15.26 *Let V be a vector space over the set $\mathbf{R}$ of real numbers. For $c \in \mathbf{R}$, the function $T : V \to V$ defined by $T(\mathbf{v}) = c\mathbf{v}$ is a linear transformation.*

Proof. Let $\mathbf{u}, \mathbf{w} \in V$. Then

$$T(\mathbf{u} + \mathbf{w}) = c(\mathbf{u} + \mathbf{w}) = c\mathbf{u} + c\mathbf{w} = T(\mathbf{u}) + T(\mathbf{w});$$

while, for $\alpha \in \mathbf{R}$,

$$T(\alpha\mathbf{u}) = c(\alpha\mathbf{u}) = (c\alpha)(\mathbf{u}) = (\alpha c)(\mathbf{u}) = \alpha(c\mathbf{u}) = \alpha T(\mathbf{u}).$$

Therefore, T is a linear transformation. ∎

For $c = 1$, the function T defined in Result 15.26 is the identity function; while for $c = 0$, the function T maps every vector into the zero vector. Consequently, both of these functions are linear transformations.

We now look at functions involving other vector spaces. For a function $f \in \mathcal{F}_{\mathbf{R}}$ and a real number r, we define the function $f + r$ by $(f + r)(x) = f(x) + r$ for all $x \in \mathbf{R}$.

Example 15.27 *Let r be a nonzero real number. Prove or disprove: The function $T : \mathcal{F}_{\mathbf{R}} \to \mathcal{F}_{\mathbf{R}}$ defined by $T(f) = f + r$ is a linear transformation.*

Solution. Let $f, g \in \mathcal{F}_{\mathbf{R}}$. Observe that

$$T(f + g) = (f + g) + r,$$

while

$$T(f) + T(g) = (f + r) + (g + r) = (f + g) + 2r.$$

Since $r \neq 0$, it follows that $T(f + g) \neq T(f) + T(g)$. Therefore, T is not a linear transformation. ♢

Example 15.28 *Let $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ be a function defined by*

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} ad & 0 \\ 0 & bc \end{bmatrix}.$$

Prove or disprove: T is a linear transformation.

Solution. Since

$$T\left(2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$2T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

T is not a linear transformation. ♢

Example 15.29 *The function $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ is defined by*

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}.$$

Prove or disprove: T is a linear transformation.

Solution. Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbf{R})$ and $\alpha \in \mathbf{R}$. Then

$$\begin{aligned} T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2 & a_1+a_2 \\ c_1+c_2 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 \\ c_2 & c_2 \end{bmatrix} \\ &= T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right); \end{aligned}$$

while

$$\begin{aligned} T\left(\alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha a_1 & \alpha a_1 \\ \alpha c_1 & \alpha c_1 \end{bmatrix} \\ &= \alpha \begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix} = \alpha T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right). \end{aligned}$$

Since T preserves both addition and scalar multiplication, T is a linear transformation. ♢

15.9 Properties of Linear Transformations

An important property of linear transformations is that the composition of any two linear transformations (when the composition is defined) is also a linear transformation. This fact has an interesting consequence as well.

Theorem 15.30 *Let V, V', and V'' be vector spaces. If $T_1 : V \to V'$ and $T_2 : V' \to V''$ are linear transformations, then the composition $T_2 \circ T_1 : V \to V''$ is a linear transformation as well.*

Proof. For $\mathbf{u}, \mathbf{v} \in V$ and a scalar α, observe that

$$\begin{aligned} (T_2 \circ T_1)(\mathbf{u}+\mathbf{v}) &= T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = (T_2 \circ T_1)(\mathbf{u}) + (T_2 \circ T_1)(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} (T_2 \circ T_1)(\alpha\mathbf{v}) &= T_2(T_1(\alpha\mathbf{v})) = T_2(\alpha T_1(\mathbf{v})) \\ &= \alpha T_2(T_1(\mathbf{v})) = \alpha(T_2 \circ T_1)(\mathbf{v}). \end{aligned}$$

Therefore, $T_2 \circ T_1$ is a linear transformation. ∎

As an example of the preceding theorem, let $T_1 : \mathbf{R}^3 \to \mathbf{R}^2$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $T_1(a,b,c) = (a+2b-c, 3b+2c)$ and $T_2(a,b) = (b, 2a, a+b)$. Then $T_2 \circ T_1 : \mathbf{R}^3 \to \mathbf{R}^3$ is given by

$$\begin{aligned} (T_2 \circ T_1)(a,b,c) &= T_2(T_1(a,b,c)) \\ &= T_2(a+2b-c, 3b+2c) \\ &= (3b+2c, 2a+4b-2c, a+5b+c). \end{aligned}$$

From what we mentioned earlier, T_1 and T_2 can also be defined by

$$T_1\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 3 & 2 \end{array}\right]\left[\begin{array}{c} a \\ b \\ c \end{array}\right] \text{ and } T_2\left(\left[\begin{array}{c} a \\ b \end{array}\right]\right) = \left[\begin{array}{cc} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} a \\ b \end{array}\right].$$

Interestingly enough,

$$(T_2 \circ T_1)\left(\left[\begin{array}{c} a \\ b \\ c \end{array}\right]\right) = \left[\begin{array}{cc} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{array}\right]\left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & 3 & 2 \end{array}\right]\left[\begin{array}{c} a \\ b \\ c \end{array}\right],$$

that is, the composition $T_2 \circ T_1$ can be obtained by multiplying the matrices that describe T_1 and T_2. Therefore, if we represent the linear transformations T_1 and T_2 by matrices A_1 and A_2, respectively, then the matrix that represents $T_2 \circ T_1$ is A_2A_1. This also explains why the definition of matrix multiplication, though curious at first, is actually quite logical.

Two fundamental properties of a linear transformation are given in the next theorem.

Theorem 15.31 *Let V and V' be vector spaces with respective zero vectors $\mathbf{z}$ and $\mathbf{z}'$. If $T : V \to V'$ is a linear transformation, then*

(i) $T(\mathbf{z}) = \mathbf{z}'$ *and*

(ii) $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v} \in V$.

Proof. We first verify (i). Since T preserves scalar multiplication,

$$T(\mathbf{z}) = T(0\mathbf{z}) = 0T(\mathbf{z}) = \mathbf{z}'.$$

Next we verify (ii). Let $\mathbf{v} \in V$. Then

$$T(\mathbf{v}) + T(-\mathbf{v}) = T(\mathbf{v} + (-\mathbf{v})) = T(\mathbf{z}) = \mathbf{z}',$$

the last equality following by (i). Since the vector $T(\mathbf{v})$ in V' has a unique negative, namely $-T(\mathbf{v})$, we conclude that $T(-\mathbf{v}) = -T(\mathbf{v})$. ∎

If $T : V \to V'$ is a linear transformation, then it is often of interest to know how T acts on subspaces of V. Let's recall some terminology and notation from functions. In a linear transformation $T : V \to V'$, the set V is the **domain** of T and the set V' is the **codomain** of T. If W is a subset of V, then $T(W) = \{T(\mathbf{w}) \ : \ \mathbf{w} \in W\}$ is the **image** of W under T. In particular, $T(V)$ is the **range** of T.

Theorem 15.32 *Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. If W is a subspace of V, then $T(W)$ is a subspace of V'.*

Proof. Let $\mathbf{z}$ and $\mathbf{z}'$ be the zero vectors in V and V', respectively. Since $\mathbf{z} \in W$ and $T(\mathbf{z}) = \mathbf{z}'$ by Theorem 15.31, it follows that $\mathbf{z}' \in T(W)$ and so $T(W) \neq \emptyset$. Thus we need only show that $T(W)$ is closed under addition and scalar multiplication. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $T(W)$. Hence, there exist vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y}$. Then

$$\mathbf{x} + \mathbf{y} = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}).$$

Since $\mathbf{u}, \mathbf{v} \in W$ and W is a subspace of V, it follows that $\mathbf{u} + \mathbf{v} \in W$. Hence $\mathbf{x} + \mathbf{y} = T(\mathbf{u} + \mathbf{v}) \in T(W)$.

Next let α be a scalar and $\mathbf{x} \in T(W)$. We show that $\alpha\mathbf{x} \in T(W)$. Since $\mathbf{x} \in T(W)$, there exists $\mathbf{u} \in W$ such that $T(\mathbf{u}) = \mathbf{x}$. Now

$$\alpha\mathbf{x} = \alpha T(\mathbf{u}) = T(\alpha\mathbf{u}).$$

Since $\alpha\mathbf{u} \in W$, it follows that $\alpha\mathbf{x} = T(\alpha\mathbf{u}) \in T(W)$. By the Subspace Test, $T(W)$ is a subspace of V'. ∎

To illustrate Theorem 15.32, let's return to the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined in Result 15.24 by $T(a, b, c) = (2a + c, 3c - b)$. Let $W = \{(a, b, 0) \ : \ a, b \in \mathbf{R}\}$. We use the Subspace Test to show that W is a subspace of $\mathbf{R}^3$. Since $(0, 0, 0) \in W$, it follows that $W \neq \emptyset$. Let $(a_1, b_1, 0), (a_2, b_2, 0) \in W$ and let $\alpha \in \mathbf{R}$. Then

$$(a_1, b_1, 0) + (a_2, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0) \in W \text{ and } \alpha(a_1, b_1, 0) = (\alpha a_1, \alpha b_1, 0) \in W.$$

Since W is closed under addition and scalar multiplication, W is a subspace of $\mathbf{R}^3$. By Theorem 15.32, $T(W) = \{(2a, -b) \ : \ a, b \in \mathbf{R}\}$ is a a subspace of $\mathbf{R}^2$. We show in fact that $T(W) = \mathbf{R}^2$. Certainly, $\mathbf{R}^2 = \langle (1, 0), (0, 1) \rangle$. Hence to show that $T(W) = \mathbf{R}^2$, it suffices, by Corollary 15.16, to show that $(1, 0)$ and $(0, 1)$ belong to $T(W)$. Letting $a = 1/2$ and $b = 0$, we see that $(1, 0) \in T(W)$; while letting $a = 0$ and $b = -1$, we see that $(0, 1) \in T(W)$.

For this same linear transformation T, we saw that $T(1, -6, -2) = (0, 0)$ and $T(0, 0, 0) = (0, 0)$. Hence both $(1, -6, -2)$ and $(0, 0, 0)$ map into the zero vector of $\mathbf{R}^2$. The fact that $(0, 0, 0)$ maps into $(0, 0)$ is not surprising, of course, since Theorem 15.31 guarantees this.

If $T : V \to V'$ is a linear transformation and W' is a subset of V', then

$$T^{-1}(W') = \{\mathbf{v} \in V \ : \ T(\mathbf{v}) \in W'\}$$

is called the **inverse image** of W' under T. If $W' = \{\mathbf{z}'\}$, where $\mathbf{z}'$ is the zero vector of V', then $T^{-1}(W')$ is called the **kernel** of T and is denoted by $\ker(T)$. That is, the kernel of $T : V \to V'$ is the set

$$\ker(T) = T^{-1}(\{\mathbf{z}'\}) = \{v \in V \ : \ T(v) = \mathbf{z}'\}.$$

An interesting feature of the kernel lies in the following theorem.

Theorem 15.33 *Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. Then the kernel of T is a subspace of V.*

Proof. Let $\mathbf{z}$ and $\mathbf{z}'$ be the zero vectors of V and V', respectively. Since $T(\mathbf{z}) = \mathbf{z}'$, it follows that $\mathbf{z} \in \ker(T)$ and so $\ker(T) \neq \emptyset$. Now let $\mathbf{u}, \mathbf{v} \in \ker(T)$ and $\alpha \in \mathbf{R}$. Then

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(v) = \mathbf{z}' + \mathbf{z}' = \mathbf{z}'$$

and

$$T(\alpha\mathbf{u}) = \alpha T(\mathbf{u}) = \alpha\mathbf{z}' = \mathbf{z}'.$$

This implies that $\mathbf{u} + \mathbf{v} \in \ker(T)$ and $\alpha\mathbf{u} \in \ker(T)$. By the Subspace Test, $\ker(T)$ is a subspace of V. ∎

Returning once again to the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ in Result 15.24 defined by $T(a, b, c) = (2a + c, 3c - b)$, we see that

$$\ker(T) = \{(a, b, c) \ : \ 2a + c = 0 \text{ and } 3c - b = 0\}$$

is a subspace of $\mathbf{R}^3$. Since $2a + c = 0$ and $3c - b = 0$, it follows that $a = -c/2$ and $b = 3c$. Thus $\ker(T) = \{(-c/2, 3c, c) \ : \ c \in \mathbf{R}\}$. In other words, $\ker(T)$ is the subspace of $\mathbf{R}^3$ consisting of all scalar multiples of $(-1/2, 3, 1)$.

Exercises for Chapter 15

15.1 Prove that the set $\mathcal{C} = \{a + bi \ : \ a, b \in \mathbf{R}\}$ of complex numbers is a vector space under the addition $(a + bi) + (c + di) = (a + c) + (b + d)i$ and scalar multiplication $\alpha(a + bi) = \alpha a + \alpha bi$, where $\alpha \in \mathbf{R}$.

15.2 Although we have taken $\mathbf{R}$ to be the set of scalars in a vector space, this need not always be the case. Let $V = \{([a], [b]) \ : \ [a], [b] \in \mathbf{Z}_3\}$ and let $\mathbf{Z}_3$ be the set of scalars.

(a) Show that V is a vector space over the set $\mathbf{Z}_3$ of scalars under the addition $([a], [b]) + ([c], [d]) = ([a + c], [b + d])$ and scalar multiplication $[c]([a], [b]) = ([ca], [cb])$.

(b) Write out precisely the elements of V. (Hence a vector space can have more than one vector and be finite.)

15.3 Addition or scalar multiplication is defined in $\mathbf{R}^3$ in each of the following. (Each operation not defined is taken as the standard one.) Under these operations, determine whether $\mathbf{R}^3$ is a vector space.

(a) $(a, b, c) + (d, e, f) = (a, b, c)$

(b) $(a, b, c) + (d, e, f) = (a - d, b - e, c - f)$

(c) $(a, b, c) + (d, e, f) = (0, 0, 0)$

(d) $\alpha(a, b, c) = (a, b, c)$

(e) $\alpha(a, b, c) = (b, c, a)$

(f) $\alpha(a, b, c) = (0, 0, 0)$

(g) $\alpha(a, b, c) = (\alpha a, 3\alpha b, \alpha c)$

15.4 Let V be a vector space, where $\mathbf{u}, \mathbf{v} \in V$. Prove that there exists a unique vector $\mathbf{x}$ in V such that $\mathbf{u} + \mathbf{x} = \mathbf{v}$.

15.5 Let V be a vector space with $\mathbf{v} \in V$ and $\alpha \in \mathbf{R}$. Prove that $\alpha(-\mathbf{v}) = (-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$.

15.6 (a) Let V be a vector space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. (This is the cancellation property for addition of vectors.)

(b) Use (a) to prove Theorem 15.3.

15.7 Prove or disprove:

(a) No vector is its own negative.

(b) Every vector is the negative of some vector.

(c) Every vector space has at least two vectors.

15.8 Let V be a vector space containing nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Prove that if $\mathbf{u} \neq \alpha\mathbf{v}$ for each $\alpha \in \mathbf{R}$, then $\mathbf{u} \neq \beta(\mathbf{u} + \mathbf{v})$ for each $\beta \in \mathbf{R}$.

15.9 Determine which of following subsets of $\mathbf{R}^4$ are subspaces of $\mathbf{R}^4$.

(a) $W_1 = \{(a, a, a, a) \ : \ a \in \mathbf{R}\}$

(b) $W_2 = \{(a, 2b, 3a, 4b) \ : \ a, b \in \mathbf{R}\}$

(c) $W_3 = \{(a, 0, 0, 1) \ : \ a \in \mathbf{R}\}$

(d) $W_4 = \{(a, a^2, 0, 0) \ : \ a \in \mathbf{R}\}$

(e) $W_5 = \{(a, b, a + b, b) \ : \ a, b \in \mathbf{R}\}$

15.10 Let $\mathcal{F}_{\mathbf{R}}$ be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Determine which of the following subsets of $\mathcal{F}_{\mathbf{R}}$ are subspaces of $\mathcal{F}_{\mathbf{R}}$.

(a) W_1 consists of all functions f such that $f(1) = 0 = f(2)$.

(b) W_2 consists of all functions f such that $f(1) = 0$ or $f(2) = 0$.

(c) W_3 consists of all functions f such that $f(2) = 2f(1)$.

(d) W_4 consists of all functions f such that $f(1) \neq f(2)$.

(d) W_5 consists of all functions f such that $f(1) \neq 0$.

15.11 Recall that the set $\mathbf{R}[x]$ of polynomial functions is a subspace of $\mathcal{F}_{\mathbf{R}}$. Now determine which of the following subsets of $\mathbf{R}[x]$ are subspaces of $\mathbf{R}[x]$.

(a) $U_1 = \{f \ : \ f(x) = a \text{ for a fixed real number } a\}$ (The set of all constant polynomials)

(b) $U_2 = \{f \ : \ f(x) = a + bx + cx^2 + dx^3, \ a, b, c, d \in \mathbf{R}, d \neq 0\}$

(c) $U_3 = \{f \ : \ f(x) = a + bx + cx^2 + dx^3, a, b, c, d \in \mathbf{R}\}$

(d) $U_4 = \{f \ : \ f(x) = a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2m}x^{2m}, m \geq 0, \text{ and } a_i \in \mathbf{R} \text{ for } 0 \leq i \leq m\}$

(e) $U_5 = \{f \ : \ f(x) = (x^3 + 1)g(x) \text{ for some } g \in \mathbf{R}[x]\}$

15.12 Let $M_2(\mathbf{R})$ be the vector space of 2×2 matrices whose entries are real numbers. Determine which of the following subsets of $M_2(\mathbf{R})$ are subspaces of $M_2(\mathbf{R})$.

(a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ ad - bc = 0 \right\}$

(b) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d = 0 \right\}$, where α_1, α_2, α_3, α_4 are fixed real numbers.

15.13 Prove that

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} \ : \ a_i \in \mathbf{R} \text{ for } 1 \leq i \leq 6 \right\}$$

is a subspace of the vector space $M_3[\mathbf{R}]$.

15.14 Let U and W be subspaces of a vector space V. Prove that $U \cap W$ is a subspace of V.

15.15 The graph of the function$f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \frac{3}{5}x$ is a straight line in $\mathbf{R}^2$ passing through the origin. Each point (x, y) on this graph is a solution of the equation $3x - 5y = 0$. Prove that the set S of solutions of this equation is a subspace of $\mathbf{R}^2$.

15.16 Determine the following linear combinations:

(a) $4 \cdot (1, -2, 3) + (-2) \cdot (1, -1, 0)$

(b) $(-1)\begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 5\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

15.17 In $\mathbf{R}^3$, write $\mathbf{i} = (1, 0, 0)$ as a linear combination of $\mathbf{u}_1 = (0, 1, 1), \mathbf{u}_2 = (1, 0, 1)$, and $\mathbf{u}_3 = (1, 1, 0)$.

15.18 Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (0, 1, 2)$, and $\mathbf{w} = (3, 1, -1)$ be vectors in $\mathbf{R}^3$.

(a) Show that $\mathbf{w}$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) Show that the vector $\mathbf{x} = (8, 5, 2)$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in more than one way.

15.19 Let V be a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_m$. Let $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ and $W' = \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \rangle$. Prove that if each vector v_i $(1 \leq i \leq n)$ is a linear combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, then $W \subseteq W'$.

15.20 Prove that $\langle (1, 2, 3), (0, 4, 1) \rangle = \langle (1, 6, 4), (1, -2, 2) \rangle$ in $\mathbf{R}^3$

15.21 Let V be a vector space and let $\mathbf{u}$ and $\mathbf{v}$ in V. Prove that

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$

15.22 Determine which sets S of vectors are linearly independent in the indicated vector space V.

(a) $S = \{(1, 1, 1), (1, -2, 3), (2, 5, -1)\}; V = \mathbf{R}^3$.

(b) $S = \{(1, 0, -1), (2, 1, 1), (0, 1, 3)\}; V = \mathbf{R}^3$.

(c) $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}; V = M_2(\mathbf{R})$.

15.23 For the vectors $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 0, 2)$, find a vector $\mathbf{w}$ such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent in $\mathbf{R}^3$. Verify that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

15.24 Prove or disprove: If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent vectors in a vector space V, then $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, 2\mathbf{u}_3$ are linearly independent vectors in V.

15.25 Determine which sets S of vectors in $\mathcal{F}_{\mathbf{R}}$ are linearly independent.

(a) $S = \{1, \sin^2 x, \cos^2 x\}$

(b) $S = \{1, \sin x, \cos x\}$

(c) $S = \{1, e^x, e^{-x}\}$

(d) $S = \{1, x, x/(x^2+1)\}$.

15.26 Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a linearly dependent set of $n \geq 2$ vectors in a vector space V. Prove that if each subset of S consisting of $n-1$ vectors is linearly independent, then there exist nonzero scalars $c_1, c_2, \ldots, c_n$ such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = \mathbf{z}$.

15.27 Prove that if $T : V \to V'$ is a linear transformation, then

$$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \ldots + \alpha_n T(\mathbf{v}_n),$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$.

15.28 Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. Prove that if W' is a subspace of V', then $T^{-1}(W')$ is a subspace of V.

15.29 Prove that there exists a bijective linear transformation $T : \mathbf{R}^2 \to \mathcal{C}$, where $\mathcal{C} = \{a + bi \ : \ a, b \in \mathbf{R}\}$ is the set of complex numbers.

15.30 For vector spaces V and V', let T_1 and T_2 be linear transformations from V to V'. Define $T_1 + T_2 : V \to V'$ as

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v}).$$

Prove that $T_1 + T_2$ is also a linear transformation.

15.31 Let $W = \left\{ \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix} \ : \ a, b \in \mathbf{R} \right\}$.

(a) Prove that W is a subspace of $M_2(\mathbf{R})$

(b) Prove that there exists a bijective linear transformation $T : \mathbf{R}^2 \to W$.

15.32 For the 2×3 matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \end{bmatrix}$, a function $T : \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $T(\mathbf{u}) = A\mathbf{u}$, where $\mathbf{u}$ is a 3×1 column vector in $\mathbf{R}^3$.

(a) Determine $T(\mathbf{u})$ for $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$.

(b) Prove that T is a linear transformation.

15.33 Let $D : \mathbf{R}[x] \to \mathbf{R}[x]$ be the differentiation linear transformation defined by

$$D(c_0 + c_1x + \ldots + c_nx^n) = c_1 + 2c_2x + \ldots + nc_nx^{n-1}.$$

Determine each of the following.

(a) $D(W)$, where $W = \{a + bx \ : \ a, b \in \mathbf{R}\}$.

(b) $D(W)$, where $W = \mathbf{R}$.

(c) $\ker(D)$.

15.34 Let $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

and consider the subset $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \ : \ a, d \in \mathbf{R} \right\}$ of $M_2(\mathbf{R})$.

(a) Prove that W is a subspace of $M_2(\mathbf{R})$.

(b) Determine the subspace $T(W)$ of $M_2(\mathbf{R})$.

(c) Determine the subspace $\ker(T)$ of $M_2(\mathbf{R})$.

15.35 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let V be a vector space. If $\mathbf{u}$ is a vector of V such that $\mathbf{u}+\mathbf{v} = \mathbf{v}$ for some $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

Proof. Assume that $\mathbf{u}+\mathbf{v} = \mathbf{v}$ for some $\mathbf{v} \in V$. Then we also know that $\mathbf{z}+\mathbf{v} = \mathbf{v}$, where $\mathbf{z}$ is the zero vector of V. Hence $\mathbf{u} + \mathbf{v} = \mathbf{z} + \mathbf{v}$. By Exercise 15.6, $\mathbf{u} = \mathbf{z}$ and so $\mathbf{u} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$. ∎

Chapter 16

Proofs in Topology

Recall from calculus that a function $f : X \to \mathbf{R}$, where $X \subseteq \mathbf{R}$, is **continuous** at $a \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. When we write $|x - a|$, we are referring to how far apart x and a are, that is, the distance between them. Similarly, $|f(x) - f(a)|$ is the distance between $f(x)$ and $f(a)$. It is not surprising that distance enters the picture here since when we say that f is continuous at a, we mean that if x is a number that is close to a, then $f(x)$ is close to $f(a)$. The term "close" only has meaning once we understand how we are measuring the distances between the two pairs of numbers involved. It might seem obvious that the distance between two real numbers x and y is $|x-y|$; however, it turns out that the distance between x and y need not be defined as $|x-y|$, although it is certainly the most common definition. Furthermore, when the continuity of a function $f : A \to B$ is being considered, it is not essential that A and B be sets of real numbers. That is, it is possible to place these concepts of calculus in a more general setting. The area of mathematics that deals with this is topology.

16.1 Metric Spaces

We have already mentioned that the distance between two real numbers x and y is given by $|x - y|$. There are four properties that this distance has, which will turn out to be especially interesting to us:

$$\begin{aligned} &(1) \quad |x-y| \geq 0 \text{ for all } x, y \in \mathbf{R}; \\ &(2) \quad |x-y| = 0 \text{ if and only if } x = y \text{ for all } x, y \in \mathbf{R}; \\ &(3) \quad |x-y| = |y-x| \text{ for all } x, y \in \mathbf{R}; \\ &(4) \quad |x-z| \leq |x-y| + |y-z| \text{ for all } x, y, z \in \mathbf{R}. \end{aligned} \tag{16.1}$$

Many of the fundamental results from calculus depend on these four properties. Using these properties as our guide, we now define distance in a more general manner.

Let X be a nonempty set and let $d : X \times X \to \mathbf{R}$ be a function from the Cartesian product $X \times X$ to the set $\mathbf{R}$ of real numbers. Hence for each ordered pair $(x, y) \in X \times X$, it follows that $d((x, y))$ is a real number. For simplicity, we write $d(x, y)$ rather than $d((x, y))$ and refer to $d(x, y)$ as the **distance** from x to y. The distance d is called a **metric** on X if it satisfies the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$;

(2) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \mathbf{R}$;

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (**symmetric property**);

(4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (**triangle inequality**).

A set X together with a metric d defined on X is called a **metric space** and is denoted by (X, d). Since the set $\mathbf{R}$ of real numbers together with the distance d defined on $\mathbf{R}$ by $d(x, y) = |x - y|$ satisfies the properties listed in (16.1), it follows that $(\mathbf{R}, d)$ is a metric space.

We now consider two other ways of defining the distance between two real numbers.

Example 16.1 *For $X = \mathbf{R}$, let the distance $d : X \times X \to \mathbf{R}$ be defined by $d(x, y) = x - y$. Determine which of the four properties of a metric are satisfied by this distance.*

Solution. Since $d(1, 2) = -1$, property 1 is not satisfied. On the other hand, since $d(x, y) = x - y = 0$ if and only if $x = y$, property 2 is satisfied. Because $d(2, 1) = 1$, it follows that $d(1, 2) \neq d(2, 1)$ and so the symmetric property (property 3) is not satisfied. Finally,

$$d(x, z) = x - z = (x - y) + (y - z) = d(x, y) + d(y, z),$$

and the triangle inequality (property 4) holds. $\diamondsuit$

In our next example, we present a distance function that is actually a metric on $\mathbf{R}$.

Result 16.2 *For $X = \mathbf{R}$, let $d : X \times X \to \mathbf{R}$ be defined by*

$$d(x, y) = |2^x - 2^y|.$$

Then (X, d) is a metric space.

Proof. Clearly, $d(x, y) = |2^x - 2^y| \geq 0$ and $d(x, y) = 0$ if and only if $2^x = 2^y$. Certainly, if $x = y$, then $2^x = 2^y$. Assume next that $2^x = 2^y$. If we take logarithms to the base 2 of both 2^x and 2^y, then we have $x = y$. Thus, $d(x, y) = 0$ if and only if $x = y$. Since $d(x, y) = |2^x - 2^y| = |2^y - 2^x| = d(y, x)$, it follows that d satisfies the symmetric property. Finally, by property 4 in (16.1),

$$d(x, z) = |2^x - 2^z| = |(2^x - 2^y) + (2^y - 2^z)| \leq |2^x - 2^y| + |2^y - 2^z| = d(x, y) + d(y, z)$$

and the triangle inequality holds. ∎

Another set on which you have undoubtedly seen a distance defined is $\mathbf{R} \times \mathbf{R} = \mathbf{R}^2$. Hence an element $P \in \mathbf{R}^2$ can be expressed as (x, y), where $x, y \in \mathbf{R}$. Here we are discussing points in the Cartesian plane, as you saw in the study of analytic geometry. There the (Euclidean) distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

This distance is actually a metric on $\mathbf{R}^2$. That the first three properties are satisfied depends only on the following facts for real numbers a and b: (1) $a^2 \geq 0$, (2) $a^2 + b^2 = 0$ if and only if $a = b = 0$, (3) $a^2 = (-a)^2$. The triangle inequality is more difficult to verify, however, and its proof depends on the following lemma, which is a special case of a result commonly called **Schwarz's Inequality**.

Lemma 16.3 *If $a, b, c, d \in \mathbf{R}$, then*

$$ab + cd \leq \sqrt{(a^2 + c^2)(b^2 + d^2)}.$$

Proof. Certainly, $(ab + cd)^2 + (ad - bc)^2 \geq (ab + cd)^2$. Since

$$\begin{aligned}(ab + cd)^2 + (ad - bc)^2 &= (a^2b^2 + 2abcd + c^2d^2) + (a^2d^2 - 2abcd + b^2c^2) \\ &= a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 \\ &= (a^2 + c^2)(b^2 + d^2),\end{aligned}$$

the desired inequality follows. ∎

We can now show that this distance is a metric.

Result 16.4 *For $X = \mathbf{R}^2$, let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in $\mathbf{R}^2$ and let $d : X \times X \to \mathbf{R}$ be defined by*

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Then (X, d) is a metric space.

Proof. We have already mentioned that the first three properties of a metric are satisfied, so only the triangle inequality remains to be verified. Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$. Thus using Lemma 16.3, where $a = x_1 - x_2$, $b = x_2 - x_3$, $c = y_1 - y_2$, and $d = y_2 - y_3$, we have

$$\begin{aligned}[d(P_1, P_3)]^2 &= (x_1 - x_3)^2 + (y_1 - y_3)^2 \\ &= [(x_1 - x_2) + (x_2 - x_3)]^2 + [(y_1 - y_2) + (y_2 - y_3)]^2 \\ &= (x_1 - x_2)^2 + (x_2 - x_3)^2 + 2(x_1 - x_2)(x_2 - x_3) + \\ &\quad 2(y_1 - y_2)(y_2 - y_3) + (y_1 - y_2)^2 + (y_2 - y_3)^2 \\ &\leq (x_1 - x_2)^2 + (x_2 - x_3)^2 + \\ &\quad 2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} + \\ &\quad (y_1 - y_2)^2 + (y_2 - y_3)^2 \\ &= \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}\right)^2 \\ &= [d(P_1, P_2) + d(P_2, P_3)]^2,\end{aligned}$$

which gives us the desired result. ∎

There is a metric defined on $\mathbf{N} \times \mathbf{N} = \mathbf{N}^2$ which goes by the name of the **Manhattan metric** or **taxicab metric**. For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{N}^2$, the distance $d(P_1, P_2)$ is defined by

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|.$$

For example, consider the points $P_1 = (2, 2)$ and $P_2 = (4, 6)$ shown in Figure 16.1 (a). The taxicab distance between these two points is $d(P_1, P_2) = |2 - 4| + |2 - 6| = 6$. Thinking of the points (x, y) as street intersections in a certain city (Manhattan), we have a minimum of 6 blocks to travel (by taxicab). Two such routes are shown in Figure 16.1 (b), (c).

Not only is the Manhattan metric a metric on $\mathbf{N}^2$, it is also a metric on $\mathbf{Z}^2$ and on $\mathbf{R}^2$. A proof of the following result is left as an exercise (Exercise 16.2).

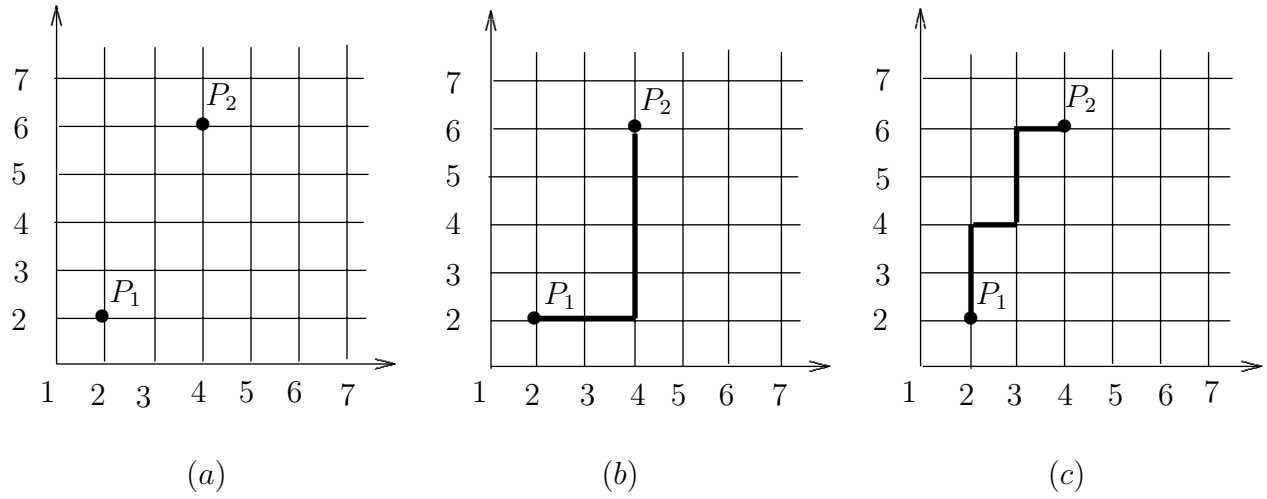

Figure 16.1: The Manhattan metric

Result 16.5 *For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, the distance $d(P_1, P_2)$ defined by*

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

is a metric on $\mathbf{R}^2$ (the Manhattan metric).

We have seen that there is more than one metric on both $\mathbf{R}$ and $\mathbf{R}^2$. The metric spaces $(\mathbf{R}, d)$, where $d(x, y) = |x - y|$, and $(\mathbf{R}^2, d)$, where $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, are called **Euclidean spaces** and the associated metrics are the **Euclidean metrics**. These are certainly the most familiar metrics on $\mathbf{R}$ and $\mathbf{R}^2$.

For every nonempty set A, it is always possible to define a distance $d : A \times A \to \mathbf{R}$ that is a metric. For $x, y \in A$, the distance

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is called the **discrete metric** on A.

Result 16.6 *The discrete metric d defined on a nonempty set A is a metric.*

Proof. By definition, $d(x, y) \geq 0$ for all $x, y \in A$ and $d(x, y) = 0$ if and only if $x = y$. Also, by the definition of this distance, $d(x, y) = d(y, x)$ for all $x, y \in A$. Now let $x, y, z \in A$. If $x = z$, then certainly $0 = d(x, z) \leq d(x, y) + d(y, z)$. If $x \neq z$, then $d(x, z) = 1$. Since $x \neq y$ or $y \neq z$, it follows that $d(x, y) + d(y, z) \geq 1 = d(x, z)$. In any case, the triangle inequality holds. ∎

16.2 Open Sets in Metric Spaces

Returning to our discussion of a real-valued function f from calculus, we said that f is continuous at a real number a in the domain of f if for every $\epsilon > 0$, there exists a number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. This, of course, is what led us to rethink what we meant by distance and which then led us to metric spaces. However, continuity itself can be described in a somewhat different manner. A function f is continuous at a if for every $\epsilon > 0$,

there exists a number $\delta > 0$ such that if x is a number in the open interval $(a - \delta, a + \delta)$, then $f(x)$ is a number in the open interval $(f(a) - \epsilon, f(a) + \epsilon)$. That is, continuity can be defined in terms of open intervals. What are some properties of open intervals? Of course, an open interval is a certain kind of subset of the set of real numbers. But each open interval has a property that can be generalized in a very useful manner. An open interval I of real numbers has the property that for every $x \in I$, there exists a real number $r > 0$ such that $(x - r, x + r) \subseteq I$, that is, for every $x \in I$, there is an open interval I_1 centered at x that is contained in I.

Let (X, d) be a metric space. Also, let $a \in X$ and let a real number $r > 0$ be given. The subset of X consisting of those points (elements) $x \in X$ such that $d(x, a) < r$ is called the **open sphere with center** a **and radius** r and is denoted by $S_r(a)$. Thus $x \in S_r(a)$ if and only if $d(x, a) < r$. For example, the open sphere $S_r(a)$ in the Euclidean space ($\mathbf{R}$, d) is the open interval $(a - r, a + r)$ with mid-point a and length $2r$. Conversely, each open interval in ($\mathbf{R}$, d) is an open sphere according to this definition. So the open spheres in ($\mathbf{R}$, d) are precisely the open intervals of the form (a, b), where $a < b$ and $a, b \in \mathbf{R}$. In the Euclidean space $(\mathbf{R}^2, d)$, the open sphere $S_r(P)$ is the interior of the circle with center P and radius r. In the Manhattan metric space $(\mathbf{R}^2, d)$, where the distance between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, the open sphere $S_3(P)$ for $P = (5, 4)$ is the interior of the square shown in Figure 16.2.

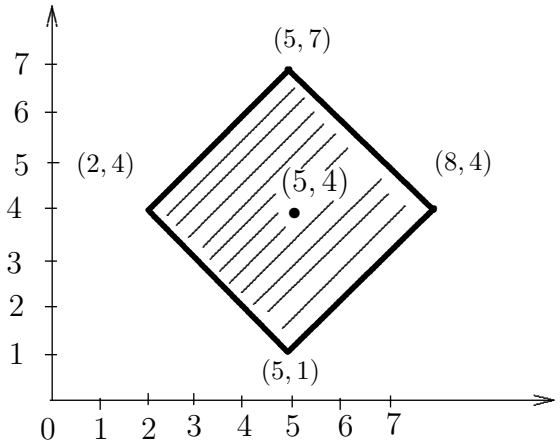

Figure 16.2: An open sphere $S_3(P)$ for $P = (5, 4)$

Since every point in a metric space (X, d) belongs to an open sphere in X (indeed, it is the center of an open sphere), it is immediate that every two distinct points of X belong to distinct open spheres. In fact, they belong to disjoint open spheres.

Theorem 16.7 *Every two distinct points in a metric space belong to disjoint open spheres.*

Proof. Let a and b be distinct points in a metric space (X, d) and suppose that $d(a, b) = r$. Necessarily, $r > 0$. Consider the open spheres $S_{\frac{r}{2}}(a)$ and $S_{\frac{r}{2}}(b)$ having radius $r/2$ centered at a and b, respectively. We claim that $S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b) = \emptyset$. Assume, to the contrary, that $S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b) \neq \emptyset$. Then there exists $c \in S_{\frac{r}{2}}(a) \cap S_{\frac{r}{2}}(b)$. Thus $d(c, a) < r/2$ and $d(c, b) < r/2$. By the triangle inequality, $r = d(a, b) \leq d(a, c) + d(c, b) < r/2 + r/2 = r$, which is a contradiction. ■

A subset O of a metric space (X, d) is defined to be **open** if for every point a of O, there exists a positive real number r such that $S_r(a) \subseteq O$, that is, each point of O is the center of an open sphere contained in O. In the Euclidean space ($\mathbf{R}$, d), each open interval (a, b), where

$a < b$, is an open set. To see this, for each $x \in (a, b)$, let $r = \min(x - a, b - x)$. Then the open sphere $S_r(x) = (x - r, x + r)$ is contained in (a, b). In fact, the set $(-\infty, a) \cup (a, \infty)$ is open in $(\mathbf{R}, d)$ for each $a \in \mathbf{R}$. On the other hand, the half-open set (or half-closed set) $(a, b]$ is not open since there exists no open sphere centered at b and contained in $(a, b]$. Similarly, the sets $[a, b]$, $[a, b)$, $(-\infty, a]$, and $[a, \infty)$ are not open in $(\mathbf{R}, d)$.

Every metric space contains some open sets, as we now show.

Theorem to Prove In a metric space (X, d),

(i) the empty set $\emptyset$ and the set X are open, and

(ii) every open sphere is an open set.

Proof Strategy To show that a subset A of X is open, it is required to show that if a is a point of A, then a is the center of an open sphere contained in A. The empty set satisfies this condition vacuously and X satisfies this condition trivially; so we concentrate on verifying (ii).

We begin with an open sphere $S_r(a)$ having center a and radius r. For an arbitrary element $x \in S_r(a)$, we need to show that there is an open sphere centered at x and with an appropriate radius that is contained in $S_r(a)$. Since the theorem concerns an arbitrary metric space (X, d), there is not necessarily any geometric appearance to the open sphere $S_r(a)$. On the other hand, it is helpful to visualize $S_r(a)$ as the interior of circle (see Figure 16.3).

Since $d(x, a) < r$, it follows that $r' = r - d(x, a)$ is a positive real number. It appears likely that $S_{r'}(x) \subseteq S_r(a)$. To show this, it remains to show that if $y \in S_{r'}(x)$, then $y \in S_r(a)$; that is, if $d(y, x) < r'$, then $d(y, a) < r$. It is natural to use the triangle inequality in an attempt to verify this. $\diamondsuit$

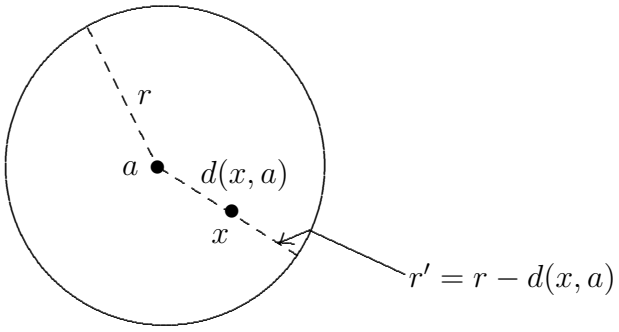

Figure 16.3: A diagram indicating an open sphere $S_r(a)$ in a metric space (X, d)

Theorem 16.8 *In a metric space (X, d),*

(i) *the empty set $\emptyset$ and the set X are open, and*

(ii) *every open sphere is an open set.*

Proof. Since there is no point in $\emptyset$, the statement that $\emptyset$ is open is true vacuously. For each point $a \in X$, every open sphere centered at a is contained in X. Thus X is open and (i) is verified.

To verify (ii), let $S_r(a)$ be an open sphere in (X, d) and let $x \in S_r(a)$. We show that there exists an open sphere centered at x and contained in $S_r(a)$. Since $d(x, a) < r$, it follows that

$r' = r - d(x,a) > 0$. We show that $S_{r'}(x) \subseteq S_r(a)$. Let $y \in S_{r'}(x)$. Since $d(y,x) < r'$ and $d(x,a) = r - r'$, it follows by the triangle inequality that

$$d(y,a) \leq d(y,x) + d(x,a) < r' + (r - r') = r.$$

Therefore, $y \in S_r(a)$ and so $S_{r'}(x) \subseteq S_r(a)$. ∎

To illustrate Theorem 16.8, we return to the metric space (X,d) described in Result 16.2, namely, $X = \mathbf{R}$ with $d(x,y) = |2^x - 2^y|$ for $x, y \in \mathbf{R}$. Thus $\emptyset$ and $X = \mathbf{R}$ are open sets as are all open spheres $S_r(a)$, where $a \in \mathbf{R}$ and $r > 0$. One such open sphere is $S_1(0) = \{x \in \mathbf{R} : |2^x - 2^0| < 1\}$. The inequality $|2^x - 2^0| < 1$ is equivalent to the inequalities $-1 < 2^x - 1 < 1$ and $0 < 2^x < 2$. Since $2^x > 0$ for all x, it follows that $0 < 2^x < 2$ is satisfied for all real numbers in the infinite interval $(-\infty, 1)$, and so $(-\infty, 1)$ is the open sphere with center 0 and radius 1 (according to the given metric). We also consider the open sphere $S_6(1) = \{x \in \mathbf{R} : |2^x - 2^1| < 6\}$. Here $|2^x - 2^1| < 6$ is equivalent to the inequalities $-6 < 2^x - 2 < 6$ and $-4 < 2^x < 8$ and so $S_6(1)$ is the open sphere $(-\infty, 3)$ with center 1 and radius 6.

We are now prepared to present a characterization of open sets in any metric space.

Theorem 16.9 *A subset O of a metric space is open if and only if it is a (finite or infinite) union of open spheres.*

Proof. Let (X,d) be a metric space. First let O be an open set in (X,d). We show that O is a union of open spheres. If $O = \emptyset$, then O is the union of zero open spheres. So we may assume that $O \neq \emptyset$. Let $x \in O$. Since O is open, there exists a positive number r_x such that $S_{r_x}(x) \subseteq O$. This implies that $\bigcup_{x \in O} S_{r_x}(x) \subseteq O$. On the other hand, if $x \in O$, then $x \in S_{r_x}(x) \subseteq \bigcup_{x \in O} S_{r_x}(x)$, implying that $O \subseteq \bigcup_{x \in O} S_{r_x}(x)$. Therefore, $O = \bigcup_{x \in O} S_{r_x}(x)$.

Next we show that if O is a subset of (X,d) that is a union of open spheres, then O is open. If $O = \emptyset$, then O is open. Hence we may assume that $O \neq \emptyset$. Let $x \in O$. Since O is a union of open spheres, x belongs to some open sphere, say $S_r(a)$. Since $S_r(a)$ is open, there exists $r' > 0$ (as we saw in the proof of Theorem 16.8) such that $S_{r'}(x) \subseteq S_r(a) \subseteq O$. Therefore, O is open. ∎

Two important properties of open sets are established in the next theorem.

Theorem 16.10 *Let (X,d) be a metric space. Then*

(*i*) *the intersection of any finite number of open sets in X is open, and*

(*ii*) *the union of any number of open sets in X is open.*

Proof. We first verify (*i*). Let $O_1, O_2, \cdots, O_k$ be k open sets in X, and let $O = \cap_{i=1}^k O_i$. If O is empty, then O is open by Theorem 16.8(*i*). Thus, we may assume that O is nonempty and let $x \in O$. We show that x is the center of an open sphere that is contained in O. Since $x \in O$, it follows that $x \in O_i$ for all i $(1 \leq i \leq k)$. Because each set O_i is open, there exists an open sphere $S_{r_i}(x) \subseteq O_i$, where $1 \leq i \leq k$. Let $r = \min\{r_1, r_2, \cdots, r_k\}$. Then $r > 0$ and $S_r(x) \subseteq S_{r_i}(x) \subseteq O_i$ for each i $(1 \leq i \leq k)$. Therefore, $S_r(x) \subseteq \cap_{i=1}^k O_i = O$. Thus O is open.

Next we verify (*ii*). Let $\{O_\alpha\}_{\alpha \in I}$ be an indexed collection of open sets in X, and let $O = \cup_{\alpha \in I} O_\alpha$. We show that O is open. If $O = \emptyset$, then again O is open. So we assume that $O \neq \emptyset$. By Theorem 16.9, each open set O_α $(\alpha \in I)$ is the union of open spheres. Thus, O is a union of open spheres. It again follows by Theorem 16.9 that O is open. ∎

For the Euclidean space $(\mathbf{R}, d)$, each open interval $I_n = \left(-1 - \frac{1}{n}, 1 + \frac{1}{n}\right), n \in \mathbf{N}$, is an open set. By Theorem 16.10, $\bigcup_{n=1}^{\infty} I_n = (-2, 2)$ is an open set, as is $\bigcap_{n=1}^{100} I_n = \left(-\frac{101}{100}, \frac{101}{100}\right)$. However, Theorem 16.10 does not guarantee that $\bigcap_{n=1}^{\infty} I_n$ is open. Indeed, $\bigcap_{n=1}^{\infty} I_n$ is the closed interval $[-1, 1]$, which is not an open set. The open interval $J_n = \left(0, \frac{1}{n}\right), n \in \mathbf{N}$, is an open set as well. Thus $\bigcup_{n=1}^{\infty} J_n = (0, 1)$ is an open set. In this case, $\bigcap_{n=1}^{\infty} J_n = \emptyset$, which is also an open set.

We now turn to the Euclidean space $(\mathbf{R}^2, d)$. Let $P_0 = (0, 0)$. For $n \in \mathbf{N}$, the open sphere $S_n(P_0)$ centered at $(0, 0)$ and having radius n is an open set. Here $\bigcup_{n=1}^{\infty} S_n(P_0) = \mathbf{R}^2$, which is open; while $\bigcap_{n=1}^{\infty} S_n(P_0) = S_1(P_0)$, which is open. In $(\mathbf{R}^2, d)$, where d is the discrete metric, $S_1(P_0) = \{P_0\}$, while $S_2(P_0) = \mathbf{R}^2$. Of course, all sets are open in a discrete metric space.

There is another important class of sets in metric spaces that arise naturally from open sets. Let (X, d) be a metric space. A subset F of X is called **closed** if its complement $\overline{F}$ is open. For example, in the Euclidean space $(\mathbf{R}, d)$, each closed interval $[a, b]$ where $a < b$, is closed since its complement $(-\infty, a) \cup (b, \infty)$ is open. Let a be a point in a metric space and let $S_r[a]$ consist of those points $x \in X$ such that $d(x, a) \leq r$. The set $S_r[a]$ is called a **closed sphere** with center a and radius r. Not surprisingly, $S_r[a]$ is closed, as we show next. Moreover, $\emptyset$ and X are both open and closed.

Theorem 16.11 *In a metric space (X, d),*

(i) $\emptyset$ and X are closed, and

(ii) every closed sphere is closed.

Proof. Since $\emptyset$ and X are complements of each other and each is open, it follows that each is closed. To verify *(ii)*, let $S_r[a]$ be a closed sphere in (X, d), where $a \in X$. We show that its complement $\overline{S_r[a]}$ is open. We may assume that $\overline{S_r[a]}$ is nonempty and a proper subset of X. Let $x \in \overline{S_r[a]}$. Thus $d(x, a) > r$ and $r^* = d(x, a) - r > 0$. We show that $S_{r^*}(x) \subseteq \overline{S_r[a]}$, that is, if $y \in S_{r^*}(x)$, then $y \notin S_r[a]$. Let $y \in S_{r^*}(x)$. Since $d(x, y) < r^* = d(x, a) - r$, it then follows by the triangle inequality that

$$d(y, a) \geq d(x, a) - d(x, y) > d(x, a) - r^* = r$$

and so $d(y, a) > r$. Hence $y \in \overline{S_r[a]}$, which implies that $S_{r^*}(x) \subseteq \overline{S_r[a]}$. ∎

Some other useful facts about closed sets follow immediately from Theorem 16.10. First, it is useful to recall from Result 9.15 and Exercise 9.24 that if $A_1, A_2, \ldots, A_n$ are $n \geq 2$ sets, then

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i} \quad \text{and} \quad \overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}.$$

These are DeMorgan's Laws for any finite number of sets. There is a more general form of DeMorgan's Laws.

Theorem 16.12 (*Extended DeMorgan Laws*) *For an indexed collection* $\{A_\alpha\}_{\alpha\in I}$ *of sets,*

$$(a)\ \overline{\bigcup_{\alpha\in I} A_\alpha} = \bigcap_{\alpha\in I} \overline{A_\alpha} \qquad \text{and} \qquad (b)\ \overline{\bigcap_{\alpha\in I} A_\alpha} = \bigcup_{\alpha\in I} \overline{A_\alpha}.$$

We present the proof of (a) only, leaving the proof of (b) as an exercise (Exercise 16.14).

Proof of Theorem 16.12 (a). First we show that $\overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$. Let $x \in \overline{\bigcup_{\alpha\in I} A_\alpha}$. Then $x \notin \bigcup_{\alpha\in I} A_\alpha$. Hence $x \notin A_\alpha$ for each $\alpha \in I$, which implies that $x \in \overline{A_\alpha}$ for all $\alpha \in I$. Consequently, $x \in \bigcap_{\alpha\in I} \overline{A_\alpha}$ and so $\overline{\bigcup_{\alpha\in I} A_\alpha} \subseteq \bigcap_{\alpha\in I} \overline{A_\alpha}$.

Next we show that $\bigcap_{\alpha\in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}$. Let $x \in \bigcap_{\alpha\in I} \overline{A_\alpha}$. Then $x \in \overline{A_\alpha}$ for each $\alpha \in I$. Thus $x \notin A_\alpha$ for all $\alpha \in I$. This implies, however, that $x \notin \bigcup_{\alpha\in I} A_\alpha$ and hence that $x \in \overline{\bigcup_{\alpha\in I} A_\alpha}$. Therefore, $\bigcap_{\alpha\in I} \overline{A_\alpha} \subseteq \overline{\bigcup_{\alpha\in I} A_\alpha}$. ∎

Corollary 16.13 *Let* (X, d) *be a metric space. Then*

(*i*) *the union of any finite number of closed sets in* X *is closed, and*

(*ii*) *the intersection of any number of closed sets in* X *is closed.*

Proof. Let $F_1, F_2, \cdots, F_k$ be k closed sets in X and let $F = \bigcup_{i=1}^{k} F_i$. Then $\overline{F} = \overline{\bigcup_{i=1}^{k} F_i} = \bigcap_{i=1}^{k} \overline{F_i}$. Since each set F_i $(1 \le i \le k)$ is closed, each set $\overline{F_i}$ is open. By Theorem 16.10, $\overline{F}$ is open and so F is closed. This verifies (*i*).

Next we verify (*ii*). Let $\{F_\alpha\}_{\alpha\in I}$ be an indexed collection of closed sets in X, and let $F = \bigcap_{\alpha\in I} F_\alpha$. Then $\overline{F} = \overline{\bigcap_{\alpha\in I} F_\alpha} = \bigcup_{\alpha\in I} \overline{F_\alpha}$ by Theorem 16.12. Since each set F_α $(\alpha \in I)$ is closed, each set $\overline{F_\alpha}$ is open. By Theorem 16.10, $\overline{F}$ is open and so F is closed. ∎

16.3 Continuity in Metric Spaces

We have seen then in calculus that defining a function f to be continuous at a real number can be formulated in terms of distance or in terms of open intervals, each of which can be generalized. Now we generalize the concept of continuity itself.

Let (X, d) and (Y, d') be metric spaces, and let $a \in X$. A function $f : X \to Y$ is said to be **continuous at the point** a if for every positive real number ϵ, there exists a positive real number δ such that if $x \in X$ and $d(x, a) < \delta$, then $d'(f(x), f(a)) < \epsilon$. The function $f : X \to Y$ is **continuous** on X if it is continuous at each point of X. If $X = Y = \mathbf{R}$ and $d = d'$ is defined by $d(x, y) = |x - y|$ for all $x, y \in \mathbf{R}$, then we are giving the standard definition of continuity in calculus.

We now consider some examples of continuous functions in this more general setting.

Result 16.14 *Let* $(\mathbf{R}^2, d)$ *be the Manhattan metric space whose distance* $d(P_1, P_2)$ *between two points* $P_1 = (x_1, y_1)$ *and* $P_2 = (x_2, y_2)$ *in* $\mathbf{R}^2$ *is defined by* $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, *and let* $(\mathbf{R}, d')$ *be the Euclidean space, where* $d'(a, b) = |a - b|$. *Then*

(*i*) *the function* $f : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $f((x, y)) = f(x, y) = x + y$ *is continuous.*

(*ii*) *the function* $g : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $g(x, y) = d'(x, y) = |x - y|$ *is continuous.*

Proof. We first verify (*i*). Let $\epsilon > 0$ be given and let $P_0 = (x_0, y_0) \in \mathbf{R}^2$. We choose $\delta = \epsilon$. Now let $P = (x, y) \in \mathbf{R}^2$ such that $d(P, P_0) = |x - x_0| + |y - y_0| < \delta$. Then

$$\begin{aligned} d'(f(x, y), f(x_0, y_0)) &= d'(x + y, x_0 + y_0) = |(x + y) - (x_0 + y_0)| \\ &= |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \delta = \epsilon. \end{aligned}$$

Therefore, f is continuous.

We now verify (*ii*). Again, let $\epsilon > 0$ be given and let $P_0 = (x_0, y_0) \in \mathbf{R}^2$. For a given $\epsilon > 0$, choose $\delta = \epsilon$. Let $P = (x, y) \in \mathbf{R}^2$ such that $d(P, P_0) = |x - x_0| + |y - y_0| < \delta$. We show that

$$d'(g(P), g(P_0)) = d'(|x - y|, |x_0 - y_0|) = ||x - y| - |x_0 - y_0|| < \epsilon,$$

which is equivalent to $-\epsilon < |x - y| - |x_0 - y_0| < \epsilon$. Observe, by the triangle inequality, that

$$\begin{aligned} |x - y| - |x_0 - y_0| &= |(x - x_0) + (x_0 - y_0) + (y_0 - y)| - |x_0 - y_0| \\ &\leq |x - x_0| + |x_0 - y_0| + |y_0 - y| - |x_0 - y_0| \\ &= |x - x_0| + |y_0 - y| < \delta = \epsilon. \end{aligned}$$

Similarly, $|x_0 - y_0| - |x - y| \leq |x - x_0| + |x - y| + |y_0 - y| - |x - y| = |x - x_0| + |y_0 - y| < \epsilon$. ∎

Proof Analysis Let's review how Theorem 16.14(*ii*) was proved. The main goal was to show that $||x-y|-|x_0-y_0|| < \epsilon$ given that $|x-x_0|+|y_0-y| < \delta$. Letting $a = |x-y|$ and $b = |x_0-y_0|$, we have the inequality $|a - b| < \epsilon$ to verify, which is equivalent to $-\epsilon < a - b < \epsilon$, which, in turn, is equivalent to

$$a - b < \epsilon \text{ and } b - a < \epsilon.$$

Thus one of the inequalities we wish to establish is $|x - y| - |x_0 - y_0| < \epsilon$. Since we know that $|x-x_0|+|y_0-y| < \delta$, this suggests working the expression $|x-x_0|+|y_0-y|$ into the expression $|x-y|-|x_0-y_0|$. This can be accomplished by adding and subtracting the appropriate quantities. Observe that

$$\begin{aligned} |x - y| - |x_0 - y_0| &= |(x - x_0) + (x_0 - y_0) + (y_0 - y)| - |x_0 - y_0| \\ &\leq |x - x_0| + |x_0 - y_0| + |y_0 - y| - |x_0 - y_0| \\ &= |x - x_0| + |y_0 - y| < \delta. \end{aligned}$$

This suggests choosing $\delta = \epsilon$. Of course, we must be certain that with this choice of δ, we can also show that $|x_0 - y_0| - |x - y| < \epsilon$. ♢

The function $i : \mathbf{R} \to \mathbf{R}$ defined by $i(x) = x$ for all $x \in \mathbf{R}$ is, of course, the identity function. It would probably seem that this function must surely be continuous. However, this depends on the metrics being used.

Example 16.15 *Let* $(\mathbf{R}, d)$ *be the discrete metric space and* $(\mathbf{R}, d')$ *the Euclidean space with* $d'(x, y) = |x - y|$ *for all* $x, y \in \mathbf{R}$. *Then*

(*i*) *the function* $f : (\mathbf{R}, d) \to (\mathbf{R}, d')$ *defined by* $f(x) = x$ *for all* $x \in \mathbf{R}$ *is continuous, and*

(*ii*) *the function* $g : (\mathbf{R}, d') \to (\mathbf{R}, d)$ *defined by* $g(x) = x$ *for all* $x \in \mathbf{R}$ *is not continuous.*

Solution. First we verify (*i*). Let $a \in \mathbf{R}$ and let $\epsilon > 0$ be given. Choose $\delta = 1/2$. Let $x \in \mathbf{R}$ such that $d(x, a) < \delta = 1/2$. We show that $d'(f(x), f(a)) < \epsilon$. Since d is the discrete metric and $d(x, a) < 1/2$, it follows that $x = a$. Hence $d'(f(x), f(a)) = |f(x) - f(a)| = |x - a| = |a - a| = 0 < \epsilon$.

Next we verify (*ii*). Let $a \in \mathbf{R}$ and choose $\epsilon = 1/2$. Let δ be any positive real number. Let $x = a + \delta/2 \in \mathbf{R}$. Then $d'(x, a) = |x - a| = |(a + \delta/2) - a| = \delta/2 < \delta$. Since $x \neq a$, $d(g(x), g(a)) = d(x, a) = 1 > \epsilon$. Hence for $\epsilon = 1/2$, there is no $\delta > 0$ such that if $d'(x, a) < \delta$, then $d(g(x), g(a)) < \epsilon$. Therefore, g is not continuous at a. $\diamondsuit$

Continuity of functions defined from one metric space to another can also be described by means of open sets. To do this, we need additional definitions and notation. Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. If A is a subset of X, then its **image** $f(A)$ is that subset of Y defined by

$$f(A) = \{f(x) \ : \ x \in A\}.$$

Similarly, if B is a subset of Y, then its **inverse image** $f^{-1}(B)$ is defined by

$$f^{-1}(B) = \{x \in X \ : \ f(x) \in B\}.$$

To illustrate these concepts, consider a function $f : \mathbf{R} \to \mathbf{R}$, for some metric d on $\mathbf{R}$, where f is defined by $f(x) = x^2$ for all $x \in \mathbf{R}$. Then $f(x)$ is a polynomial (whose graph is a parabola). Let $A = (-1, 2]$, $B = [-2, 2]$, and $C = [0, 4]$. Then $f(A) = C$, while $f^{-1}(C) = B$.

Now let (X, d) and (Y, d') be metric spaces, let $f : X \to Y$, and let $a \in X$. Suppose that for each $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in X$ and $d(x, a) < \delta$, then $d'(f(x), f(a)) < \epsilon$. Then f is continuous at a. Equivalently, f is continuous at a if whenever $x \in S_\delta(a)$, then $f(x) \in S_\epsilon(f(a))$. Hence f is continuous at a if for each $\epsilon > 0$, there exists $\delta > 0$ such that $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$. We now present a characterization of those functions f that are continuous on the entire set X.

Theorem to Prove Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. Then f is continuous on X if and only if for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X.

Proof Strategy Let's begin with the implication: If f is continuous on X, then for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X. Using a direct proof, we would assume that f is continuous and that O is an open set in Y. If $f^{-1}(O) = \emptyset$, then $f^{-1}(O)$ is an open set in X; while if $f^{-1}(O) \neq \emptyset$, then we are required to show that every element $x \in f^{-1}(O)$ is the center of an open sphere contained in $f^{-1}(O)$. So let $x \in f^{-1}(O)$. Therefore, $f(x) \in O$. We know that O is open; so there is some open sphere $S_\epsilon(f(x))$ contained in O. However, f is continuous at x; so there exists $\delta > 0$ such that $f(S_\delta(x)) \subseteq S_\epsilon(f(x))$. Hence $S_\delta(x) \subseteq f^{-1}(O)$.

We also attempt a direct proof to verify the converse. We begin then by assuming that for each open set O in Y, the set $f^{-1}(O)$ is open in X. Our goal is to show that f is continuous on X. We let $a \in X$ and $\epsilon > 0$ be given. The open sphere $S_\epsilon(f(a))$ is an open set in Y. By hypothesis, $f^{-1}(S_\epsilon(f(a)))$ is an open set in X. Furthermore, $a \in f^{-1}(S_\epsilon(f(a)))$. Therefore, there exists $\delta > 0$ such that $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$ and f is continuous on X. $\diamondsuit$

We now give a more concise proof.

Theorem 16.16 *Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$. Then f is continuous on X if and only if for each open set O in Y, the inverse image $f^{-1}(O)$ is an open set in X.*

Proof. Assume first that f is continuous on X. Let O be an open set in Y. We show that $f^{-1}(O)$ is open in X. If $f^{-1}(O) = \emptyset$, then $f^{-1}(O)$ is open; so we may assume that $f^{-1}(O) \neq \emptyset$. Let $x \in f^{-1}(O)$. Since $x \in f^{-1}(O)$, it follows that $f(x) \in O$. Because O is open, there exists an open sphere $S_\epsilon(f(x))$ that is contained in O. Since f is continuous at x, there exists $\delta > 0$ such that $f(S_\delta(x)) \subseteq S_\epsilon(f(x)) \subseteq O$. Thus, $S_\delta(x) \subseteq f^{-1}(O)$, as desired.

For the converse, assume that for each open set O of Y, the inverse image $f^{-1}(O)$ is an open set of X. We show that f is continuous on X. Let a be an arbitrary point in X. Let $\epsilon > 0$ be given. The set $S_\epsilon(f(a))$ is open in Y and so its inverse image $f^{-1}(S_\epsilon(f(a)))$ is open in X and contains a. Then there exists $\delta > 0$ such that the open sphere $S_\delta(a) \subseteq f^{-1}(S_\epsilon(f(a)))$. Therefore, $f(S_\delta(a)) \subseteq S_\epsilon(f(a))$ and so f is continuous at a. Hence f is continuous on X. ∎

With the aid of Theorem 16.16, it can now be shown that any constant function from one metric space to another is continuous.

Result 16.17 *Let (X, d) and (Y, d') be metric spaces and let $f : X \to Y$ be a constant function, that is, $f(x) = c$ for some $c \in Y$. Then f is continuous.*

Proof. Let O be an open set in Y. Then $f^{-1}(O) = \emptyset$ if $c \notin O$; otherwise $f^{-1}(O) = X$. In any case, $f^{-1}(O)$ is open. By Theorem 16.16, f is continuous on X. ∎

16.4 Topological Spaces

In the previous section, we introduced the concept of a continuous function from one metric space to another, and the definition was formulated in terms of the metrics on the spaces involved. However, Theorem 16.16 shows that the continuity of a function on a metric space can be established in terms of open sets only, without any direct reference to metrics. This suggests the possibility of discarding metrics altogether, replacing them by open sets, and describing continuity in an even more general setting. This gives rise to another mathematical structure, called a topological space.

Let X be a nonempty set, and let τ (the Greek letter "tau") be a collection of subsets of X. Then (X, τ) is called a **topological space**, and τ itself is called a **topology** on X, if the following properties are satisfied:

(1) $X \in \tau$ and $\emptyset \in \tau$.

(2) If $O_1, O_2, \cdots, O_n \in \tau$, where $n \in \mathbf{N}$, then $\cap_{i=1}^{n} O_i \in \tau$.

(3) If, for an index set I, $O_\alpha \in \tau$ for each $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha \in \tau$.

In a topological space (X, τ), we refer to each element of τ as an **open set** of X. Property (1) states that X and the empty set are open. Property (2) states that the intersection of any finite number of open sets is open; while property (3) states the union of any number of open sets is open. For example, for a nonempty set X, let $\tau_1 = \{\emptyset, X\}$ and $\tau_2 = \mathcal{P}(X)$, the set of all subsets of X. Then for $i = 1, 2$, (X, τ_i) is a topological space. The topology τ_1 is called the **trivial topology** on X, while τ_2 is the **discrete topology** on X. In (X, τ_1), the only open sets are X and $\emptyset$; while in (X, τ_2), every subset of X is open.

It follows immediately from the definition of a topological space and the properties of open sets in a metric space that every metric space is a topological space. The converse is not true however. When we say that a topological space (X, τ) is a metric space, we mean that it is possible to define a metric d on X such that the set of open sets of (X, d) is τ.

Example 16.18 *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a topological space that is not a metric space.*

Solution. To see that (X, τ) is a topological space, it suffices to observe that the union or intersection of any elements of τ also belongs to τ.

We now show that (X, τ) is not a metric space, that is, there is no way to define a metric on X such that the resulting open sets are precisely the elements of τ. We verify this by contradiction. Assume, to the contrary, that there exists a metric d such that the open sets in (X, d) are the elements of τ. Let $r = \min\{d(a, b), d(b, c)\}$. Necessarily, $r > 0$. Then

$$S_r(b) = \{x \in X \ : \ d(x, b) < r\} = \{b\},$$

which, however, does not belong to τ, a contradiction. $\diamondsuit$

We now present two other examples of topological spaces, the first of which is suggested by the preceding result.

Result 16.19 *Let X be a nonempty set. For $a \in X$, let τ consist of $\emptyset$ and each subset of X containing a. Then (X, τ) is a topological space.*

Proof. Since $a \in X$, it follows that $X \in \tau$. Furthermore, $\emptyset \in \tau$; so property (1) is satisfied. Let $O_1, O_2, \cdots, O_n$ be n elements of τ. If $O_i = \emptyset$ for some i $(1 \leq i \leq n)$, then $\cap_{i=1}^n O_i = \emptyset$ and so $\cap_{i=1}^n O_i \in \tau$. Otherwise, $a \in O_i$ for all i with $1 \leq i \leq n$. Thus $a \in \cap_{i=1}^n O_i$, implying that $\cap_{i=1}^n O_i \in \tau$. Finally, for an index set I, let $\{O_\alpha\}_{\alpha \in I}$ be a collection of elements of τ. If $O_\alpha = \emptyset$ for all $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha = \emptyset$ and so $\cup_{\alpha \in I} O_\alpha \in \tau$. Otherwise, $a \in O_\alpha$ for some $\alpha \in I$ and so $a \in \cup_{\alpha \in I} O_\alpha$. Therefore, $\cup_{\alpha \in I} O_\alpha \in \tau$. Hence (X, τ) is a topological space. ∎

Our next example of a topological space uses the Extended DeMorgan Laws (Theorem 16.12).

Result to Prove Let X be a nonempty set, and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is finite. Then (X, τ) is a topological space.

Proof Strategy If X is a finite set, then τ consists of all subsets of X. In this case, τ is the discrete topology on X, and (X, τ) is a topological space. Hence we need only be concerned with the case when X is infinite. We already known that $\emptyset \in \tau$. Also $\overline{X} = \emptyset$, which is finite; so $X \in \tau$ as well. So (X, τ) satisfies property (1) required of a topological space.

In order to show that (X, τ) satisfies property (2), we let $O_1, O_2, \cdots, O_n \in \tau$ for $n \in \mathbf{N}$. We are required to show that $\cap_{i=1}^n O_i \in \tau$. If any of the open sets $O_1, O_2, \cdots, O_n$ is empty, then $\cap_{i=1}^n O_i = \emptyset$ and so $\cap_{i=1}^n O_i$ belongs to τ. Hence it suffices to assume that $O_i \neq \emptyset$ for all i $(1 \leq i \leq n)$. It is necessary to show that $\overline{\cap_{i=1}^n O_i}$ is finite. However, $\overline{\cap_{i=1}^n O_i} = \cup_{i=1}^n \overline{O_i}$ by DeMorgan's law. Since each set $\overline{O_i}$ is finite $(1 \leq i \leq n)$, the union of these sets is finite as well. Therefore, $\cap_{i=1}^n O_i \in \tau$ and property (2) is satisfied.

To show that property (3) is satisfied, we begin with an indexed family $\{O_\alpha\}_{a \in I}$ of open sets in X and are required to show that $\cup_{a \in I} O_\alpha \in \tau$. We can proceed in a manner similar to the verification of property (2). $\diamondsuit$

Result 16.20 *Let X be a nonempty set, and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is finite. Then (X, τ) is a topological space.*

Proof. If X is finite, then τ is the discrete topology. Hence we may assume that X is infinite. Since the complement of X is $\emptyset$, it follows that $X \in \tau$. Since $\emptyset \in \tau$ as well, (1) holds. Let $O_1, O_2, \cdots, O_n$ be n elements of τ. If $O_i = \emptyset$ for some i $(1 \leq i \leq n)$, then $\cap_{i=1}^n O_i = \emptyset \in \tau$. Hence we may assume that $O_i \neq \emptyset$ for all i $(1 \leq i \leq n)$. Then each set $\overline{O_i}$ is finite. By DeMorgan's law, $\overline{\cap_{i=1}^n O_i} = \cup_{i=1}^n \overline{O_i}$. Since $\overline{\cap_{i=1}^n O_i}$ is a finite union of finite sets, it is finite. Thus $\cap_{i=1}^n O_i \in \tau$ and so (2) is satisfied. To verify (3), let $\{O_\alpha\}_{a \in I}$ be any collection of elements of τ. Again, by DeMorgan's law,

$$\overline{\bigcup_{a \in I} O_\alpha} = \bigcap_{a \in I} \overline{O_\alpha}.$$

If $O_\alpha = \emptyset$ for all $\alpha \in I$, then $\overline{O_\alpha} = X$ and so $\overline{\bigcup_{a \in I} O_\alpha} = X$. Thus we may assume that there is some $\beta \in I$ such that $O_\beta \neq \emptyset$. Hence $\overline{O_\beta}$ is finite and $\bigcap_{a \in I} \overline{O_\alpha} \subseteq \overline{O_\beta}$. So $\bigcap_{a \in I} \overline{O_\alpha}$ is finite as well. Therefore, $\cup_{a \in I} O_\alpha \in \tau$ and (3) is satisfied. ∎

We saw in Theorem 16.7 that every two distinct points in a metric space (X, d) belong to disjoint open spheres in X. Since open spheres are open sets in X, it follows that two distinct points in X belong to disjoint open sets. This is often a useful property for a topological space to have.

A topological space (X, τ) is called a **Hausdorff space** (named for the mathematician Felix Hausdorff) if for each pair a, b of distinct points of X, there exist disjoint open sets O_a and O_b of X containing a and b, respectively. The following result is a consequence of Theorem 16.7.

Corollary 16.21 *Every metric space is a Hausdorff space.*

On the other hand, not every topological space is a Hausdorff space and not every Hausdorff space is a metric space. We verify the first of these. The second of these is a deeper question in topology.

Example 16.22 *Let X be an infinite set and let τ be the set consisting of $\emptyset$ and every subset of X whose complement is finite. Then (X, τ) is a topological space that is not a Hausdorff space.*

Solution. We saw in Result 16.20 that (X, τ) is a topological space; so it remains only to show that (X, τ) is not a Hausdorff space. Let a and b be any two distinct elements of X. We claim that there do not exist two disjoint open sets, one containing a and the other b. Assume, to the contrary, that there exist (nonempty) open sets O_a and O_b containing a and b, respectively, such that $O_a \cap O_b = \emptyset$. Then, by DeMorgan's law, $\overline{O_a \cap O_b} = X = \overline{O_a} \cup \overline{O_b}$. Since X is infinite, at least one of $\overline{O_a}$ and $\overline{O_b}$ is infinite. This implies that at least one of O_a and O_b is not open, which is a contradiction. ♢

16.5 Continuity in Topological Spaces

By Theorem 16.16, if (X, d) and (Y, d') are metric spaces, then a function $f : X \to Y$ is continuous if and only if $f^{-1}(O)$ is an open set in X for each open set O in Y. Hence, instead of defining a function f to be continuous in terms of distances in the two metric spaces (as we did), we could have defined f to be continuous in terms of open sets. Since it would be meaningless

to define a function from one topological space to another to be continuous in terms of distance, we have a logical alternative.

Let (X, τ) and (Y, τ') be two topological spaces. A function $f : X \to Y$ is defined to be **continuous** if $f^{-1}(O)$ is an open set in X for every open set O in Y. Let's see how this definition works in practice.

Result 16.23 *Let (X, τ) and (Y, τ') be two topological spaces.*

(*i*) *If τ is the discrete topology on X, then every function $f : X \to Y$ is continuous.*

(*ii*) *Let τ be the trivial topology on X and let $f : X \to Y$ be a surjective function. Then f is continuous if and only if τ' is the trivial topology on Y.*

Proof. First we verify (*i*). Let O be an open set in Y. Since $f^{-1}(O)$ is a subset of X, it follows that $f^{-1}(O)$ is an open set in X and so f is continuous.

Next we verify (*ii*). Assume first that τ' is the trivial topology on Y. Then Y and $\emptyset$ are the only open sets in Y. Since $f^{-1}(Y) = X$ and $f^{-1}(\emptyset) = \emptyset$ are open sets in X, it follows that f is continuous. For the converse, assume that τ' is a topology on Y that is not the trivial topology. Then there exists some open set O in Y distinct from Y and $\emptyset$. Since f is surjective, $f^{-1}(O)$ is distinct from X and $\emptyset$. Thus $f^{-1}(O)$ is not an open set in X, implying that f is not continuous. ■

Result 16.24 *Let (X, τ) and (Y, τ') be topological spaces.*

(*i*) *The identity function $i : X \to X$ (defined by $i(x) = x$ for all $x \in X$) is continuous.*

(*ii*) *If $g : X \to Y$ is a constant function, that is, if $g(x) = c$ for all $x \in X$, where $c \in Y$, then g is continuous.*

Proof. We first verify (*i*). Let O be an open set in X. Since $i^{-1}(O) = O$ is an open set in X, the function i is continuous.

Next we verify (*ii*). Let O be an open set in Y. If $c \in O$, then $g^{-1}(O) = X$; while if $c \notin O$, then $g^{-1}(O) = \emptyset$. In either case, $g^{-1}(O)$ is an open set in X and so g is continuous. ■

Example 16.25 *Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$ and let $f : X \to X$ be defined by $f(a) = b$, $f(b) = c$, and $f(c) = a$. Determine whether f is continuous.*

Solution. Since $O = \{a\}$ is an open set in X and $f^{-1}(O) = \{b\}$ is not an open set in X, the function f is not continuous. ♢

Based on the definition given of a continuous function from one metric space to another, it might appear more natural, for topological spaces (X, τ) and (Y, τ'), to define a function $f : X \to Y$ to be continuous if, for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$. This is equivalent to our definition, as we are about to see. First, a lemma is useful.

Lemma 16.26 *Let X and Y be nonempty sets and let $f : X \to Y$ be a function. For every subset B of Y,*

$$f\left(f^{-1}(B)\right) \subseteq B.$$

Proof. Let $y \in f\left(f^{-1}(B)\right)$. Then there is $x \in f^{-1}(B)$ such that $f(x) = y$. This implies that $y \in B$. ∎

Result to Prove Let (X, τ) and (Y, τ') be topological spaces. Then $f : X \to Y$ is continuous if and only if for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$.

Proof Strategy Assume first that f is continuous. Let $x \in X$ and let O be an open set in Y containing $y = f(x)$. What we are required to do is to find an open set U of X containing x such that $f(U) \subseteq O$. There is an obvious choice for U, however, namely, $f^{-1}(O)$. An application of Lemma 16.26 will complete the proof of this implication.

Next, we consider the converse. Assume that for every $x \in X$ and every open set O of Y containing $f(x)$, there is an open set U of X containing x such that $f(U) \subseteq O$. Since our goal is to show that f is continuous, we need to show that for every open set B of Y, the set $f^{-1}(B)$ is open in X. Of course, if $f^{-1}(B) = \emptyset$, then $f^{-1}(B)$ is an open set; so we assume that $f^{-1}(B) \neq \emptyset$. If we can show that $f^{-1}(B)$ is the union of open sets, then $f^{-1}(B)$ is open. Let $x \in f^{-1}(B)$. Then $f(x) \in B$. By hypothesis, there is an open set U_x in X containing x such that $f(U_x) \subseteq B$. This implies that $f^{-1}(B)$ is a union of open sets in X. ♢

Result 16.27 *Let (X, τ) and (Y, τ') be topological spaces. Then $f : X \to Y$ is continuous if and only if for every $x \in X$ and every open set O of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subseteq O$.*

Proof. Assume first that f is continuous. Let $x \in X$ and let O be an open set in Y that contains $f(x)$. Since f is continuous, $f^{-1}(O)$ is an open set in X containing x. Let $U = f^{-1}(O)$. By Lemma 16.26, $f(U) = f\left(f^{-1}(O)\right) \subseteq O$.

For the converse, assume that for every $x \in X$ and every open set O of Y containing $f(x)$, there is an open set U of X containing x such that $f(U) \subseteq O$. Let B be an open set in Y. We show that $f^{-1}(B)$ is an open set in X. If $f^{-1}(B) = \emptyset$, then $f^{-1}(B)$ is open in X. So we may assume that $f^{-1}(B) \neq \emptyset$. For each $x \in f^{-1}(B)$, the set B is an open set in Y containing $f(x)$. By assumption, there is an open set U_x in X containing x such that $f(U_x) \subseteq B$. Thus $U_x \subseteq f^{-1}(B)$. However, then, $f^{-1}(B) = \bigcup_{x \in f^{-1}(B)} U_x$ and so $f^{-1}(B)$ is an open set in X as well. ∎

Exercises for Chapter 16

16.1 In each of the following, a distance is defined on the set $\mathbf{R}$ of real numbers. Determine which of the four properties of a metric space are satisfied by d. Verify your answers.

(a) $d(x, y) = y - x$	**(b)** $d(x, y) = (x - y) + (y - x)$
(c) $d(x, y) = \lvert x - y \rvert + \lvert y - x \rvert$	(d) $d(x, y) = x^2 + y^2$
(e) $d(x, y) = \lvert x^2 - y^2 \rvert$	(f) $d(x, y) = \lvert x^3 - y^3 \rvert$

16.2 For points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, the Manhattan metric $d(P_1, P_2)$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. Prove that the Manhattan metric is, in fact, a metric on $\mathbf{R}^2$.

16.3 Let (X, d) be a metric space. For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in X^2, define $d' : X \times X \to \mathbf{R}$ by $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2)$. Which of the four properties of a metric space are satisfied by d'?

16.4 Let (X, d) be a metric space. For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in X^2, define $d^* : X \times X \to \mathbf{R}$ by $d^*(P_1, P_2) = \sqrt{[d(x_1, x_2)]^2 + [d(y_1, y_2)]^2}$. Which of the four properties of a metric space are satisfied by d^*?

16.5 Let A be a set and let a and b be two distinct elements of A. A distance $d : A \times A \to \mathbf{R}$ is defined as follows:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } \{x, y\} = \{a, b\} \\ 2 & \text{if } x \neq y \text{ and } \{x, y\} \neq \{a, b\}. \end{cases}$$

Which of the four properties of a metric space are satisfied by this distance?

16.6 Let (X, d) be a metric space.

(a) Define $d_1(x, y) = d(x, y)/[1 + d(x, y)]$. Prove that d_1 is a metric for X.

(b) Define $d_2(x, y) = \min\{1, d(x, y)\}$. Prove that d_2 is a metric for X.

16.7 In each part that follows, a distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is defined on the Cartesian product $\mathbf{R}^2$. Determine which of the four properties of a metric space is satisfied by each distance d. For those distances that are metrics, describe the associated open spheres.

(a) $d(P_1, P_2) = \min\{|x_1 - x_2|,\ |y_1 - y_2|\}$

(b) $d(P_1, P_2) = \max\{|x_1 - x_2|,\ |y_1 - y_2|\}$

(c) $d(P_1, P_2) = (|x_1 - x_2| + |y_1 - y_2|)/2$

16.8 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is given by $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Prove that the set $S = \{(x, y) : -1 < x < 1 \text{ and } -1 < y < 1\}$ is open in $(\mathbf{R}^2, d)$.

16.9 Let $(\mathbf{R}^2, d)$ and $(\mathbf{R}^2, d')$ be metric spaces, where for two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$, $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $d'(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$. Prove each of the following.

(a) Every open set in $(\mathbf{R}^2, d)$ is open in $(\mathbf{R}^2, d')$.

(b) Every open set in $(\mathbf{R}^2, d')$ is open in $(\mathbf{R}^2, d)$.

16.10 In the metric space $(\mathbf{R}, d)$, where $d(x, y) = |x - y|$, determine which of the following sets are open, closed, or neither and verify your answers.

(a)	$(0, 1]$	(b)	$[0, 1]$	(c)	$(-\infty, 1]$
(d)	$(0, \infty)$	(e)	$(0, 2) - \{1\}$	(f)	$\mathbf{Q}$
(g)	$\mathbf{I}$	(h)	$\{\frac{1}{n} \mid n \in \mathbf{N}\}$	(i)	$\{\frac{1}{n} \mid n \in \mathbf{N}\} \cup \{0\}$

16.11 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$ is defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$, and let $(\mathbf{R}, d')$ be the metric space with $d'(a, b) = |a - b|$.

Verify each of the following.

(a) The function $f : (\mathbf{R}^2, d) \to (\mathbf{R}, d')$ defined by $f(x, y) = \frac{1}{2}(x - y)$ is continuous.

(b) The function $g : (\mathbf{R}^2, d) \to (\mathbf{R}, d')$ defined by $g(x, y) = x$ is continuous.

16.12 Let $(\mathbf{R}^2, d)$ be the metric space whose distance $d(P_1, P_2)$ between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbf{R}^2$ is defined by $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and let d' be the discrete metric, that is,

$$d'(P_1, P_2) = \begin{cases} 0 & \text{if } P_1 = P_2 \\ 1 & \text{if } P_1 \neq P_2. \end{cases}$$

Verify each of the following.

(a) The function $f : (\mathbf{R}^2, d) \to (\mathbf{R}^2, d')$ defined by $f(x, y) = (x, y)$ is continuous.

(b) The function $g : (\mathbf{R}^2, d') \to (\mathbf{R}^2, d)$ defined by $g(x, y) = (x, y)$ is not continuous.

16.13 Let $X = \{a, b, c, d\}$. Determine which of the following collections of subsets of X are topologies on X. Verify your answers.

(a) $S_1 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$

(b) $S_2 = \{\emptyset, X, \{a, b\}, \{a, c\}\}$

(c) $S_3 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, d\}, \{a, b, d\}\}$

16.14 Prove the Extended DeMorgan Law in Theorem 16.12(b).

16.15 let X be a nonempty set and let $S \subseteq X$. Let τ consist of $\emptyset$ and each subset of X containing S. Prove that (X, τ) is a topological space.

16.16 Let (X, τ) be a topological space. Prove that if $\{x\}$ is an open set for every $x \in X$, then τ is the discrete topology.

16.17 Let (X, τ) be a topological space, where X is finite. Prove that (X, τ) is a metric space if and only if τ is the discrete topology on X.

16.18 (a) For a set X with $a \in X$, let τ consists of X together with all sets S such that $a \notin S$. Prove that (X, τ) is a topological space.

(b) State and prove a generalization of the result in (a).

16.19 Let X be a nonempty set and let τ be the set consisting of $\emptyset$ and each subset of X whose complement is countable. Prove that (X, τ) is a topological space.

16.20 Let a, b, c be three distinct elements in a Hausdorff space (X, τ). Prove that there exist pairwise disjoint open sets O_a, O_b, and O_c containing a, b, and c, respectively.

16.21 Let τ be the set consisting of $\emptyset$, $\mathbf{R}$, and each interval (a, ∞), where $a \in \mathbf{R}$. It is known that $(\mathbf{R}, \tau)$ is a topological space. (Don't attempt to prove this.) Show that $(\mathbf{R}, \tau)$ is not a Hausdorff space.

16.22 Prove that if (X, τ) is a topological space with the discrete topology, then (X, τ) is a Hausdorff space.

16.23 Let $(\mathbf{N}, \tau)$ be a topological space, where τ consists of $\emptyset$ and $\{S : S \subseteq \mathbf{N}, 1 \in S\}$, and let $f : \mathbf{N} \to \mathbf{N}$ be a continuous permutation. Determine $f(1)$.

16.24 Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Determine all continuous functions from X to X.

16.25 Let (X, τ_1), (Y, τ_2), and (Z, τ_3) be topological spaces, and let $f : X \to Y$ and $g : Y \to Z$ be functions. Prove that if f and g are continuous, then the composition $g \circ f$ is a continuous function from X to Z.

16.26 Let τ be the trivial topology on a nonempty set X. Prove that if $f : X \to X$ is continuous, then f is a constant function.

16.27 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let X be an infinite set and let τ consists of $\emptyset$ and all infinite subsets of X. Then (X, τ) is a topological space.

Proof. Since X is an infinite subset of X, it follows that $X \in \tau$. Since $\emptyset \in \tau$, property (1) of a topological space is satisfied. Let $O_1, O_2, \ldots, O_n$ be elements of τ for $n \in \mathbf{N}$. We show that $\cap_{i=1}^n O_i \in \tau$. If $O_i = \emptyset$ for some i with $1 \leq i \leq n$, then $\cap_{i=1}^n O_i = \emptyset$ and $\cap_{i=1}^n O_i \in \tau$. Otherwise, O_i is infinite for all i ($\leq i \leq n$). Hence $\cap_{i=1}^n O_i$ is infinite and so $\cap_{i=1}^n O_i \in \tau$. Thus property (2) is satisfied. Next, let $\{O_\alpha\}_{\alpha \in I}$ be an indexed family of open sets. If $O_\alpha = \emptyset$ for each $\alpha \in I$, then $\cup_{\alpha \in I} O_\alpha = \emptyset$ and so $\cup_{\alpha \in I} O_\alpha \in \tau$. Otherwise, O_α is infinite for some $\alpha \in I$ and so $\cup_{\alpha \in I} O_\alpha$ is infinite. Hence $\cup_{\alpha \in I} O_\alpha \in \tau$. Therefore, τ is a topology on X. ∎

16.28 Let (X, τ) and (Y, τ') be two topological spaces. According to Result 16.23(i), if τ is the discrete topology on X, then every function $f : X \to Y$ is continuous. The converse of Result 16.23(i) is stated as follows together with a "proof".

Converse of Result 16.23(i): Let (X, τ) and (Y, τ') be two topological spaces. If every function from X to Y is continuous, then τ is the discrete topology on X.

"**Proof.**" Suppose that every function $f : X \to Y$ is continuous and assume, to the contrary, that τ is not the discrete topology on X. Then there exists some subset S of X such that S is not open in X. So S is distinct from X and $\emptyset$. Let T be an open set in Y and let $a, b \in Y$ such that $a \in T$ and $b \notin T$. Define a function $f : X \to Y$ by

$$f(x) = \begin{cases} a & \text{if } x \in S \\ b & \text{if } x \notin S. \end{cases}$$

Since T is open in Y and $f^{-1}(T) = S$ is not open in X, it follows that f is not continuous, which is a contradiction. ∎

(a) Is the proposed proof of the converse correct?

(b) If the answer to (a) is yes, then state Result 16.23(i) and its converse using "if and only if". If the answer to (a) is no, then revise the hypothesis of the converse so that it is true (with attached proof).

16.29 Let X be a set with at least two elements, and let $a \in X$. Prove or disprove:

(a) If (X, d) is a metric space, then $X - \{a\}$ is an open set.

(b) If (X, d) is a topological space, then $X - \{a\}$ is an open set.

16.30 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let (X, d) be a metric space. For every open set O in X such that $O \neq \emptyset$, and every element $b \in \overline{O}$, there exists an open sphere $S_r(b)$ in X such that $S_r(b)$ and O are disjoint.

Proof. Let $r = \min\{d(b, x) : x \in O\}$. Consider the open sphere $S_r(b)$. We claim that $S_r(b) \cap O = \emptyset$. Assume, to the contrary, that $S_r(b) \cap O \neq \emptyset$. Then there exists $y \in S_r(b) \cap O$. Since $y \in S_r(b)$, it follows that $d(b, y) < r$. However, since $y \in O$, this contradicts the fact that r is the minimum distance between b and an element of O. ∎

16.31 Prove or disprove: Let (X, d) be a metric space. For every open set O in X such that $O \neq \emptyset$, there exist $b \in \overline{O}$ and an open sphere $S_r(b)$ in X such that $S_r(b)$ and O are disjoint.

Answers and Hints to Selected Odd-Numbered Exercises in Chapters 14-16

Chapter 14

14.1 (a) **Proof.** Let $a, b \in k\mathbf{Z}$. Then $a = kx$ and $b = ky$ for some $x, y \in \mathbf{Z}$. Note that $a + b = kx + ky = k(x + y)$ and $ab = (kx)(ky) = k(kxy)$. Since $x + y, kxy \in \mathbf{Z}$, it follows that $a + b, ab \in k\mathbf{Z}$; so the addition and multiplication defined are binary operations on $k\mathbf{Z}$. Since $k\mathbf{Z} \subseteq \mathbf{Z}$ and the binary operations in $k\mathbf{Z}$ are the same as those in $\mathbf{Z}$, properties R1, R2, R5, and R6 are automatically satisfied. Moreover, since $0 = k \cdot 0$ and $0 \in \mathbf{Z}$, it follows that $k\mathbf{Z}$ has an additive identity. To show that property R4 is also satisfied, let $a \in k\mathbf{Z}$. So $a = kx$, where $x \in \mathbf{Z}$. Then $-a = -(kx) = k(-x)$. Since $-x \in \mathbf{Z}$, it follows that $-a \in k\mathbf{Z}$. ∎

14.3 (a) **Solution** We show that $(S, *, \circ)$ is not a ring. Certainly, $*$ and $\circ$ are binary operations on S. However, property R6 is not satisfied. To see this, let $a = b = c = 0$. Then $a \circ (b * c) = 0 \circ 1 = 0$ and $(a \circ b) * (a \circ c) = 0 * 0 = 1$. ♢

14.7 **Proof.** Let $a \in \mathbf{R}$. Then $a^2 = a$. Thus $(a + a)^2 = (a + a)(a + a) = a(a + a) + a(a + a) = (a^2 + a^2) + (a^2 + a^2) = (a + a) + (a + a)$. Since $(a + a)^2 = a + a$, it follows that $(a + a) + (a + a) = (a + a) + 0$. Applying the Cancellation Law of Addition (Theorem 14.10), we obtain $a + a = 0$. Therefore, $-a = a$.

14.9 (a) Since the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to S, it follows that $S \neq \emptyset$. Let $M_1, M_2 \in S$. Thus $M_1 = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$ and $M_2 = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$, where $a_i, b_i \in \mathbf{R}$ for $i = 1, 2$. Then $M_1 - M_2 = \begin{bmatrix} a_1 - a_2 & 0 \\ 0 & b_1 - b_2 \end{bmatrix}$ and $M_1 M_2 = \begin{bmatrix} a_1 a_2 & 0 \\ 0 & b_1 b_2 \end{bmatrix}$ belong to S. By the Subring Test, S is a subring of $M_2(\mathbf{R})$.

14.11 **Solution** The set $2G$ of even Gaussian integers is a subring of G.

Proof. Since $0 \in 2\mathbf{Z}$, it follows that $0 = 0 + 0i \in 2G$ and so $2G \neq \emptyset$. Let $x, y \in 2G$. Then $x = a_1 + b_1 i$ and $y = a_2 + b_2 i$, where $a_i, b_i \in 2\mathbf{Z}$ for $i = 1, 2$. Then $x - y = (a_1 - a_2) + (b_1 - b_2)i$ and $xy = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$. Since $a_1 - a_2, b_1 - b_2, a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1 \in 2\mathbf{Z}$, it follows by the Subring Test that $2G$ is a subring of G. ∎

14.13 (a) Since the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ belongs to S, it follows that $S \neq \emptyset$. Let $M_1, M_2 \in S$. Thus $M_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$, where $a_i, b_i \in \mathbf{R}$ for $1 \leq i \leq 2$. Then $M_1 - M_2 = \begin{bmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & 0 \end{bmatrix}$ and $M_1 M_2 = \begin{bmatrix} a_1 a_2 & a_1 b_2 \\ 0 & 0 \end{bmatrix}$ belong to S. By the Subring Test, S is a subring of $M_2(\mathbf{R})$.

(b) Let $E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ be an arbitrary element of S. Then $EA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$. Let $C = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \in S$. Then $CE = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \neq C$.

14.15 **Proof.** First we show that $(2\mathbf{Z}, +, \circ)$ is a ring. Certainly, $2\mathbf{Z}$ is closed under addition. Let $a, b, c \in 2\mathbf{Z}$. Then $a = 2x$, $b = 2y$, and $c = 2z$, where $x, y, z \in 2\mathbf{Z}$. So $a \circ b = (2x)(2y)/2 = 2(xy)$. Since xy is an integer, $2\mathbf{Z}$ is closed under this multiplication. Because $(2\mathbf{Z}, +, \cdot)$ is a ring, where $\cdot$ is ordinary multiplication, $(2\mathbf{Z}, +, \circ)$ satisfies properties R1–R4 and the integer 0 is the zero element. Now $a \circ (b \circ c) = a \circ (bc/2) = a(bc)/4 = (ab)c/4 = (ab/2) \circ c = (a \circ b) \circ c$; so $(2\mathbf{Z}, +, \circ)$ satisfies property R5. Finally, $a \circ (b+c) = a(b+c)/2 = (ab/2) + (ac/2) = (a \circ b) + (a \circ c)$, and so $(2\mathbf{Z}, +, \circ)$ satisfies property R6. Therefore, $(2\mathbf{Z}, +, \circ)$ is a ring. Since $a \circ b = ab/2 = ba/2 = b \circ a$, the ring $(2\mathbf{Z}, +, \circ)$ is commutative. Because $a \circ 2 = (a \cdot 2)/2 = a$ and $2 \in 2\mathbf{Z}$, the integer 2 is a unity for $(2\mathbf{Z}, +, \circ)$. Next, suppose that $a \circ b = 0$, where $a, b \in \mathbf{Z}$. Then $ab/2 = 0$ and so $ab = 0$, implying that $a = 0$ or $b = 0$. Therefore, $(2\mathbf{Z}, +, \circ)$ is an integral domain. ∎

14.19 Hint: Consider the following rings R and subrings S:

(a) $R = M_2(\mathbf{R})$; $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbf{R} \right\}$.

(b) $R = \mathbf{R}[x]$; $S = \{f \in \mathbf{R}[x] : f \text{ is a constant function}\}$.

(c) $R = \mathbf{Q} \times \mathbf{Z}$; $S = \mathbf{Q} \times \{0\}$.

14.21 Hint: First show that $\mathbf{Q}[i]$ is a subring of $\mathcal{C}$. Then show that every nonzero element of $\mathbf{Q}[i]$ is a unit.

14.23 (a) $\mathbf{Z}_n$ $(n \geq 2)$

(b) $\mathbf{Z}$

(c) $M_2(\mathbf{Z}_2)$

(d) $M_2(\mathbf{R})$

14.25 (3) occurs. Now explain your answer with justification.

Chapter 15

15.1 **Proof.** Let $\mathbf{u}, \mathbf{v} \in \mathcal{C}$ and $\alpha, \beta \in \mathbf{R}$. Then $\mathbf{u} = a + bi$ and $\mathbf{v} = c + di$, where $a, b, c, d \in \mathbf{R}$. Then $\mathbf{u} + \mathbf{v} = (a + bi) + (c + di) = (a + c) + (b + d)i$ and $\alpha\mathbf{u} = \alpha(a + bi) = \alpha a + \alpha bi$. Since $a+c, b+d, \alpha a, \alpha b \in \mathbf{R}$, it follows that $\mathbf{u}+\mathbf{v} \in \mathcal{C}$ and $\alpha\mathbf{u} \in \mathcal{C}$. Now $\mathbf{u}+\mathbf{v} = (a+c)+(b+d)i = (c+a)+(d+b)i = \mathbf{v}+\mathbf{u}$, and property 1 is satisfied. Let $\mathbf{w} = e+fi$, where $e, f \in \mathbf{R}$. Then $(\mathbf{u}+\mathbf{v})+\mathbf{w} = [(a+c)+(b+d)i]+(e+fi) = [(a+c)+e]+[(b+d)+f]i = [a+(c+e)]+[b+(d+f)]i = (a+bi)+[(c+e)+(d+f)i] = (a+bi)+[(c+di)+(e+fi)] = \mathbf{u}+(\mathbf{v}+\mathbf{w})$; so property 2 is satisfied.

Let $\mathbf{z} = 0 + 0i$. Since $\mathbf{u} + \mathbf{z} = (a + bi) + (0 + 0i) = a + bi = \mathbf{u}$, property 3 is satisfied. Let $-\mathbf{u} = (-a) + (-b)i$. Then $\mathbf{u} + (-\mathbf{u}) = (a + bi) + [(-a) + (-b)i] = 0 + 0i = \mathbf{z}$, and property 4 is satisfied. Because $\alpha(\mathbf{u} + \mathbf{v}) = \alpha[(a + bi) + (c + di)] = \alpha[(a + c) + (b + d)i] = (\alpha a+\alpha c)+(\alpha b+\alpha d)i = (\alpha a+\alpha bi)+(\alpha c+\alpha di) = \alpha(a+bi)+\alpha(c+di) = \alpha\mathbf{u}+\alpha\mathbf{v}$, property 5 is satisfied. Now $(\alpha+\beta)\mathbf{u} = (\alpha+\beta)(a+bi) = (\alpha+\beta)a+(\alpha+\beta)bi = \alpha a+\beta a+\alpha bi+\beta bi = (\alpha a+\alpha bi)+(\beta a+\beta bi) = \alpha(a+bi)+\beta(a+bi) = \alpha\mathbf{u}+\beta\mathbf{u}$. Thus property 6 is satisfied. Since $(\alpha\beta)\mathbf{u} = (\alpha\beta)(a + bi) = (\alpha\beta)a + (\alpha\beta)bi = \alpha(\beta a) + \alpha(\beta bi) = \alpha(\beta a + \beta bi) = \alpha(\beta(a + bi)) = \alpha(\beta\mathbf{u})$, property 7 is satisfied. Finally, $1 \cdot \mathbf{u} = 1(a + bi) = 1 \cdot a + 1 \cdot bi = a + bi = \mathbf{u}$, and so property 8 is satisfied. ∎

15.3 (a) Since $(1,0,0)+(0,1,0)=(1,0,0)$ and $(0,1,0)+(1,0,0)=(0,1,0)$, property 1 is not satisfied and so $\mathbf{R}^3$ is not a vector space.

(c) Let $\mathbf{v}=(1,0,0)$ and let $\mathbf{z}=(a,b,c)$ be the zero vector, where $a,b,c\in\mathbf{R}$. Then $\mathbf{v}+\mathbf{z}=(0,0,0)\neq\mathbf{v}$; so property 3 is not satisfied and $\mathbf{R}^3$ is not a vector space.

(e) Let $\mathbf{v}=(1,0,0)$. Since $1\mathbf{v}=(0,0,1)\neq\mathbf{v}$, property 8 is not satisfied and $\mathbf{R}^3$ is not a vector space.

15.5 **Proof.** Observe that $\alpha(-\mathbf{v})=\alpha((-1)\mathbf{v})=(\alpha(-1))\mathbf{v}=(-\alpha)\mathbf{v}=((-1)\alpha)\mathbf{v}=(-1)(\alpha\mathbf{v})=-(\alpha\mathbf{v})$. ∎

15.7 (a) The statement is false. Since $\mathbf{z}+\mathbf{z}=\mathbf{z}$, it follows that $-\mathbf{z}=\mathbf{z}$. ♢

15.9 (a) The set W_1 is a subspace of $\mathbf{R}^4$. **Proof.** Since $(0,0,0,0)\in W_1$, it follows that $W_1\neq\emptyset$. Let $\mathbf{u},\mathbf{v}\in W_1$ and $\alpha\in\mathbf{R}$. Then $\mathbf{u}=(a,a,a,a)$ and $\mathbf{v}=(b,b,b,b)$ for some $a,b\in\mathbf{R}$. Then $\mathbf{u}+\mathbf{v}=(a+b,a+b,a+b,a+b)$ and $\alpha\mathbf{u}=(\alpha a,\alpha a,\alpha a,\alpha a)$. Because $\mathbf{u}+\mathbf{v},\alpha\mathbf{u}\in W_1$, it follows that W_1 is a subspace of $\mathbf{R}^4$ by the Subspace Test. ∎

(c) Since $(0,0,0,1)\in W_3$ but $2(0,0,0,1)\notin W_3$, it follows that W_1 is not closed under scalar multiplication and so W_3 is not a subspace of $\mathbf{R}^4$. ♢

15.11 (a) The set U_1 is a subspace of $\mathbf{R}[x]$. **Proof.** Since the zero function f_0 defined by $f_0(x)=0$ for all $x\in\mathbf{R}$ belongs to $\mathbf{R}[x]$, it follows that $U_1\neq\emptyset$. Let $f,g\in U_1$ and $\alpha\in\mathbf{R}$. Then there exist constants a and b such that $f(x)=a$ and $g(x)=b$ for all $x\in\mathbf{R}$. Then $(f+g)(x)=f(x)+g(x)=a+b$ and $(\alpha f)(x)=\alpha f(x)=\alpha a$. Since $f+g,\alpha f\in U_1$, it follows by the Subspace Test that U_1 is a subspace of $\mathbf{R}[x]$. ∎

(b) Since the function h defined by $h(x)=x^3$ for all $x\in\mathbf{R}$ belongs to U_2, but $(0\cdot h)(x)=0\cdot h(x)=0\cdot x^3=0$ does not belong to U_2, it follows that U_2 is not closed under scalar multiplication and so U_2 is not a subspace of $\mathbf{R}[x]$. ♢

15.15 **Proof.** Since $(0,0)$, that is, $x=0$ and $y=0$, is a solution of the equation, $(0,0)\in S$ and so $S\neq\emptyset$. Let $(x_1,y_1),(x_2,y_2)\in S$ and $\alpha\in\mathbf{R}$. Then $3x_1-5y_1=0$ and $3x_2-5y_2=0$. However, $3(x_1+x_2)-5(y_1+y_2)=(3x_1-5y_1)+(3x_2-5y_2)=0$. Thus $(x_1+x_2,y_1+y_2)\in S$. Furthermore, $3(\alpha x_1)-5(\alpha y_1)=\alpha(3x_1-5y_1)=\alpha\cdot 0=0$, and so $\alpha(x_1,y_1)=(\alpha x_1,\alpha y_1)\in S$. Therefore, S is a subspace of $\mathbf{R}^2$ by the Subspace Test. ∎

15.17 $i=-\frac{1}{2}\mathbf{u}_1+\frac{1}{2}\mathbf{u}_2+\frac{1}{2}\mathbf{u}_3$.

15.19 **Proof.** Let $\mathbf{v}\in W$. Thus $\mathbf{v}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_n\mathbf{v}_n$, where $c_i\in\mathbf{R}$ for $1\leq i\leq n$. Furthermore, let $\mathbf{v}_i=a_{i1}\mathbf{w}_1+a_{i2}\mathbf{w}_2+\ldots+a_{im}\mathbf{w}_m$, where $a_{ij}\in\mathbf{R}$ for $1\leq i\leq n$ and $1\leq j\leq m$. Then

$$\mathbf{v}=[c_1\ c_2\ \ldots\ c_n]\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\\ \vdots\\ \mathbf{v}_n\end{bmatrix}=[c_1\ c_2\ \ldots\ c_n]\,A\begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\\ \vdots\\ \mathbf{w}_m\end{bmatrix},$$

where $A=\begin{bmatrix}a_{11}&a_{12}&\ldots&a_{1m}\\ a_{21}&a_{22}&\ldots&a_{2m}\\ \vdots&\vdots&\vdots&\vdots\\ a_{n1}&a_{n2}&\ldots&a_{nm}\end{bmatrix}$. Hence $\mathbf{v}$ is a linear combination of $\mathbf{w}_1,\mathbf{w}_2,\ldots,\mathbf{w}_m$ and so $\mathbf{v}\in W'$.

15.21 **Proof.** We first show that $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$. Observe that $\mathbf{u} \in \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$ and $\mathbf{v} = (-2)\mathbf{u} + 1 \cdot (2\mathbf{u} + \mathbf{v}) \in \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$. By Exercise 15.19, $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$.

Next, we show that $\langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$. Since $\mathbf{u} \in \langle \mathbf{u}, \mathbf{v} \rangle$ and $2\mathbf{u}+\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, it follows that $\langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$, again by Exercise 15.19. ∎

15.23 Hint: One possibility is to choose $\mathbf{w} = (1, 0, 0)$. Now consider $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (0, 0, 0)$, where $a, b, c \in \mathbf{R}$.

15.25 (a) The set S is not linearly independent since $1 + (-1)\sin^2 x + (-1)\cos^2 x = 0$ for all $x \in \mathbf{R}$.

(b) The set S is linearly independent. **Proof.** Let $a, b, c \in \mathbf{R}$ such that $a \cdot 1 + b \cdot \sin x + c \cdot \cos x = 0$. We show that $a = b = c = 0$. Letting $x = 0$, $x = \pi/2$, and $x = -\pi/2$, we obtain $a + c = 0$, $a + b = 0$, and $a - b = 0$, respectively. Solving these equations simultaneously, we obtain $a = b = c = 0$. ∎

15.27 Hint: Consider a proof by mathematical induction.

15.29 **Proof.** Define the mapping $T : \mathbf{R}^2 \to \mathcal{C}$ by $T(a, b) = a + bi$. Assume that $T(a, b) = T(c, d)$. Then $a + bi = c + di$, which implies that $a = c$ and $b = d$. Thus $(a, b) = (c, d)$. Hence T is one-to-one. Next let $a + bi \in \mathcal{C}$. Since $T(a, b) = a + bi$, the mapping T is onto. Therefore, T is bijective. Let $\mathbf{u} = (a, b)$ and $\mathbf{v} = (c, d)$ be vectors in $\mathbf{R}^2$ and let $\alpha \in \mathbf{R}$. Then $T(\mathbf{u} + \mathbf{v}) = T(a + c,\ b + d) = (a + c) + (b + d)i = (a + bi) + (c + di) = T(\mathbf{u}) + T(\mathbf{v})$. Also, $T(\alpha\mathbf{u}) = T(\alpha a, \alpha b) = (\alpha a) + (\alpha b)i = \alpha(a + bi) = \alpha T(\mathbf{u})$. Since T preserves both addition and scalar multiplication, T is a linear transformation. ∎

15.33 (a) $D(W) = \mathbf{R}$

(b) $D(W) = \{0\}$

(c) $\ker(T) = \mathbf{R}$.

15.35 (1) occurs.

Chapter 16

16.1 (a) property (1) is not satisfied. For example, $d(2, 1) = -1 < 0$.
property (2) is satisfied since $d(x, y) = y - x = 0$ if and only if $x = y$.
property (3) is not satisfied. For example, $d(2, 1) = -1$ and $d(1, 2) = 1$.
property (4) is satisfied since $d(x, y) + d(y, z) = (y - x) + (z - y) = z - x = d(x, z)$.

(b) Since $d(x, y) = (x - y) + (y - x) = 0$, property (1) is satisfied.
Since $d(1, 2) = 0$ and $1 \neq 2$, property (2) is not satisfied.
Since $d(x, y) = d(y, x) = 0$, property (3) is satisfied.
Since $d(x, y) + d(z, x) = d(x, z) = 0$, property (4) is satisfied.

16.3 Hint: For $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2) \geq 0 + 0 = 0$, so (1) is satisfied. If $P_1 = P_2$, then $x_1 = x_2$ and $y_1 = y_2$. Thus $d'(P_1, P_2) = d(x_1, x_2) + d(y_1, y_2) = 0 + 0 = 0$. Conversely, if $d'(P_1, P_2) = 0$, then $d(x_1, x_2) + d(y_1, y_2) = 0$. Since $d(x_1, x_2) \geq 0$ and $d(y_1, y_2) \geq 0$, it follows that $d(x_1, x_2) = 0$ and $d(y_1, y_2) = 0$. So $x_1 = x_2$ and $y_1 = y_2$, it follows that $P_1 = P_2$, and so (2) is satisfied. Now properties (3) and (4) remain to be considered.

16.5 Hint: It is straightforward to show that properties (1)-(3) are satisfied. So only property (4) needs to be investigated. Consider $d(x, y)$ for various pairs x, y of elements of A.

16.7 (a) Let $P_1 = (1, 2)$ and $P_2 = (1, 3)$. Since $d(P_1, P_2) = 0$ and $P_1 \neq P_2$, it follows that $(\mathbf{R}^2, d)$ is not a metric space.

16.9 (a) Hint: Consider beginning a proof as follows: Let O be an open set in $(\mathbf{R}^2, d)$. To show that O is open in $(\mathbf{R}^2, d')$, we show that every point $P_0 = (x_0, y_0)$ is the center of an open sphere in $(\mathbf{R}^2, d')$ that is contained in O. Since O is open in $(\mathbf{R}^2, d)$, there exists a real number $r > 0$ such that $S_r(P_0) \subseteq O$. It remains to show that $S'_r(P_0) = \{P \in \mathbf{R}^2 : d'(P, P_0) < r\}$ is contained in $S_r(P_0)$.

16.11 (a) Hint: Consider beginning a proof as follows: Let $P_0 = (x_0, y_0) \in \mathbf{R}^2$, and let $\epsilon > 0$ be given. We show that there exists $\delta > 0$ such that if $d(P, P_0) < \delta$, where $P = (x, y)$, then $d'(f(P), f(P_0)) < \epsilon$. Notice that $d(P, P_0) = |x - x_0| + |y - y_0|$ and $d'(f(P), f(P_0)) = \left|\frac{1}{2}(x - y) - \frac{1}{2}(x_0 - y_0)\right|$.

16.13 (a) No, since $X \notin S_1$.

(b) No, since $\{a, b\} \cap \{a, c\} = \{a\} \notin S_2$.

(c) Yes.

16.17 Hint: Consider beginning a proof as follows: Observe that the result is true if $|X| \leq 1$. So we may assume that $|X| \geq 2$. Assume that (X, τ) is a metric space, say (X, d). First we show that $\{a\}$ is open for every $a \in X$. Let $r = \min\{d(x, a) : x \in X - \{a\}\}$. Since $X - \{a\}$ is finite, $r > 0$. Then $S_r(a) = \{a\}$ is open. Now complete the proof of this implication.

For the converse, assume that (X, τ) is a discrete topological space. Then $\tau = \mathcal{P}(X)$. We define the "discrete" metric d on X by $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. It remains to show that every subset of X is open in (X, d).

16.19 Hint: It is useful to prove the **Lemma**: If $O_1, O_1, \ldots, O_n$ are countable sets, where $n \in \mathbf{N}$, then $\cup_{i=1}^{n} O_i$ is countable.

16.21 Let a and b be distinct real numbers, where, say $a < b$, and let O_a and O_b be open sets containing a and b, respectively. Since $(a, \infty) \subset O_a$ and $(b, \infty) \subset O_b$, it follows that $(b, \infty) \subset O_a \cap O_b$. So $O_a \cap O_b \neq \emptyset$.

16.23 We claim that $f(1) = 1$.

Proof. Let $f(a) = 1$. Since $\{1\}$ is an open set and f is continuous, it follows that $f^{-1}(\{1\}) = \{a\}$ is open. Since $1 \in \{a\}$, it follows that $a = 1$. Thus $f(1) = 1$. ∎

16.27 (3) occurs. The fact that $O_1, O_2, \ldots, O_n$ $(n \in \mathbf{N})$ are infinite sets does not imply that $\cap_{\alpha \in I} O_\alpha$ is infinite. For example, let $X = \mathbf{Z}$, $n = 2$, $O_1 = \{k \in \mathbf{Z} : k \geq 0\}$ and $O_2 = \{k \in \mathbf{Z} : k \leq 0\}$. Then O_1 and O_2 are infinite, but $O_1 \cap O_2 = \{0\}$.

16.29 (a). **Solution**: The statement is true.

Proof. Let $b \in X - \{a\}$, and let $d(b, a) = r$. Then the open sphere $S_r(b)$ is contained in $X - \{a\}$. ∎

16.31 The statement is false. Now a counterexample must be found.